Context-Sensitive Dependency Pairs*

Beatriz Alarcón, Raúl Gutiérrez, and Salvador Lucas

DSIC, Universidad Politécnica de Valencia, Spain
{balarcon, rgutierrez, slucas}@dsic.upv.es

Abstract. Termination is one of the most interesting problems when dealing with context-sensitive rewrite systems. Although there is a good number of techniques for proving termination of context-sensitive rewriting (*CSR*), the dependency pair approach, one of the most powerful techniques for proving termination of rewriting, has not been investigated in connection with proofs of termination of *CSR*. In this paper, we show how to use dependency pairs in proofs of termination of *CSR*. The implementation and practical use of the developed techniques yield a novel and powerful framework which improves the current state-of-the-art of methods for proving termination of *CSR*.

Keywords: Dependency pairs, term rewriting, program analysis, termination.

1 Introduction

A *replacement map* is a mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f of a signature $\mathcal{F}$ [Luc98]. We use them to discriminate the argument positions on which the rewriting steps are allowed. In this way, for a given Term Rewriting System (TRS [Ohl02,Ter03]), we obtain a restriction of rewriting which we call *context-sensitive rewriting* (*CSR* [Luc98,Luc02]). In *CSR* we only rewrite μ-replacing subterms: t_i is a μ-replacing subterm of $f(t_1, \ldots, t_k)$ if $i \in \mu(f)$; every term t (as a whole) is μ-replacing by definition. With *CSR* we can *achieve* a terminating behavior with non-terminating TRSs, by pruning (all) infinite rewrite sequences. Proving termination of *CSR* has been recently recognized as an interesting problem with several applications in the fields of term rewriting and programming languages (see [DLMMU06,GM04,Luc02,Luc06] for further motivation).

Several methods have been developed for proving termination of *CSR* under a replacement map μ for a given TRS $\mathcal{R}$ (i.e., for proving the *μ-termination* of $\mathcal{R}$). In particular, a number of transformations which permit to treat termination of *CSR* as a standard termination problem have been described (see [GM04,Luc06] for recent surveys). Direct techniques like polynomial orderings

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and the context-sensitive version of the recursive path ordering have also been investigated [BLR02,GL02,Luc04b,Luc05]. Up to now, however, the *dependency pairs method* [AG00,GAO02,GTS04,HM04], one of the most powerful techniques for proving termination of rewriting, has not been investigated in connection with proofs of termination of *CSR*. In this paper, we address this problem.

Roughly speaking, given a TRS $\mathcal{R}$, the dependency pairs associated to $\mathcal{R}$ conform a new TRS $\mathsf{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so called *dependency chains* whose finiteness or infiniteness characterize termination of $\mathcal{R}$. Given a rewrite rule $l \to r$, we get dependency pairs $l^\sharp \to s^\sharp$ for all subterms s of r which are rooted by a defined symbol[1]; the notation $t^\sharp$ for a given term t means that the root symbol f of t is *marked* thus becoming $f^\sharp$ (often just capitalized: F). A chain of dependency pairs is a sequence $u_i \to v_i$ of dependency pairs such that $\sigma(v_i)$ rewrites to $\sigma(u_{i+1})$ for some substitution σ and $i \geq 1$. The dependency pairs can be presented as a *dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph. These basic intuitions are valid for *CSR*, although some important differences arise.

Example 1. Consider the following TRS $\mathcal{R}$ [GM99, Example 1]:

```
c → a        f(a,b,X) → f(X,X,X)
c → b
```

together with $\mu(\mathtt{f}) = \{3\}$. As shown by Giesl and Middeldorp, among all existing transformations for proving termination of *CSR*, only the *complete* Giesl and Middeldorp's transformation [GM04] (yielding a TRS $\mathcal{R}^\mu_C$) could be used in this case, but no concrete proof of termination for $\mathcal{R}^\mu_C$ is known yet. Furthermore, $\mathcal{R}^\mu_C$ has 13 dependency pairs and the dependency graph contains many cycles. In contrast, $\mathcal{R}$ has only *one* context-sensitive (CS-)dependency pair

```
F(a,b,X) -> F(X,X,X)
```

and the corresponding dependency graph has *no* cycle (due to the replacement restrictions, since we extend μ by $\mu(\mathtt{F}) = \{3\}$). As we show below, a direct (and automatic) proof of μ-termination of $\mathcal{R}$ is easy now.

Basically, the subterms in the right-hand sides of the rules which are considered to build the CS-dependency pairs must be *μ-replacing* terms. However, this is not sufficient to obtain a correct approximation. The following example shows the need of a new kind of dependency pairs.

Example 2. Consider the following TRS $\mathcal{R}$:

```
a -> c(f(a))
f(c(X)) -> X
```

together with $\mu(\mathtt{c}) = \varnothing$ and $\mu(\mathtt{f}) = \{1\}$. There is no μ-replacing subterm s in the right-hand sides of the rules which is rooted by a defined symbol. Thus, there is no 'regular' dependency pair. We could wrongly conclude that $\mathcal{R}$ is μ-terminating, which is not true:

$$\mathtt{f}(\underline{\mathtt{a}}) \hookrightarrow_\mu \underline{\mathtt{f(c(f(a)))}}\ \mathtt{f}(\underline{\mathtt{a}}) \hookrightarrow_\mu \cdots$$

[1] A symbol f is said to be *defined* in a TRS $\mathcal{R}$ if $\mathcal{R}$ contains a rule $f(l_1, \ldots, l_k) \to r$.

Indeed, we must add the following dependency pair

```
F(c(X)) -> X
```

which would not be allowed in Arts and Giesl's approach because the right-hand side is a variable.

After some preliminaries in Section 2, Section 3 introduces the general framework to compute and use context-sensitive dependency pairs for proving termination of *CSR*. The introduction of a new kind of dependency pairs (as in Example 2) leads to a new notion of context-sensitive dependency *chain*. We prove the correctness and completeness of the new approach, i.e., our dependency pairs approach fully characterize termination of *CSR*. We also show how to use term orderings for proving termination of *CSR* by means of the new approach. Furthermore, we are properly extending Arts and Giesl's approach: whenever $\mu(f) = \{1, \ldots, k\}$ for all k-ary symbols $f \in \mathcal{F}$, *CSR* and ordinary rewriting coincide; coherently, our results boil down into the standard results for the dependency pair approach. Section 4 shows how to compute the (estimated) context-sensitive dependency graph and investigates how to use term orderings together with the dependency graph to achieve automatic proofs of termination of *CSR* within the dependency pairs approach. Section 5 adapts Hirokawa and Middeldorp's subterm criterion [HM04] to *CSR*. Section 6 provides an experimental evaluation of our techniques. Section 7 concludes. The proofs are given in an appendix.

2 Preliminaries

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Given positions p, q, we denote their concatenation as $p.q$. If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. We denote the topmost position by Λ. The set of positions of a term t is $\mathcal{P}os(t)$. Positions of non-variable symbols in t are denoted as $\mathcal{P}os_{\mathcal{F}}(t)$, and $\mathcal{P}os_{\mathcal{X}}(t)$ are the positions of variables. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s. We write $t \trianglerighteq s$ if $s = t|_p$ for some $p \in \mathcal{P}os(t)$ and $t \triangleright s$ if $t \trianglerighteq s$ and $t \neq s$. The symbol labelling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with zero or more 'holes' $\Box$ (a fresh constant symbol).

