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Generalizing Traub's method to a parametric iterative class for solving multidimensional nonlinear problems

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In this work, we modify the iterative structure of Traub's method to include a real parameter α. A parametric family of iterative methods is obtained as a generalization of Traub, which is also a member of it. The cubic order of convergence is proved for any value of α. Then, a dynamical analysis is performed after applying the family for solving a system cubic polynomials by means of multidimensional real dynamics. This analysis allows to select the best members of the family in terms of stability as a preliminary study to be generalized to any nonlinear function. Finally, some iterative schemes of the family are used to check numerically the previous developments when they are used to approximate the solutions of academic nonlinear problems and a chemical diffusion reaction problem.

KEYWORDS
basins of attraction, iterative methods, nonlinear systems, stability analysis

MSC CLASSIFICATION
65H10, 65P40

1 | INTRODUCTION

One of the classical problems whose resolution is mainly carried out by means of numerical analysis techniques is finding the solution of nonlinear problems due to the presence of nonlinearity in most physical processes. Nowadays, the vast majority of engineering processes are modeled mathematically by means of partial derivatives equations, integral equations, matrix equations, and so forth. These processes require the calculation of the solutions of a system of nonlinear equations of large dimensions because of the high volume of data involved and whose resolution is usually complex. The exact solution is difficult to find using analytical processes. Iterative techniques approximate the solutions.

This problem consists of finding the solution $x^* \in \mathbb{R}^n$ of a system of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and F is a sufficiently Fréchet differentiable function in an open convex set D. Fixed point iterative methods obtain an approximation to x^*. They can be described as

$$x^{(k+1)} = \Phi(x^{(k)}), \qquad k = 0, 1, 2, \ldots$$

Starting from an initial estimation $x^{(0)} \in \mathbb{R}^n$ close the solution x^*, a sequence $\{x^{(k)}\}$ of approximations to the solution is obtained.

The most commonly used iterative scheme is Newton's method with quadratic convergence. Over the last decades, several iterative algorithms [1–5] have been proposed to solve this type of multidimensional problems with higher order of convergence and, therefore, requiring fewer iterations to converge to the solution of the problem.

However, recent research focuses not only on improving the speed of convergence of the methods, but also their stability. For this purpose, iterative schemes are analyzed using complex or real dynamical tools. This analysis allows to determine the best initial estimates of the iterative process and also the most stable members of a family of methods. Cordero et al. [1] introduce a novel procedure to perform this analysis. In this study, the complex dynamical tools used so far in the analysis of scalar schemes are adapted to study the stability of iterative methods for multidimensional problems performing a multidimensional real dynamical study.

Abbasbandy et al. [6] and Darvishi et al. [7] present methods for solving systems of nonlinear equations with cubic order of convergence. We highlight the classical Traub's method [8], whose iterative expression for multidimensional problems is

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1}F(y^{(k)}), \end{aligned} \qquad k = 0, 1, 2, \ldots ,$$

and also has cubic order of convergence.

In this work, we focus on improving the stability of Traub's method. For this aim, a generalization of its iterative structure is performed resulting in a family of iterative classes described in Section 2. We perform a stability analysis of the family, comparing the basins of attraction of different members of the family with those of Traub's method and determining the most stable schemes in terms of the initial estimates considered. This analysis, using multidimensional real dynamics, is carried out in Section 3. Section 4 is devoted to verify the results obtained theoretically by means of numerical experiments. Finally, the conclusions of this work are shown.

2 | FAMILY OF ITERATIVE METHODS Tα

In this section, we generalize the iterative expression of Traub's method to a family of iterative schemes holding its order of convergence. Then, we define T_α family including a parameter $\alpha \in \mathbb{R}$, $\alpha \neq 0$, as follows:

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} + \alpha(y^{(k)} - x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - \frac{1}{\alpha^2}[F'(x^{(k)})]^{-1}\left((\alpha - 1)F(x^{(k)}) + F(z^{(k)})\right), \end{aligned} \qquad k = 0, 1, 2, \ldots \tag{1}$$

Let us note that Traub's method is the particular case $\alpha = 1$ of (1). The order of convergence of family (1) is proved in the following result.

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and let us denote by $x^* \in D$ a solution of $F(x) = 0$, such that F' is continuous and nonsingular in x^*. Then, if the initial estimation $x^{(0)}$ is close enough to x^*, the members of family T_α converge to x^* with order of convergence three for any value of $\alpha \neq 0$, being its error equation*

$$e^{(k+1)} = \left(2C_2^2 + (\alpha - 1)C_3\right) e^{(k)^3} + \mathcal{O}(e^{(k)^4}),$$

where $e^{(k)} = x^{(k)} - x^*$ is the error in each iteration and $C_j = \frac{1}{j!}[F'(x^*)]^{-1}F^{(j)}(x^*)$, $j \geq 2$.

Proof. Taylor series at x^* is

$$F(x^{(k)}) = F'(x^*)\left[e^{(k)} + C_2e^{(k)^2} + C_3e^{(k)^3} + C_4e^{(k)^4}\right] + \mathcal{O}(e^{(k)^5}), \tag{2}$$

and its derivative can be expressed as

$$F'(x^{(k)}) = F'(x^*)\left[I + 2C_2e^{(k)} + 3C_3e^{(k)^2} + 4C_4e^{(k)^3}\right] + \mathcal{O}(e^{(k)^4}).$$

From the equality $[F'(x^{(k)})]^{-1}F'(x^{(k)}) = I$,

$$[F'(x^{(k)})]^{-1} = [I - 2C_2e^{(k)} + (-3C_3 + 4C_2^2)e^{(k)^2} + (-4C_4 + 6C_2C_3 + 6C_3C_2 - 8C_2^3)e^{(k)^3}][F'(x^*)]^{-1} + \mathcal{O}(e^{(k)^4}). \quad (3)$$

From (2) and (3), the first step of family T_α is

$$\begin{aligned} y^{(k)} - x^* &= e^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ &= C_2e^{(k)^2} + (-2C_2^2 + 2C_3)e^{(k)^3} + (3C_4 - 4C_2C_3 - 3C_3C_2 + 4C_2^3)e^{(k)^4} + \mathcal{O}(e^{(k)^5}). \end{aligned}$$

