Context-Sensitive Dependency Pairs ⋆

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Abstract

Termination is one of the most interesting problems when dealing with context-sensitive rewrite systems. Although there is a good number of techniques for proving termination of context-sensitive rewriting (*CSR*), the dependency pair approach, one of the most powerful techniques for proving termination of rewriting, has not been investigated in connection with proofs of termination of *CSR*. In this paper, we show how to use dependency pairs in proofs of termination of *CSR*. The implementation and practical use of the developed techniques yield a novel and powerful framework which improves the current state-of-the-art of methods for proving termination of *CSR*.

Key words: Dependency pairs, term rewriting, program analysis, termination.

1 Introduction

Most computational systems whose operational principle is based on reducing expressions can be described and analyzed by using notions and techniques coming from the abstract model of Term Rewriting Systems (TRSs [BN98,TeR03]). Such computational systems (e.g., functional, algebraic, and equational programming languages as well as theorem provers based on rewriting techniques) often incorporate a predefined reduction strategy which is used to break down the nondeterminism which is inherent to reduction relations.

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Eventually, this can rise problems, as each kind of strategy only behaves properly (i.e., it is normalizing, optimal, etc.) for particular classes of programs. For this reason, the designers of programming languages have developed some features and language constructs aimed at giving the user more flexible control of the program execution. For instance, *syntactic annotations* (which are associated to arguments of symbols) have been used in programming languages such as Clean [NSEP92], Haskell [HPW92], Lisp [McC60], Maude [CDEL+07], OBJ2 [FGJM85], OBJ3 [GWMFJ00], CafeOBJ [FN97], etc., to improve the termination and efficiency of computations. Lazy languages (e.g., Haskell, Clean) interpret them as *strictness annotations* in order to become 'more eager' and efficient. Eager languages (e.g., Lisp, Maude, OBJ2, OBJ3, CafeOBJ) use them as *replacement restrictions* to become 'more lazy' thus (hopefully) avoiding nontermination.

Context-sensitive rewriting (*CSR* [Luc98,Luc02]) is a restriction of rewriting that forbids reductions on some subexpressions and that has proved useful to model and analyze such programming language features at different levels, see, e.g., [BM06,DLM+04,DLM+08,GM04,Luc01,LM08a]. Such a restriction of the rewriting computations is formalized at a very simple syntactic level: that of the arguments of function symbols f in the signature $\mathcal{F}$. As usual, by a *signature* we mean a set of function symbols $f_1, \ldots, f_n, \ldots$ together with an *arity* function $ar : \mathcal{F} \to \mathbb{N}$ which establishes the number of 'arguments' associated to each symbol. A *replacement map* is a mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f in the signature $\mathcal{F}$ [Luc98]. We use them to discriminate the argument positions on which the rewriting steps are allowed. In *CSR* we only rewrite μ-replacing subterms: every term t (as a whole) is μ-replacing by definition; and t_i (as well as all its μ-replacing subterms) is a μ-replacing subterm of $f(t_1, \ldots, t_k)$ if $i \in \mu(f)$.

Example 1 *The following nonterminating TRS $\mathcal{R}$ can be used to compute the list of prime numbers by using the well-known Erathostenes sieve*[1] *[GM99]:*

$$
\begin{array}{rcl}
\mathtt{primes} & \to & \mathtt{sieve}(\mathtt{from}(\mathtt{s}(\mathtt{s}(\mathtt{0})))) \\
\mathtt{from}(x) & \to & \mathtt{cons}(x, \mathtt{from}(\mathtt{s}(x))) \\
\mathtt{head}(\mathtt{cons}(x, y)) & \to & x \\
\mathtt{sieve}(\mathtt{cons}(x, y)) & \to & \mathtt{cons}(x, \mathtt{filt}(x, \mathtt{sieve}(y))) \\
\mathtt{tail}(\mathtt{cons}(x, y)) & \to & y \\
\mathtt{if}(\mathtt{true}, x, y) & \to & x \\
\mathtt{if}(\mathtt{false}, x, y) & \to & y \\
\mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), \mathtt{cons}(y, z)) & \to & \mathtt{if}(\mathtt{div}(\mathtt{s}(\mathtt{s}(x)), y), \mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), z), \mathtt{cons}(y, \mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), z)))
\end{array}
$$

[1] Without appropriate rules for defining symbol `div`, the TRS has no complete computational meaning. However, we took it here as given in [GM99] for the purpose of comparing different techniques for proving termination of *CSR* by transformation.

Consider the replacement map μ for the signature $\mathcal{F}$ given by:

$\mu(\texttt{cons}) = \mu(\texttt{if}) = \{1\}$ *and* $\mu(f) = \{1, \ldots, ar(f)\}$ *for all* $f \in \mathcal{F} - \{\texttt{cons}, \texttt{if}\}$.

This replacement map exemplifies two of the most typical applications of context-sensitive rewriting as a computational mechanism:

(1) The declaration $\mu(\texttt{if}) = \{1\}$ *allows us to forbid reductions on the two* alternatives *s and t of* if-then-else *expressions* $\texttt{if}(b, s, t)$ *whereas it is still possible to perform reductions on the* boolean *part b, as required to implement the usual semantics of the operator.*
(2) The declaration $\mu(\texttt{cons}) = \{1\}$ *disallows reductions on the* list *part of the list constructor* `cons`*, thus making possible a kind of* lazy evaluation *of lists. We can still use projection operators as* `tail` *to continue the evaluation when needed.*

1.1 Termination of context-sensitive rewriting

Termination is one of the most interesting practical problems in computation and software engineering. A program or computational system is said to be *terminating* if it does not lead to any infinite computation for any possible call or input data. Ensuring termination is often a prerequisite for essential program properties like correctness. Messages reporting (a neverending) "processing", "waiting for an answer", or even "abnormal termination" (which are often raised during the execution of software applications) usually correspond to nonterminating computations arising from bugs in the program. Thus, being able to *automatically prove* termination of programs is a key issue in modern software development.

Termination is also one of the most interesting problems when dealing with *CSR*. With *CSR* we can *achieve* a terminating behavior with nonterminating TRSs by pruning (all) infinite rewrite sequences. For instance, as we prove below, all context-sensitive computations for the TRS $\mathcal{R}$ in Example 1 are terminating when the replacement map μ in the example is considered. Recently, proving termination of *CSR* has been recognized as an interesting problem with several applications in the fields of term rewriting and programming languages [DLM$^+$04,DLM$^+$08,GM04,Luc02,Luc06].

Several methods have been developed for proving termination of *CSR* under a replacement map μ for a given TRS $\mathcal{R}$ (i.e., for proving the *μ-termination* of $\mathcal{R}$). A number of transformations which permit to treat termination of *CSR* as a standard termination problem have been described (see [GM04,Luc06] for recent surveys). Polynomial orderings and the context-sensitive version of the recursive path ordering have also been investigated [BLR02,GL02,Luc04b,Luc05].

The *dependency pairs method* [AG00,GAO02,GTS04,GTSF06,HM04,HM05], one of the most powerful techniques for proving termination of rewriting, has not been investigated in connection with proofs of termination of *CSR*. In this paper, we address this problem.

1.2 Dependency pairs for context-sensitive rewriting

Roughly speaking, given a TRS $\mathcal{R}$, the dependency pairs associated to $\mathcal{R}$ conform a new TRS $\mathsf{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so-called *dependency chains* whose finiteness or infiniteness characterize termination or nontermination of $\mathcal{R}$. Given a rewrite rule $l \to r$, we get dependency pairs $l^\sharp \to s^\sharp$ for all subterms s of r which are rooted by a defined symbol [2]; the notation $t^\sharp$ for a given term t means that the root symbol f of t is *marked* thus becoming $f^\sharp$ (often just capitalized: F, as done in our examples).

Example 2 *Consider the TRS $\mathcal{R}$ in Example 1. According to [AG00], the set $\mathsf{DP}(\mathcal{R})$ of dependency pairs in $\mathcal{R}$ consists of the following pairs:*

$$\begin{array}{rlr}
\mathtt{PRIMES} \to \mathtt{SIEVE}(\mathtt{from}(\mathtt{s}(\mathtt{s}(\mathtt{0})))) & & (1)\\
\mathtt{PRIMES} \to \mathtt{FROM}(\mathtt{s}(\mathtt{s}(\mathtt{0}))) & & (2)\\
\mathtt{FROM}(x) \to \mathtt{FROM}(\mathtt{s}(x)) & & (3)\\
\mathtt{SIEVE}(\mathtt{cons}(x,y)) \to \mathtt{SIEVE}(y) & & (4)\\
\mathtt{SIEVE}(\mathtt{cons}(x,y)) \to \mathtt{FILT}(x,\mathtt{sieve}(y)) & & (5)\\
\mathtt{FILT}(\mathtt{s}(\mathtt{s}(x)),\mathtt{cons}(y,z)) \to \mathtt{FILT}(\mathtt{s}(\mathtt{s}(x)),z) & & (6)\\
\mathtt{FILT}(\mathtt{s}(\mathtt{s}(x)),\mathtt{cons}(y,z)) \to \mathtt{FILT}(x,\mathtt{sieve}(y)) & & (7)\\
\mathtt{FILT}(\mathtt{s}(\mathtt{s}(x)),\mathtt{cons}(y,z)) \to \mathtt{SIEVE}(y) & & (8)
\end{array}$$

$$\mathtt{FILT}(\mathtt{s}(\mathtt{s}(x)),\mathtt{cons}(y,z)) \to \mathtt{IF}(\mathtt{div}(\mathtt{s}(\mathtt{s}(x)),y),\mathtt{filt}(\mathtt{s}(\mathtt{s}(x)),z),\mathtt{cons}(y,\mathtt{filt}(\mathtt{s}(\mathtt{s}(x)),z))) \quad (9)$$

A chain of dependency pairs is a sequence $u_i \to v_i$ of dependency pairs together with a substitution σ such that $\sigma(v_i)$ rewrites to $\sigma(u_{i+1})$ for all $i \geq 1$. The dependency pairs can be presented as a *dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph. For instance, the dependency graph which corresponds to the TRS $\mathcal{R}$ in Example 1 is depicted in Figure 1. The cycle consisting of the node (3) together with the arc going from this node to itself witnesses the *nontermination* of $\mathcal{R}$ (viewed as an ordinary rewrite system, without any restriction on its rewriting relation).

In general, these intuitions are valid for *CSR*: the subterms s of the right-hand sides r of the rules $l \to r$ which are considered to build the *context-sensitive dependency pairs* $l^\sharp \to s^\sharp$ must be *μ-replacing* terms now.

Example 3 *Consider $\mathcal{R}$ and μ as in Example 1. Only the dependency pairs* (1)*,* (2)*, and* (9) *in Example 2 are also context-sensitive dependency pairs.*

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[2] A symbol f is said to be *defined* in a TRS $\mathcal{R}$ if $\mathcal{R}$ contains a rule $f(l_1,\ldots,l_k) \to r$.

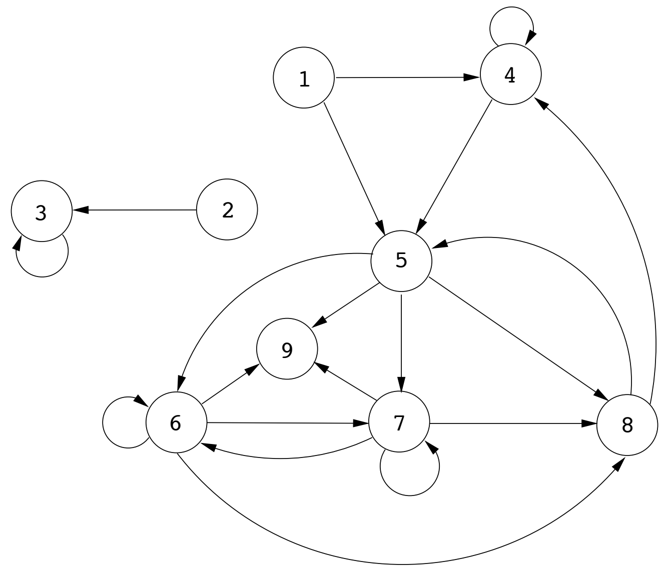

Fig. 1. Dependency Graph for the TRS $\mathcal{R}$ in Example 1

However, this is not sufficient to obtain a correct approach. The following example shows the need of a new kind of dependency pairs.

Example 4 *Consider the following TRS $\mathcal{R}$:*

$$\mathtt{a} \to \mathtt{c}(\mathtt{f}(\mathtt{a})) \qquad\qquad \mathtt{f}(\mathtt{c}(x)) \to x$$

together with $\mu(\mathtt{c}) = \varnothing$ and $\mu(\mathtt{f}) = \{1\}$. No μ-replacing subterm s in the right-hand sides of the rules is rooted by a defined symbol. Thus, there is no 'regular' dependency pair. If no other dependency pairs are considered, we could wrongly conclude that $\mathcal{R}$ is μ-terminating, which is not true:

$$\mathtt{f}(\underline{\mathtt{a}}) \hookrightarrow_\mu \underline{\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a})))} \hookrightarrow_\mu \mathtt{f}(\underline{\mathtt{a}}) \hookrightarrow_\mu \cdots$$

Indeed, we must add the following collapsing *dependency pair:*

$$\mathtt{F}(\mathtt{c}(x)) \to x$$

which would not be allowed in Arts and Giesl's approach [AG00] because the right-hand side is a variable.

1.3 Plan of the paper

After some preliminaries in Section 2, we develop the material in the paper in three main parts:

(1) We investigate the structure of infinite context-sensitive rewrite sequences.

This analysis is essential to provide an appropriate definition of context-sensitive dependency pair, and the related notions of chains, graph, etc. Section 3 provides appropriate notions of *minimal* non-μ-terminating terms and introduces the main properties of such terms. Section 4 introduces the notion of *hidden term* in a CS-TRS. This notion turns to be essential for the appropriate treatment of collapsing dependency pairs. Section 5 investigates the structure of infinite context-sensitive rewrite sequences starting from minimal non-μ-terminating terms.

(2) We define the notions of *context-sensitive dependency pair* and *context-sensitive chain of pairs* and show how to use them to *characterize* termination of *CSR*. Sections 6 and 7 introduce the general framework to compute and use context-sensitive dependency pairs for proving termination of *CSR*. The introduction of a new kind of dependency pairs (the *collapsing* dependency pairs, as in Example 4) leads to a notion of context-sensitive dependency *chain*, which is quite differente from the standard one. In Section 8 we prove that our *context-sensitive dependency pairs approach* fully characterize termination of *CSR*.

(3) We describe a suitable *framework* for dealing with proofs of termination of *CSR* by using the previous results. Section 9 provides an adaptation of the *dependency pair framework* [GTS04,GTSF06] to *CSR* by defining appropriate notions of *CS-termination problem* and *CS-processor* which rely in the notions and results investigated in the second part of the paper. Section 10 introduces the notion of *context-sensitive (dependency) graph* and the associated CS-processor which formalizes the usual practice of analyzing the absence of infinite (minimal) chains by considering the (maximal) cycles in the dependency graph. As in the standard case, the CS-dependency graph is not computable, so we show how to obtain the *estimated* CS-dependency graph which is a computable overstimation of it. Section 11 describes some CS-processors for removing or transforming collapsing pairs from CS-termination problems in some particular cases. Section 12 investigates the use of term orderings to achieve proofs of termination of *CSR* within the context-sensitive dependency pairs framework. We introduce the notion of *μ-reduction pair*, which is the straightforward adaptation of reduction pairs used for dealing with dependency pairs in the standard case. Nevertheless, some important differences with the standard case arise when collapsing pairs are considered due to the need of imposing some additional conditions. Section 13 adapts Hirokawa and Middeldorp's *subterm criterion* [HM04] to *CSR*. Section 14 adapts *narrowing transformation* of pairs in [GTSF06] to *CSR*.

The paper ends with an experimental evaluation of our techniques in Section 15 and a discussion about related work in Section 16, including a detailed comparison between the material in this paper and the results in its predecessors [AGL06,AGL07]. Section 17 concludes.

2 Preliminaries

This section collects a number of definitions and notations about term rewriting. More details and missing notions can be found in [BN98,Ohl02,TeR03].

Let A be a set and $\mathsf{R} \subseteq \mathsf{A} \times \mathsf{A}$ be a binary relation on A. We denote the transitive closure of R by R^+ and its reflexive and transitive closure by R^*. We say that R is *terminating* (*strongly normalizing*) if there is no infinite sequence $a_1 \mathrel{\mathsf{R}} a_2 \mathrel{\mathsf{R}} a_3 \cdots$. A reflexive and transitive relation R is a quasi-ordering.

2.1 Signatures, Terms, and Positions

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. A term is *ground* if it contains no variable. A term is said to be *linear* if it has no multiple occurrences of a single variable.

Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. We denote the empty chain by Λ. Given positions p, q, we denote their concatenation as $p.q$. Positions are ordered by the standard prefix ordering: $p \leq q$ if $\exists q'$ such that $q = p.q'$ If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. The set of positions of a term t is $\mathcal{P}os(t)$. Positions of nonvariable symbols in t are denoted as $\mathcal{P}os_{\mathcal{F}}(t)$, and $\mathcal{P}os_{\mathcal{X}}(t)$ are the positions of variables. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s.