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Context-sensitive rewriting. A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [Luc98]. Let $M_{\mathcal{F}}$ be the set of all

$\mathcal{F}$-maps (or $M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $(\mathcal{F}, R)$). A binary relation R on terms is μ-monotonic if $t\,R\,s$ implies $f(t_1,\ldots,t_{i-1},t,\ldots,t_k)\,R\,f(t_1,\ldots,t_{i-1},s,\ldots,t_k)$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and $t,s,t_1,\ldots,t_k \in \mathcal{T}(\mathcal{F},\mathcal{X})$. The set of *$\mu$-replacing positions* $\mathcal{P}os^\mu(t)$ of $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ is: $\mathcal{P}os^\mu(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^\mu(t) = \{\Lambda\} \cup \bigcup_{i\in\mu(root(t))} i.\mathcal{P}os^\mu(t|_i)$, if $t \notin \mathcal{X}$. The set of *replacing* variables of t is $\mathcal{V}ar^\mu(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^\mu(t), t|_p = x\}$. The μ-replacing subterm relation $\trianglerighteq_\mu$ is given by $t \trianglerighteq_\mu s$ if there is $p \in \mathcal{P}os^\mu(t)$ such that $s = t|_p$. We write $t \triangleright_\mu s$ if $t \trianglerighteq_\mu s$ and $t \neq s$. In *context-sensitive rewriting* (*CSR* [Luc98]), we (only) contract *replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_\mu s$ (or $t \hookrightarrow_{\mathcal{R},\mu} s$ and even $t \hookrightarrow s$), if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^\mu(t)$. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_\mu$ is terminating. A term t is μ-terminating if there is no infinite μ-rewrite sequence $t = t_1 \hookrightarrow_\mu t_2 \hookrightarrow_\mu \cdots \hookrightarrow_\mu t_n \hookrightarrow_\mu \cdots$ starting from t. A pair $(\mathcal{R},\mu)$ where $\mathcal{R}$ is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS.

Dependency pairs. Given a TRS $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ a new TRS $\mathsf{DP}(\mathcal{R}) = (\mathcal{F}^\sharp, D(R))$ of *dependency pairs* for $\mathcal{R}$ is given as follows: if $f(t_1,\ldots,t_m) \to r \in R$ and $r = C[g(s_1,\ldots,s_n)]$ for some defined symbol $g \in \mathcal{D}$ and $s_1,\ldots,s_n \in \mathcal{T}(\mathcal{F},\mathcal{X})$, then $f^\sharp(t_1,\ldots,t_m) \to g^\sharp(s_1,\ldots,s_n) \in D(R)$, where $f^\sharp$ and $g^\sharp$ are new fresh symbols (called *tuple* symbols) associated to defined symbols f and g respectively [AG00]. Let $\mathcal{D}^\sharp$ be the set of tuple symbols associated to symbols in $\mathcal{D}$ and $\mathcal{F}^\sharp = \mathcal{F} \cup \mathcal{D}^\sharp$. As usual, for $t = f(t_1,\ldots,t_k) \in \mathcal{T}(\mathcal{F},\mathcal{X})$, we write $t^\sharp$ to denote the *marked* term $f^\sharp(t_1,\ldots,t_k)$. Conversely, given a marked term $t = f^\sharp(t_1,\ldots,t_k)$, where $s_1,\ldots,s_k \in \mathcal{T}(\mathcal{F},\mathcal{X})$, we write $t^\natural$ to denote the term $f(t_1,\ldots,t_k) \in \mathcal{T}(\mathcal{F},\mathcal{X})$. Given $T \subseteq \mathcal{T}(\mathcal{F},\mathcal{X})$, let $T^\sharp$ be the set $\{t^\sharp \mid t \in T\}$.

A reduction pair $(\succsim, \sqsupset)$ consists of a stable and weakly monotonic quasi-ordering $\succsim$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\succsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succsim \subseteq \sqsupset$. Note that *monotonicity is not required* for $\sqsupset$.

3 Context-Sensitive Dependency Pairs

Let $\mathcal{M}_{\infty,\mu}$ be a set of minimal non-μ-terminating terms in the following sense: t belongs to $\mathcal{M}_{\infty,\mu}$ if t is non-μ-terminating and every strict *μ-replacing* subterm s (i.e., $t \triangleright_\mu s$) is μ-terminating. Obviously, if $t \in \mathcal{M}_{\infty,\mu}$, then $root(t)$ is a defined symbol. The following proposition establishes that, given a minimal non-μ-terminating term $t \in \mathcal{M}_{\infty,\mu}$, there are two ways for an infinite μ-rewrite sequence to proceed. The first one is by using 'visible' parts of the rules which correspond to μ-replacing subterms in the right-hand sides which are rooted by a defined symbol. The second one is by showing up 'hidden' non-μ-terminating subterms which are activated by *migrating* variables in a rule $l \to r$, i.e., variables $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ which are *not*μ-replacing in the left-hand side l but become μ-replacing in the right-hand side r.

Proposition 1. *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Then for all $t \in \mathcal{M}_{\infty,\mu}$, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{M}_{\infty,\mu}$ such that $t \overset{>\Lambda}{\hookrightarrow^*} \sigma(l) \xrightarrow{\Lambda} \sigma(r) \trianglerighteq_\mu u$ and either*

1. *there is a μ-replacing subterm s of r such that $u = \sigma(s)$, or*
2. *there is $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$ such that $\sigma(x) \trianglerighteq_{\mu} u$.*

Proposition 1 motivates the following.

Definition 1. *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) \cup \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ be the set of* context-sensitive *dependency pairs (CS-DPs) where:*

$$\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) = \{l^{\sharp} \to s^{\sharp} \mid l \to r \in R, r \trianglerighteq_{\mu} s, root(s) \in \mathcal{D}, l \not\triangleright_{\mu} s\}$$

and

$$\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \{l^{\sharp} \to x \mid l \to r \in R, x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)\}$$

where $\mu^{\sharp}(f) = \mu(f)$ if $f \in \mathcal{F}$, and $\mu^{\sharp}(f^{\sharp}) = \mu(f)$ if $f \in \mathcal{D}$.

A rule $l \to r$ of a CS-TRS $(\mathcal{R},\mu)$ is μ-conservative if $\mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)$, i.e., it does not contain migrating variables; $(\mathcal{R},\mu)$ is μ-conservative if all its rules are (see [Luc06]). The following result is immediate from Definition 1.

Proposition 2. *For μ-conservative CS-TRSs $(\mathcal{R},\mu)$, we have that $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$.*

Therefore, in order to deal with μ-conservative TRSs $\mathcal{R}$ we only need to consider the 'classical' dependency pairs in $\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ (which are a subset of Arts and Giesl's dependency pairs: $\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) \subseteq \mathsf{DP}(\mathcal{R})$).

Example 3. Consider the TRS $\mathcal{R}$:

```
g(X) -> h(X)       h(d) -> g(c)       c -> d
```

together with $\mu(\mathtt{g}) = \mu(\mathtt{h}) = \varnothing$ [Zan97, Example 1]. $\mathsf{DP}(\mathcal{R},\mu)$ is:

```
G(X) -> H(X)       H(d) -> G(c)
```

with $\mu^{\sharp}(\mathtt{G}) = \mu^{\sharp}(\mathtt{H}) = \varnothing$.

If the TRS $\mathcal{R}$ contains non-μ-conservative rules, then we also need to consider dependency pairs with variables in the right-hand side.

Example 4. Consider the TRS $\mathcal{R}$ [Zan97, Example 5]:

```
if(true,X,Y) -> X        f(X) -> if(X,c,f(true))
if(false,X,Y) -> Y
```

with $\mu(\mathtt{if}) = \{1,2\}$. Then, $\mathsf{DP}(\mathcal{R},\mu)$ is:

```
F(X) -> IF(X,c,f(true))        IF(false,X,Y) -> Y
```

with $\mu^{\sharp}(\mathtt{F}) = \{1\}$ and $\mu(\mathtt{IF}) = \{1,2\}$.

Now we introduce the notion of chain of CS-DPs.

Definition 2 (Chain of CS-DPs). *Given a CS-TRS $(\mathcal{P},\mu^{\sharp})$ of dependency pairs $\mathcal{P} \subseteq \mathsf{DP}(\mathcal{R},\mu)$ associated to a CS-TRS $(\mathcal{R},\mu)$, an $(\mathcal{R},\mathcal{P},\mu^{\sharp})$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in \mathcal{P}$, for $i \geq 1$ such that there is a substitution σ satisfying both:*

1. $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$, *if* $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, *and*
2. *if* $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, *then there is* $s_i \in \mathcal{T}(\mathcal{F},\mathcal{X})$ *such that* $\sigma(x_i) \trianglerighteq_\mu s_i$ *and* $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$.

for $i \geq 1$. *Here, as usual we assume that different occurrences of dependency pairs do not share any variable (renamings are used if necessary).*

We say that an $(\mathcal{R},\mathcal{P},\mu^\sharp)$*-chain with* $u_1 \to v_1 \in \mathcal{P}$ *as heading dependency pair is* minimal *if* $\sigma(u_1)^\natural \in \mathcal{M}_{\infty,\mu}$ *and all dependency pairs in* $\mathcal{P}$ *occur infinitely often.*

Remark 1. When an $(\mathcal{R},\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp)$-chain is written for a given substitution σ, we write $\sigma(u) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} \sigma(v)$ for steps which use a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ but we rather write $\sigma(u) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} s^\sharp$ for steps which use a dependency pair $u \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, where s is as in Definition 2.