Similarly, we can write

$$z^{(k)} - x^* = (1-\alpha)e^{(k)} + \alpha C_2e^{(k)^2} + \alpha(-2C_2^2 + 2C_3)e^{(k)^3} + \alpha(3C_4 - 4C_2C_3 - 3C_3C_2 + 4C_2^3)e^{(k)^4} + \mathcal{O}(e^{(k)^5}),$$

and then we get

$$\begin{aligned} F(z^{(k)}) &= F'(x^*)[z^{(k)} - x^* + C_2(z^{(k)} - x^*)^2 + C_3(z^{(k)} - x^*)^3] + \mathcal{O}(e^{(k)^5}) \\ &= F'(x^*)[(1-\alpha)e^{(k)} + (\alpha C_2 + (1-\alpha)^2C_2)e^{(k)^2} + (\alpha(-2C_2^2 + 2C_3) + 2(1-\alpha)\alpha C_2^2 + (1-\alpha)^3C_3)e^{(k)^3} \\ &\quad + (\alpha(3C_4 - 4C_2C_3 - 3C_3C_2 + 4C_2^3) + \alpha(1-\alpha)(-4C_2^3 + 4C_2C_3) + \alpha^2C_2^3 + 3(1-\alpha)^2\alpha C_3C_2 + (1-\alpha)^4C_4)e^{(k)^4}] \\ &\quad + \mathcal{O}(e^{(k)^5}). \end{aligned}$$

From the previous series, we obtain

$$\begin{aligned} [F'(x^{(k)})]^{-1}\left((\alpha-1)F(x^{(k)}) + F(z^{(k)})\right) &= \alpha^2C_2e^{(k)^2} + (\alpha^2(3-\alpha)C_3 - 4\alpha^2C_2^2)e^{(k)^3} + ((\alpha^4 - 4\alpha^3 + 6\alpha^2)C_4 \\ &\quad + \alpha^2((2\alpha - 10)C_2C_3 + 13C_2^3 + (-9 + 3\alpha)C_3C_2))e^{(k)^4} + \mathcal{O}(e^{(k)^5}). \end{aligned}$$

Finally, the error equation of family T_α turns

$$\begin{aligned} e^{(k+1)} &= y^{(k)} - x^* - \frac{1}{\alpha^2}[F'(x^{(k)})]^{-1}\left((\alpha-1)F(x^{(k)}) + F(z^{(k)})\right) \\ &= \left(2C_2^2 + (\alpha-1)C_3\right)e^{(k)^3} + ((4\alpha - \alpha^2 - 3)C_4 - 9C_2^3 + (6 - 2\alpha)C_2C_3 + (6 - 3\alpha)C_3C_2)e^{(k)^4} + \mathcal{O}(e^{(k)^5}), \end{aligned}$$

so family T_α is third-order convergent for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$. □

3 | STABILITY ANALYSIS

This section is devoted to analyze the dynamical behavior of family T_α. In the previous section, it has been shown that all the methods of the family have cubic order of convergence independently of the value of parameter $\alpha \neq 0$. However, the value of α plays an important role for selecting the best methods of the family in terms of stability. For this purpose, we will perform a stability analysis of the family when it is applied on a low-degree polynomial system using tools from real multidimensional dynamics. First of all, the basic concepts required for the study are presented.

3.1 | Basics on multidimensional real dynamics

The stability analysis of an iterative method for solving scalar problems is commonly carried out using complex dynamics. There is a classical literature [9, 10], and more recent works [11, 12] devoted to this study. The dynamics for the multidimensional case is a more recent area of research, and this analysis is approached by making a generalization from the tools of complex dynamics [1].

In order to analyze the stability of family T_α, we propose a real multidimensional dynamical analysis to determine its convergence depending on the initial estimations. For this development, we describe first the basics to understand this study. Further information and similar studies can be found in [13, 14].

The fixed-point function obtained when an iterative scheme is applied to a n-variable polynomial $p(x)$, $p:\mathbb{R}^n\to\mathbb{R}^n$, is studied. Let $R:\mathbb{R}^n\to\mathbb{R}^n$ denote the vectorial rational function associated to an iterative method on polynomial $p(x)$. We define the orbit of a point $x^{(0)}\in\mathbb{R}^n$ as the set of the successive applications of R:

$$\{x^{(0)}, R(x^{(0)}), R^2(x^{(0)}), \dots\}.$$

We classify the dynamical behavior of the orbit of a point in $\mathbb{R}^n$ depending on its asymptotic behavior. In this sense, $x^*\in\mathbb{R}^n$ is a fixed point of R when $R(x^*)=x^*$. Fixed points different from the roots of polynomial $p(x)$ are called strange fixed points. In a similar way, $x^T\in\mathbb{R}^n$ is a periodic point of orbit $T\geq 1$ when $R^T(x^T)=x^T$ and $R^t(x^T)\neq x^T$ for $t<T$.

To analyze the stability of the fixed points, we recall the following result [15]:

Theorem 2. *Let $R:\mathbb{R}^n\to\mathbb{R}^n$ be C^2. Assuming x^T is a T-periodic point and let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of $R'(x^T)$.*

1. *If all the eigenvalues λ_j have $|\lambda_j|<1$, then x^T is attracting.*
2. *If one eigenvalue λ_{j_0} satisfies $|\lambda_{j_0}|>1$, x^T is unstable, that is, repelling or saddle.*
3. *If all the eigenvalues λ_j have $|\lambda_j|>1$, then x^T is repelling.*

In addition, a fixed point is called hyperbolic when all the eigenvalues λ_j satisfy $|\lambda_j|\neq 1$. In particular, if there exists an eigenvalue λ_i such that $|\lambda_i|<1$ and an eigenvalue λ_j such that $|\lambda_j|>1$, the hyperbolic point is called saddle point. Furthermore, if all the eigenvalues are equal to zero, then the fixed point is superattracting.