We write $t \trianglerighteq s$, read *s is a subterm of t*, if $s = t|_p$ for some $p \in \mathcal{P}os(t)$ and $t \triangleright s$ if $t \trianglerighteq s$ and $t \neq s$. We write $t \ntrianglerighteq s$ and $t \ntriangleright s$ for the negation of the corresponding properties. The symbol labeling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with a 'hole' $\Box$ (a fresh constant symbol). We write $C[\,]_p$ to denote that there is a (usually single) hole $\Box$ at position p of C. Generally, we write $C[\,]$ to denote an arbitrary context and make explicit the position of the hole only if necessary. $C[\,] = \Box$ is called the *empty* context.

2.2 Substitutions

A substitution is a mapping $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. Denote as ε the 'identity' substitution: $\varepsilon(x) = x$ for all $x \in \mathcal{X}$. The set $\mathcal{D}om(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is called the *domain* of σ.

Remark 1 *In this paper, we do* not *impose that the domain of the substitutions is finite. This is usual practice in the dependency pairs approach, where a single substitution is used to instantiate an infinite number of variables coming from renamed versions of the dependency pairs (see below).*

Whenever $\mathcal{D}om(\sigma) \cap \mathcal{D}om(\sigma') = \varnothing$, for substitutions σ, σ', we denote by $\sigma \cup \sigma'$, a substitution such that $(\sigma \cup \sigma')(x) = \sigma(x)$ if $x \in \mathcal{D}om(\sigma)$ and $(\sigma \cup \sigma')(x) = \sigma'(x)$ if $x \in \mathcal{D}om(\sigma')$.

2.3 Renamings and unifiers

A *renaming* is an injective substitution ρ such that $\rho(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$. For renamings, we assume that $\mathcal{V}ar(\rho)$ is finite (which is the usual practice) and also idempotency, i.e., $\rho(\rho(x)) = \rho(x)$ for all $x \in \mathcal{X}$.

The quasi-ordering of subsumption $\leq$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is $t \leq t' \Leftrightarrow \exists \sigma.\ t' = \sigma(t)$. We denote as $\sigma \leq \sigma'$ the fact that $\sigma(x) \leq \sigma'(x)$ for all $x \in \mathcal{X}$, thus extending the quasi-ordering to substitutions.

A substitution σ such that $\sigma(s) = \sigma(t)$ for two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is called a *unifier* of s and t; we also say that s and t unify (with substitution σ). If two terms s and t unify, then there is a unique (up to renaming of variables) *most general unifier (mgu)* θ which is *minimal* (w.r.t. the subsumption quasi-ordering $\leq$) among all other unifiers of s and t.

A relation $\mathsf{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$ on terms is *stable* if for all terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and substitutions σ, we have $\sigma(s) \,\mathsf{R}\, \sigma(t)$ whenever $s \,\mathsf{R}\, t$.

2.4 Rewrite Systems and Term Rewriting

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A rewrite rule $l \to r$ is said to be *collapsing* if $r \in \mathcal{X}$. A *Term Rewriting System* (TRS) is a pair $\mathcal{R} = (\mathcal{F}, R)$, where R is a set of rewrite rules. Given TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{R}' = (\mathcal{F}', R')$, we let $\mathcal{R} \cup \mathcal{R}'$ be the TRS $(\mathcal{F} \cup \mathcal{F}', R \cup R')$. An instance $\sigma(l)$ of a *lhs* l of a rule is called a *redex*. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Example 5 *Consider again the TRS in Example 1. The symbols* `primes`*,* `sieve`*,* `from`*,* `head`*,* `tail`*,* `if` *and* `filt` *are defined, and* `s`*,* `0`*,* `cons`*,* `true`*,* false *and* div *are constructors.*

For simplicity, we often write $l \to r \in \mathcal{R}$ instead of $l \to r \in R$ to express that the rule $l \to r$ is a rule of $\mathcal{R}$.

A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position p), written $t \xrightarrow{p}_{\mathcal{R}} s$ (or just $t \to s$, or $t \to_{\mathcal{R}} s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ. We write $t \xrightarrow{>p}_{\mathcal{R}} s$ if $t \xrightarrow{q}_{\mathcal{R}} s$ for some $q > p$. A TRS $\mathcal{R}$ is terminating if its one step rewrite relation $\to_{\mathcal{R}}$ is terminating.

2.5 Context-Sensitive Rewriting

A mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [Luc98]. Let $M_{\mathcal{F}}$ be the set of all $\mathcal{F}$-maps (or $M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $(\mathcal{F}, R)$). Let $\mu_{\top}$ be the replacement map given by $\mu_{\top}(f) = \{1, \ldots, ar(f)\}$ for all $f \in \mathcal{F}$ (i.e., no replacement restrictions are specified).

A binary relation R on terms is μ-monotonic if whenever $t \mathrel{\mathsf{R}} s$ we have that $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) \mathrel{\mathsf{R}} f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and $t, s, t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If R is $\mu_{\top}$-monotonic, we just say that R is *monotonic.*

The set of *μ-replacing positions* $\mathcal{P}os^{\mu}(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^{\mu}(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^{\mu}(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^{\mu}(t|_i)$, if $t \notin \mathcal{X}$. When no replacement map is made explicit, the μ-replacing positions are often called *active*; and the non-μ-replacing ones are often called *frozen.* The following result about *CSR* is often used without any explicit mention.

Proposition 1 [Luc98] *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $p = q.q' \in \mathcal{P}os(t)$. Then $p \in \mathcal{P}os^{\mu}(t)$ iff $q \in \mathcal{P}os^{\mu}(t) \wedge q' \in \mathcal{P}os^{\mu}(t|_q)$*

The μ-replacing subterm relation $\trianglerighteq_{\mu}$ is given by $t \trianglerighteq_{\mu} s$ if there is $p \in \mathcal{P}os^{\mu}(t)$ such that $s = t|_p$. We write $t \triangleright_{\mu} s$ if $t \trianglerighteq_{\mu} s$ and $t \neq s$. We write $t \triangleright_{\not\mu} s$ to denote that s is a non-μ-replacing (hence strict) subterm of t: $t \triangleright_{\not\mu} s$ if there is $p \in \mathcal{P}os(t) - \mathcal{P}os^{\mu}(t)$ such that $s = t|_p$. The set of *μ-replacing* variables of a term t, i.e., variables occurring at some μ-replacing position in t, is $\mathcal{V}ar^{\mu}(t) = \{x \in \mathcal{V}ar(t) \mid t \trianglerighteq_{\mu} x\}$. The set of non-$\mu$-replacing variables of t, i.e., variables occurring at some non-μ-replacing position in t, is $\mathcal{V}ar^{\not\mu}(t) = \{x \in \mathcal{V}ar(t) \mid t \triangleright_{\not\mu} x\}$. Note that $\mathcal{V}ar^{\mu}(t)$ and $\mathcal{V}ar^{\not\mu}(t)$ do not need to be disjoint.

A pair $(\mathcal{R}, \mu)$ where $\mathcal{R}$ is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS. In *context-sensitive rewriting*, we (only) contract *μ-replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_{\mu} s$ (or $t \hookrightarrow_{\mathcal{R},\mu} s$ and even $t \hookrightarrow s$), if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^{\mu}(t)$.

Example 6 *Consider $\mathcal{R}$ and μ as in Example 1. Then, we have:*

$$\underline{\texttt{from}(\texttt{0})} \hookrightarrow_\mu \texttt{cons}(\texttt{0}, \underline{\texttt{from}(\texttt{s}(\texttt{0}))}) \not\hookrightarrow_\mu \texttt{cons}(\texttt{0}, \texttt{cons}(\texttt{s}(\texttt{0}), \texttt{from}(\texttt{s}(\texttt{s}(\texttt{0})))))$$

Since the second argument of `cons` *is not μ-replacing, we have that $2 \notin \mathcal{P}os^\mu(\texttt{cons}(\texttt{0}, \texttt{from}(\texttt{s}(\texttt{0}))))$, and the redex $\texttt{from}(\texttt{s}(\texttt{0}))$ cannot be μ-rewritten.*

A term t is μ-terminating (or $(\mathcal{R}, \mu)$-terminating, if we want an explicit reference to the involved TRS $\mathcal{R}$) if there is no infinite μ-rewrite sequence $t = t_1 \hookrightarrow_\mu t_2 \hookrightarrow_\mu \cdots \hookrightarrow_\mu t_n \hookrightarrow_\mu \cdots$ starting from t. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_\mu$ is terminating.

A term t *μ-narrows* to a term s (written $t \leadsto_{\mathcal{R},\mu,\theta} s$), if there is a nonvariable μ-replacing position $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$ and a rule $l \to r$ in $\mathcal{R}$ (sharing no variable with t) such that $t|_p$ and l unify with most general unifier θ and $s = \theta(t[r]_p)$.

3 Minimal non-μ-terminating terms and infinite μ-rewrite sequences

Given a TRS $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$, the *minimal* nonterminating terms associated to $\mathcal{R}$ are nonterminating terms t whose proper subterms u (i.e., $t \rhd u$) are terminating; $\mathcal{T}_\infty$ is the set of minimal nonterminating terms associated to $\mathcal{R}$ [HM04,HM07]. Minimal nonterminating terms have two important properties:

(1) Every nonterminating term s contains a minimal nonterminating term $t \in \mathcal{T}_\infty$ (i.e., $s \unrhd t$), and
(2) minimal nonterminating terms t are always rooted by a *defined* symbol $f \in \mathcal{D}$: $\forall t \in \mathcal{T}_\infty, root(t) \in \mathcal{D}$.

Considering the structure of the infinite rewrite sequences starting from a minimal nonterminating term $t = f(t_1, \ldots, t_k) \in \mathcal{T}_\infty$ is helpful to come to the notion of dependency pair. Such sequences proceed as follows (see, e.g., [HM04]):

(1) a finite number of reductions can be performed *below* the root of t, thus rewriting t into t'; then
(2) a rule $f(l_1, \ldots, l_k) \to r$ applies *at the root* of t' (i.e., $t' = \sigma(f(l_1, \ldots, l_k))$ for some substitution σ); and
(3) there is a minimal nonterminating term $u \in \mathcal{T}_\infty$ (hence $root(u) \in \mathcal{D}$) at some position p of $\sigma(r)$ satisfying that $p \in \mathcal{P}os_\mathcal{F}(r)$, (i.e., p is a *nonvariable position of* r) which 'continues' the infinite sequence initiated by t in a similar way.

This means that considering the occurrences of defined symbols in the right-hand sides of the rewrite rules suffices to 'catch' *every possible infinite rewrite* *sequence starting from* $\sigma(r)$. In particular, no infinite sequence can be issued *below* the variables of r (more precisely: all bindings $\sigma(x)$ are *terminating* terms). This is summarized as follows:

Proposition 2 [HM04, Lemma 1] *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS. For all $t \in \mathcal{T}_\infty$, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{T}_\infty$ such that $root(u) \in \mathcal{D}$, $t \xrightarrow{>\Lambda}{}^* \sigma(l) \xrightarrow{\Lambda} \sigma(r) \trianglerighteq u$ and there is a nonvariable subterm v of r, $r \trianglerighteq v$, such that $u = \sigma(v)$.*

The standard definition of dependency pair relies on (2) and (3) above: after marking $t = f(t_1, \ldots, t_k)$ as $t^\sharp = f^\sharp(t_1, \ldots, t_k)$, only reductions below the root of t are possible; then, such rewritings transform $t^\sharp$ into $\sigma(f^\sharp(l_1, \ldots, l_k))$ for some substitution σ and rule $f(l_1, \ldots, l_k) \to r$ of the TRS. The set of dependency pairs $f^\sharp(l_1, \ldots, l_k) \to v_i^\sharp$ for $1 \leq i \leq n$ associated to such a rule represent all possible ways to continue the infinite sequence initiated by t with a minimal nonterminating term $\sigma(v_i)$.

3.1 Minimal non-μ-terminating terms

Before starting our discussion about (minimal) non-μ-terminating terms, we provide an obvious auxiliary result about μ-terminating terms.

Lemma 1 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_\mathcal{F}$, and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If t is μ-terminating, then:*

(1) If $t \trianglerighteq_\mu s$, then s is μ-terminating.
(2) If $t \hookrightarrow^*_{\mathcal{R},\mu} s$, then s is μ-terminating.

Given a TRS $\mathcal{R} = (\mathcal{F}, R)$ and a replacement map $\mu \in M_\mathcal{F}$, maybe the most straightforward definition of minimal non-μ-terminating terms is the following: let $\mathcal{T}_{\infty,\mu}$ be a set of minimal non-μ-terminating terms in the following sense: t belongs to $\mathcal{T}_{\infty,\mu}$ if t is non-μ-terminating and every strict subterm u (i.e., $t \triangleright u$) is μ-terminating. It is obvious that $root(t) \in \mathcal{D}$ for all $t \in \mathcal{T}_{\infty,\mu}$. We also have:

Lemma 2 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_\mathcal{F}$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If s is not μ-terminating, then there is a subterm t of s ($s \trianglerighteq t$) such that $t \in \mathcal{T}_{\infty,\mu}$.*

Proof. By structural induction. If s is a constant symbol, it is obvious: take $t = s$. If $s = f(s_1, \ldots, s_k)$, then we proceed by contradiction. If there is no subterm t of s such that $t \in \mathcal{T}_{\infty,\mu}$, then in particular $s \notin \mathcal{T}_{\infty,\mu}$, i.e., (since s is not μ-terminating) there is a strict subterm t of s ($s \triangleright t$) which is not μ-terminating. By the Induction Hypothesis, there is $t' \in \mathcal{T}_{\infty,\mu}$ such that $t \trianglerighteq t'$. Then, we have $s \triangleright t'$, thus leading to a contradiction. $\square$

Unfortunately, there can be non-μ-terminating terms having no *μ-replacing* subterm in $\mathcal{T}_{\infty,\mu}$.

Example 7 *Consider the CS-TRS $(\mathcal{R},\mu)$ in Example 4 and $s = \mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a})))$. Note that s is not μ-terminating, but $s \notin \mathcal{T}_{\infty,\mu}$ because $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a}))) \rhd \mathtt{f}(\mathtt{a})$ and $\mathtt{f}(\mathtt{a})$ is not μ-terminating. Note that $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a}))) \not\rhd_{\mu} \mathtt{f}(\mathtt{a})$. The only μ-replacing strict subterm of s is $\mathtt{c}(\mathtt{f}(\mathtt{a}))$, which is μ-terminating, i.e., $\mathtt{c}(\mathtt{f}(\mathtt{a})) \notin \mathcal{T}_{\infty,\mu}$.*

Therefore, this kind of minimal non-μ-terminating terms are not the most natural ones because they could occur at non-μ-replacing positions, where no μ-rewriting step is possible. So, this simple notion would not lead to an appropriate generalization of Proposition 2 to *CSR*. Still, we use them advantageously below; for this reason we pay them some attention here.

There is a suitable generalization of Proposition 2 to *CSR* (see Proposition 4 below) based on the following notion.

Definition 1 (Minimal non-μ-terminating term) *Let $\mathcal{M}_{\infty,\mu}$ be a set of minimal non-μ-terminating terms in the following sense: t belongs to $\mathcal{M}_{\infty,\mu}$ if t is non-μ-terminating and every strict* μ-replacing *subterm s of t (i.e., $t \rhd_{\mu} s$) is μ-terminating.*

Note that $\mathcal{T}_{\infty,\mu} \subseteq \mathcal{M}_{\infty,\mu}$. In the following we often say that terms in $\mathcal{T}_{\infty,\mu}$ are *strongly minimal* non-μ-terminating terms. Now we have the following.

Lemma 3 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$. If s is not μ-terminating, then there is a μ-replacing subterm t of s such that $t \in \mathcal{M}_{\infty,\mu}$.*

Proof. By structural induction. If s is a constant symbol, it is obvious: take $t = s$. If $s = f(s_1,\ldots,s_k)$, then we proceed by contradiction. If there is no μ-replacing subterm t of s such that $t \in \mathcal{M}_{\infty,\mu}$, then in particular $s \notin \mathcal{M}_{\infty,\mu}$, i.e., there is a strict μ-replacing subterm t of s which is not μ-terminating. By the Induction Hypothesis, t contains a μ-replacing subterm t' which belongs to $\mathcal{M}_{\infty,\mu}$. But, since t is a μ-replacing subterm of s (i.e., $t = s|_p$ for some $p \in \mathcal{P}os^{\mu}(s)$), t' itself is also a μ-replacing subterm of s (because $t' = t|_q$ for some $q \in \mathcal{P}os^{\mu}(t)$ and $p.q \in \mathcal{P}os^{\mu}(s)$ by Proposition 1) which belongs to $\mathcal{M}_{\infty,\mu}$, thus leading to a contradiction. $\square$

Obviously, if $t \in \mathcal{M}_{\infty,\mu}$, then $root(t)$ is a defined symbol. Since μ-terminating terms are preserved under μ-rewriting (Lemma 1), it follows that $\mathcal{M}_{\infty,\mu}$ is preserved under *inner* μ-rewritings in the following sense.