In the following, we use $\mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ to denote the subset of dependency pairs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ whose migrating variables occur on non-μ-replacing immediate subterms in the left-hand side:

$$\mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu) = \{f^\sharp(u_1,\ldots,u_k) \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \mid \exists i, 1 \leq i \leq k, i \notin \mu(f^\sharp), x \in \mathcal{V}ar(u_i)\}$$

For instance, $\mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ for the CS-TRS $(\mathcal{R},\mu)$ in Example 4. For this subset of CS-dependency pairs, we have the following.

Proposition 3. *There is no infinite* $(\mathcal{R},\mathcal{P},\mu^\sharp)$*-chain with* $\mathcal{P} \subseteq \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$.

The following result establishes the correctness of the context-sensitive depenency pairs approach.

Theorem 1 (Correctness). *Let* $\mathcal{R}$ *be a TRS and* $\mu \in M_{\mathcal{R}}$. *If there is no infinite* $(\mathcal{R},\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp)$*-chain, then* $\mathcal{R}$ *is* μ*-terminating.*

As an immediate consequence of Theorem 1 and Proposition 3, we have the following.

Corollary 1. *Let* $\mathcal{R}$ *be a TRS and* $\mu \in M_{\mathcal{R}}$. *If* $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, *then* $\mathcal{R}$ *is* μ*-terminating.*

Example 5. Consider the following TRS $\mathcal{R}$ [Luc98, Example 15]

```
and(true,X) -> X            first(0,X) -> nil
and(false,Y) -> false       first(s(X),cons(Y,Z)) -> cons(Y,first(X,Z))
if(true,X,Y) -> X           from(X) -> cons(X,from(s(X)))
if(false,X,Y) -> Y
add(0,X) -> X
add(s(X),Y) -> s(add(X,Y))
```

with $\mu(\texttt{cons}) = \mu(\texttt{s}) = \mu(\texttt{from}) = \varnothing$, $\mu(\texttt{add}) = \mu(\texttt{and}) = \mu(\texttt{if}) = \{1\}$, and $\mu(first) = \{1,2\}$. Then, $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ is:

```
ADD(0,X) -> X          IF(true,X,Y) -> X
AND(true,X) -> X       IF(false,X,Y) -> Y
```

Thus, by Corollary 1 we conclude the μ-termination of $\mathcal{R}$.

Now we prove that the previous CS-dependency pairs approach is not only correct but also complete for proving termination of *CSR*.

Theorem 2 (Completeness). *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then there is no infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain.*

Corollary 2 (Characterization of μ-termination). *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. $\mathcal{R}$ is μ-terminating iff there is no infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain.*

In the dependency pairs approach, the absence of infinite chains is checked by finding a *reduction pair* $(\succeq, \sqsupset)$ which is compatible with the rules and the dependency pairs [AG00]. In our setting, we can relax the monotonicity requirements and use *μ-reduction pairs* $(\gtrsim, \sqsupset)$ where $\gtrsim$ is a stable and μ-monotonic quasi-ordering which is compatible with the well-founded and stable ordering $\sqsupset$, i.e., $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$. The following result shows how to use μ-reduction pairs for proving μ-termination. This is the context-sensitive counterpart of [AG00, Theorem 7]; however, a number of remarkable differences arise due to the treatment of the dependency pairs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$. Basically, we need to ensure that the quasi-ordering is able to 'look' for a μ-replacing subterm inside the instantiation $\sigma(x)$ of a migrating variable x (hence we require $\trianglerighteq_\mu \subseteq \gtrsim$) and also connect a term which is rooted by defined symbol f and the corresponding dependency pair which is rooted by $f^\sharp$ (hence the requirement $f(x_1,\ldots,x_k) \gtrsim f^\sharp(x_1,\ldots,x_k)$).

Theorem 3. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$. Then, $\mathcal{R}$ is μ-terminating if and only if there is a μ-reduction pair $(\gtrsim, \sqsupset)$ such that,*

1. *$l \gtrsim r$ for all $l \to r \in R$,*
2. *$u \sqsupset v$ for all $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, and*
3. *whenever $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \neq \varnothing$ we have that $\trianglerighteq_\mu \subseteq \gtrsim$, where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$, and*
   (a) $u\ (\gtrsim \cup \sqsupset)\ v$ for all $u \to v \in \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, $u \sqsupset v$ for all $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) - \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, and $f(x_1,\ldots,x_k) \gtrsim f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$, or
   (b) $u(\gtrsim \cup \sqsupset)v$ for all $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ and $f(x_1,\ldots,x_k) \sqsupset f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$.

4 Context-sensitive dependency graph

As noticed by Arts and Giesl, the analysis of infinite sequences of dependency pairs can be made by looking at (the cycles $\mathfrak{C}$ of) the *dependency graph* associated to the TRS $\mathcal{R}$. The nodes of the dependency graph are the dependency pairs in $\mathsf{DP}(\mathcal{R})$; there is an arc from a dependency pair $u \to v$ to a dependency pair $u' \to v'$ if there are substitutions σ and θ such that $\sigma(v) \to^*_{\mathcal{R}} \theta(u')$.

Similarly, in the *context-sensitive (CS-)dependency graph*:

1. There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ to a dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R},\mu)$ if there are substitutions σ and θ such that $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \theta(u')$.

2. There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ to *each* dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R},\mu)$.

Note that the use of $\mu^\sharp$ (which restricts reductions on the arguments of the dependency pair symbols $f^\sharp$) is essential: given a set of dependency pairs associated to a CS-TRS $(\mathcal{R},\mu)$, we have less arcs between them due to the presence of such replacement restrictions.

Example 6. Consider the CS-TRS in Example 1. $\mathsf{DP}(\mathcal{R},\mu)$ is:

```
F(a,b,X) -> F(X,X,X)
```

with $\mu^\sharp(\texttt{F}) = \{3\}$. Although the dependency graph contains a cycle (due to $\sigma(\texttt{F(X,X,X)}) \to^* \sigma(\texttt{F(a,b,Y)})$ for $\sigma(X) = \sigma(Y) = \texttt{c}$), the CS-dependency graph contains *no* cycle because it is *not* possible to $\mu^\sharp$-reduce $\theta(\texttt{F(X,X,X)})$ into $\theta(\texttt{F(a,b,Y)})$ for any substitution θ (due to $\mu^\sharp(\texttt{F}) = \{3\}$).

As noticed by Arts and Giesl, the presence of an infinite chain of dependency pairs correspond to a cycle in the dependency graph (but not vice-versa).

Again, as an immediate consequence of Theorem 1 and Proposition 3, we have the following.

Corollary 3. *Let $\mathcal{R}$ be a TRS, $\mu \in M_{\mathcal{R}}$ and $\mathfrak{C} \subseteq \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$. Then, there is no minimal $(\mathcal{R},\mathfrak{C},\mu^\sharp)$-chain.*

According to this, and continuing Example 6, we conclude the μ-termination of $\mathcal{R}$ in Example 1.