We define the basin of attraction of an attracting fixed point x^*, denoted by $\mathcal{A}(x^*)$, as the set of points whose orbit tends to x^*,

$$\mathcal{A}(x^*)=\{x^{(0)}\in\mathbb{R}^n : R^m(x^{(0)})\to x^*, m\to\infty\}.$$

Finally, a critical point $x\in\mathbb{R}^n$ of R satisfies $\frac{\partial r_i(x)}{\partial x_j}=0$, for all $i,j\in\{1,2,\dots,n\}$, being $r_i(x)$ the coordinate functions of the vectorial rational operator R. Then, a superattracting fixed point will be also a critical point. The critical points that do not match with the roots of $p(x)$ are called free critical points.

3.2 | Dynamical analysis of T_α family

Now, we propose to analyze the real dynamics when family T_α is applied on a system of polynomials. We particularize this study to two dimensions in order to visualize graphically the basins of attraction in the real plane. However, the dynamical properties of an iterative method on a two-dimensional polynomial can be extended to a similar n-dimensional one. The considered system for the stability analysis is the following polynomial

$$p(x):\begin{cases} x_1^3-1=0,\\ x_2^3-1=0,\end{cases}$$

whose only real root is $(1,1)$. When family T_α is applied on the cubic polynomial $p(x)$, the resulting vectorial rational operator has the expression

$$R(x_1,x_2)=\begin{bmatrix}\frac{-\alpha+(\alpha+45)x_1^9-3(\alpha-15)x_1^6+3(\alpha-3)x_1^3}{81x_1^8}\\ \frac{-\alpha+(\alpha+45)x_2^9-3(\alpha-15)x_2^6+3(\alpha-3)x_2^3}{81x_2^8}\end{bmatrix}.$$

Proposition 1 describes the fixed points and the strange fixed points of $R(x_1,x_2)$.

Proposition 1. *The real root of $p(x)$, that is, $(1,1)$, is a fixed point of $R(x_1,x_2)$ for all $\alpha\in\mathbb{R}-\{0\}$. In addition, the operator has the following strange fixed points:*

(i) When $\alpha>-\frac{3}{4}$, $\alpha\neq 36$, there exist eight strange fixed points: $(1,s_j)$ and $(s_j,1)$, for $j=1,2$, and (s_j,s_k), for $j,k=1,2$, being s_1 and s_2 the real roots of the sixth-degree polynomial:

$$q_6(x,\alpha)=(\alpha-36)x^6+(9-2\alpha)x^3+\alpha.$$

(ii) When $\alpha = -\frac{3}{4}$*, there exist three strange fixed points:* $\left(1, \frac{1}{\sqrt[3]{7}}\right)$, $\left(\frac{1}{\sqrt[3]{7}}, 1\right)$*, and* $\left(\frac{1}{\sqrt[3]{7}}, \frac{1}{\sqrt[3]{7}}\right)$.

(iii) For $\alpha = 36$*, the strange fixed points of* $R(x_1, x_2)$ *are* $\left(1, \sqrt[3]{\frac{4}{7}}\right)$, $\left(\sqrt[3]{\frac{4}{7}}, 1\right)$*, and* $\left(\sqrt[3]{\frac{4}{7}}, \sqrt[3]{\frac{4}{7}}\right)$.

Proof. The fixed points are obtained by solving the system:

$$\left.\begin{array}{l} \dfrac{-\alpha + (\alpha+45)x_1^9 - 3(\alpha-15)x_1^6 + 3(\alpha-3)x_1^3}{81x_1^8} = x_1 \\ \dfrac{-\alpha + (\alpha+45)x_2^9 - 3(\alpha-15)x_2^6 + 3(\alpha-3)x_2^3}{81x_2^8} = x_2 \end{array}\right\} \tag{4}$$

Developing (4), we obtain for $j = 1, 2$:

$$\frac{\left(x_j^3 - 1\right)\left((\alpha-36)x_j^6 + (9-2\alpha)x_j^3 + \alpha\right)}{81x_j^8} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_j^3 - 1 = 0 \\ (\alpha-36)x_j^6 + (9-2\alpha)x_j^3 + \alpha = 0 \end{cases}$$

Therefore, $(1, 1)$ is a fixed point of $R(x_1, x_2)$ and the strange fixed points are the points $\left(x_1^F, x_2^F\right)$ different from the root of $p(x)$ and whose components are $x_j^F = 1$ $(j = 1, 2)$ and the real roots of $q_6(x, \alpha) = (\alpha-36)x_j^6 + (9-2\alpha)x_j^3 + \alpha$. Moreover, $q_6(x, \alpha)$ only has two real roots when $\alpha \geq -\frac{3}{4}$ denoted by s_1 and s_2. In addition, we have $q_6\left(x, -\frac{3}{4}\right) = -\frac{3}{4}\left(1 - 7x^3\right)^2$ and $q_6(x, 36) = 63x^3 + 36$, so $s_1 = s_2$ when $\alpha = \left\{-\frac{3}{4}, 36\right\}$ and there exist only three strange fixed points for these values of parameter α. □

From Proposition 1, the rational vectorial operator $R(x_1, x_2)$ does not have strange fixed points when $\alpha < -\frac{3}{4}$. In this case, the only fixed point is the real root of the cubic polynomial system.

The asymptotic behavior of the fixed points requires calculating the Jacobian matrix of the operator, whose expression is:

$$R'(x_1, x_2) = \begin{bmatrix} \frac{\left(x_1^3-1\right)^2\left(8\alpha+(45+\alpha)x_1^3\right)}{81x_1^9} & 0 \\ 0 & \frac{\left(x_2^3-1\right)^2\left(8\alpha+(45+\alpha)x_2^3\right)}{81x_2^9} \end{bmatrix}. \tag{5}$$

Then, the eigenvalues of $R'(x_1, x_2)$ are the components of its diagonal:

$$\lambda_1(x_1, x_2) = \frac{\left(x_1^3 - 1\right)^2\left(8\alpha + (45+\alpha)x_1^3\right)}{81x_1^9},$$

$$\lambda_2(x_1, x_2) = \frac{\left(x_2^3 - 1\right)^2\left(8\alpha + (45+\alpha)x_2^3\right)}{81x_2^9}.$$

We have $\lambda_1(1, 1) = \lambda_2(1, 1) = 0$, so $(1, 1)$ is a superattracting fixed point. The stability of the strange fixed points depends on the value of parameter α. Table 1 shows the asymptotic behavior of them depending on α.