Lemma 4 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $t \in \mathcal{M}_{\infty,\mu}$. If $t \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} u$ and u is non-μ-terminating, then $u \in \mathcal{M}_{\infty,\mu}$.*

PROOF. All μ-rewritings below the root are issued on μ-replacing and μ-terminating terms which remain μ-terminating by Lemma 1. Then, if u is not μ-terminating, all its μ-replacing subterms (which are the ones which can be originated or transformed by μ-rewritings from t to u) have to be μ-terminating as well. Hence, $u \in \mathcal{M}_{\infty,\mu}$. □

Lemma 4 does *not* hold for $\mathcal{T}_{\infty,\mu}$: consider the CS-TRS $(\mathcal{R}, \mu)$ in Example 4. We have that $\mathtt{f}(\mathtt{a}) \in \mathcal{T}_{\infty,\mu}$. Now, $\mathtt{f}(\mathtt{a}) \hookrightarrow_\mu \mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a})))$ and $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a})))$ is not μ-terminating. However, $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a}))) \notin \mathcal{T}_{\infty,\mu}$ as shown in Example 7.

4 Hidden terms in minimal μ-rewrite sequences

Given a CS-TRS $(\mathcal{R}, \mu)$ the *hidden terms* are nonvariable terms occurring on some frozen position in the right-hand side of some rule of $\mathcal{R}$. As we show in the next section they play an important role in infinite minimal μ-rewrite sequences associated to $\mathcal{R}$.

Definition 2 (Hidden symbols and terms) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. We say that $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}$ is a* hidden term *if there is a rule $l \to r \in R$ such that $r \rhd_{\not\mu} t$. Let $\mathcal{HT}(\mathcal{R}, \mu)$ (or just $\mathcal{HT}$, if $\mathcal{R}$ and μ are clear for the context) be the set of all hidden terms in $(\mathcal{R}, \mu)$. We say that $f \in \mathcal{F}$ is a* hidden symbol *if it occurs in a hidden term. Let $\mathcal{H}(\mathcal{R}, \mu)$ (or just $\mathcal{H}$) be the set of all hidden symbols in $(\mathcal{R}, \mu)$.*

Example 8 *For $\mathcal{R}$ and μ as in Example 1, the maximal hidden terms are* $\mathtt{from}(\mathtt{s}(x))$, $\mathtt{filt}(x, \mathtt{sieve}(y))$, $\mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), z)$, *and* $\mathtt{cons}(y, \mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), z))$. *The hidden symbols are* `from`, `filt`, `sieve`, `cons` *and* `s`.

In the following, we also use $\mathcal{DHT} = \{t \in \mathcal{HT} \mid root(t) \in \mathcal{D}\}$ for the set of hidden terms which are rooted by a *defined* symbol.

The following lemma says that frozen subterms t in the contractum $\sigma(r)$ of a redex $\sigma(l)$ which do *not* contain t, are (at least partly) 'introduced' by a hidden term in the right-hand side r of the involved rule $l \to r$.

Lemma 5 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution. If there is a rule $l \to r \in R$ such that $\sigma(l) \not\rhd t$ and $\sigma(r) \rhd_{\not\mu} t$, then there is no $x \in \mathcal{V}ar(r)$ such that $\sigma(x) \trianglerighteq t$. Furthermore, there is a term $t' \in \mathcal{HT}$ such that $r \rhd_{\not\mu} t'$ and $\sigma(t') = t$.*

PROOF. By contradiction. If there is $x \in \mathcal{V}ar(r)$ such that $\sigma(x) \trianglerighteq t$, then since variables in l are always below some function symbol we have $\sigma(l) \rhd t$, leading to a contradiction.

Since there is no $x \in \mathcal{V}ar(r)$ such that $\sigma(x) \trianglerighteq t$ but we have that $\sigma(r) \triangleright_{\not\mu} t$, then there is a nonvariable and non-μ-replacing position $p \in \mathcal{P}os_{\mathcal{F}}(r) - \mathcal{P}os^{\mu}(r)$ of r, such that $\sigma(r|_p) = t$. Then, we let $t' = r|_p$. Note that $t' \in \mathcal{HT}$. □

The following lemma establishes that minimal non-μ-terminating and non-μ-replacing subterms occurring in a μ-rewrite sequence involving only minimal terms directly come from the first term in the sequence or are instances of a hidden term.

Lemma 6 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let A be a μ-rewrite sequence $t_1 \hookrightarrow t_2 \hookrightarrow \cdots \hookrightarrow t_n$ with $t_i \in \mathcal{M}_{\infty,\mu}$ for all i, $1 \leq i \leq n$ and $n \geq 1$. If there is a term $t \in \mathcal{M}_{\infty,\mu}$ such that $t_1 \not\trianglerighteq t$ and $t_n \triangleright_{\not\mu} t$, then $t = \sigma(s)$ for some $s \in \mathcal{DHT}$ and substitution σ.*

Proof. By induction on n:

1. If $n = 1$, then it is vacuously true because $t_1 \not\trianglerighteq t$ and $t_1 \triangleright_{\not\mu} t$ do not simultaneously hold.
2. If $n > 1$, then we assume that $t_1 \not\trianglerighteq t$ and $t_n \triangleright_{\not\mu} t$. Let $l \to r \in R$ be such that $t_{n-1} = C[\sigma(l)]$ and $t_n = C[\sigma(r)]$ for some context $C[\,]$ and substitution σ. We consider two cases: either $t_{n-1} \triangleright_{\not\mu} t$ holds or not.
   (a) If $t_{n-1} \triangleright_{\not\mu} t$, then by the induction hypothesis the conclusion follows.
   (b) If $t_{n-1} \triangleright_{\not\mu} t$ does not hold, then, since assuming $t_{n-1} \triangleright_{\mu} t$ leads to a contradiction (because $t_{n-1} \in \mathcal{M}_{\infty,\mu}$ in the hypothesis implies that $t \notin \mathcal{M}_{\infty,\mu}$), we have that $t_{n-1} \not\trianglerighteq t$. In particular, $\sigma(l) \not\trianglerighteq t$; then, since $t_n \triangleright_{\not\mu} t$ there must be $\sigma(r) \triangleright_{\not\mu} t$. Thus, by Lemma 5 we conclude that $t = \sigma(s)$ for some $s \in \mathcal{HT}$ and substitution σ. Since $t \in \mathcal{M}_{\infty,\mu}$, it follows that $root(t) = root(s) \in \mathcal{D}$. Thus, $s \in \mathcal{DHT}$.

□

We use the previous results to investigate infinite sequences that combine μ-rewriting steps on minimal non-μ-terminating terms and the extraction of such subterms as μ-replacing subterms of (instances of) right-hand sides of the rules.

Proposition 3 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let A be a finite or infinite sequence of the form $t_1 \stackrel{\Lambda}{\hookrightarrow} s_1 \trianglerighteq_{\mu} t'_2 \stackrel{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} t_2 \stackrel{\Lambda}{\hookrightarrow} s_2 \trianglerighteq_{\mu} t'_3 \stackrel{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} t_3 \cdots$ with t_i, $t'_i \in \mathcal{M}_{\infty,\mu}$ for all $i \geq 1$. If there is a term $t \in \mathcal{M}_{\infty,\mu}$ such that $t_i \triangleright_{\not\mu} t$ for some $i \geq 1$, then $t_1 \triangleright_{\not\mu} t$ or $t = \sigma(s)$ for some $s \in \mathcal{DHT}$ and substitution σ.*

Proof. By induction on i:

1. If $i = 1$, it is trivial.
2. If $i > 1$ and $t_i \triangleright_{\not\mu} t$, then we consider two cases: either $t_{i-1} \triangleright_{\not\mu} t$ holds or

not.

(a) If $t_{i-1} \rhd_{\not\mu} t$, then by the induction hypothesis the conclusion follows.

(b) If $t_{i-1} \rhd_{\not\mu} t$ does not hold, then let $l \to r \in R$ and σ be such that $t_{i-1} = \sigma(l)$ and $s_{i-1} = \sigma(r) \trianglerighteq_\mu t'_i$. Since $t_{i-1} \rhd_\mu t$ leads to a contradiction (because $t_{i-1} \in \mathcal{M}_{\infty,\mu}$ implies that $t \notin \mathcal{M}_{\infty,\mu}$), we have that $t_{i-1} \not\rhd t$. Then we consider two cases: either $t'_i \rhd t$ or $t'_i \not\rhd t$.

(A) If $t'_i \rhd t$, since $t'_i, t \in \mathcal{M}_{\infty,\mu}$ the case $t'_i \rhd_\mu t$ is excluded and the only possibility is that $t'_i \rhd_{\not\mu} t$. Then, since $\sigma(l) = t_{i-1} \not\rhd t$ and $\sigma(r) \trianglerighteq_\mu t'_i \rhd_{\not\mu} t$, i.e. $\sigma(r) \rhd_{\not\mu} t$, by Lemma 5 we conclude that $t = \sigma(s)$ for some $s \in \mathcal{HT}$ and substitution σ. Since $t \in \mathcal{M}_{\infty,\mu}$, it follows that $root(t) = root(s) \in \mathcal{D}$. Thus, $s \in \mathcal{DHT}$.

(B) If $t'_i \not\rhd t$, then, by applying Lemma 4 and Lemma 6 to the μ-rewrite sequence $t'_i \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} t_i$ the conclusion follows.

□

5 Infinite μ-rewrite sequences starting from minimal terms

The following proposition establishes that, given a minimal non-μ-terminating term $t \in \mathcal{M}_{\infty,\mu}$, there are only two ways for an infinite μ-rewrite sequence to proceed. The first one is by using 'visible' parts of the rules which correspond to μ-replacing nonvariable subterms in the right-hand sides which are rooted by a defined symbol. The second one is by showing up 'hidden' non-μ-terminating subterms which are activated by *migrating* variables in a rule $l \to r$, i.e., variables $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ which are *not* μ-replacing in the left-hand side l but become μ-replacing in the right-hand side r.

Proposition 4 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Then for all $t \in \mathcal{M}_{\infty,\mu}$, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{M}_{\infty,\mu}$ such that $t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma(l) \overset{\Lambda}{\hookrightarrow} \sigma(r) \trianglerighteq_\mu u$ and either*

(1) there is a μ-replacing subterm s of r, $r \trianglerighteq_\mu s$, such that $u = \sigma(s)$, or
(2) there is $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ such that $\sigma(x) \trianglerighteq_\mu u$.

Proof. Consider an infinite μ-rewrite sequence starting from t. By definition of $\mathcal{M}_{\infty,\mu}$, all proper μ-replacing subterms of t are μ-terminating. Therefore, t has an inner reduction to an instance $\sigma(l)$ of the left-hand side of a rule $l \to r$ of $\mathcal{R}$: $t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma(l) \overset{\Lambda}{\hookrightarrow} \sigma(r)$ and $\sigma(r)$ is not μ-terminating. Thus, we can write $t = f(t_1, \ldots, t_k)$ and $\sigma(l) = f(l_1, \ldots, l_k)$ for some k-ary defined symbol f, and $t_i \hookrightarrow^* \sigma(l_i)$ for all i, $1 \leq i \leq k$. Since all t_i are μ-terminating for $i \in \mu(f)$, by Lemma 1, $\sigma(l_i)$ and all its μ-replacing subterms also are. In particular, $\sigma(x)$ is μ-terminating for all μ-replacing variable x in l: $x \in \mathcal{V}ar^\mu(l)$. Since $\sigma(r)$ is non-μ-terminating, by Lemma 3 it contains a μ-replacing subterm $u \in \mathcal{M}_{\infty,\mu}$:

$\sigma(r) \trianglerighteq_\mu u$, i.e., there is a position $p \in \mathcal{P}os^\mu(\sigma(r))$ such that $\sigma(r)|_p = u$. We consider two cases:

(1) If $p \in \mathcal{P}os_\mathcal{F}(r)$ is a nonvariable position of r, then there is a μ-replacing subterm s of r, such that $u = \sigma(s)$.
(2) If $p \notin \mathcal{P}os_\mathcal{F}(r)$, then there is a μ-replacing variable position $q \in \mathcal{P}os^\mu(r) \cap \mathcal{P}os_\mathcal{X}(r)$ such that $q \leq p$. Let $x \in \mathcal{V}ar^\mu(r)$ be such that $r|_q = x$. Then, $\sigma(x) \trianglerighteq_\mu u$ and $\sigma(x)$ is not μ-terminating (by assumption, $u \in \mathcal{M}_{\infty,\mu}$ is not μ-terminating: by Lemma 1, $\sigma(x)$ cannot be μ-terminating either). Since $\sigma(l_i)$ is μ-terminating for all $i \in \mu(f)$, and $\sigma(x)$ is also μ-terminating for all μ-replacing variables in l, we conclude that $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$.

□

Proposition 4 entails the following result, which establishes some properties of infinite sequences starting from minimal non-μ-terminating terms.

Corollary 1 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_\mathcal{F}$. For all $t \in \mathcal{M}_{\infty,\mu}$, there is an infinite sequence*

$$t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow} \sigma_1(r_1) \trianglerighteq_\mu t_1 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow} \sigma_2(r_2) \trianglerighteq_\mu t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$ are rewrite rules, σ_i are substitutions, and terms $t_i \in \mathcal{M}_{\infty,\mu}$ are minimal non-μ-terminating terms such that either

(1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$, or
(2) $\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$.

Remark 2 *The $(\hookrightarrow_\mu \cup \trianglerighteq_\mu)$-sequence in Corollary 1 can be easily viewed as an infinite μ-rewrite sequence by just introducing appropriate contexts $C_i[\,]_{p_i}$ with μ-replacing holes: since $\sigma_i(r_i) \trianglerighteq_\mu t_i$, there is $p_i \in \mathcal{P}os^\mu(\sigma_i(r_i))$ such that $\sigma_i(r_i) = \sigma_i(r_i)[t_i]_{p_i}$; just take $C_i[\,]_{p_i} = \sigma(r_i)[\Box]_{p_i}$. Then we get:*

$$t \hookrightarrow^*_{\mathcal{R},\mu} \sigma_1(l_1) \hookrightarrow_{\mathcal{R},\mu} C_1[t_1]_{p_1} \hookrightarrow^*_{\mathcal{R},\mu} C_1[\sigma_2(l_2)]_{p_1} \hookrightarrow_{\mathcal{R},\mu} C_1[C_2[t_2]_{p_2}]_{p_1} \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

Note that, e.g., $p_1.p_2 \in \mathcal{P}os^\mu(C_1[C_2[t_2]_{p_2}]_{p_1})$ (use Proposition 1).

5.1 Infinite μ-rewrite sequences starting from strongly minimal terms

In the following, we consider a function $\textsc{Ren}^\mu$ which *independently* renames all *occurrences* of μ-replacing variables within a term t by using new fresh variables which are not in $\mathcal{V}ar(t)$:

- $\textsc{Ren}^\mu(x) = y$ if x is a variable, where y is intended to be a fresh new variable which has not yet been used (we could think of y as the 'next' variable in an infinite list of variables); and

- $\text{REN}^\mu(f(t_1,\ldots,t_k)) = f([t_1]^f_1,\ldots,[t_k]^f_k)$ for evey k-ary symbol f, where given a term $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$, $[s]^f_i = \text{REN}^\mu(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$.

Note that $\text{REN}^\mu(t)$ renames all μ-replacing positions of variables in t by new fresh variables y but keeps variables at non-μ-replacing positions untouched. Note that, in contrast to a *renaming* substitution (often denoted by ρ), REN^μ is *not* a substitution: it will replace different μ-replacing occurrences of the same variable by different variables.

Proposition 5 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_\mathcal{F}$. Let $t \in \mathcal{T}(\mathcal{F},\mathcal{X})-\mathcal{X}$ be a nonvariable term and σ be a substitution. If $\sigma(t) \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma(l)$ for some (probably renamed) rule $l \to r \in R$, then $\text{REN}^\mu(t)$ is μ-narrowable.*

PROOF. We can write the sequence from $\sigma(t)$ to $\sigma(l)$ as follows: $\sigma(t) = t_1 \overset{>\Lambda}{\hookrightarrow} t_2 \overset{>\Lambda}{\hookrightarrow} \cdots \overset{>\Lambda}{\hookrightarrow} t_m = \sigma(l)$ for some $m \geq 1$. We proceed by induction on m.

(1) If $m = 1$, then $\sigma(t) = \sigma(l)$. Since $t \notin \mathcal{X}$, t is μ-narrowable (at the root position) using the rule $l \to r$. Therefore, $\text{REN}^\mu(t)$ is also μ-narrowable by using the unifier σ' which, for each $x_i \in \mathcal{V}ar^\mu(t)$ identifies the new fresh variables $y_1,\ldots,y_{M_i}$ introduced by REN^μ for the M_i different μ-replacing positions of x_i in t: $\sigma'(x) = \sigma(x)$ for all $x \in \mathcal{V}ar(l)$, $\sigma'(y_j) = \sigma'(x_i) = \sigma(x_i)$ for all $x_i \in \mathcal{V}ar^\mu(t)$ and $1 \leq j \leq M_i$, and $\sigma'(x) = \sigma(x)$ for all $x \in \mathcal{V}ar(t) - \mathcal{V}ar^\mu(t)$. Clearly, $\sigma'(\text{REN}^\mu(t)) = \sigma'(t) = \sigma'(l) = \sigma(l)$, i.e., $\text{REN}^\mu(t)$ is narrowable at the root position using the same rule $l \to r$.