4.1 Estimating the CS-dependency graph

In general, the (context-sensitive) dependency graph of a TRS is *not* computable and we need to use some approximation of it. Following [AG00], we describe how to approximate the CS-dependency graph of a CS-TRS $(\mathcal{R},\mu)$. Let $\textsc{Cap}^\mu$ be given as follows: let D be a set of defined symbols (in our context, $D = \mathcal{D} \cup \mathcal{D}^\sharp$):

$$\textsc{Cap}^\mu(x) = x \quad \text{if } x \text{ is a variable}$$
$$\textsc{Cap}^\mu(f(t_1,\ldots,t_k)) = \begin{cases} y & \text{if } f \in D \\ f([t_1]^f_1,\ldots,[t_k]^f_1) & \text{otherwise} \end{cases}$$

where y is intended to be a new, fresh variable which has not yet been used and given a term s, $[s]^f_i = \textsc{Cap}^\mu(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$. Let $\textsc{Ren}^\mu$ given by: $\textsc{Ren}^\mu(x) = y$ if x is a variable and $\textsc{Ren}^\mu(f(t_1,\ldots,t_k)) = f([t_1]^f_1,\ldots,[t_k]^f_k)$ for evey k-ary symbol f, where given a term $s \in \mathcal{T}^\sharp(\mathcal{F},\mathcal{X})$, $[s]^f_i = \textsc{Ren}^\mu(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$. Then, we have an arc from $u_i \to v_i$ to $u_j \to v_j$ if $\textsc{Ren}^\mu(\textsc{Cap}^\mu(v_i))$ and u_j unify; following [AG00], we say that v_i and u_j are *μ-connectable*. The following result whose proof is similar to that of [AG00, Theorem 21] (we only need to take into account the replacement restrictions indicated by the replacement map μ) formalizes the correctness of this approach.

Proposition 4. *Let $(\mathcal{R},\mu)$ be a CS-TRS. If there is an arc from $u \to v$ to $u' \to v'$ in the CS-dependency graph, then v and u' are μ-connectable.*

Example 7. (Continuing Ex. 6) Since $\textsc{Ren}^{\mu^\sharp}(\textsc{Cap}^{\mu^\sharp}(\texttt{F(X,X,X)})) = \texttt{F(X,X,Z)}$ and `F(a,b,Y)` do not unify, we conclude (and this can be easily implemented) that the CS-dependency graph for the CS-TRS $(\mathcal{R},\mu)$ in Example 1 contains no cycles.

4.2 Checking μ-termination with the dependency graph

For the cycles in the dependency graph, the absence of infinite chains is checked by finding (possibly different) *reduction pairs* $(\succeq_{\mathfrak{C}}, \sqsupset_{\mathfrak{C}})$ for each cycle $\mathfrak{C}$ [GAO02, Theorem 3.5]. In our setting, we use μ-reduction pairs.

Theorem 4 (Use of the CS-dependency graph). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$. Then, $\mathcal{R}$ is μ-terminating if and only if for each cycle $\mathfrak{C}$ in the context-sensitive dependency graph there is a μ-reduction pair $(\gtrsim_{\mathfrak{C}}, \sqsupset_{\mathfrak{C}})$ such that, $R \subseteq \gtrsim_{\mathfrak{C}}$, $\mathfrak{C} \subseteq \gtrsim_{\mathfrak{C}} \cup \sqsupset_{\mathfrak{C}}$, and*

1. *If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \varnothing$, then $\mathfrak{C} \cap \sqsupset_{\mathfrak{C}} \neq \varnothing$*
2. *If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \neq \varnothing$, then $\trianglerighteq_\mu \subseteq \gtrsim_{\mathfrak{C}}$ (where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$), and*
   (a) $\mathfrak{C} \cap \sqsupset_{\mathfrak{C}} \neq \varnothing$ and $f(x_1,\ldots,x_k) \gtrsim_{\mathfrak{C}} f^\sharp(x_1,\ldots,x_k)$ for all $f^\sharp$ in $\mathfrak{C}$, or
   (b) $f(x_1,\ldots,x_k) \sqsupset_{\mathfrak{C}} f^\sharp(x_1,\ldots,x_k)$ for all $f^\sharp$ in $\mathfrak{C}$.

Following Hirokawa and Middeldorp, the practical use of Theorem 4 concerns the so-called *strongly connected components*(SCCs) of the dependency graph, rather than the cycles themselves (which are exponentially many) [HM04,HM05]. A strongly connected component in the (CS-)dependency graph is a *maximal cycle*, i.e., it is not contained in any other cycle. According to Hirokawa and Middeldorp, the adaptation of the subterm criterion to the CS-dependency graph recursively applies as follows: when considering an SCC $\mathfrak{C}$, we *remove* from $\mathfrak{C}$ those pairs $u \to v$ satisfying $u \sqsupset v$. Then, we recompute the SCCs with the remaining pairs in the CS-dependency graph and start again (see [HM05, Section 4]). In our setting, it is not difficult to see that, if the condition $f(x_1,\ldots,x_k) \sqsupset_{\mathfrak{C}} f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$ holds for a given cycle $\mathfrak{C}$, then we can remove from $\mathfrak{C}$ *all* dependency pairs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, thus continuing from $\mathfrak{C} - \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$.

Example 8. Consider the CS-TRS $(\mathcal{R},\mu)$ in Example 4 and $\mathsf{DP}(\mathcal{R},\mu)$:

```
F(X) -> IF(X,c,f(true))
IF(false,X,Y) -> Y
```

with $\mu^\sharp(\texttt{F}) = \{1\}$ and $\mu^\sharp(\texttt{IF}) = \{1,2\}$. These two CS-dependency pairs form the only cycle in the CS-dependency graph. The μ-reduction pair $(\geq, >)$ induced by the polynomial interpretation

$$[\texttt{c}] = [\texttt{true}] = 0 \qquad [\texttt{f}](x) = x \qquad [\texttt{F}](x) = x$$
$$[\texttt{false}] = 1 \qquad [\texttt{if}](x,y,z) = x+y+z \qquad [\texttt{IF}](x,y,z) = x+z$$

can be used to prove the μ-termination of $\mathcal{R}$. First, as required by Theorem 4:

$$\begin{array}{rcl} [\texttt{f(X)}] = & X & \geq X = [\texttt{X}] \\ [\texttt{if(X,Y,Z)}] = & X+Y+Z & \geq X = [\texttt{X}] \\ [\texttt{if(X,Y,Z)}] = & X+Y+Z & \geq Y = [\texttt{Y}] \\ [\texttt{f(X)}] = & X & \geq X = [\texttt{F(X)}] \\ [\texttt{if(X,Y,Z)}] = & X+Y+Z & \geq X+Z = [\texttt{IF(X,Y,Z}] \end{array}$$

Now we have:

$$\begin{array}{rcl} [\texttt{f(X)}] = & X & \geq X = [\texttt{if(X,c,f(true))}] \\ [\texttt{if(true,X,Y)}] = & X+Y & \geq X = [\texttt{X}] \\ [\texttt{if(false,X,Y)}] = & X+Y & \geq Y = [\texttt{Y}] \\ [\texttt{F(X)}] = & X & \geq X = [\texttt{IF(X,c,f(true))}] \\ [\texttt{IF(false,X,Y)}] = & Y+1 & > Y = [\texttt{Y}] \end{array}$$

This permits to remove the dependency pair `IF(false,X,Y) -> Y` from the cycle. The dependency pair `F(X) -> IF(X,c,f(true))` itself conforms no cycle. Thus, the μ-termination of $\mathcal{R}$ is proved.

On the other hand, the use of *argument filterings*, which is standard in the current formulations of the dependency pairs method, also adapts without changes to the context-sensitive setting. This is a simple consequence of [AG00, Theorem 11] (using μ-monotonicity instead of monotonicity for the quasi-orderings is not a problem).

5 Subterm criterion

In [HM04], Hirokawa and Middeldorp introduce a very interesting *subterm criterion* which permits to ignore certain cycles of the dependency graph.

Definition 3. [HM04] *Let $\mathcal{R}$ be a TRS and $\mathfrak{C} \subseteq \mathsf{DP}(\mathcal{R})$ such that every dependency pair symbol in $\mathfrak{C}$ has positive arity. A* simple projection *for $\mathfrak{C}$ is a mapping π that assigns to every k-ary dependency pair symbol $f^\sharp$ in $\mathfrak{C}$ an argument position $i \in \{1,\dots,k\}$. The mapping that assigns to every term $f^\sharp(t_1,\dots,t_k) \in \mathcal{T}^\sharp(\mathcal{F},\mathcal{X})$ with $f^\sharp$ a dependency pair symbol in $\mathcal{R}$ its argument position $\pi(f^\sharp)$ is also denoted by π.*

In the following result, for a simple projection π and $\mathfrak{C} \subseteq \mathsf{DP}(\mathcal{R},\mu)$, we let $\pi(\mathfrak{C}) = \{\pi(u) \to \pi(v) \mid u \to v \in \mathfrak{C}\}$. Note that $u, v \in \mathcal{T}^\sharp(\mathcal{F},\mathcal{X})$, but $\pi(u), \pi(v) \in \mathcal{T}(\mathcal{F},\mathcal{X})$.