Table 1 reveals there is a large range of values for the parameter $\alpha \geq -\frac{3}{4}$ where there are no strange fixed points whose behavior is attracting. In particular, when $\alpha \notin \left(-\frac{3}{4}, -0.74025\right)$ and $\alpha \notin (13.6778, 36)$ all the strange fixed points are repelling, saddle or non hyperbolic points.

Our aim in this dynamical analysis is to select the best members belonging to family T_α in terms of stability, that is, the values of α whose corresponding iterative methods have great stability properties. For this purpose, the information provided by the parameters lines [16] is essential, as these figures show the methods of the family converging to the roots of a nonlinear equation. Parameter lines are represented using a free critical point as initial estimation, so the critical points of the rational operator are studied first in Proposition 2.

TABLE 1 Number of strange fixed points of $R(x_1, x_2)$ being attracting (A), repelling (R), saddle (S), or non hyperbolic (NH).

α	A	R	S	NH
$\alpha = -\frac{3}{4}$	$\nexists$	$\nexists$	$\nexists$	3
$\alpha \in \left(-\frac{3}{4}, -0.74025\right)$	3	1	4	$\nexists$
$\alpha = -0.74025$	$\nexists$	1	2	5
$\alpha \in (-0.74025, 0)$	$\nexists$	3	5	$\nexists$
$\alpha \in (0, 13.6778)$	$\nexists$	3	5	$\nexists$
$\alpha = 13.6778$	$\nexists$	1	2	5
$\alpha \in (13.6778, 36)$	3	1	4	$\nexists$
$\alpha = 36$	$\nexists$	1	2	$\nexists$
$\alpha > 36$	$\nexists$	3	5	$\nexists$

FIGURE 1 Parameters line of $R(x_1, x_2)$.

Proposition 2. *The critical points of the rational operator $R(x_1, x_2)$ agree with the real root of polynomial $p(x)$ for any value of parameter α. In addition, when $\alpha \neq \{-45, -5\}$ there exist three free critical points:*

$$cr_1 = \left(1, \sqrt[3]{\frac{-8\alpha}{45+\alpha}}\right), \qquad cr_2 = \left(\sqrt[3]{\frac{-8\alpha}{45+\alpha}}, 1\right), \qquad cr_3 = \left(\sqrt[3]{\frac{-8\alpha}{45+\alpha}}, \sqrt[3]{\frac{-8\alpha}{45+\alpha}}\right).$$

Proof. From (5), the critical points are calculated solving the system:

$$\frac{\left(x_j^3 - 1\right)^2 \left(8\alpha + (45+\alpha)x_j^3\right)}{81x_j^9} = 0, \qquad j = 1, 2.$$

On the one hand, from $x_j^3 - 1 = 0$, the real root of $p(x)$ is a critical point of $R(x_1, x_2)$. On the other hand, from $8\alpha + (45+\alpha)x_j^3 = 0$, we obtain the real root $x_j = \sqrt[3]{\frac{-8\alpha}{45+\alpha}}$ when $\alpha \neq -45$. When $\alpha = -5$, we have $8\alpha + (45+\alpha)x_j^3 = x_j^3 - 1$ and then $x_j = 1$ and there are no free critical points. □

According to Proposition 2, the iterative schemes of T_α family corresponding to $\alpha = -45$ and $\alpha = -5$ do not have free critical points.

When the rational operator has free critical points, we can represent the parameters line. This plot depicts the values of parameter α whose associated methods converge to the roots of the polynomial. They are generated taking as initial estimation a free critical point and then applying successively the rational operator for each value of the parameter. When there is convergence to an attracting fixed point, the point in the real line is depicted in white color. In other case, it is represented in black. Then, we can select the values of the parameter that provide the best members of a family in terms of stability.

From Proposition 2, $R(x_1, x_2)$ has three free critical points. However, the parameter lines obtained for each of them are equal. Figure 1 shows the parameters line corresponding to any of the free critical points of the family for values of the parameter $\alpha \in [-50,50]$. The convergence is set when the difference between an iteration and $(1, 1)$ is less than 10^{-3} with a maximum of 50 iterations. As we can observe in the parameters line, when $\alpha > 0$, there is no convergence to the root of the polynomial. However, for $\alpha < 0$, there is a wide set of values of α represented in white. Then, the most stable members of family T_α are mainly those who correspond to negative values of the parameter.

In order to visualize the basin of attraction of the real root of $p(x)$, we select different values of α and represent the associated dynamical planes. In this plot, the X-axis and Y-axis represent a set of initial iterations of the method. Each point (x_1, x_2) in the plane corresponds to an initial estimation to iterate the associated method with the value of α that belongs to family T_α fixed previously. In the dynamical plane, we can observe the Fatou set, composed by all the points whose orbit tends to an attracting fixed point, and the Julia set, its complementary in the plane.

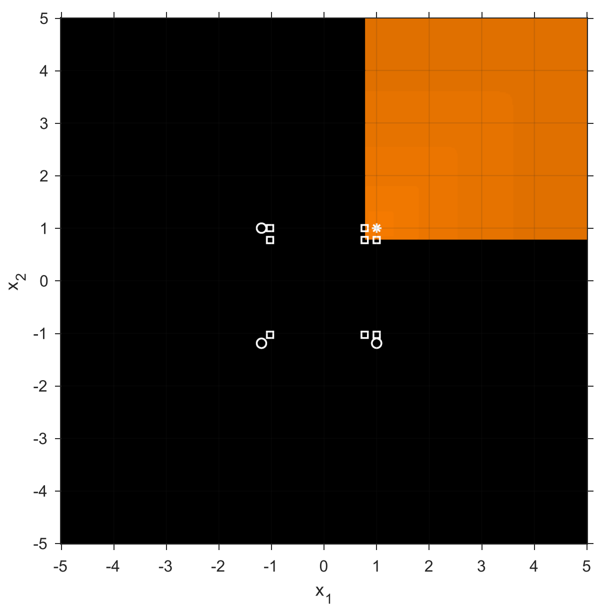
According to Propositions 1 and 2, the asymptotic behavior of the strange fixed points and the number of free critical points depends on α. Then, we have select different values for the parameter in order to compare the stability properties of different methods of family T_α. Figures 2–5 show these dynamical planes for a set of initial estimations $(x_1, x_2) \in [-5, 5] \times [-5, 5]$.