(2) If $m > 1$, then we have $t_1 \overset{>\Lambda}{\hookrightarrow} t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} \sigma(l)$. We consider two cases according to the position $p \in \mathcal{P}os^\mu(t_1)$ where the μ-rewrite step $t_1 \overset{>\Lambda}{\hookrightarrow} t_2$ is performed (note that $t_1 = \sigma(t)$ by assumption).
   (a) If $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$, then there is a rule $l' \to r'$ and a substitution θ such that $\sigma(t)|_p = \sigma(t|_p) = \theta(l')$. Again, w.l.o.g. we can write $\sigma(t|_p) = \sigma(l')$, i.e., t is μ-narrowable at position p using rule $l' \to r'$ and (reasoning as above), we conclude that $\text{REN}^\mu(t)$ also is.
   (b) If $p \notin \mathcal{P}os^\mu_\mathcal{F}(t)$, then there is a μ-replacing variable position $q \in \mathcal{P}os^\mu_\mathcal{X}(t)$ of t such that $t|_q = x \in \mathcal{V}ar^\mu(t)$, $q \leq p$ and $\sigma(x) \hookrightarrow_\mu t_2|_q$. Therefore, $t_1 = \sigma(t[x]_q) = \sigma(t)[\sigma(x)]_q$ and $t_2 = \sigma(t)[t_2|_q]_q = \sigma'(t')$ for a term $t' = t[y]_q$ where y is a new fresh variable $y \notin \mathcal{V}ar(t)$ and a substitution σ' given by $\sigma'(y) = t_2|_q$ and $\sigma'(z) = \sigma(z)$ for all $z \in \mathcal{V}ar(t)$ (including x). Clearly,

$$\sigma'(t') = \sigma'(t[y]_q) = \sigma'(t)[\sigma'(y)]_q = \sigma(t)[t_2|_q]_q = t_2.$$

   By the induction hypothesis, $\text{REN}^\mu(t')$ is μ-narrowable. Since t and t' only differ in a variable, we can assume that $\text{REN}^\mu(t') = \text{REN}^\mu(t)$. Thus, we conclude that $\text{REN}^\mu(t)$ is μ-narrowable as well.

$\square$

Corollary 2 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}$ be a nonvariable term and σ be a substitution such that $\sigma(t) \in \mathcal{M}_{\infty,\mu}$. Then, $\textsc{Ren}^{\mu}(t)$ is μ-narrowable.*

Proof. By Proposition 4, there is a rule $l \to r$ and a substitution σ such that $\sigma(t) \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma(l)$ (since we can assume that variables in l and variables in t are disjoint we can apply the same substitution σ to t and l without any problem). By Proposition 5, the conclusion follows. $\square$

In the following, we write $\textsc{Narr}^{\mu}(t)$ to indicate that t is μ-narrowable (w.r.t. the intended TRS $\mathcal{R}$). We also let

$$\mathcal{NHT}(\mathcal{R}, \mu) = \{t \in \mathcal{DHT} \mid \textsc{Narr}^{\mu}(\textsc{Ren}^{\mu}(t))\}$$

be the set of *hidden terms* which are rooted by a *defined* symbol, and that, after applying $\textsc{Ren}^{\mu}$, become *μ-narrowable*. As a consequence of the previous results, we have the following main result which we will use later.

Theorem 1 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. For all $t \in \mathcal{T}_{\infty,\mu}$, there is an infinite sequence*

$$t = t_0 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow} \sigma_1(r_1) \trianglerighteq_{\mu} t_1 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow} \sigma_2(r_2) \trianglerighteq_{\mu} t_2 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$ are rewrite rules, σ_i are substitutions, and terms $t_i \in \mathcal{M}_{\infty,\mu}$ are minimal non-μ-terminating terms such that either

(1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_{\mu} s_i$, or
(2) $\sigma_i(x_i) \trianglerighteq_{\mu} t_i$ for some $x_i \in \mathcal{V}ar^{\mu}(r_i) - \mathcal{V}ar^{\mu}(l_i)$ and $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{NHT}$ and substitution θ_i.

Proof. Since $\mathcal{T}_{\infty,\mu} \subseteq \mathcal{M}_{\infty,\mu}$, by Corollary 1, we have a sequence

$$t = t_0 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow} \sigma_1(r_1) \trianglerighteq_{\mu} t_1 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow} \sigma_2(r_2) \trianglerighteq_{\mu} t_2 \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$, σ_i are substitutions, $t_i \in \mathcal{M}_{\infty,\mu}$, and either (1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_{\mu} s_i$ or (2) $\sigma_i(x_i) \trianglerighteq_{\mu} t_i$ for some $x_i \in \mathcal{V}ar^{\mu}(r_i) - \mathcal{V}ar^{\mu}(l_i)$ (and hence $\sigma(l_i) \rhd_{\not\mu} t_i$ and $\sigma(r_i) \trianglerighteq_{\mu} t_i$ as well). We only need to prove that terms t_i are instances of hidden terms in $\mathcal{NHT}$ whenever the second condition holds. By Proposition 3, for all such terms t_i, we have that either (A) $\sigma_1(l_1) \rhd_{\not\mu} t_i$ or (B) $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{DHT}$ and substitution θ_i. In the second case (B), we are done by just considering Corollary 2, which ensures that $t'_i \in \mathcal{NHT}$. In the first one (A), since $t \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathcal{R},\mu} \sigma_1(l_1)$ and $\sigma_1(l_1)$ is not μ-terminating, by Lemma 4 all terms u_j in the μ-rewrite sequence

$$t = u_1 \overset{>\Lambda}{\hookrightarrow} u_2 \overset{>\Lambda}{\hookrightarrow} \cdots \overset{>\Lambda}{\hookrightarrow} u_m = \sigma_1(l_1)$$

for $m \geq 1$, belong to $\mathcal{M}_{\infty,\mu}$: $u_j \in \mathcal{M}_{\infty,\mu}$ for all j, $1 \leq j \leq m$. Since $t \in \mathcal{T}_{\infty,\mu}$, all its strict subterms (disregarding their μ-replacing character) are μ-terminating. Therefore, $t \not\trianglerighteq t_i$ (because t_i is not μ-terminating) and by Lemma 6, $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{DHT}$ and substitution θ_i. Again, by Corollary 2 we have $t'_i \in \mathcal{NHT}$. $\square$

6 Context-Sensitive Dependency Pairs

Lemma 2 and Theorem 1 are the basis for our definition of *Context-Sensitive Dependency Pairs* (and the corresponding *chains*). Together, they show that every non-μ-terminating term s has an associated infinite μ-rewrite sequence starting from a strongly minimal subterm $t \in \mathcal{T}_{\infty,\mu}$ (i.e., $s \trianglerighteq t$). Such a sequence proceeds by first performing some μ-rewriting steps *below the root* of t to obtain a term t' (i.e., $t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} t'$) and then applying a rule $l \to r$ at the *topmost* position of t' (i.e., $t' = \sigma(l)$ for some matching substitution σ). According to Proposition 4, the application of such a rule either

(1) *introduces* a new minimal non-μ-terminating subterm u having a prefix s which is a μ-replacing subterm of r (i.e., $r \trianglerighteq_\mu s$ and $u = \sigma(s)$). Furthermore, by Corollary 2, $\textsc{Ren}^\mu(s)$ must be μ-narrowable; or else
(2) *takes* a minimal non-μ-terminating and non-μ-replacing subterm u of t' (i.e., $t' \rhd_{\not\mu} u$) and
   (a) brings it up to an *active* position by means of the binding $\sigma(x)$ (i.e., $\sigma(x) \trianglerighteq_\mu u$) for some *migrating variable* x in $l \to r$ (i.e., $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$).
   (b) At this point, we know that u, which is rooted by a defined symbol due to $u \in \mathcal{M}_{\infty,\mu}$, is an instance of a hidden term $u' \in \mathcal{NHT}$: $u = \theta(u')$ for some substitution θ.
   (c) Afterwards, further *inner* μ-rewritings on u lead to match the left-hand-side l' of a new rule $l' \to r'$ and everything starts again.

This process is abstracted in the following definition of *context-sensitive dependency pairs* and in the definition of chain below.

Given a signature $\mathcal{F}$ and $f \in \mathcal{F}$, we let $f^\sharp$ be a new fresh symbol (often called *tuple* symbol or DP-symbol) associated to a symbol f [AG00]. Let $\mathcal{F}^\sharp$ be the set of tuple symbols associated to symbols in $\mathcal{F}$. As usual, for $t = f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^\sharp$ to denote the *marked* term $f^\sharp(t_1, \ldots, t_k)$. Conversely, given a marked term $t = f^\sharp(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^\natural$ to denote the term $f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Let $\mathcal{T}^\sharp(\mathcal{F}, \mathcal{X}) = \{t^\sharp \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}\}$ be the set of marked terms.

Definition 3 (Context-Sensitive Dependency Pairs) *Let* $\mathcal{R} = (\mathcal{F}, R) =$

$(\mathcal{C} \uplus \mathcal{D}, R)$ *be a TRS and* $\mu \in M_{\mathcal{F}}$. *We define* $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) \cup \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ *to be the set of* context-sensitive *dependency pairs (CSDPs) where:*

$$\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) = \{l^\sharp \to s^\sharp \mid l \to r \in R, r \trianglerighteq_\mu s, root(s) \in \mathcal{D}, l \ntriangleright_\mu s, \textsc{Narr}^\mu(\textsc{Ren}^\mu(s))\}$$
$$\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \{l^\sharp \to x \mid l \to r \in R, x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)\}$$

We extend $\mu \in M_{\mathcal{F}}$ *into* $\mu^\sharp \in M_{\mathcal{F}\cup\mathcal{D}^\sharp}$ *by* $\mu^\sharp(f) = \mu(f)$ *if* $f \in \mathcal{F}$, *and* $\mu^\sharp(f^\sharp) = \mu(f)$ *if* $f \in \mathcal{D}$.

The CSDPs $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ in Definition 3, consisting of collapsing rules only, are called the *collapsing* CSDPs.

A rule $l \to r$ of a TRS $\mathcal{R}$ is μ-conservative if $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$, i.e., it does not contain migrating variables; $\mathcal{R}$ is μ-conservative if all its rules are (see [Luc96,Luc06]). The following fact is obvious from Definition 3.

Proposition 6 *If* $\mathcal{R}$ *is a* μ*-conservative TRS, then* $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$.

Therefore, in order to deal with μ-conservative TRSs $\mathcal{R}$ we only need to consider the 'classical' dependency pairs in $\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$.

Example 9 *Consider the following TRS* $\mathcal{R}$*:*

$$\mathtt{g}(x) \to \mathtt{h}(x) \qquad\qquad \mathtt{h}(\mathtt{d}) \to \mathtt{g}(\mathtt{c})$$
$$\mathtt{c} \to \mathtt{d}$$

together with $\mu(\mathtt{g}) = \mu(\mathtt{h}) = \varnothing$ *[Zan97, Example 1]. Note that* $\mathcal{R}$ *is* μ*-conservative.* $\mathsf{DP}(\mathcal{R},\mu)$ *consists of the following (noncollapsing) CSDPs:*

$$\mathtt{G}(x) \to \mathtt{H}(x) \qquad\qquad \mathtt{H}(\mathtt{d}) \to \mathtt{G}(\mathtt{c})$$

with $\mu^\sharp(\mathtt{G}) = \mu^\sharp(\mathtt{H}) = \varnothing$.

If the TRS $\mathcal{R}$ contains non-μ-conservative rules, then we also need to consider dependency pairs with variables in the right-hand side.

Example 10 *For the CS-TRS* $(\mathcal{R},\mu)$ *in Example 1, we have six CSDPs:* (1)*,* (2)*, and* (9) *as in Example 2 plus the following three* collapsing *CSDPs:*

$$\mathtt{TAIL}(\mathtt{cons}(x,y)) \to y \tag{10}$$
$$\mathtt{IF}(\mathtt{true},x,y) \to x \tag{11}$$
$$\mathtt{IF}(\mathtt{false},x,y) \to y \tag{12}$$

7 Chains of CSDPs

An essential property of the dependency pairs method is that it provides a *characterization* of termination of TRSs $\mathcal{R}$ as the absence of infinite (minimal) *chains of dependency pairs* [AG00,GTSF06]. As we prove in Section 8, this is also true for *CSR* when CSDPs are considered. First, we have to introduce a suitable notion of chain which can be used with CSDPs. As in the DP-framework [GTS04,GTSF06], where the procedence of *pairs* does not matter, we rather think of another TRS $\mathcal{P}$ which is used together with $\mathcal{R}$ to build the chains. Once this more abstract notion of chain is introduced, it can be particularized to be used with CSDPs, by just taking $\mathcal{P} = \mathsf{DP}(\mathcal{R}, \mu)$.

Definition 4 (Chain of pairs - Minimal chain) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. A $(\mathcal{P}, \mathcal{R}, \mu)$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in \mathcal{P}$, together with a substitution σ satisfying that, for all $i \geq 1$:*

(1) if $v_i \notin \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$, then $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$, and
(2) if $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$, then there is $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(v_i) \trianglerighteq_{\mu} s_i$ and $s_i^{\sharp} \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$.

As usual, we assume that different occurrences of dependency pairs do not share any variable (renaming substitutions are used if necessary).

A $(\mathcal{P}, \mathcal{R}, \mu)$-chain is called minimal *if for all $i \geq 1$,*

(1) if $v_i \notin \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$, then $\sigma(v_i)$ is $(\mathcal{R}, \mu)$-terminating, and
(2) if $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$, then $s_i^{\sharp}$ is $(\mathcal{R}, \mu)$-terminating and $\exists \bar{s}_i \in \mathcal{NHT}(\mathcal{R}, \mu)$ such that $s_i = \sigma(\bar{s}_i)$.

Note that the condition $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$ in Definition 4 implies that v_i is a variable. Furthermore, since each $u_i \to v_i \in \mathcal{P}$ is a rewrite rule (i.e., $\mathcal{V}ar(v_i) \subseteq \mathcal{V}ar(u_i)$), v_i is a *migrating variable* in the rule $u_i \to v_i$.

Remark 3 (Conventions about $\mathcal{P}$) *The following conventions about the component $\mathcal{P} = (\mathcal{G}, P)$ of our chains will be observed during our development:*

(1) According to the usual terminology [GTSF06], we often call pairs *to the rules $u \to v \in \mathcal{P}$.*
(2) Marking is part of the definition of chain: we have to mark terms $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ before connecting *them to the instance $\sigma(u_{i+1})$ of the left-hand side of the next pair. Since marked symbols $f^{\sharp}$ are* fresh *(w.r.t. the signature $\mathcal{F}$ of the TRS $\mathcal{R}$), we also assume that $\mathcal{D}^{\sharp} \cap \mathcal{F} = \varnothing$ and $\mathcal{D}^{\sharp} \subseteq \mathcal{G}$ (since we only mark defined symbols, we do not need to extend the marking to $\mathcal{F}$).*

(3) *We also silently assume that $\mathcal{P}$ contains a* finite *set of rules. This is essential in many proofs.*

In the following, the pairs in a CS-TRS $(\mathcal{P},\mu)$, where $\mathcal{P}=(\mathcal{G},P)$, are partitioned according to its role in Definition 4 as follows:

$$P_{\mathcal{X}} = \{u \to v \in P \mid v \in \mathcal{V}ar(u) - \mathcal{V}ar^{\mu}(u)\} \text{ and } P_{\mathcal{G}} = P - P_{\mathcal{X}}$$

Remark 4 (Collapsing pairs) *Note that all pairs in $\mathcal{P}_{\mathcal{X}} = (\mathcal{G},P_{\mathcal{X}})$ are* collapsing*. The rules in $\mathcal{P}_{\mathcal{G}} = (\mathcal{G},P_{\mathcal{G}})$* can be collapsing *as well: a rewrite rule $f(x) \to x \in \mathcal{P}$ with $\mu(f)=\{1\}$ does* not *belong to $\mathcal{P}_{\mathcal{X}}$ but rather to $\mathcal{P}_{\mathcal{G}}$ because x is not a migrating variable.*

Despite this fact, we refer to $\mathcal{P}_{\mathcal{X}}$ as the set of collapsing *pairs in $\mathcal{P}$ because its intended role in Definition 4 is capturing the computational behavior of collapsing CSDPs in $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$.*

Remark 5 (Notation for chains) *In general, a $(\mathcal{P},\mathcal{R},\mu)$-chain can be written as follows:*

$$\sigma(u_1) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq_{\mu}^{\sharp} t_1 \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_2) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq_{\mu}^{\sharp} t_2 \hookrightarrow^{*}_{\mathcal{R},\mu} \cdots$$

where, for all $i \geq 1$ and $u_i \to v_i \in \mathcal{P}$,

(1) if $u_i \to v_i \notin \mathcal{P}_{\mathcal{X}}$, then $t_i = \sigma(v_i)$,
(2) if $u_i \to v_i \in \mathcal{P}_{\mathcal{X}}$, then $t_i = s_i^{\sharp}$ for some term s_i such that $\sigma(v_i) \trianglerighteq_{\mu} s_i$.

The relation $\trianglerighteq_{\mu}^{\sharp}$ is defined as follows:

- *$s \trianglerighteq_{\mu}^{\sharp} t$ is equivalent to $s \trianglerighteq_{\mu} t^{\natural}$ if $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and $t \in \mathcal{T}^{\sharp}(\mathcal{F},\mathcal{X})$, and*
- *$s \trianglerighteq_{\mu}^{\sharp} t$ is equivalent to $s = t$ for $s,t \in \mathcal{T}^{\sharp}(\mathcal{F},\mathcal{X})$.*

7.1 Properties of some particular chains

In the following, we let $\mathcal{NHT}_{\mathcal{P}}(\mathcal{R},\mu) \subseteq \mathcal{NHT}(\mathcal{R},\mu)$ (or just $\mathcal{NHT}_{\mathcal{P}}$ if $\mathcal{R}$ and μ are clear from the context) be as follows:

$$\mathcal{NHT}_{\mathcal{P}}(\mathcal{R},\mu) = \{t \in \mathcal{NHT}(\mathcal{R},\mu) \mid \exists u \to v \in \mathcal{P}, \exists \theta, \theta', \theta(t^{\sharp}) \hookrightarrow^{*}_{\mathcal{R},\mu} \theta'(u)\}$$

This set contains the narrowable hidden terms which 'connect' with some pair in $\mathcal{P}$.