Theorem 5. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $\mathfrak{C} \subseteq \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ be a cycle. If there exists a simple projection π for $\mathfrak{C}$ such that $\pi(\mathfrak{C}) \subseteq \trianglerighteq_\mu$, and $\pi(\mathfrak{C}) \cap \triangleright_\mu \neq \varnothing$, then there is no minimal $(\mathcal{R},\mathfrak{C},\mu^\sharp)$-chain.*

Note that the result is restricted to cycles which do *not* include dependency pairs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$. The following result provides a kind of generalization of the subterm criterion to simple projections which only consider *non-μ-replacing* arguments of tuple symbols.

Theorem 6. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$ and $\mathfrak{C} \subseteq \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$ be a cycle. Let $\gtrsim$ be a stable quasi-ordering on terms whose strict and stable part $>$ is well-founded and π be a simple projection for $\mathfrak{C}$ such that for all $f^\sharp$ in $\mathfrak{C}$, $\pi(f^\sharp) \notin \mu^\sharp(f^\sharp)$ and $\pi(\mathfrak{C}) \subseteq \gtrsim$.*

1. *If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu) = \varnothing$ and $\mathfrak{C} \cap > \neq \varnothing$, then there is no minimal $(\mathcal{R}, \mathfrak{C}, \mu^\sharp)$-chain.*
2. *If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu) \neq \varnothing$, $\trianglerighteq_\mu \subseteq \gtrsim$ (where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F}, \mathcal{X})$), and*
   (a) $\mathfrak{C} \cap > \neq \varnothing$ and $f(x_1, \ldots, x_k) \gtrsim x_{\pi(f^\sharp)}$ for all $f \in \mathcal{D}$ such that $f^\sharp$ is in $\mathfrak{C}$, or
   (b) $f(x_1, \ldots, x_k) > x_{\pi(f^\sharp)}$ for all $f \in \mathcal{D}$ such that $f^\sharp$ is in $\mathfrak{C}$,
   then there is no minimal $(\mathcal{R}, \mathfrak{C}, \mu^\sharp)$-chain.

Example 9. Consider the CS-TRS $(\mathcal{R}, \mu)$ in Example 3. $\mathsf{DP}(\mathcal{R}, \mu)$ is:

```
G(X) -> H(X)
H(d) -> G(c)
```

where $\mu^\sharp(\mathtt{G}) = \mu^\sharp(\mathtt{H}) = \varnothing$. Note that the dependency graph contains a single cycle including both of them. The only simple projection is $\pi(\mathtt{G}) = \pi(\mathtt{H}) = 1$. Since $\pi(\mathtt{G(X)}) = \pi(\mathtt{H(X)})$, we only need to guarantee that $\pi(\mathtt{H(d)}) = \mathtt{d} > \mathtt{c} = \pi(\mathtt{G(c)})$ holds for a stable and well-founded ordering $>$. This is easily fulfilled by, e.g., a polynomial ordering.

6 Experiments

We have implemented the techniques described in the previous sections as part of the tool MU-TERM [Luc04a]. Following Theorem 1, given a CS-TRS $(\mathcal{R}, \mu)$, the tool automatically generates the dependency pairs in $\mathsf{DP}(\mathcal{R}, \mu)$ and tries to prove that no cycle $\mathfrak{C}$ in the (estimated) context-sensitive dependency graph induces an infinite $(\mathcal{R}, \mathfrak{C}, \mu^\sharp)$-minimal chain by using Theorems 4, 5, 6, and eventually Corollary 3. Since our current implementation is based on the use of polynomial orderings, we actually do not compute any argument filtering function when using Theorem 4; it is well-known that they are somehow implicit in the computation of the polynomial interpretations (see, e.g., [AG00]). The following URL:

`http://www.dsic.upv.es/~slucas/csr/termination/examples`

collects a good number of examples and shows the computed proofs based on CS-DPs. We have also considered the examples in the 2006 Termination Competition (TRSs, subcategory Context-Sensitive Rewriting) available through the URL:

`http://www.lri.fr/~marche/termination-competition/2006`

which collects 90 examples (out from around 20 different references) of CS-TRSs. We have used our prototype implementation of the CS-dependency pairs techniques described above to obtain a comparison of different techniques for proving termination of CSR. We consider the CSRPO [BLR02], the polynomial orderings generated by MU-TERM according to [Luc05], and the transformations introduced by Giesl and Middeldorp [GM04] and Zantema [Zan97] which, according to the analysis in [Luc06], have the most successful (combined) behavior among all existing ones. Regarding the concrete settings for CS-DP-based automatic proofs which we have used in our experiments, besides the subterm criterion, we used μ-reduction pairs generated by linear polynomial interpretations whose coefficients take value on $\{0, \frac{1}{2}, 1, 2\}$ (see [Luc05] for further details about the use and generation of such polynomial interpretations) and a timeout of 1 minute. The termination proofs for the transformed systems $\mathcal{R}^{\mu}_{GM}$ and $\mathcal{R}^{\mu}_{Z}$ which are obtained by MU-TERM also use the dependency pairs techniques described in this paper but applied to TRSs rather than CS-TRSs (a TRS $\mathcal{R}$ can also be seen as a CS-TRS $(\mathcal{R}, \mu_{\top})$, where $\mu_{\top}(f) = \{1, \ldots, k\}$ for all k-ary symbols $f \in \mathcal{F}$).

The following table summarizes our results (the number of successful proofs is below each considered technique):

CS-DPs	CSRPO	POL	GM	Z	GM+Z
39	33	27	12	27	34

Note from the table that the use of the current techniques developed for the CS-DPs approach improves on the use of the analogous techniques for the transformed systems (which are TRSs). Even combining the proofs which can be achieved by using Giesl and Middeldorp's transformation or Zantema's transformation (last column), we get better performance with CS-DPs. This suggests that the direct use of CS-DPs performs better than using DPs on the transformed TRSs.

Of course, the state-of-the-art of DP-based techniques for proving termination of rewriting is much more evolved than CS-DPs which we are just introducing in this paper. In fact, we have also tried the transformed systems (with GM and Z transformations) corresponding to the examples above by using AProVE [GTSF04]. AProVE is currently the most powerful tool for proving termination of TRSs and implements most existing results and techniques regarding DPs and related techniques. AProVE succeeded (again with a timeout of 1 minute) on 58 examples by using Giesl and Middeldorp's transformation (on 37 of them) or Zantema's transformation (on 43 of them) and then using the impressive amount of implemented techniques for proving termination of rewriting, see [GTSF04,GTS04]. Thus summarizing, all these experimental results suggest that further research on the CS-DP approach will dramatically improve the current state-of-the-art of techniques for proving termination of *CSR*.

7 Conclusions

We have shown how to use dependency pairs in proofs of termination of *CSR*. The implementation and practical use of the developed techniques yield a novel and powerful framework which improves the current state-of-the-art of methods for proving termination of *CSR*. Some interesting differences arise which can be summarized as follows: in sharp contrast to the standard dependency pairs approach, where all dependency pairs have tuple symbols $f^\sharp$ both in the left- and right-hand sides, we have dependency pairs having a single *variable* in the right-hand side. These variables reflect the effect of the *migrating* variables into the termination behavior of *CSR*. Given a rule $l \to r$, we say that a replacing variable x in r is migrating if there is no replacing position of x in l. This leads to a new definition of chain of context-sensitive dependency pairs which also differs from the standard approach in that we have to especially deal with such migrating variables. As in Arts and Giesl's approach, the presence or absence of infinite chains of dependency pairs from $\mathsf{DP}(\mathcal{R}, \mu)$ characterizes the μ-terminaton of $\mathcal{R}$ (Theorems 1 and 2). Furthermore, we are also able to use term orderings to ensure the absence of infinite chains of context-sensitive dependency pairs (Theorem 3). In fact, we are properly extending Arts and Giesl's approach: whenever $\mu(f) = \{1, \ldots, k\}$ for all k-ary symbols $f \in \mathcal{F}$, *CSR* and ordinary rewriting coincide and all these results and techniques boil down into well-known results and techniques for the dependency pairs approach.