For the implementation of the dynamical planes, we set the convergence when $\|(x_1, x_2) - (1, 1)\| < 10^{-3}$ or the orbit of a point reaches 50 iterations without convergence to the root of the polynomial, which has been represented in a white star. When the orbit of a point converges to the root of $p(x)$, the point in the plane is depicted in orange, while the convergence to an strange fixed point is represented in white and otherwise, in black color. In addition, the strange fixed points and the free critical points have been represented with white squares and circles, respectively.

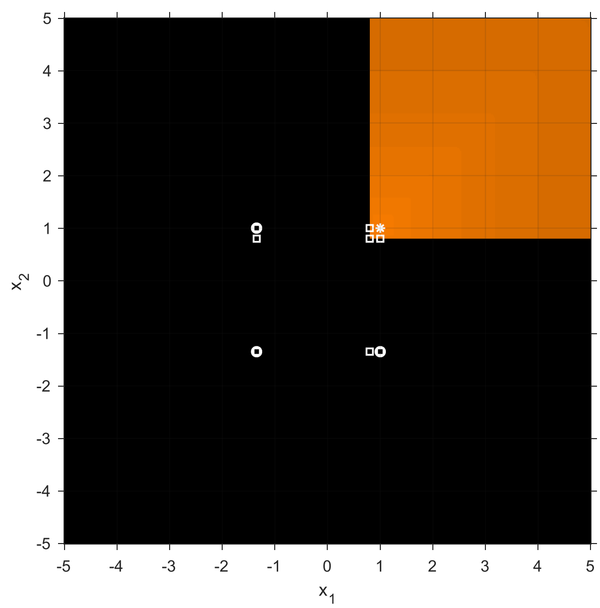
First of all, we have represented in Figure 2 the dynamical planes associated with $\alpha = 12$ and $\alpha = 20$. In both cases, the corresponding methods have eight strange fixed points, and some of them belong to the black region. In addition, there exist three free critical points, and they do not belong to the basin of attraction of the root of $p(x)$. The basin of attraction showed in the dynamical planes in Figure 2 is quite small, so the associated methods are not as stable as desirable. Furthermore, in Figure 3, it is represented on the dynamical plane associated to $\alpha = 12$ the orbit of the free critical points in yellow color. We observe that they belong to periodic orbits and do not converge to the root. Similar results are obtained for $\alpha = 20$.

Figures 2 and 3 agree with the results of Proposition 1 and 2. As it was expected, there exist attractors different to the solution, and as a consequence, the vast majority of the points are represented in black.

As family T_α is a generalization of Traub's method, it is interesting to compare the basins of attraction of this method with those of other schemes in the family to see if we can improve their stability. In Figure 4, the dynamical plane of Traub's method ($\alpha = 1$) is shown. We can see that the strange fixed points remain in the borders between the basins of



(A) $\alpha = 12$



(B) $\alpha = 20$

FIGURE 2 Dynamical planes for different values of α. [Colour figure can be viewed at wileyonlinelibrary.com]

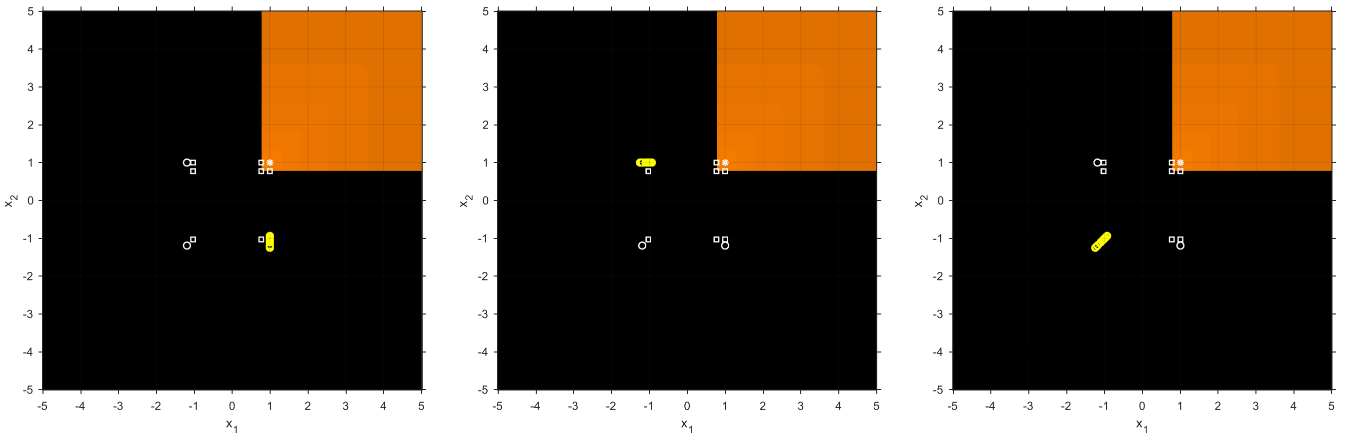


FIGURE 3 Periodic orbits of the free critical points for $\alpha = 12$. [Colour figure can be viewed at wileyonlinelibrary.com]