Remark 6 *Note that $\mathcal{NHT}_{\mathcal{P}}(\mathcal{R},\mu)$ is not computable, in general, due to the need of checking the reachability of $\theta'(u)$ from $\theta(t^{\sharp})$ using* CSR*. Suitable (over)*approximations *are discussed below.*

We let $\mathcal{P}^1_\mathcal{X}$ denote the subTRS of $\mathcal{P}_\mathcal{X}$ containing the rules whose migrating variables occur on non-μ-replacing immediate subterms in the left-hand side:

$$\mathcal{P}^1_\mathcal{X} = \{f(u_1,\ldots,u_k) \to x \in \mathcal{P}_\mathcal{X} \mid \exists i, 1 \leq i \leq k, i \notin \mu(f), x \in \mathcal{V}ar(u_i)\}$$

Proposition 7 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

(1) If $\mathcal{NHT}_\mathcal{P} = \varnothing$, then every infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain is an infinite minimal $(\mathcal{P}_\mathcal{G},\mathcal{R},\mu)$-chain and there is no infinite minimal $(\mathcal{P}_\mathcal{X},\mathcal{R},\mu)$-chain.
(2) If $\mathcal{P} = \mathcal{P}^1_\mathcal{X}$, then there is no infinite $(\mathcal{P},\mathcal{R},\mu)$-chain.

Proof.

(1) By contradiction. Assume that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain containing any $u_i \to v_i \in \mathcal{P}_\mathcal{X}$. By Definition 4, such a pair must be followed by a pair $u_{i+1} \to v_{i+1} \in \mathcal{P}$ such that $\theta_i(\bar{s}^\sharp_i) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$ for some $\bar{s}_i \in \mathcal{NHT}$ and substitution θ_i. Therefore, $t'_i \in \mathcal{NHT}_\mathcal{P}$, but $\mathcal{NHT}_\mathcal{P} = \varnothing$, leading to a contradiction.
(2) By contradiction. Assume that there is an infinite chain which only uses dependency pairs $u_i \to x_i \in \mathcal{P}^1_\mathcal{X}$ for all $i \geq 1$. Let $f_i = root(u_i)$ for $i \geq 1$. Then, by definition of $\mathcal{P}^1_\mathcal{X}$, for all $i \geq 1$ there is $j_i \in \{1,\ldots,ar(f_i)\} - \mu(f_i)$ such that $u_i|_{j_i} \trianglerighteq x_i$. According to Definition 4, we have that $\sigma(u_i)|_{j_i} \trianglerighteq \sigma(x_i) \trianglerighteq_\mu s_i$ for some term s_i such that $s^\sharp_i \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$, with $root(s^\sharp_i) = root(u_{i+1}) = f_{i+1}$ and $j_{i+1} \notin \mu(f_{i+1})$. Since no μ-rewriting step is possible on the j_{i+1}-th immediate subterm $s_i|_{j_{i+1}}$ of s_i, it follows that $s_i|_{j_{i+1}} = \sigma(u_{i+1})|_{j_{i+1}} \trianglerighteq \sigma(x_{i+1})$, i.e., $\sigma(x_i) \triangleright \sigma(x_{i+1})$ for all $i \geq 1$. We get an infinite sequence $\sigma(x_1) \triangleright \sigma(x_2) \triangleright \cdots$ which contradicts well-foundedness of $\triangleright$.

□

The following proposition establishes some important 'basic' cases of (absence of) infinite context-sensitive chains of pairs which are used later.

Proposition 8 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

(1) If $P = \varnothing$, then there is no $(\mathcal{P},\mathcal{R},\mu)$-chain.
(2) If $R = \varnothing$, then there is no infinite $(\mathcal{P}_\mathcal{X},\mathcal{R},\mu)$-chain.
(3) Let $u \to v \in \mathcal{P}_\mathcal{G}$ be such that $v' = \theta(u)$ for some substitution θ and renamed version v' of v. Then, there is an infinite $(\mathcal{P},\mathcal{R},\mu)$-chain.

Proof.

(1) Obvious, by Definition 4.
(2) By contradiction. If there is an infinite $(\mathcal{P}_\mathcal{X},\mathcal{R},\mu)$-chain, then, since there

is no rule in $\mathcal{R}$, there is a substitution σ such that

$$\sigma(u_1) \hookrightarrow_{\mathcal{P},\mu} \sigma(x_1) \trianglerighteq^\sharp_\mu t_1 = \sigma(u_2) \hookrightarrow_{\mathcal{P},\mu} \sigma(x_2) \trianglerighteq^\sharp_\mu t_2 = \sigma(u_3) \cdots$$

where $t_i = s_i^\sharp$ for some terms $s_i \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ for $i \geq 1$. Since $x_i \in \mathcal{V}ar(u_i)$ and u_i is not a variable, we have $u_i \triangleright x_i$, hence $\sigma(u_i) \triangleright \sigma(x_i)$ (by stability of $\triangleright$), and also $\sigma(u_i) \triangleright s_i$ for all $i \geq 1$. Since s_i and $\sigma(u_{i+1})$ only differ in the root symbol, we can actually say that $s_i \triangleright s_{i+1}$ for all $i \geq 1$. Thus, we obtain an infinite sequence $s_1 \triangleright s_2 \triangleright \cdots$ which contradicts the well-foundedness of $\triangleright$.

(3) Since we always deal with *renamed* versions $u_i \to v_i$ of the pair $u \to v \in \mathcal{P}$, for each $x \in \mathcal{V}ar(u)$, we write x_i to denote the 'name' of the variable x in $u_i \to v_i$. According to our hypothesis, we can assume the existence of substitutions θ_{i+1} such that $v_i = \theta_{i+1}(u_{i+1})$. Note that, for all $x \in \mathcal{V}ar(u)$ and $i \geq 1$, $\mathcal{V}ar(\theta_{i+1}(u_{i+1})) \subseteq \mathcal{V}ar(v_i) \subseteq \mathcal{V}ar(u_i)$. We can define an infinite $(\varnothing, \{u \to v\}, \mu)$-chain (hence an $(\mathcal{P},\mathcal{R},\mu)$-chain) by using the renamed versions $u_i \to v_i$ of $u \to v$ for $i \geq 1$ together with σ given (inductively) as follows: for all $x \in \mathcal{V}ar(u)$, $\sigma(x_1) = x_1$ and $\sigma(x_i) = \sigma(\theta_i(x_i))$ for all $i > 1$. Note that $\sigma(v_i) = \sigma(\theta_{i+1}(u_{i+1})) = \sigma(u_{i+1})$ for all $i \geq 1$.

□

The following example shows that Proposition 8(2) does not hold for TRSs $\mathcal{P}$ with arbitrary rules.

Example 11 *Consider $\mathcal{P} = \{\mathtt{F}(x) \to x, \mathtt{G}(x) \to \mathtt{F}(\mathtt{g}(x))\}$ together with a TRS $\mathcal{R}$ with an emtpy set of rules: $\mathcal{R} = (\mathcal{F}, \varnothing)$. Let μ be given by $\mu(f) = \varnothing$ for all $f \in \mathcal{F} \cup \mathcal{G}$. Note that $\mathcal{P}_\mathcal{X}$ consists of the pair $\mathtt{F}(x) \to x$ because $x \in \mathcal{V}ar(\mathtt{F}(x)) - \mathcal{V}ar^\mu(\mathtt{F}(x))$. Then, we have an infinite chain*

$$\mathtt{F}(\mathtt{g}(x)) \hookrightarrow_{\mathcal{P},\mu} \mathtt{g}(x) \trianglerighteq^\sharp_\mu \mathtt{G}(x) \hookrightarrow_{\mathcal{P},\mu} \mathtt{F}(\mathtt{g}(x)) \hookrightarrow_{\mathcal{R},\mu} \cdots$$

Note that this chain is not *minimal because $\mathcal{NHT} = \varnothing$, hence $\mathtt{g}(x)$ is not an instance of any term in $\mathcal{NHT}$.*

8 Characterizing termination of *CSR* using chains of CSDPs

The following result establishes the soundness of the context-sensitive dependency pairs approach. As usual, in order to fit the requirement of variable-disjointness among two arbitrary pairs in a chain of pairs, we assume that appropriately renamed CSDPs are available when necessary.

Theorem 2 (Soundness) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If there is no infinite minimal $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain, then $\mathcal{R}$ is μ-terminating.*

Proof. By contradiction. If $\mathcal{R}$ is not μ-terminating, then by Lemma 2 there is $t \in \mathcal{T}_{\infty,\mu}$. By Theorem 1, there are rules $l_i \to r_i \in \mathcal{R}$, matching substitutions σ_i, and terms $t_i \in \mathcal{M}_{\infty,\mu}$, for $i \geq 1$ such that

$$t = t_0 \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \sigma_1(l_1) \xhookrightarrow{\Lambda} \sigma_1(r_1) \trianglerighteq_\mu t_1 \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \sigma_2(l_2) \xhookrightarrow{\Lambda} \sigma_2(r_2) \trianglerighteq_\mu t_2 \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \cdots$$

where either (D1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$ or (D2) $\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$ and $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{NHT}$. Furthermore, since $t_{i-1} \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \sigma_i(l_i)$ and $t_{i-1} \in \mathcal{M}_{\infty,\mu}$ (in particular, $t_0 = t \in \mathcal{T}_{\infty,\mu} \subseteq \mathcal{M}_{\infty,\mu}$), by Lemma 4, $\sigma_i(l_i) \in \mathcal{M}_{\infty,\mu}$ for all $i \geq 1$. Note that, since $t_i \in \mathcal{M}_{\infty,\mu}$, we have that $t_i^\sharp$ is μ-terminating (with respect to $\mathcal{R}$), because all μ-replacing subterms of t_i (hence of $t_i^\sharp$ as well) are μ-terminating and $root(t^\sharp)$ is not a defined symbol of $\mathcal{R}$.

First, note that $\mathsf{DP}(\mathcal{R},\mu)$ is a TRS $\mathcal{P}$ over the signature $\mathcal{G} = \mathcal{F} \cup \mathcal{D}^\sharp$ and $\mu^\sharp \in M_{\mathcal{F}\cup\mathcal{G}}$ as required by Definition 4. Furthermore, $\mathcal{P}_\mathcal{G} = \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$ and $\mathcal{P}_\mathcal{X} = \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$. We can define an infinite minimal $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain using CSDPs $u_i \to v_i$ for $i \geq 1$, where $u_i = l_i^\sharp$ and

(1) $v_i = s_i^\sharp$ if (D1) holds. Since $t_i \in \mathcal{M}_{\infty,\mu}$, we have that $root(s_i) \in \mathcal{D}$ and, because $t_i = \sigma_i(s_i)$, by Corollary 2 $\textsc{Ren}^\mu(s_i)$ is μ-narrowable. Furthermore, if we assume that s_i is a μ-replacing subterm of l_i (i.e., $l_i \triangleright_\mu s_i$), then $\sigma_i(l_i) \triangleright_\mu \sigma_i(s_i)$ which (since $\sigma_i(s_i) = t_i \in \mathcal{M}_{\infty,\mu}$) contradicts that $\sigma_i(l_i) \in \mathcal{M}_{\infty,\mu}$. Thus, $l_i \not\trianglerighteq_\mu s_i$. Hence, $u_i \to v_i \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$. Furthermore, $t_i^\sharp = \sigma_i(v_i)$ is μ-terminating. Finally, since $t_i = \sigma_i(s_i) \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \sigma_{i+1}(l_{i+1})$ and $\mu^\sharp$ extends μ to $\mathcal{F} \cup \mathcal{D}^\sharp$ by $\mu^\sharp(f^\sharp) = \mu(f)$ for all $f \in \mathcal{D}$, we also have that $\sigma_i(v_i) = \sigma_i(s_i^\sharp) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma_{i+1}(u_{i+1})$.

(2) $v_i = x_i$ if (D2) holds. Clearly, $u_i \to v_i \in \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$. As discussed above, $t_i^\sharp$ is μ-terminating. Since $\sigma_i(x_i) \trianglerighteq_\mu t_i$, we have that $\sigma_i(v_i) \trianglerighteq_\mu t_i$. Finally, since $t_i \xhookrightarrow{>\Lambda}{}^*_{\mathcal{R},\mu} \sigma_{i+1}(l_{i+1})$, again we have that $u_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma_{i+1}(u_{i+1})$. Furthermore, $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{NHT}$.

Regarding σ, w.l.o.g. we can assume that $\mathcal{V}ar(l_i) \cap \mathcal{V}ar(l_j) = \varnothing$ for all $i \neq j$, and therefore $\mathcal{V}ar(u_i) \cap \mathcal{V}ar(u_j) = \varnothing$ as well. Then, σ is given by $\sigma(x) = \sigma_i(x)$ whenever $x \in \mathcal{V}ar(u_i)$ for $i \geq 1$. From the discussion in points (1) and (2) above, we conclude that the CSDPs $u_i \to v_i$ for $i \geq 1$ together with σ define an infinite minimal $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain which contradicts our initial assumption. □

As for arbitrary pairs, we use $\mathsf{DP}^1_\mathcal{X}$ to denote the subset of dependency pairs in $\mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$ whose migrating variables occur on non-μ-replacing immediate subterms in the left-hand side.

As an immediate consequence of Theorem 2 and Propositions 7 and 8, we have the following.

Corollary 3 (Basic μ-termination criteria) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$.*

(1) If $\mathsf{DP}(\mathcal{R},\mu) = \varnothing$, then $\mathcal{R}$ is μ-terminating.
(2) If $\mathcal{NHT}_{\mathsf{DP}(\mathcal{R},\mu)}(\mathcal{R},\mu) = \varnothing$ and $\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu) = \varnothing$, then $\mathcal{R}$ is μ-terminating.
(3) If $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$, then $\mathcal{R}$ is μ-terminating.

Example 12 *Consider the following TRS $\mathcal{R}$ [Luc98, Example 15]:*

$$
\begin{array}{rclrcl}
\mathtt{and}(\mathtt{true},x) & \to & x & \mathtt{add}(\mathtt{0},x) & \to & x \\
\mathtt{and}(\mathtt{false},y) & \to & \mathtt{false} & \mathtt{add}(\mathtt{s}(x),y) & \to & \mathtt{s}(\mathtt{add}(x,y)) \\
\mathtt{if}(\mathtt{true},x,y) & \to & x & \mathtt{from}(x) & \to & \mathtt{cons}(x,\mathtt{from}(\mathtt{s}(x))) \\
\mathtt{if}(\mathtt{false},x,y) & \to & y & \mathtt{first}(\mathtt{0},x) & \to & \mathtt{nil} \\
\mathtt{first}(\mathtt{s}(x),\mathtt{cons}(y,z)) & \to & \mathtt{cons}(y,\mathtt{first}(x,z)) & & &
\end{array}
$$

with $\mu(\mathtt{cons}) = \mu(\mathtt{s}) = \mu(\mathtt{from}) = \varnothing$, $\mu(\mathtt{add}) = \mu(\mathtt{and}) = \mu(\mathtt{if}) = \{1\}$, and $\mu(\mathtt{first}) = \{1,2\}$. Then, $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ is:

$$
\begin{array}{rclrcl}
\mathtt{AND}(\mathtt{true},x) & \to & x & \mathtt{IF}(\mathtt{true},x,y) & \to & x \\
\mathtt{ADD}(\mathtt{0},x) & \to & x & \mathtt{IF}(\mathtt{false},x,y) & \to & y
\end{array}
$$

Note also that $\mathcal{NHT}_{\mathsf{DP}(\mathcal{R},\mu)} = \varnothing$. Thus, by any of the last two statements of Corollary 3, we conclude the μ-termination of $\mathcal{R}$.

The following example shows that Corollary 3(3) does *not* hold for chains consisting of *arbitrary* collapsing CSDPs.

Example 13 *Consider the CS-TRS $(\mathcal{R},\mu)$ in Example 4. Note that $\mathsf{DP}(\mathcal{R},\mu) = \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ (both $\mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ and $\mathsf{DP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ are empty!). We have the following infinite $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain:*

$$\mathtt{F}(\underline{\mathtt{a}}) \hookrightarrow_{\mathcal{R},\mu^\sharp} \underline{\mathtt{F}(\mathtt{c}(\mathtt{f}(\mathtt{a})))} \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} \mathtt{F}(\underline{\mathtt{a}}) \hookrightarrow_{\mathcal{R},\mu^\sharp} \cdots$$

Now we prove that the previous CS-dependency pairs approach is not only correct but also complete for proving termination of *CSR*.