Regarding the practical use of the CS-dependency pairs in proofs of termination of *CSR*, we have shown how to build and use the corresponding CS-dependency graph to either prove that the rules of the TRS and the cycles in the CS-dependency graph are compatible with some reduction pair (Theorem 4) or to prove that there are cycles which do not need to be considered at all (Theorems 5 and 6). We have implemented these ideas as part of the termination tool MU-TERM; after a thorough comparison with other techniques for proving termination of *CSR*, in particular those implemented by other termination tools like AProVE (see Section 6), we can conclude that the CS-dependency pairs can play in *CSR* the role than dependency pairs play in rewriting.

There are many other aspects of the dependency pairs approach which are also worth to be considered and eventually extended to *CSR* (e.g., narrowing refinements, modularity issues, innermost computations, usable rules, ...). These aspects provide an interesting subject for future work.

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Proofs

The following twp basic results about *CSR* will be often used without any explicit mention.

Proposition 5. [Luc98] *Let $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and $p = q.q' \in \mathcal{P}os(t)$. Then $p \in \mathcal{P}os^\mu(t)$ iff $q \in \mathcal{P}os^\mu(t) \wedge q' \in \mathcal{P}os^\mu(t|_q)$*

The following proposition establishes that the replacing nature of a position within a term does not depend on the context surrounding that position. Here, the chain of symbols lying on positions above/on $p \in \mathcal{P}os(t)$ is $prefix_t(\Lambda) = root(t)$, $prefix_t(i.p) = root(t).prefix_{t_i}(p)$. The strict prefix *sprefix* is $sprefix_t(\Lambda) = \Lambda$, $sprefix_t(p.i) = prefix_t(p)$.

Proposition 6. [Luc98] *If $p \in \mathcal{P}os(t) \cap \mathcal{P}os(s)$ and $sprefix_t(p) = sprefix_s(p)$, then $p \in \mathcal{P}os^\mu(t) \Leftrightarrow p \in \mathcal{P}os^\mu(s)$.*

Lemma 1. *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS, $\mu \in M_\mathcal{R}$ and $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$. If t is not μ-terminating, then there is a μ-replacing subterm s of t such that $s \in \mathcal{M}_{\infty,\mu}$.*

Proof. By structural induction. If t is a constant symbol, it is obvious: take $s = t$. If $t = f(t_1,\ldots,t_k)$, then we proceed by contradiction. If there is no μ-replacing subterm s of t such that $s \in \mathcal{M}_{\infty,\mu}$, then in particular $t \notin \mathcal{M}_{\infty,\mu}$, i.e., there is a strict μ-replacing subterm s of t which is not μ-terminating. By the Induction Hypothesis, s contains a μ-replacing subterm s' which belongs to $\mathcal{M}_{\infty,\mu}$. But s' itself is a μ-replacing subterm of t which belongs to $\mathcal{M}_{\infty,\mu}$, thus leading to a contradiction.

Lemma 2. *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS, $\mu \in M_\mathcal{R}$, and $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$. If t is μ-terminating and $t \hookrightarrow^*_\mu s$, then s is μ-terminating.*

Proposition 1 *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_\mathcal{R}$. Then for all $t \in \mathcal{M}_{\infty,\mu}$, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{M}_{\infty,\mu}$ such that $t \stackrel{>\Lambda}{\hookrightarrow^*} \sigma(l) \stackrel{\Lambda}{\to} \sigma(r) \trianglerighteq_\mu u$ and either*

1. *there is a μ-replacing subterm s of r such that $u = \sigma(s)$, or*
2. *there is $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ such that $\sigma(x) \trianglerighteq_\mu u$.*

Proof. Consider an infinite μ-rewrite sequence starting from t. By definition of $\mathcal{M}_{\infty,\mu}$, all proper μ-replacing subterms of t are μ-terminating. Therefore, t has an inner reduction to an instance $\sigma(l)$ of the left-hand side of a rule $l \to r$ of $\mathcal{R}$: $t \stackrel{>\Lambda}{\hookrightarrow^*} \sigma(l) \stackrel{\Lambda}{\to} \sigma(r)$. Thus, we can write $t = f(t_1,\ldots,t_k)$, $\sigma(l) = f(l_1,\ldots,l_k)$ for some k-ary defined symbol f, and $t_i \hookrightarrow^* \sigma(l_i)$ for all i, $1 \leq i \leq k$. Since all t_i are μ-terminating for $i \in \mu(f)$, by Lemma 2, $\sigma(l_i)$ and all its μ-replacing subterms also are. Since $\sigma(r)$ is non-μ-terminating, by Lemma 1 it contains a μ-replacing subterm $u \in \mathcal{M}_{\infty,\mu}$: $\sigma(r) \trianglerighteq_\mu u$, i.e., there is a position $p \in \mathcal{P}os^\mu(\sigma(r))$ such that $\sigma(r)|_p = u$. We consider two cases:

1. If $p \in \mathcal{P}os_{\mathcal{F}}(r)$, then there is a μ-replacing subterm s of r, such that $u = \sigma(s)$.
2. If $p \notin \mathcal{P}os_{\mathcal{F}}(r)$, then there is a μ-replacing variable position $q \in \mathcal{P}os^{\mu}(r) \cap \mathcal{P}os_{\mathcal{X}}(r)$ such that $q \leq p$. Let $x \in \mathcal{V}ar^{\mu}(r)$ be such that $r|_q = x$. Then, $\sigma(x) \trianglerighteq_{\mu} u$ and $\sigma(x)$ is not μ-terminating. Since $\sigma(l_i)$ is terminating for all $i \in \mu(f)$ and $\sigma(x)$ is also terminating for all μ-replacing variables in l, we conclude that $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$.

Proposition 3 *There is no infinite $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain with $\mathcal{P} \subseteq \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R}, \mu)$.*

Proof. By contradiction. Assume that there is an infinite chain which only uses such dependency pairs $u_i \to x_i$ for $i \geq 1$. Then, for all $i \geq 1$, $u_i|_{j_i} \trianglerighteq x_i$ for some $j_i \notin \mu^\sharp(f_i^\sharp)$. Therefore, by definition of $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$-chain, we have that $\sigma(u_i)|_{j_i} \trianglerighteq \sigma(x_i) \trianglerighteq_{\mu} s_i$ for some term s_i such that $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$, with $root(s^\sharp) = root(u_{i+1}) = f^\sharp_{i+1}$ and $j_{i+1} \notin \mu^\sharp(f^\sharp_{i+1})$. Thus, it follows that $s_i|_{j_{i+1}} = \sigma(u_{i+1})|_{j_{i+1}} \trianglerighteq \sigma(x_{i+1})$, i.e., $\sigma(x_i) \triangleright \sigma(x_{i+1})$ for all $i \geq 1$. Therefore, we get an infinite sequence $\sigma(x_1) \triangleright \sigma(x_2) \triangleright \cdots$ which contradicts well-foundedness of $\triangleright$.

Theorem 1 *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If there is no infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain, then $\mathcal{R}$ is μ-terminating.*

Proof. By contradiction. If $\mathcal{R}$ is not μ-terminating, then there is $t \in \mathcal{M}_{\infty,\mu}$. By Proposition 1, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{M}_{\infty,\mu}$ such that $t \stackrel{>\Lambda}{\hookrightarrow^*} \sigma(l) \stackrel{\Lambda}{\to} \sigma(r) \trianglerighteq_{\mu} u$ and either

1. there is a μ-replacing subterm s of r such that $u = \sigma(s)$, or
2. there is $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$ such that $\sigma(x) \trianglerighteq_{\mu} u$.

In the first case above, we have a dependency pair $l^\sharp \to s^\sharp \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$ such that $u = \sigma(s) \in \mathcal{M}_{\infty,\mu}$, i.e., we can start an $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain beginning with $\sigma(l^\sharp) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} \sigma(s^\sharp)$.