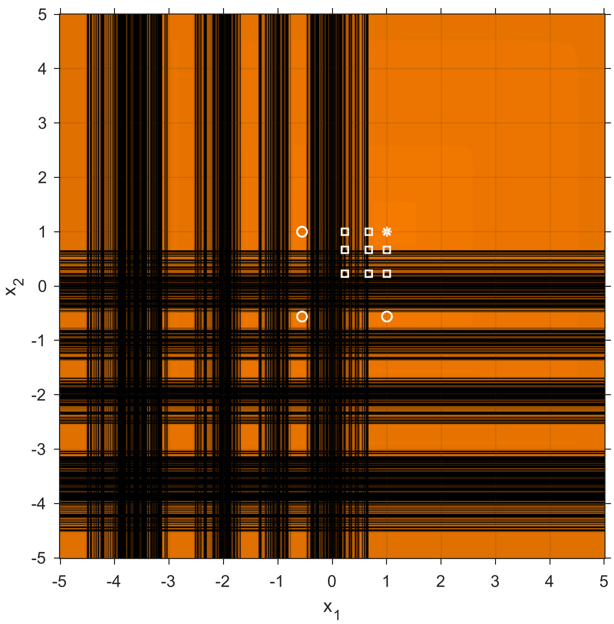


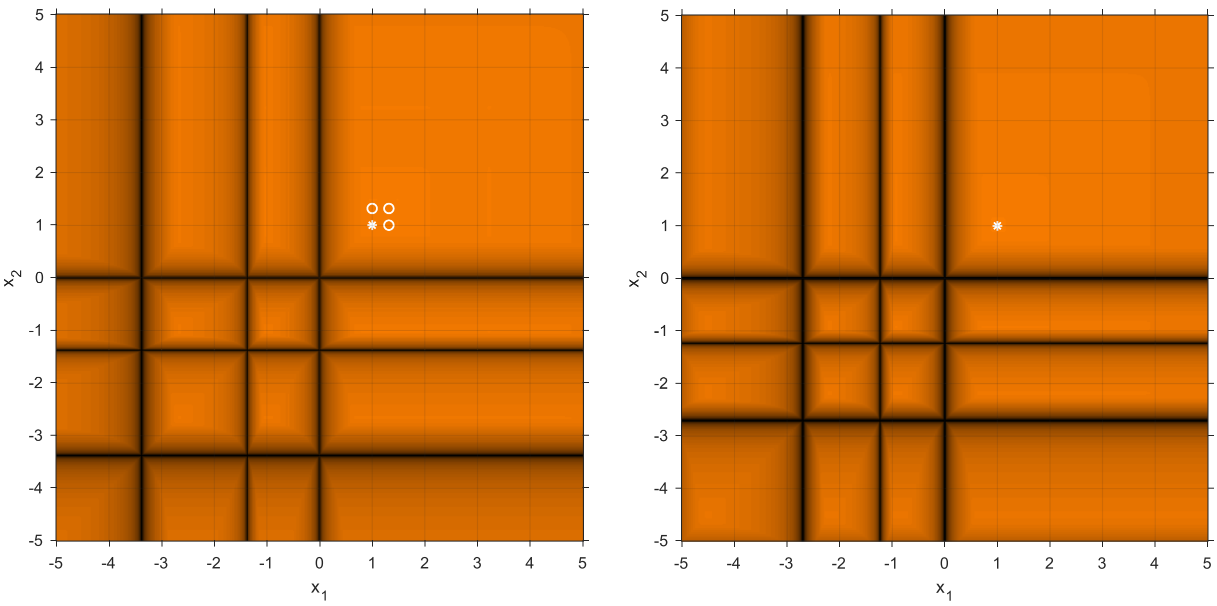
FIGURE 4 Dynamical plane of Traub's method. [Colour figure can be viewed at wileyonlinelibrary.com]

attraction (Julia set) while the critical points belong to the basin of the root of $p(x)$ (Fatou set). The set of initial estimations converging to the root is larger than for $\alpha = 12$ and $\alpha = 20$ (Figure 2), but there are still quite a few points whose orbit is divergent.

Finally, we have shown in Figure 5 the dynamical planes associated with negative values of the parameter for which there do not exist strange fixed points. Also, these values were represented in white in the parameters line (Figure 1). In addition, for $\alpha = -10$, there exist three free critical points, but they are represented in orange, and for $\alpha = -5$, there is only a fixed point and that is the root of the polynomial. In both cases, the convergence of Traub's method has been improved. Black points correspond to slow convergence, so all the points in the plane converge to the solution. This fact shows the stability of these iterative schemes belonging to family T_α, improving the previous methods and in particular Traub's one.

4 | NUMERICAL EXPERIMENTS

In this section, we test the performance of T_α family for solving different nonlinear problems. We use the numerical experiments in order to verify the results obtained in the previous sections. Let us remind that $\alpha = 1$ corresponds to Traub's iterative method. In addition, we have studied in the dynamical analysis of Section 3 that there exists a wide set of values of the parameter that improve the stability of Traub's scheme. As we can see in Figure 5, the methods corresponding to



(A) $\alpha = -10$ (B) $\alpha = -5$

FIGURE 5 Dynamical planes for different values of α. [Colour figure can be viewed at wileyonlinelibrary.com]

$\alpha = -10$ and $\alpha = -5$ have greater basins of attraction than Traub. However, Figure 4 shows that $\alpha = 12$ and $\alpha = 20$ are values whose associated methods of T_α family do not have as good dynamical properties as the previous ones. Now, we are going to compare them numerically.

In addition, to comparing the performance of different methods of the family, we also perform a numerical estimation of the cubic order of convergence proved in Section 2 by means of the approximated computational order of convergence (ACOC) presented in [17] and defined as

$$p \approx \frac{\ln\left(\|x^{(k+1)} - x^{(k)}\| \,/\, \|x^{(k)} - x^{(k-1)}\|\right)}{\ln\left(\|x^{(k)} - x^{(k-1)}\| \,/\, \|x^{(k-1)} - x^{(k-2)}\|\right)}, \qquad k = 2, 3, \ldots$$

For the numerical comparison, we have selected three nonlinear problems. The first one corresponds to a chemical application modeled by a differential equation. The other two examples correspond to academic systems of nonlinear equations.