Theorem 3 (Completeness) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then there is no infinite $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain.*

Proof. By contradiction. If there is an infinite $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain, then there is a substitution σ and dependency pairs $u_i \to v_i \in \mathsf{DP}(\mathcal{R},\mu)$ such that

(1) $\sigma(v_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$, if $u_i \to v_i \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$, and
(2) if $u_i \to v_i = u_i \to x_i \in \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$, then there is $s_i \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$.

for $i \geq 1$. Now, consider the first dependency pair $u_1 \to v_1$ in the sequence:

(1) If $u_1 \to v_1 \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$, then $v_1^\natural$ is a μ-replacing subterm of the right-hand-side r_1 of a rule $l_1 \to r_1$ in $\mathcal{R}$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os^\mu(r_1)$ and we can perform the μ-rewriting step $t_1 = \sigma(u_1) \hookrightarrow_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[\sigma(v_1^\natural)]_{p_1} = s_1$, where $\sigma(v_1^\natural)^\sharp = \sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ and $\sigma(u_2)$ initiates an infinite $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain. Note that $p_1 \in \mathcal{P}os^\mu(s_1)$.
(2) If $u_1 \to x \in \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$, then there is a rule $l_1 \to r_1$ in $\mathcal{R}$ such that $u_1 = l_1^\sharp$, and $x \in \mathcal{V}ar^\mu(r_1) - \mathcal{V}ar^\mu(l_1)$, i.e., $r_1 = C_1[x]_{q_1}$ for some $q_1 \in \mathcal{P}os^\mu(r_1)$. Furthermore, since there is a subterm s such that $\sigma(x) \trianglerighteq_\mu s$ and $s^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$, we can write $\sigma(x) = C_1'[s]_{p_1'}$ for some $p_1' \in \mathcal{P}os^\mu(\sigma(x))$ and context $C_1'[\,]_{p_1'}$. Therefore, we can perform the μ-rewriting step $t_1 = \sigma(l_1) \hookrightarrow_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[C_1'[s]_{p_1'}]_{q_1} = s_1$ where $s^\sharp \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ (hence $s \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} u_2^\natural$) and $\sigma(u_2)$ initiates an infinite $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain. Note that $p_1 = q_1.p_1' \in \mathcal{P}os^\mu(s_1)$ (use Proposition 1).

Since $\mu^\sharp(f^\sharp) = \mu(f)$, and $p_1 \in \mathcal{P}os^\mu(s_1)$, we have that $s_1 \hookrightarrow^*_{\mathcal{R},\mu} t_2[\sigma(u_2)]_{p_1} = t_2$ and $p_1 \in \mathcal{P}os^\mu(t_2)$. Therefore, we can build in that way an infinite μ-rewrite sequence

$$t_1 \hookrightarrow_{\mathcal{R},\mu} s_1 \hookrightarrow^*_{\mathcal{R},\mu} t_2 \hookrightarrow_{\mathcal{R},\mu} \cdots$$

which contradicts the μ-termination of $\mathcal{R}$. $\square$

According to this, Proposition 8(3) suggests a simple checking of *non-μ-termination*.

Corollary 4 (Non-μ-termination criterion) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_\mathcal{F}$. If there is $u \to v \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$ such that $v' = \theta(u)$ for some substitution θ and renamed version v' of v, then $\mathcal{R}$ is not μ-terminating.*

As a corollary of Theorems 2 and 3, we have.

Corollary 5 (Characterization of μ-termination) *Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. Then, $\mathcal{R}$ is μ-terminating if and only if there is no infinite minimal $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain.*

9 Mechanizing proofs of μ-termination using CSDPs

During the last ten years, the dependency pairs method has evolved to a powerful technique for proving termination of TRSs in practice. From the already classical Arts and Giesl's article [AG00] to the last developments corresponding to the so-called *dependency pair framework* [GTS04,GTSF06,Thi07] many new improvements have been introduced.

In the DP-approach [AG00], the starting point is a TRS $\mathcal{R}$ from which a set of dependency pairs $\mathsf{DP}(\mathcal{R})$ is obtained. Then, such dependency pairs are organized in a dependency graph $\mathsf{DG}(\mathcal{R})$ and the cycles of the graph are analyzed to show that no infinite chains of DPs can be obtained from them. The dependency pairs approach emphasizes (at least theoretically) a 'linear' (although somehow *modular*, see [GAO02]) procedure for proving termination. In this sense, the treatment of strongly connected components of the graph (SCCs) instead of cycles, as suggested by Hirokawa and Middeldorp [HM04,HM05], brought an important improvement in its practical use because it provides a way to make the proofs more incremental without running out of the basic DP-approach. In the DP-approach, dependency pairs are considered as components of the chains (or cycles). Since they only make sense when an underlying TRS is given as the source of the dependency pairs, transforming DPs is possible (the *narrowing* transformation is already described in [AG00]) but only as a final step because, afterwards, they are not dependency pairs of the original TRS anymore.

The dependency pair framework solves these problems in a clean way, leading to a more powerful mechanization of termination proofs.

9.1 Mechanizing termination proofs with the dependency pair framework

An appealing aspect of the DP-framework [GTS04,GTSF06] is that the procedence of *pairs* does not matter; they can be *independent* from the considered TRS. The notion of chain is parametric on both a TRS $\mathcal{R}$ and a set of pairs $\mathcal{P}$ (a TRS, actually) which are connected by using $\mathcal{R}$-rewrite sequences. Regarding termination proofs, the central notion now is that of *DP-termination problem*: given a TRS $\mathcal{R}$ and a set of pairs $\mathcal{P}$, the goal is checking the absence (or presence) of infinite (minimal) chains. Termination of a TRS $\mathcal{R}$ is addressed as a DP-termination problem where $\mathcal{P} = \mathsf{DP}(\mathcal{R})$. The most important notion regarding mechanization of the proofs is that of *processor*. A (correct) processor basically transforms DP-termination problems into (hopefully) *simpler* ones, in such a way that the existence of an infinite chain in the original DP-termination problem implies the existence of an infinite chain in the transformed one. Here 'simpler' usually means that fewer pairs are involved. Often, processors are not only correct but also *complete*, i.e., there is an infinite minimal chain in the original DP-termination problem if and only if there is an infinite minimal chain in the transformed problem. This is essential if we are interested in *disproving* termination.

Processors are used in a pipe (more precissely, a *tree*) to incrementally simplify the original DP-termination problem as much as possible, possibly decomposing it into smaller pieces which are then independently treated in the very same way. The trivial case of this *iterative* process comes when the set of pairs $\mathcal{P}$ becomes empty. Then, no infinite chain is possible and we can provide a *positive* answer yes to the DP-termination problem which is propagated upwards to the original problem in the root of the tree. In some cases it is also possible to witness the existence of infinite chains for a given DP-termination problem; then a *negative* answer no can be provided and propagated upwards. Of course, DP-termination problems are undecidable (in general), thus don't know answers can also be generated (for instance by a time-out system which interrupts the usually complex search processes which are involved in the proofs). When all answers are collected, a final conclusion about the whole DP-termination problem can be given:

(1) if we have positive answers (yes) for all problems in the leaves of the tree, then we conclude yes as well;
(2) if a negative answer (no) was raised somewhere and the DP-processors which were used in the path from the root to the node producing the negative answer were *complete*, then we conclude no as well;
(3) Otherwise, the conclusion is don't know.

The notions of graph, cycles, SCCs, etc., are also part of the framework but (1) they are incorporated as *processors* now, and (2) they do *not* refer to dependency pairs anymore, but rather to the pairs in the (different) sets of pairs which are obtained through the process sketched above. In this way, we obtain a much more flexible framework to mechanize termination proofs and also to benefit from new future developments which could lead to the introduction of new processors.

In the following, we adapt these ideas to *CSR* to provide a suitable framework for mechanizing proofs of termination of *CSR* using CSDPs.

9.2 CS-termination problems and processors

The following definition adapts the notion of (DP-)termination problem in [GTSF06] to *CSR*. In our definition, we prefer to avoid '*DP*' because, as discussed above, dependency pairs (as such) are relevant in the theoretical framework only for investigating a particular problem (termination of TRSs), whereas some transformations can yield sets of pairs which are not dependency pairs of the underlying TRS anymore.

Definition 5 (CS-termination problems) *A CS-termination problem τ is a tuple $\tau = (\mathcal{P}, \mathcal{R}, \mu)$, where $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ are TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. A CS-termination problem is finite if there is no infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain.*

Finite CS-termination problems correspond to those generating a positive answer yes in the full proof process sketched above. Accordingly, CS-termination problems which are *not* finite correspond to a negative answer no.

Remark 7 *According to Corollary 5, we can say now that a TRS $\mathcal{R}$ is μ-terminating if and only if the CS-termination problem $(\mathsf{DP}(\mathcal{R}, \mu), \mathcal{R}, \mu^\sharp)$ is finite.*

According to our previous results (Propositions 7 and 8), for some specific CS-termination problems it is easy to say whether they are finite or not.

Proposition 9 (Basic CS-termination problems) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

(1) If $P = \varnothing$, or $\mathcal{P} = \mathcal{P}^1_{\mathcal{X}}$, or $R = \varnothing$ and $\mathcal{P} = \mathcal{P}_{\mathcal{X}}$, then the CS-termination problem $(\mathcal{P}, \mathcal{R}, \mu)$ is finite.
(2) If there is $u \to v \in \mathcal{P}_{\mathcal{G}}$ such that $v' = \theta(u)$ for some substitution θ and renamed version v' of v, then the CS-termination problem $(\mathcal{P}, \mathcal{R}, \mu)$ is not finite.

The CS-termination problems in Proposition 9 provide the necessary *base* cases for our proofs of termination. The following definition adapts the notion of *processor* [GTSF06] to *CSR*.

Definition 6 (CS-processor) *A CS-processor* Proc *is a mapping from CS-termination problems into sets of CS-termination problems. A CS-processor* Proc *is*

- sound *if for all CS-termination problems τ, τ is finite whenever τ' is finite for all $\tau' \in \mathsf{Proc}(\tau)$.*
- complete *if for all CS-termination problems τ, whenever τ is finite, then τ' is finite for all $\tau' \in \mathsf{Proc}(\tau)$.*

In the following sections we describe a number of sound and (most of them) complete CS-processors.

10 Context-Sensitive Dependency Graph

CS-termination problems focus our attention on the analysis of *infinite minimal* $(\mathcal{P},\mathcal{R},\mu)$*-chains*. In general, an infinite sequence $S = a_1, a_2, \ldots, a_n, \ldots$ of objects a_i belonging to a set A can be represented as a path in a *graph* G whose nodes are the objects in A, and whose arcs among them are appropriately established to represent S (in particular, an arc from a_i to a_{i+1} should be established if we want to be able to capture the sequence above). Actually, if A is *finite*, then the infinite sequence S defines at least one *cycle* in G: since there is a finite number of different objects $a_i \in A$ in S, there is an infinite tail $S' = a_m, a_{m+1}, \ldots$ of S where *all objects* a_i *occur infinitely many times* for all $i \geq m$. This clearly corresponds to a cycle in G.

In the dependency pairs approach [AG00], a *dependency graph* $\mathsf{DG}(\mathcal{R})$ is associated to the TRS $\mathcal{R}$. The nodes of the dependency graph are the dependency pairs in $\mathsf{DP}(\mathcal{R})$; there is an arc from a dependency pair $u \to v$ to a dependency pair $u' \to v'$ if there are substitutions θ and θ' such that $\theta(v) \to^*_{\mathcal{R}} \theta'(u')$.

In more recent approaches, the analysis of infinite chains of dependency pairs as such is just a starting point. Many often, chains of dependency pairs are *transformed* into chains of more general *pairs* which cannot be considered dependency pairs anymore. This is the case for the *narrowing* or *instantiation* transformations, among others, see [GTSF06] for instance. Still, the analysis of the cycles in the graph build out from such pairs is useful to investigate the existence of infinite (minimal) chains of pairs. Thus, a more general notion of *graph of pairs* $\mathsf{DG}(\mathcal{P},\mathcal{R})$ associated to a set of pairs $\mathcal{P}$ and a TRS $\mathcal{R}$ is considered; the pairs in $\mathcal{P}$ are used now as the nodes of the graph but they are connected by $\mathcal{R}$-rewriting in the same way [GTSF06, Definition 7].

In the following section we take into account these points to provide an appropriate definition of context-sensitive (dependency) graph.

10.1 Definition of the context-sensitive dependency graph

According to the discussion above, our starting point are two TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ togheter with a replacement map $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Our aim is obtaining a notion of graph which is able to represent all infinite *minimal* chains of pairs as given in Definition 4.

When considering pairs $u \to v \in \mathcal{P}_{\mathcal{G}}$, we can proceed as in the standard case to define appropriate connections to other pairs $u' \to v' \in \mathcal{P}$: there is an arc from $u \to v$ to $u' \to v'$ if $\theta(v) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(u')$ for some substitutions θ and θ'. When considering collapsing pairs $u \to v \in \mathcal{P}_{\mathcal{X}}$, we know that such pairs can only be followed by a pair $u' \to v' \in \mathcal{P}$ such that $\theta(t^\sharp) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(u')$ for some $t \in \mathcal{NHT}$ and substitutions θ and θ' (see Definition 4).

Definition 7 (Context-Sensitive Graph of Pairs) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. The* context-sensitive (CS-)graph *associated to $\mathcal{R}$ and $\mathcal{P}$ (denoted $\mathsf{G}(\mathcal{P}, \mathcal{R}, \mu)$) has $\mathcal{P}$ as the set of nodes and arcs which connect them as follows:*

(1) There is an arc from $u \to v \in \mathcal{P}_\mathcal{G}$ to $u' \to v' \in \mathcal{P}$ if there are substitutions θ and θ' such that $\theta(v) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(u')$.

(2) There is an arc from $u \to v \in \mathcal{P}_\mathcal{X}$ to $u' \to v' \in \mathcal{P}$ if there is $t \in \mathcal{NHT}(\mathcal{R}, \mu)$ and substitutions θ and θ' such that $\theta(t^\sharp) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(u')$.

In termination proofs, we are concerned with the so-called *strongly connected components* (SCCs) of the dependency graph, rather than with the cycles themselves (which are exponentially many) [HM05]. A strongly connected component in a graph is a *maximal cycle*, i.e., a cycle which is not contained in any other cycle. The following result justifies the use of SCCs for proving the absence of infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chains.

Theorem 4 (SCC processor) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Then, the processor Proc_{SCC} given by*

$$\mathsf{Proc}_{SCC}(\mathcal{P}, \mathcal{R}, \mu) = \{(\mathcal{Q}, \mathcal{R}, \mu) \mid \mathcal{Q} \textit{ contains the pairs of an SCC in } \mathsf{G}(\mathcal{P}, \mathcal{R}, \mu)\}$$

is sound and complete.

Proof. We prove soundness by contradiction. Assume that Proc_{SCC} is not sound. Then, there is a CS-termination problem $\tau = (\mathcal{P}, \mathcal{R}, \mu)$ such that, for all $\tau' \in \mathsf{Proc}_{SCC}(\tau)$, τ' is finite but τ is not finite. Thus, there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain A. Since $\mathcal{P}$ contains a finite number of pairs, there is $\mathcal{P}' \subseteq \mathcal{P}$ and a tail B of A which is an infinite minimal $(\mathcal{P}', \mathcal{R}, \mu)$-chain where all pairs in $\mathcal{P}'$ are infinitely often used. According to Definition 7, this means that $\mathcal{P}'$ is a cycle in $\mathsf{G}(\mathcal{P}, \mathcal{R}, \mu)$, hence it belongs to some SCC with nodes in $\mathcal{Q}$, i.e., $\mathcal{P}' \subseteq \mathcal{Q}$. Hence B is an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu)$-chain, i.e., $\tau' = (\mathcal{Q}, \mathcal{R}, \mu)$ is not finite. Since $\tau' \in \mathsf{Proc}_{SCC}(\tau)$, we obtain a contradiction.

Regarding completenes, since $\mathcal{Q} \subseteq \mathcal{P}$ for some SCC in $\mathsf{G}(\mathcal{P}, \mathcal{R}, \mu)$ with nodes in $\mathcal{Q}$, every infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu)$-chain is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain, hence the processor is complete as well. $\square$

As a consequence of this theorem, we can *separately* work with the strongly connected components of $\mathsf{G}(\mathcal{P}, \mathcal{R}, \mu)$, disregarding other parts of the graph.

Now we can use these notions to introduce the context-sensitive dependency graph.

Definition 8 (Context-Sensitive Dependency Graph) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. The* Context-Sensitive Dependency Graph *associated to $\mathcal{R}$ and μ is $\mathsf{DG}(\mathcal{R}, \mu) = \mathsf{G}(\mathsf{DP}(\mathcal{R}, \mu), \mathcal{R}, \mu^\sharp)$.*

10.2 Estimating the CS-dependency graph

In general, the (context-sensitive) dependency graph of a TRS is *not* computable: it involves reachability of $\theta'(u')$ from $\theta(v)$ (for $u \to v \in \mathcal{P}_{\mathcal{G}}$) or $\theta(t^\sharp)$ (for $t \in \mathcal{NHT}_{\mathcal{P}}$) using *CSR*; as in the unrestricted case, the reachability problem for *CSR* is undecidable. So, we need to use some approximation of it. Following [AG00], we describe how to approximate the CS-dependency graph of a CS-TRS.