In the second case above, since $u \in \mathcal{M}_{\infty,\mu}$, by Proposition 1 there is a rule $\lambda \to \rho$ such that $u \stackrel{>\Lambda}{\hookrightarrow^*} \sigma(\lambda)$ (since we can assume that the variables in this rule do not occur in l, we can use the same –conveniently extended– substitution σ) and $\sigma(\rho)$ contains a subterm in $\mathcal{M}_{\infty,\mu}$. Hence, $u^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(\lambda^\sharp)$. Furthermore, there is a dependency pair $l^\sharp \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ such that $\sigma(x) \trianglerighteq_{\mu} u$; thus, according to Definition 2, we can start an $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain beginning with

$$\sigma(l^\sharp) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} u^\sharp$$

and then continuing with a dependency pair $u' \to v'$ such that $u' = \lambda^\sharp$ and $u^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u')$.

Thus, in both cases we can start an $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain which could be infinitely extended in a similar way by starting from $u^\sharp$. This contradicts our initial assumption.

Theorem 2 *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then there is no infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain.*

Proof. By contradiction. If there is an infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain, then there is a substitution σ and dependency pairs $u_i \to v_i \in \mathsf{DP}(\mathcal{R},\mu)$ such that

1. $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$, if $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, and
2. if $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, then there is $s_i \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$.

for $i \geq 1$. Now, consider the first dependency pair $u_1 \to v_1$ in the sequence:

1. If $u_1 \to v_1 \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, then $v_1^\natural$ is a μ-replacing subterm of the right-hand-side r_1 of a rule $l_1 \to r_1$ in $\mathcal{R}$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os^\mu(r_1)$ and we can perform the μ-rewriting step $t_1 = \sigma(u_1) \hookrightarrow_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[\sigma(v_1^\natural)]_{p_1} = s_1$, where $\sigma(v_1^\natural)^\sharp = \sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ and $\sigma(u_2)$ initiates an infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain. Note that $p_1 \in \mathcal{P}os^\mu(s_1)$.
2. If $u_1 \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, then there is a rule $l_1 \to r_1$ in $\mathcal{R}$ such that $u_1 = l_1^\sharp$, and $x \in \mathcal{V}ar^\mu(r_1) - \mathcal{V}ar^\mu(l_1)$, i.e., $r_1 = C_1[x]_{q_1}$ for some $q_1 \in \mathcal{P}os^\mu(r_1)$. Furthermore, since there is a subterm s such that $\sigma(x) \trianglerighteq_\mu s$ and $s^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$, we can write $\sigma(x) = C_1'[s]_{p_1'}$ for some $p_1' \in \mathcal{P}os^\mu(\sigma(x))$. Therefore, we can perform the μ-rewriting step $t_1 = \sigma(l_1) \hookrightarrow_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[C_1'[s]_{p_1'}]_{q_1} = s_1$ where $s^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ (hence $s \overset{>\Lambda}{\hookrightarrow^*} u_2^\natural$) and $\sigma(u_2)$ initiates an infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain. Note that $p_1 = q_1.p_1' \in \mathcal{P}os^\mu(s_1)$.

Since $\mu^\sharp(f^\sharp) = \mu(f)$, and $p_1 \in \mathcal{P}os^\mu(s_1)$, we have that $s_1 \hookrightarrow^*_{\mathcal{R},\mu} t_2[\sigma(u_2)]_{p_1} = t_2$ and $p_1 \in \mathcal{P}os^\mu(t_2)$. Therefore, we can build in that way an infinite μ-rewrite sequence

$$t_1 \hookrightarrow_{\mathcal{R},\mu} s_1 \hookrightarrow^*_{\mathcal{R},\mu} t_2 \hookrightarrow_{\mathcal{R},\mu} \cdots$$

which contradicts the μ-termination of $\mathcal{R}$.

Theorem 3 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$. Then, $\mathcal{R}$ is μ-terminating if and only if there is a μ-reduction pair $(\gtrsim, \sqsupset)$ such that,*

1. *$l \gtrsim r$ for all $l \to r \in R$,*
2. *$u \sqsupset v$ for all $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, and*
3. *whenever $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \neq \varnothing$ we have that $\trianglerighteq_\mu \subseteq \gtrsim$, where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$, and*
   - *(a) $u \ (\gtrsim \cup \sqsupset)\ v$ for all $u \to v \in \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, $u \sqsupset v$ for all $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) - \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, and $f(x_1,\ldots,x_k) \gtrsim f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$, or*
   - *(b) $u(\gtrsim \cup \sqsupset)v$ for all $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ and $f(x_1,\ldots,x_k) \sqsupset f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$.*

Proof. For the *if* part of the proof, we proceed by contradiction. If $\mathcal{R}$ is not μ-terminating, then by Theorem 1 there is an infinite $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain:

$$\sigma(u_1) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} s_2^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_3) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} \cdots$$

for some substitution σ, where $s_i^\sharp = \sigma(v_i)$ if $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ and $s_i^\sharp$ is such that $\sigma(x_i) \trianglerighteq_\mu s_i$ if $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, for $i \geq 1$. By Proposition 3, the chain contains an infinite number of pairs in $\mathsf{DP}(\mathcal{R},\mu) - \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$. For any dependency pair $u_i \to v_i$ which is used in this sequence, we have that either

1. $u_i \sqsupset v_i$ if $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, or
2. $u_i\ (\gtrsim \cup \sqsupset)\ v_i$ if $u_i \to v_i \in \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, or
3. if $u_i \to v_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) - \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, then
   (a) $u_i \sqsupset v_i$ and $f(x_1,\ldots,x_k) \gtrsim f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$, or
   (b) $u_i\ (\gtrsim \cup \sqsupset)\ v_i$ and $f(x_1,\ldots,x_k) \sqsupset f^\sharp(x_1,\ldots,x_k)$ for all $f \in \mathcal{D}$.

By stability, we have $\sigma(u_i) \sqsupset \sigma(v_i)$ (respectively $\sigma(u_i) \gtrsim \sigma(v_i)$). Now, we distinguish two cases:

1. If $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, then $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$ and by stability, μ-monotonicity and transitivity of $\gtrsim$ we have that $\sigma(v_i) \gtrsim \sigma(u_{i+1})$.
2. If $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, then $\sigma(x_i) \trianglerighteq_\mu s_i$. Since $\trianglerighteq_\mu \subseteq \gtrsim$ we have $\sigma(x_i) \gtrsim s_i$. Furthermore, since $f(x_1,\ldots,x_k) \gtrsim f^\sharp(x_1,\ldots,x_k)$ (or $f(x_1,\ldots,x_k) \sqsupset f^\sharp(x_1,\ldots,x_k)$) for all $f \in \mathcal{D}$, by stability we have that $s_i \gtrsim s_i^\sharp$ (respectively $s_i \sqsupset s_i^\sharp$). Finally, since $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$ for all $i \geq 1$, by stability, μ-monotonicity and transitivity of $\gtrsim$ we have that $s_i^\sharp \gtrsim \sigma(u_{i+1})$.

By using the compatibility conditions of the μ-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$.

The *only if* part follows the proof of [AG00, Theorem 7]. It is possible to show that the μ-termination of $\mathcal{R}$ implies the $\mu^\sharp$-termination of

$$\mathcal{R}' = \mathcal{R} \cup \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) \cup \mathcal{P}_{\mathcal{X}}(\mathcal{R},\mu) \cup \mathcal{S}(\mathcal{F},\mu)$$

where: $\mathcal{P}_{\mathcal{X}} = \{u \to \mathit{subterm}(x) \mid u \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)\}$ and

$$\begin{aligned}\mathcal{S}(\mathcal{F},\mu) = &\{\mathit{subterm}(f(x_1,\ldots,x_k)) \to \mathit{subterm}(x_i) \mid f \in \mathcal{F}, i \in \mu(f)\}\\ &\cup \{\mathit{subterm}(f(x_1,\ldots,x_k)) \to f^\sharp(x_1,\ldots,x_k) \mid f \in \mathcal{D}\}\end{aligned}$$

and we assume $\mu^\sharp(\mathit{subterm}) = \varnothing$. The TRS $\mathcal{S}(\mathcal{F},\mu)$ simulates the μ-replacing subterm relation on terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$: for all $t,s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, $\mathit{subterm}(t) \hookrightarrow^*_{\mathcal{S}(\mathcal{F},\mu),\mu^\sharp} \mathit{subterm}(s)$ if and only if $t \trianglerighteq_\mu s$. The role of the last subset of rules is 'marking' defined symbols as to enable the connection of the minimal subterm s which is below $\sigma(x)$ after the application of a rule $l^\sharp \to x \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$. Thus, we can use the μ-rewrite relation $\hookrightarrow^*_{\mathcal{R}',\mu^\sharp}$ to get the required μ-reduction pair $(\gtrsim, >)$, (where $\lfloor t \rfloor \gtrsim \lfloor s \rfloor$ if $t \hookrightarrow^*_{\mathcal{R}',\mu^\sharp} s$ and $\lfloor\_\rfloor$ removes the occurrences of *subterm* everywhere) which fulfills the required properties.