Example 1 (**Diffusion reaction**). One of the problems that arise in chemical engineering is to predict the overall reaction rate of a porous catalyst pellet. From the conservation of mass in a spherical domain, we have

$$D\left[\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dc}{dR}\right)\right] = kf(c),\ 0 < R < r_p,$$

under conditions

$$\frac{dc}{dR}(0) = 0, \qquad c(r_p) = c_0,$$

where R is the radial coordinate, D the diffusivity, c is the concentration of a given chemical, k is the rate constant, and $f(c)$ is the reaction rate function. Consider now a sphere of γ-alumina with 5 mm in diameter on which platinum is dispersed to catalyze the dehydrogenation of cyclohexane. Then, we define

$$u = \frac{\text{concentration of cyclohexane}}{\text{concentration of cyclohexane at the surface of the sphere}}$$

and we denote by r the dimensionless radial coordinate based on the radius of the sphere ($r_p = 2.5$ mm). If we assume that the spherical pellet is isothermal, the conservation of mass equation for cyclohexane is

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = \Phi^2 u^2, \qquad 0 < r < 1, \tag{6}$$

with conditions

$$\frac{du}{dr}(0) = 0, \; u(1) = 1,$$

where Φ is set in this case as

$$\Phi = r_p\sqrt{\frac{k}{D}} \approx 2.236.$$

By using finite differences to approximate the derivatives, we are going to transform the boundary value problem (6) in a system of nonlinear equations, which will be solved by using different methods of T_α family.

First of all, the independent variable $r \in (0, 1)$ is discretized by defining a set of equispaced nodes r_i in $(0, 1)$ as

$$r_i = 0 + ih, \qquad i = 0, 1, 2, \ldots, N+1,$$

where $N+1$ is the number of subintervals defined for the independent variable after the discretization and $h = \frac{1-0}{N+1}$. Let us note that $r_0 = 0$ and $r_{N+1} = 1$.

Then, we approximate the derivatives of function u at each point r_i using central finite differences:

$$u'(r_i) \approx \frac{u(r_i+h) - u(r_i-h)}{2h}, \qquad u''(r_i) \approx \frac{u(r_i+h) - 2u(r_i) + u(r_i-h)}{h^2},$$

If we denote u_i the approximated solution at r_i, that is, $u_i \approx u(r_i)$, $i = 0, 1, \ldots, N+1$, from (6), the resulting scheme for the approximated problem after using finite differences turns into

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{2}{r_i}\frac{u_{i+1} - u_{i-1}}{2h} = \Phi^2 u_i^2.$$

After some algebraic manipulations, we obtain a system of $N+1$ nonlinear equations whose general expression is

$$\left(1 + \frac{h}{r_i}\right)u_{i+1} - 2u_i + \left(1 - \frac{h}{r_i}\right)u_{i-1} - h^2\Phi^2 u_i^2 = 0, \qquad i = 0, \ldots, N. \tag{7}$$

Let us consider in (7) the first equation ($i = 0$):

$$\left(1 + \frac{h}{r_0}\right)u_1 - 2u_0 + \left(1 - \frac{h}{r_0}\right)u_{-1} - h^2\Phi^2 u_0^2 = 0. \tag{8}$$

The value of u_{-1} can be obtained from the boundary conditions using central finite differences:

$$u'(r_0) \approx \frac{u_1 - u_{-1}}{2h} \;\Leftrightarrow\; 0 = \frac{u_1 - u_{-1}}{2h} \;\Leftrightarrow\; u_{-1} = u_1$$

so Equation (8) turns into

$$2u_1 - 2u_0 - h^2\Phi^2 u_0^2 = 0.$$

Similarly, the boundary condition $u(1) = u_{N+1} = 1$ is replaced in the last equation of the system, so that we obtain for $i = N$:

$$\left(1 + \frac{h}{r_N}\right) - 2u_N + \left(1 - \frac{h}{r_N}\right)u_{N-1} - h^2\Phi^2 u_N^2 = 0.$$

Taking into account these assumptions, the nonlinear system (7) can be written in matrix form as

$$Au - h^2\Phi^2 u^2 + B = 0, \tag{9}$$

being

$$A = \begin{pmatrix} -2 & 2 & 0 & \dots & 0 & 0 \\ 1-\frac{h}{r_1} & -2 & 1+\frac{h}{r_1} & \dots & 0 & 0 \\ 0 & 1-\frac{h}{r_2} & -2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1+\frac{h}{r_{N-1}} \\ 0 & 0 & 0 & \dots & 1-\frac{h}{r_N} & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1+\frac{h}{r_N} \end{pmatrix},$$

and the vector of unknowns of the problem

$$u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}.$$

Moreover, we denote $u^2 = \left(u_0^2, u_1^2, \dots, u_N^2\right)^T$.

In (9), a system of $N+1$ nonlinear equations has been obtained. Now, we are going to approximate its solution using different members of T_α family. Table 2 shows the numerical results obtained taking $N = 100$ and $N = 200$ points in the discretization and different initial estimations. Due to the dimensions of the problem, the methods of the family with the best dynamical behavior have been selected according to Section 3. In this sense, the results for Traub's method ($\alpha = 1$), $\alpha = -5$ and $\alpha = -10$ are shown in Table 2. The convergence is set when $\|x^{(k+1)} - x^{(k)}\| < 10^{-10}$ or $\|F(x^{(k+1)})\| < 10^{-10}$ for some iteration k of the method, with a limit of 50 iterations. We also show these values in Table 2 and the CPU times (in seconds) for each method for $N = 100$ and $N = 200$.

Table 2 shows that all three methods obtain the same numerical results, providing good approximations to the solution of the problem. The value of $\|F(x^{(k+1)})\|$ shows that some approximations reach up to an accuracy of order 10^{-27} in very few iterations. When the problem dimensions are duplicated, more CPU time is required but the methods do not require more iterations and competent results are also obtained. In addition, the ACOC is closed to three as it as expected.

TABLE 2 Numerical results for Example 1.