Given a set Δ of 'defined' symbols, we let $\textsc{Cap}^\mu_\Delta$ be as follows:

$$\textsc{Cap}^\mu_\Delta(x) = x \quad \text{if } x \text{ is a variable}$$

$$\textsc{Cap}^\mu_\Delta(f(t_1, \ldots, t_k)) = \begin{cases} y & \text{if } f \in \Delta \\ f([t_1]^f_1, \ldots, [t_k]^f_1) & \text{otherwise} \end{cases}$$

where y is intended to be a new, fresh variable which has not yet been used and given a term s, $[s]^f_i = \textsc{Cap}^\mu_\Delta(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$.

Function $\textsc{Cap}^\mu_\Delta$ is intended to provide a suitable approximation of reachability problems $\theta(s) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(t)$ by means of unification. The idea is obtaining the maximal prefix context $C[\,]$ of s (i.e., $s = C[s_1, \ldots, s_n]$ for some terms $s_1, \ldots, s_n$) which we know (without any 'look-ahead' for applicable rules) that cannot be changed by any reduction starting from s. Furthermore, terms $s_1, \ldots, s_n$ above must be rooted by defined symbols (i.e., $root(s_i) \in \Delta$ for $i \in \{1, \ldots, n\}$). Now, we replace those subterms s_i which are at μ-replacing positions (i.e., $s_i = s|_{p_i}$ for some $p_i \in \mathcal{P}os^\mu(s)$) by some variable x, and we leave untouched the non-μ-replacing ones.

The following result whose proof is similar to that of [AG00, Theorem 21] (we only need to take into account the replacement restrictions indicated by the replacement map μ) formalizes the correctness of this approach.

Proposition 10 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be such that $\mathcal{V}ar(s) \cap \mathcal{V}ar(t) = \varnothing$ and θ, θ' be substitutions. If $\theta(s) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(t)$, then $\textsc{Ren}^\mu(\textsc{Cap}^\mu_{\mathcal{D}}(s))$ and t unify.*

According to Proposition 10, given terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and substitutions θ, θ', the reachability of $\theta'(t)$ from $\theta(s)$ by μ-rewriting can be *approximated* as unification of $\textsc{Ren}^\mu(\textsc{Cap}^\mu_{\mathcal{D}}(s))$ and t. So, we have the following.

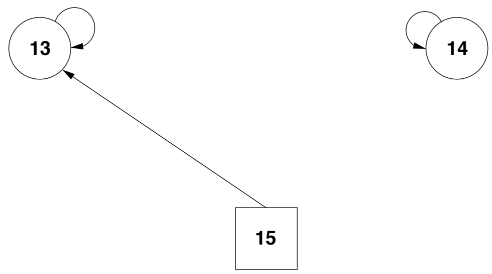

Fig. 2. Estimated CSDG for the CS-TRS $(\mathcal{R}, \mu)$ in Example 14

Definition 9 (Estimated Context-Sensitive Graph of Pairs) *Let* $\mathcal{R} = (\mathcal{F}, R)$ *and* $\mathcal{P} = (\mathcal{G}, P)$ *be TRSs and* $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. *The* estimated *CS-graph associated to* $\mathcal{R}$ *and* $\mathcal{P}$ *(denoted* $\mathsf{EG}(\mathcal{P}, \mathcal{R}, \mu)$*) has* $\mathcal{P}$ *as the set of nodes and arcs which connect them as follows:*

(1) There is an arc from $u \to v \in \mathcal{P}_{\mathcal{G}}$ *to* $u' \to v' \in \mathcal{P}$ *if* $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathcal{D}}(v))$ *and* u' *unify.*
(2) There is an arc from $u \to v \in \mathcal{P}_{\mathcal{X}}$ *to* $u' \to v' \in \mathcal{P}$ *if there is* $t \in \mathcal{NHT}(\mathcal{R}, \mu)$ *such that* $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathcal{D}}(t^\sharp))$ *and* u' *unify.*

According to Definition 7, we would have the corresponding one for the *estimated* CSDG: $\mathsf{EDG}(\mathcal{R}, \mu) = \mathsf{EG}(\mathsf{DP}(\mathcal{R}, \mu), \mathcal{R}, \mu^\sharp)$.

Example 14 *Consider the following TRS* $\mathcal{R}$ *[Zan97, Example 4]:*

$$
\begin{aligned}
\mathtt{f}(x) &\to \mathtt{cons}(x, \mathtt{f}(\mathtt{g}(x))) & \mathtt{sel}(\mathtt{0}, \mathtt{cons}(x, y)) &\to x \\
\mathtt{g}(\mathtt{0}) &\to \mathtt{s}(\mathtt{0}) & \mathtt{sel}(\mathtt{s}(x), \mathtt{cons}(y, z)) &\to \mathtt{sel}(x, z) \\
\mathtt{g}(\mathtt{s}(x)) &\to \mathtt{s}(\mathtt{s}(\mathtt{g}(x)))
\end{aligned}
$$

with $\mu(\mathtt{0}) = \varnothing$, $\mu(\mathtt{f}) = \mu(\mathtt{g}) = \mu(\mathtt{s}) = \mu(\mathtt{cons}) = \{1\}$, *and* $\mu(\mathtt{sel}) = \{1, 2\}$. *Then,* $\mathsf{DP}(\mathcal{R}, \mu)$ *is:*

$$
\begin{aligned}
\mathtt{G}(\mathtt{s}(x)) &\to \mathtt{G}(x) \quad (13) & \mathtt{SEL}(\mathtt{s}(x), \mathtt{cons}(y, z)) &\to z \quad (15) \\
\mathtt{SEL}(\mathtt{s}(x), \mathtt{cons}(y, z)) &\to \mathtt{SEL}(x, z) \quad (14)
\end{aligned}
$$

and $\mathcal{NHT} = \{\mathtt{f}(\mathtt{g}(x)), \mathtt{g}(x)\}$. *Regarding pairs* (13) *and* (14) *in* $\mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$, *there is an arc from* (13) *to itself and another one from* (14) *to itself. Regarding the only collapsing pair* (15), *we have* $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathcal{D}}(\mathtt{F}(\mathtt{g}(x)))) = \mathtt{F}(y)$ *and* $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathcal{D}}(\mathtt{G}(x))) = \mathtt{G}(y)$. *Since* $\mathtt{F}(y)$ *does not unify with the left-hand side of any pair, and* $\mathtt{G}(y)$ *unifies with the left-hand side* $\mathtt{G}(\mathtt{s}(x))$ *of* (13), *there is an arc from* (15) *to* (13), *see Figure 2. Thus, there are two cycles:* $\{(13)\}$ *and* $\{(14)\}$.

Note that Proposition 10 also provides a way to *estimate* the set $\mathcal{NHT}_{\mathcal{P}}$: if

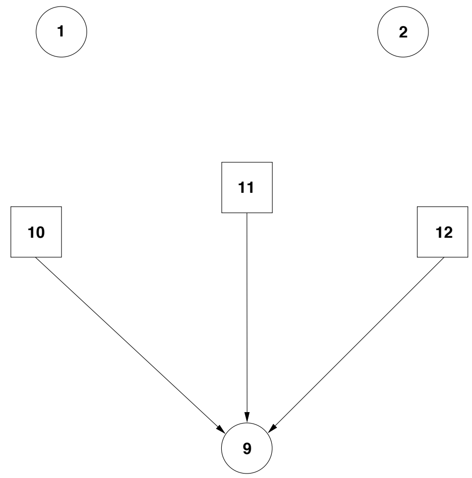

Fig. 3. Context-Sensitive Dependency Graph for the CS-TRS $(\mathcal{R}, \mu)$ in Example 1

$t \in \mathcal{NHT}_{\mathcal{P}}$, then $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathcal{D}}(t^{\sharp}))$ and u unifiy for some $u \to v \in \mathcal{P}$. In the following, our presentations of $\mathcal{NHT}_{\mathcal{P}}$ in the examples are computed in this way.

Example 15 *Consider again the CS-TRS $(\mathcal{R}, \mu)$ in Example 1. Note that*

$$\mathcal{NHT}_{\mathsf{DP}(\mathcal{R},\mu)}(\mathcal{R}, \mu^{\sharp}) = \{\mathtt{filt}(x, \mathtt{sieve}(y)), \mathtt{filt}(\mathtt{s}(\mathtt{s}(x)), z)\}$$

The CSDG is shown in Figure 3 and has no *cycle. By Theorem 4 we transform the CS-problem $(\mathsf{DP}(\mathcal{R}, \mu), \mathcal{R}, \mu^{\sharp})$ into a singleton $\{(\varnothing, \mathcal{R}, \mu^{\sharp})\}$ containing a finite CS-termination problem (use Proposition 9). Thus, we conclude that $\mathcal{R}$ is μ-terminating.*

11 Treating collapsing pairs

The following result, which is an immediate consequence of Proposition 7(1), defines a correct and complete CS-processor which removes collapsing pairs when the set of involved hidden terms $\mathcal{NHT}_{\mathcal{P}}$ is empty.

Theorem 5 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Then, the processor Proc_{eNHT} given by*

$$\mathsf{Proc}_{eNHT}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_{\mathcal{X}}, \mathcal{R}, \mu)\} & \text{if } \mathcal{NHT}_{\mathcal{P}} = \varnothing \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

The following result shows how to *safely* transform collapsing pairs into non-collapsing ones in some particular cases.

Theorem 6 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $\mathcal{P}' = (\mathcal{F}\cup\mathcal{G}, P')$ where $P' = (P - P_{\mathcal{X}}) \cup Q$ for $Q = \{u \to t^\sharp \mid u \to x \in \mathcal{P}_{\mathcal{X}}, t \in \mathcal{NHT}_{\mathcal{P}}\}$ Then, the processor Proc_{gNHT} given by*

$$\mathsf{Proc}_{gNHT}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P}', \mathcal{R}, \mu)\} \text{ if } \mathcal{NHT}_{\mathcal{P}}(\mathcal{R}, \mu) \subseteq \mathcal{T}(\mathcal{F}) \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} \text{ otherwise} \end{cases}$$

is sound.

Proof. We prove that the existence of an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain implies the existence of an infinite minimal $(\mathcal{P}', \mathcal{R}, \mu)$-chain.

First, note that $\mathcal{P}'$ is a TRS: the new rules in Q are of the form $u \to t^\sharp$ for $t \in \mathcal{NHT}_{\mathcal{P}}$, Since $\mathcal{NHT}_{\mathcal{P}} \subseteq \mathcal{T}(\mathcal{F})$, we trivially have $\mathcal{V}ar(t^\sharp) \subseteq \mathcal{V}ar(u)$, i.e., $u \to t^\sharp$ is a rewrite rule.

Consider an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain A:

$$\sigma(u_1) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu t_1 \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu t_2 \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_3) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu \cdots$$

for some substitution σ, where, for all $i \geq 1$, t_i is μ-terminating and, (1) if $u_i \to v_i \in \mathcal{P}_{\mathcal{G}}$, then $t_i = \sigma(v_i)$ and (2) if $u_i \to v_i = u_i \to x_i \in \mathcal{P}_{\mathcal{X}}$, then $t_i = s_i^\sharp$ for some s_i such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i = \theta_i(\bar{s}_i)$ for some $\bar{s}_i \in \mathcal{NHT}$ and substitution θ_i; actually, since $t_i = s_i^\sharp = \theta_i(\bar{s}_i)^\sharp = \theta_i(\bar{s}_i^\sharp)$ and $t_i \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$, we can further say that $\bar{s}_i \in \mathcal{NHT}_{\mathcal{P}}$.

In the case (2) above, since $\mathcal{NHT}_{\mathcal{P}} \subseteq \mathcal{T}(\mathcal{F})$, we have $t_i = s_i^\sharp = \theta_i(\bar{s}_i^\sharp) = \bar{s}_i^\sharp$, i.e., $t_i \in \mathcal{NHT}_{\mathcal{P}}$. Thus, we can use $u_i \to t_i \in \mathcal{Q}$ instead of $u_i \to x_i \in \mathcal{P}_{\mathcal{X}}$, because we still have $t_i \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$. By replacing in this way each $u_i \to x_i \in \mathcal{P}_{\mathcal{X}}$ by the corresponding $u_i \to t_i \in \mathcal{Q}$, each step $\sigma(u_i) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu t_i$ becomes a step $\sigma(u_i) \hookrightarrow_{\mathcal{P}',\mu} t_i$, whereas steps $\sigma(u_i) \hookrightarrow_{\mathcal{P},\mu} \sigma(v_i) = t_i$ for $u_i \to v_i \in \mathcal{P}_{\mathcal{G}}$ remain unchanged. Thus, we obtain an infinite minimal $(\mathcal{P}', \mathcal{R}, \mu)$-chain, as desired. □

Note that no pair in $\mathcal{P}'$ in Theorem 6 is collapsing. Unfortunately, Proc_{gNHT} is *not* complete.

Example 16 *Consider the following TRS:*

$$\mathtt{b} \to \mathtt{f}(\mathtt{c}(\mathtt{b}))$$
$$\mathtt{f}(x) \to x$$

together with the replacement map μ given by $\mu(\mathtt{f}) = \mu(\mathtt{c}) = \varnothing$. Then,

$\mathsf{DP}(\mathcal{R},\mu)$ *is:*

$$\mathtt{B} \to \mathtt{F}(\mathtt{c}(\mathtt{b}))$$
$$\mathtt{F}(x) \to x$$

and $\mathcal{NHT}_{\mathsf{DP}(R,\mu)} = \{\mathtt{b}\}$*. Note that* $\mathcal{R}$ *is clearly* μ*-terminating, hence there is no infinite* $(\mathcal{P},\mathcal{R},\mu^\sharp)$*-chain for* $\mathcal{P} = \mathsf{DP}(\mathcal{R},\mu)$*, i.e.,* $(\mathsf{DP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$ *is finite. However, by using* $\mathcal{P}'$ *as in Theorem 6, i.e., with*

$$\mathtt{B} \to \mathtt{F}(\mathtt{c}(\mathtt{b}))$$
$$\mathtt{F}(x) \to \mathtt{B}$$

we actually have an infinite $(\mathcal{P}',\mathcal{R},\mu^\sharp)$*-chain, i.e,* $(\mathcal{P}',\mathcal{R},\mu^\sharp)$ *is* not *finite.*

12 Use of μ-reduction pairs

A reduction pair $(\gtrsim, \sqsupset)$ consists of a stable and monotonic quasi-ordering $\gtrsim$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$ [KNT99].

The absence of infinite chains of (dependency) pairs can be ensured by finding a *reduction pair* $(\gtrsim, \sqsupset)$ which is compatible with the rules and the dependency pairs [AG00]: $l \gtrsim r$ for all rewrite rules $l \to r$ and $u \gtrsim v$ or $u \sqsupset v$ for all dependency pairs $u \to v$. In the dependency pair framework [GTS04,GTSF06] (but also in [GAO02,HM04,HM05,HM07]), they are used to obtain *smaller* sets of pairs $\mathcal{P}' \subseteq \mathcal{P}$ by removing the *strict* pairs, i.e., those pairs $u \to v \in \mathcal{P}$ such that $u \sqsupset v$.

Stability is required both for $\gtrsim$ and $\sqsupset$ because, although we only check the left- and right-hand sides of the rewrite rules $l \to r$ (with $\gtrsim$) and pairs $u \to v$ (with $\gtrsim$ or $\sqsupset$), the chains of pairs involve *instances* $\sigma(l)$, $\sigma(r)$, $\sigma(u)$, and $\sigma(v)$ of rules and pairs and we aim at concluding $\sigma(l) \gtrsim \sigma(r)$, and $\sigma(u) \gtrsim \sigma(v)$ or $\sigma(u) \sqsupset \sigma(v)$, respectively. Monotonicity is required for $\gtrsim$ to deal with the application of rules $l \to r$ to an arbitrary depth in terms. Since the pairs are 'applied' only at the root level, no monotonicity is required for $\sqsupset$ (but, for this reason, we cannot compare the rules in $\mathcal{R}$ using $\sqsupset$). Recently, Endrullis et al. noticed that *transitivity* is not necessary for the strict component $\sqsupset$ because it is somehow 'simulated' by the compatibility requirement above [EWZ08].

In our setting, since we are interested in μ-rewriting steps only, we can relax the monotonicity requirements as follows.

Definition 10 (μ-reduction pair) *Let* $\mathcal{F}$ *be a signature and* $\mu \in M_{\mathcal{F}}$*. A* μ-reduction pair $(\gtrsim, \sqsupset)$ *consists of a stable and* μ*-monotonic quasi-ordering*

$\gtrsim$ *and a well-founded stable relation* $\sqsupset$ *on terms in* $\mathcal{T}(\mathcal{F},\mathcal{X})$ *which are compatible, i.e.,* $\gtrsim \circ \sqsupset \subseteq \sqsupset$ *or* $\sqsupset \circ \gtrsim \subseteq \sqsupset$.

We say that $(\gtrsim, \sqsupset)$ *is* μ*-monotonic if* $\sqsupset$ *is* μ*-monotonic.*

The following result allows us to use a μ*-monotonic* μ-reduction pair to remove some rewrite rules from the original rewrite system $\mathcal{R}$ before starting a termination proof.