Theorem 4 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$. Then, $\mathcal{R}$ is μ-terminating if and only if for each cycle $\mathfrak{C}$ in the context-sensitive dependency graph there is a μ-reduction pair $(\gtrsim_{\mathfrak{C}}, \sqsupset_{\mathfrak{C}})$ such that, $R \subseteq \gtrsim_{\mathfrak{C}}$, $\mathfrak{C} \subseteq \gtrsim_{\mathfrak{C}} \cup \sqsupset_{\mathfrak{C}}$, and*

1. *If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \varnothing$, then $\mathfrak{C} \cap \sqsupset_{\mathfrak{C}} \neq \varnothing$*

2. *If* $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \neq \varnothing$*, then* $\trianglerighteq_\mu \subseteq \gtrsim_{\mathfrak{C}}$ *(where* $\trianglerighteq_\mu$ *is the* μ*-replacing subterm relation on* $\mathcal{T}(\mathcal{F},\mathcal{X})$*), and*
   (a) $\mathfrak{C} \cap \sqsupset_{\mathfrak{C}} \neq \varnothing$ *and* $f(x_1,\ldots,x_k) \gtrsim_{\mathfrak{C}} f^\sharp(x_1,\ldots,x_k)$ *for all* $f^\sharp$ *in* $\mathfrak{C}$*, or*
   (b) $f(x_1,\ldots,x_k) \sqsupset_{\mathfrak{C}} f^\sharp(x_1,\ldots,x_k)$ *for all* $f^\sharp$ *in* $\mathfrak{C}$*.*

Proof. The proof is completely analogous to standard ones (see, e.g., [GAO02, Theorem 3.5]). Just take into account the peculiarities of the use of μ-reduction pairs with context-sensitive dependency pairs already discussed in the proof of Theorem 3.

Theorem 5 *Let* $\mathcal{R}$ *be a TRS and* $\mu \in M_{\mathcal{R}}$*. Let* $\mathfrak{C} \subseteq \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ *be a cycle. If there exists a simple projection* π *for* $\mathfrak{C}$ *such that* $\pi(\mathfrak{C}) \subseteq \trianglerighteq_\mu$*, and* $\pi(\mathfrak{C}) \cap \triangleright_\mu \neq \varnothing$*, then there is no minimal* $(\mathcal{R},\mathfrak{C},\mu^\sharp)$*-chain.*

Proof. The proof is completely analogous to that of [HM04, Theorem 11]. The only difference is that we need to deal with the μ-subterm relation $\trianglerighteq_\mu$; this is because, we need to use the following commutation property: $\triangleright_\mu \circ \hookrightarrow_{\mathcal{R},\mu} \subseteq \hookrightarrow_{\mathcal{R},\mu} \circ \triangleright_\mu$ which does not hold if $\triangleright$ is used instead.

Theorem 6 Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$ and $\mathfrak{C}$ be a cycle in $\mathsf{DG}(\mathcal{R})$. Let $\gtrsim$ be a stable quasi-ordering on terms whose strict and stable part $>$ is well-founded and π be a simple projection for $\mathfrak{C}$ such that for all $f^\sharp$ in $\mathfrak{C}$, $\pi(f^\sharp) \notin \mu^\sharp(f^\sharp)$ and $\pi(\mathfrak{C}) \subseteq \gtrsim$.

1. **If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \varnothing$ and $\mathfrak{C} \cap > \neq \varnothing$, then there is no minimal $(\mathcal{R},\mathfrak{C},\mu^\sharp)$-chain.**
2. **If $\mathfrak{C} \cap \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) \neq \varnothing$, $\trianglerighteq_\mu \subseteq \gtrsim$ (where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$), and**
   (a) $\mathfrak{C} \cap > \neq \varnothing$ and $f(x_1,\ldots,x_k) \gtrsim x_{\pi(f^\sharp)}$ for all $f \in \mathcal{D}$ such that $f^\sharp$ is in $\mathfrak{C}$, or
   (b) $f(x_1,\ldots,x_k) > x_{\pi(f^\sharp)}$ for all $f \in \mathcal{D}$ such that $f^\sharp$ is in $\mathfrak{C}$,
   then there is no minimal $(\mathcal{R},\mathfrak{C},\mu^\sharp)$-chain.

Proof. Assume that there is a minimal $(\mathcal{R},\mathfrak{C},\mu^\sharp)$-chain

$$\sigma(u_1) \hookrightarrow_{\mathfrak{C},\mu^\sharp} s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2) \hookrightarrow_{\mathfrak{C},\mu^\sharp} s_2^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_3) \hookrightarrow_{\mathfrak{C},\mu^\sharp} \cdots$$

for some substitution σ, where $s_i^\sharp = \sigma(v_i)$ if $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ and $s_i^\sharp$ is such that $\sigma(x_i) \trianglerighteq_\mu s_i$ if $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, for $i \geq 1$. All terms $\sigma(u_i)$, $s_i^\sharp$ in this sequence have a dependency pair symbol in $\mathfrak{C}$ as root symbol for $i \geq 1$. We apply the simple projection π to this sequence. Since $\pi(f^\sharp) \notin \mu^\sharp(f^\sharp)$ for all these symbols, it follows that $\pi(s_i^\sharp) = \pi(\sigma(u_{i+1}))$ for all $i \geq 1$. For each step $\sigma(u_i) \hookrightarrow_{\mathfrak{C},\mu^\sharp} s_i^\sharp$ there is $u_i \to v_i \in \mathfrak{C}$ such that $\pi(u_i) \gtrsim \pi(v_i)$ and, by stability, $\sigma(\pi(u_i)) \gtrsim \sigma(\pi(v_i))$. Now, we distinguish two cases:

1. If $u_i \to v_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, then $\pi(\sigma(v_i)) = \pi(s_i^\sharp)$ and $\pi(\sigma(u_i)) = \sigma(\pi(u_i)) \gtrsim \sigma(\pi(v_i)) = \pi(\sigma(v_i)) = \pi(s_i^\sharp)$.

2. If $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, then $\pi(v_i) = \pi(x_i) = x_i$, and, by stability, $\pi(\sigma(u_i)) = \sigma(\pi(u_i)) \gtrsim \sigma(\pi(v_i)) = \sigma(x_i)$. Since $\sigma(x_i) \trianglerighteq_\mu s_i$, we have that $\sigma(x_i) \gtrsim s_i$ (due to $\trianglerighteq_\mu \subseteq \gtrsim$). Let $f^\sharp = root(u_{i+1})$. Since $f(x_1,\ldots,x_k) \gtrsim x_{\pi(f^\sharp)}$, and $s_i|_{\pi(f^\sharp)} = s_i^\sharp|_{\pi(f^\sharp)} = \pi(s_i^\sharp)$, by stability we have $s_i \gtrsim \pi(s_i^\sharp)$. Hence, $\pi(\sigma(u_i)) \gtrsim \pi(s_i^\sharp)$.

Thus, we always have $\pi(\sigma(u_i)) \gtrsim \pi(s_i^\sharp)$. Therefore, we obtain an infinite $\gtrsim$ sequence

$$\pi(\sigma(u_1)) \gtrsim \pi(s_1^\sharp) = \pi(\sigma(u_2)) \gtrsim \pi(s_2^\sharp) \cdots$$

which, since the dependency pairs in $\mathfrak{C}$ occur infinitely many, and according to our assumptions, contains infinitely many $>$ steps, starting from $\pi(\sigma(u_1))$. This contradicts the well-foundedness of $>$.