$x^{(0)}$	N	α	iter	$\|x^{(k+1)} - x^{(k)}\|$	$\|F(x^{(k+1)})\|$	ACOC	CPU time
$(1, 1, \dots, 1)^T$	100	1	3	1.72894e−7	2.00382e−26	2.97413	1.21999
		−5	3	1.72894e−7	2.00382e−26	2.97413	1.06999
		−10	3	1.72894e−7	2.00382e−26	2.97413	1.69999
	200	1	3	2.43011e−7	7.0977e−27	2.97363	2.98000
		−5	3	2.43011e−7	7.0977e−27	2.97363	3.05999
		−10	3	2.43011e−7	7.0977e−27	2.97363	3.00999
$(2, 2, \dots, 2)^T$	100	1	3	3.9869e−3	2.51172e−13	2.64262	1.02000
		−5	3	3.9869e−3	2.51172e−13	2.64262	1.08000
		−10	3	3.9869e−3	2.51172e−13	2.64262	1.12000
	200	1	3	5.60607e−3	8.90238e−14	2.64066	2.98000
		−5	3	5.60607e−3	8.90238e−14	2.64066	3.06999
		−10	3	5.60607e−3	8.90238e−14	2.64066	3.40999
$(-1, -1, \dots, -1)^T$	100	1	17	7.7851e−4	1.7789e−15	2.73908	3.96000
		−5	17	7.7851e−4	1.7789e−15	2.73908	3.84000
		−10	17	7.7851e−4	1.7789e−15	2.73908	4.40999
	200	1	17	1.28583e−3	1.02327e−15	2.73145	14.19999
		−5	17	1.28583e−3	1.02327e−15	2.73145	13.60999
		−10	17	1.28583e−3	1.02327e−15	2.73145	13.50999

Example 2. Next, we will try to approximate the real solution $x^* = [1, 1, 1]$ of the following polynomial system:

$$\begin{cases} x_1^2 x_2 = 1, \\ x_2^2 x_3 = 1, \\ x_3^2 x_1 = 1, \end{cases}$$

We have compared in Table 3 the results for the methods of T_α family corresponding to $\alpha \in \{20, 12, 1, -5, -10\}$. By taking initial estimations closer or away from the solution, we can observe a significant difference in the numerical results between the methods associated to $\alpha = 20$ and $\alpha = 12$ and the other ones. As it was expected, these methods require more iterations until reach the convergence and in many cases do not converge to the solution. In general, the methods that better approximate the solution of the problem are obtained for $\alpha = -5$ and $\alpha = -10$.

Example 3. The last academic example that we are going to solve numerically is the following system of nonlinear equations:

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 = 9, \\ x_1 x_2 x_3 = 1, \\ x_1 + x_2 - x_3^2 = 0. \end{cases}$$

TABLE 3 Numerical results for Example 2.

$x^{(0)}$	α	iter	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	ACOC
	20	7	3.4468e−9	3.41247e−25	2.96356
$(2.5, 2.5, 2.5)^T$	12	6	3.53247e−9	2.49783e−25	2.95692
	1	5	7.8154e−10	9.54735e−28	2.95752
	−5	4	1.38574e−7	7.0965e−28	3.64388
	−10	4	1.6475e−7	7.45282e−21	2.59328
	20	nc			
$(0.5, 0.5, 0.5)^T$	12	nc	–	–	–
	1	nc	–	–	–
	−5	7	1.91365e−8	2.58088e−31	3.70996
	−10	7	2.58086e−5	2.86543e−14	2.12856
	20	nc	–	–	–
$(-1, -1, -1)^T$	12	nc	–	–	–
	1	nc	–	–	–
	−5	6	1.58677e−10	1.22003e−39	3.82147
	−10	4	1.53382e−9	6.01406e−27	2.85118

TABLE 4 Numerical results for Example 3.

$x^{(0)}$	α	iter	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	ACOC
	20	nc	–	–	–
$(2, -1, 1)^T$	12	nc	–	–	–
	1	8	9.10664e−8	3.07005e−22	2.84549
	−5	6	2.29157e−5	2.93435e−15	2.51979
	−10	nc	–	–	–
	20	18	6.09745e−10	6.93468e−28	3.10861
$(2, 1, 3)^T$	12	nc	–	–	–
	1	4	1.88206e−10	9.33487e−30	2.86853
	−5	4	3.58795e−4	2.54159e−11	2.52643
	−10	5	1.07545e−9	7.59435e−28	2.9748
	20	6	9.84568e−8	1.58659e−21	2.9296
$(1.5, 1.5, 1.5)^T$	12	5	2.00167e−6	2.33284e−18	2.58084
	1	4	5.17859e−5	1.34402e−13	2.4762
	−5	4	1.69541e−6	2.31821e−18	2.73427
	−10	4	1.17835e−5	9.85054e−16	2.4893

The real roots of Example 3 are $x_1^* \approx [3.51979, -0.154881, -1.83437]$ and $x_2^* \approx [3.56864, 0.145403, 1.92719]$. Following the same criteria than in Example 2, the main results of Example 3 are summarized in Table 4.

In Table 4, we have obtained similar results than in Table 3, showing the stability of Traub's scheme and methods for $\alpha = -5$ and $\alpha = -10$. Although for the first initial estimate considered only are convergent the schemes for $\alpha = 1$ and $\alpha = -5$, in general, the results are of good quality since the convergent methods approximate the solution with high accuracy and few iterations.

5 | CONCLUSIONS

In this paper, a family of iterative methods has been proposed. The design of the family starts from Traub's scheme, suitably introducing a real parameter $\alpha \neq 0$. For $\alpha = 1$, we would again obtain Traub's iterative method. We have checked using Taylor series expansions that all members of the family converge cubically. Then, a dynamical analysis is performed based on the definitions of real dynamics for the multidimensional case. After this analysis we conclude that the best members of the family in terms of stability correspond to negative values of the parameter. Specifically, when $\alpha < -\frac{3}{4}$, the only fixed point of the vectorial operator under analysis is the root of the nonlinear function considered for the study. All the theoretical developments of convergence and stability are also checked numerically in the final part of the work. For this purpose, both negative and positive values of the parameter are selected to compare numerically the results obtained when the methods of the family corresponding to these values are used to approximate roots of nonlinear functions. Again, the convergence of the negative values of α improves on the other cases and a computational approximation of the order of convergence close to the theoretical cubic order is also obtained.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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