Proposition 11 *Let* $\mathcal{R}$ *be a TRS and* $\mu \in M_{\mathcal{R}}$*. Let* $(\gtrsim, \sqsupset)$ *be a* μ*-monotonic* μ*-reduction pair on* $\mathcal{T}(\mathcal{F},\mathcal{X})$ *such that* $l \ (\gtrsim \cup \sqsupset) \ r$ *for all* $l \to r \in R$*. Let* $\mathcal{S} = (\mathcal{F}, S)$ *be such that* $S = \{l \to r \in R \mid l \not\sqsupset r\}$*. Then,* $\mathcal{R}$ *is* μ*-terminating if and only if* $\mathcal{S}$ *is* μ*-terminating.*

Proof. Since $S \subseteq R$, the *if* part is obvious. For the *only if* part, we proceed by contradiction. If $\mathcal{R}$ is not μ-terminating, then there is an infinite μ-rewrite sequence A:

$$t_1 \hookrightarrow_{\mathcal{R},\mu} t_2 \hookrightarrow_{\mathcal{R},\mu} \cdots t_n \hookrightarrow_{\mathcal{R},\mu} \cdots$$

where an infinite number of rules in $\mathcal{R} - \mathcal{S}$ have been used; otherwise, there would be an infinite tail $t_m \hookrightarrow_{\mathcal{S},\mu} t_{m+1} \hookrightarrow_{\mathcal{S},\mu} \cdots$ for some $m \geq 1$ where only rules in $\mathcal{S}$ are applied, contradicting the μ-termination of $\mathcal{S}$. Let $J = \{j_1, j_2, \ldots\}$ be the infinite set of indices indicating μ-rewrite steps $t_j \hookrightarrow_{\mathcal{R},\mu} t_{j+1}$ in A, for all $j \in J$, where rules in $R - S$ have been used to perform the μ-rewriting step. Since $l \sqsupset r$ for all $l \to r \in R - S$, by stability and μ-monotonicity of $\sqsupset$, we have that $t_{j_i} \sqsupset t_{j_i+1}$. Since $l \gtrsim r$ for all $l \to r \in S$, by stability and μ-monotonicity of $\gtrsim$, we have that $t_{j_i+1} \gtrsim t_{j_{i+1}}$. By compatibility between $\gtrsim$ and $\sqsupset$, we have $t_{j_i} \sqsupset t_{j_{i+1}}$ for all $i \geq 1$. We obtain an infinite sequence $t_{j_1} \sqsupset t_{j_2} \sqsupset \cdots$ which contradicts well-foundedness of $\sqsupset$. $\square$

Reduction pairs are often used in combination with *argument filterings*, which discard subexpressions from constraints $s \gtrsim t$ or $s \sqsupset t$ in such a way that $\pi(s) \gtrsim \pi(t)$ (resp. $\pi(s) \sqsupset \pi(t)$) is often simpler to prove [AG00,GTSF06].

12.1 Argument filterings for CSR

An argument filtering π for a signature $\mathcal{F}$ is a mapping that assigns to every k-ary function symbol $f \in \mathcal{F}$ an argument position $i \in \{1, \ldots, k\}$ or a (possibly empty) list $[i_1, \ldots, i_m]$ of argument positions with $1 \leq i_1 < \cdots < i_m \leq k$ [KNT99]. In the following, by the *trivial* argument filtering $\pi_\top$ for $\mathcal{F}$, we mean the one given by $\pi_\top(f) = [1, \ldots, k]$ for each k-ary symbol $f \in \mathcal{F}$. It corresponds to the argument filtering which does nothing.

We can use an argument filtering π to 'filter' either the signature $\mathcal{F}$ or any replacement map $\mu \in M_{\mathcal{F}}$. In the following, we assume that:

(1) The signature $\mathcal{F}_\pi$ consists of all function symbols f such that $\pi(f)$ is some list $[i_1, \ldots, i_m]$, where, in $\mathcal{F}_\pi$, the arity of f is m. As usual, we give the same name to the version of $f \in \mathcal{F}$ which belongs to $\mathcal{F}_\pi$.
(2) The replacement map $\mu_\pi \in M_{\mathcal{F}_\pi}$ is given as follows: for all $f \in \mathcal{F}$ such that $f \in \mathcal{F}_\pi$ and $\pi(f) = [i_1, \ldots, i_m]$:

$$\mu_\pi(f) = \{j \in \{1, \ldots, m\} \mid i_j \in \mu(f)\}$$

An argument filtering induces a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{X})$ to $\mathcal{T}(\mathcal{F}_\pi, \mathcal{X})$, also denoted by π:

$$\pi(t) = \begin{cases} t \text{ if } t \text{ is a variable} \\ \pi(t_i) \text{ if } t = f(t_1, \ldots, t_k) \text{ and } \pi(f) = i \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) \text{ if } t = f(t_1, \ldots, t_k) \text{ and } \pi(f) = [i_1, \ldots, i_m] \end{cases}$$

Note that, for the *top* filtering $\pi_\top$, we have that $\mathcal{F}_{\pi_\top} = \mathcal{F}$, $\mu_{\pi_\top} = \mu$ for all $\mu \in M_{\mathcal{F}}$, and $\pi_\top(t) = t$ for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The following auxiliary results are used later.

Lemma 7 *Let $\mathcal{F}$ be a signature and π be an argument filtering for $\mathcal{F}$. Let σ be a substitution and σ_π be a substitution given by $\sigma_\pi(x) = \pi(\sigma(x))$ for all $x \in \mathcal{X}$. If $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, then, $\pi(\sigma(t)) = \sigma_\pi(\pi(t))$.*

Proof. By structural induction.

(1) Base case: t is a variable or a constant symbol. If $t = x \in \mathcal{X}$, then $\pi(x) = x$ and $\pi(\sigma(x)) = \sigma_\pi(x) = \sigma_\pi(\pi(x))$. If t is a constant symbol, then $\pi(t) = t$ and $\sigma(t) = t = \sigma_\pi(t)$. Hence, $\pi(\sigma(t)) = \pi(t) = t = \sigma_\pi(t) = \sigma_\pi(\pi(t))$.
(2) If $t = f(t_1, \ldots, t_k)$, then we consider the two possible cases according to $\pi(f)$:
   (a) If $\pi(f) = i$ for some $i \in \{1, \ldots, k\}$, then $\pi(t) = \pi(t_i)$. By the induction hypothesis, $\pi(\sigma(t_i)) = \sigma_\pi(\pi(t_i))$. Therefore, $\pi(\sigma(t)) = \pi(f(\sigma(t_1), \ldots, \sigma(t_k))) = \pi(\sigma(t_i)) = \sigma_\pi(\pi(t_i)) = \sigma_\pi(\pi(f(t_1 \ldots, t_k))) = \sigma_\pi(\pi(t))$.
   (b) If $\pi(f) = [i_1, \ldots, i_m]$, then $\pi(t) = f(\pi(t_{i_1}), \ldots, \pi(t_{i_m}))$. By the induction hypothesis, $\pi(\sigma(t_{i_j})) = \sigma_\pi(\pi(t_{i_j}))$ for all $j \in \{1, \ldots, m\}$. Therefore, $\pi(\sigma(t)) = \pi(f(\sigma(t_1), \ldots, \sigma(t_k))) = f(\pi(\sigma(t_{i_1})), \ldots, \pi(\sigma(t_{i_m}))) = f(\sigma_\pi(\pi(t_{i_1})), \ldots, \sigma_\pi(\pi(t_{i_m}))) = \sigma_\pi(f(\pi(t_{i_1}), \ldots, \pi(t_{i_m}))) = \sigma_\pi(\pi(t))$.

$\square$

In the following, given a substitution σ and an argument filtering π, we let σ_π be the substitution defined by $\sigma_\pi(x) = \pi(\sigma(x))$ for all $x \in \mathcal{X}$.

Proposition 12 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and π be an argument* *filtering for $\mathcal{F}$. Let $(\gtrsim, \sqsupset)$ be a μ_π-reduction pair such that $\pi(l) \gtrsim \pi(r)$ for all $l \to r \in R$, and let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If $s \hookrightarrow^*_{\mathcal{R},\mu} t$, then $\pi(s) \gtrsim \pi(t)$.*

PROOF. By induction on the length n of the μ-rewrite sequence.

(1) If $n = 0$, then $s = t$ and, trivially, $\pi(s) = \pi(t)$. Since $\gtrsim$ is reflexive, we have $\pi(s) \gtrsim \pi(t)$.

(2) If $n > 0$, we can write $s \hookrightarrow_{\mathcal{R},\mu} s' \hookrightarrow^*_{\mathcal{R},\mu} t$, where the length of the sequence from s' to t is $n - 1$. Let $p \in \mathcal{P}os^\mu(s)$ be the μ-replacing position where the μ-rewriting step $s \hookrightarrow_{\mathcal{R},\mu} s'$ is performed. We prove that $s \hookrightarrow_{\mathcal{R},\mu} s'$ implies $\pi(s) \gtrsim \pi(s')$ by induction on the structure of p.

   (a) If $p = \Lambda$, then $s = \sigma(l)$ and $s' = \sigma(r)$ for some rewrite rule $l \to r$ and matching substitution σ. By Lemma 7, $\pi(s) = \pi(\sigma(l)) = \sigma_\pi(\pi(l))$ and $\pi(s') = \pi(\sigma(r)) = \sigma_\pi(\pi(r))$. Since $\pi(l) \gtrsim \pi(r)$, by stability of $\gtrsim$ we conclude $\pi(s) = \sigma_\pi(\pi(l)) \gtrsim \sigma_\pi(\pi(r)) = \pi(s')$.

   (b) If $p = i.q$, then we can write $s = f(s_1, \ldots, s_i, \ldots, s_k)$ and $s' = f(s'_1, \ldots, s'_i, \ldots, s'_k)$ for some nonconstant symbol f (i.e., $ar(f) > 0$) and we know that $i \in \mu(f)$, $s_i \hookrightarrow_{\mathcal{R},\mu} s'_i$ and $s_j = s'_j$ for all $j \neq i$. By the induction hypothesis, $\pi(s_i) \gtrsim \pi(s'_i)$. We consider the two possible cases according to $\pi(f)$:

      (i) If $\pi(f) = j$ for some $j \in \{1, \ldots, k\}$, then $\pi(s) = \pi(s_j)$. If $i \neq j$, then $s'_j = s_j$, hence $\pi(s_j) \gtrsim \pi(s'_j)$, by reflexivity of $\gtrsim$. If $i = j$, then we know from above that $\pi(s_i) \gtrsim \pi(s'_i)$. Therefore, $\pi(s) = \pi(s_j) \gtrsim \pi(s'_j) = \pi(s')$.

      (ii) If $\pi(f) = [i_1, \ldots, i_m]$, then $\pi(s) = f(\pi(s_{i_1}), \ldots, \pi(s_{i_m}))$ and $\pi(s') = f(\pi(s'_{i_1}), \ldots, \pi(s'_{i_m}))$. Consider i_j for some $j \in \{1, \ldots, m\}$. We have two cases:

         (A) If $i_j = i$, then by the induction hypothesis, $\pi(s_{i_j}) \gtrsim \pi(s'_{i_j})$ and, by definition of μ_π, $j \in \mu_\pi(f)$.

         (B) If $i_j \neq i$, then $s'_{i_j} = s_{i_j}$ and we have $\pi(s_{i_j}) = \pi(s'_{i_j})$.

      Note that $\pi(s_{i_j})$ is the j-th immediate subterm of $\pi(s)$. Therefore, by μ_π-monotonicity of $\gtrsim$ we have

$$
\begin{aligned}
\pi(s) &= \pi(f(s_1, \ldots, s_k)) \\
&= f(\pi(s_{i_1}), \ldots, \pi(s_{i_j}), \ldots, \pi(s_{i_m})) \\
&\gtrsim f(\pi(s_{i_1}), \ldots, \pi(s'_{i_j}), \ldots, \pi(s_{i_m})) \\
&= f(\pi(s'_{i_1}), \ldots, \pi(s'_{i_j}), \ldots, \pi(s'_{i_m})) \\
&= \pi(f(s'_1, \ldots, s'_k)) \\
&= \pi(s')
\end{aligned}
$$

      where we assume that $i_j = i$ for some $j \in \{1, \ldots, k\}$. If no such j exists, then we would have $\pi(s) = \pi(s')$, which also implies $\pi(s) \gtrsim (s')$ because $\gtrsim$ is reflexive.

Thus, we have proved that $s \hookrightarrow_{\mathcal{R},\mu} s'$ implies $\pi(s) \gtrsim \pi(s')$ as desired. Since $\pi(s) \gtrsim \pi(s')$ and $\pi(s') \gtrsim \pi(t)$ by the induction hypothesis, we conclude $\pi(s) \gtrsim \pi(t)$ by transitivity of $\gtrsim$.

□

We often use argument filterings to transform (sets of) rules S as follows: $\pi(s \to t) = \pi(s) \to \pi(t)$ for a pair $s \to t$, and $\pi(S) = \{\pi(s \to t) \mid s \to t \in S\}$. Furthermore, for $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$, we write $\pi(\mathcal{R})$ and $\pi(\mathcal{P})$ instead of $\pi(R)$ and $\pi(P)$, to denote the set of filtered rules (respectively, pairs).

12.2 Removing pairs using μ-reduction orderings

For a given TRS $\mathcal{R} = (\mathcal{F}, R)$, set of pairs $\mathcal{P} = (\mathcal{G}, P)$, and replacement map $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$, checking the absence of infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chains can often be 'simplified' to checking the absence of infinite minimal $(\mathcal{P}', \mathcal{R}, \mu)$-chains for a proper subset $\mathcal{P}' \subset \mathcal{P}$ by finding appropriate *μ-reduction pairs* $(\gtrsim, \sqsupset)$. The presence of *collapsing* pairs $u \to v = u \to x \in \mathcal{P}_{\mathcal{X}}$ imposes some additional requirements on the μ-reduction pairs. Basically,

(1) We need to ensure that the quasi-ordering $\gtrsim$ is able to 'look' for a μ-replacing subterm $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ inside the instantiation $\sigma(x) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ of a migrating variable x: we know that $\sigma(x) \trianglerighteq_\mu s$. Hence we require $\trianglerighteq_\mu \subseteq \gtrsim$ where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.
(2) We need to connect the marked version $s^\sharp$ of s (which is known to be an instance of a hidden term $t \in \mathcal{NHT}_{\mathcal{P}}$, i.e., $s = \theta(t)$ for some substitution θ) with an instance $\sigma(u)$ of the left-hand side u of a pair; hence the requirement $t \gtrsim t^\sharp$ or $t \sqsupset t^\sharp$ for all $t \in \mathcal{NHT}_{\mathcal{P}}$ which, by stability, becomes $s \gtrsim s^\sharp$ or $s \sqsupset s^\sharp$.

The following theorem formalizes a generic processor to remove pairs from $\mathcal{P}$ by using argument filterings and μ-reduction pairs.

Theorem 7 (μ-reduction pair processor) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let π be an argument filtering for $\mathcal{F} \cup \mathcal{G}$ and $(\gtrsim, \sqsupset)$ be a μ_π-reduction pair such that*

(1) $\pi(\mathcal{R}) \subseteq \gtrsim$, $\pi(\mathcal{P}) \subseteq \gtrsim \cup \sqsupset$, and
(2) whenever $\mathcal{NHT}_{\mathcal{P}} \neq \varnothing$ and $\mathcal{P}_{\mathcal{X}} \neq \varnothing$, we have that
  (a) for all $f \in \mathcal{F}$, either $\pi(f) = [i_1, \ldots, i_m]$ and $\mu(f) \subseteq \pi(f)$, or $\pi(f) = i$ and $\mu(f) = \{i\}$,
  (b) $\trianglerighteq_{\mu_\pi} \subseteq \gtrsim$, where $\trianglerighteq_{\mu_\pi}$ is the μ_π-replacing subterm relation on $\mathcal{T}(\mathcal{F}_\pi, \mathcal{X})$, and
  (c) $\pi(t)\ (\gtrsim \cup \sqsupset)\ \pi(t^\sharp)$ for all $t \in \mathcal{NHT}_{\mathcal{P}}$,

Let $\mathcal{P}_{\sqsupset} = \{u \to v \in \mathcal{P} \mid \pi(u) \sqsupset \pi(v)\}$. *Then, the processor* Proc_{RP} *given by*

$$\mathsf{Proc}_{RP}(\mathcal{P},\mathcal{R},\mu) = \begin{cases} \{(\mathcal{P}-\mathcal{P}_{\sqsupset},\mathcal{R},\mu)\} & \textit{if (1) and (2) hold} \\ \{(\mathcal{P},\mathcal{R},\mu)\} & \textit{otherwise} \end{cases}$$

is sound and complete.

Proof. We have to prove that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain if and only if there is an infinite minimal $(\mathcal{P}-\mathcal{P}_{\sqsupset},\mathcal{R},\mu)$-chain. The *if* part is obvious. For the *only if* part, we proceed by contradiction. Assume that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain A, but that there is no infinite minimal $(\mathcal{P}-\mathcal{P}_{\sqsupset},\mathcal{R},\mu)$-chain. Due to the finiteness of $\mathcal{P}$, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B

$$\sigma(u_1) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_1 \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_2) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_2 \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_3) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} \cdots$$

for some substitution σ, where all pairs in $\mathcal{Q}$ are infinitely often used, and, for all $i \geq 1$, (1) if $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$, then $t_i = \sigma(v_i)$ and (2) if $u_i \to v_i = u_i \to x_i \in \mathcal{Q}_{\mathcal{X}}$, then $t_i = s_i^{\sharp}$ for some s_i such that $\sigma(x_i) \trianglerighteq_{\mu} s_i$ and $s_i = \theta_i(\bar{s}_i)$ for some $\bar{s}_i \in \mathcal{NHT}$ and substitution θ_i; actually, since $t_i = s_i^{\sharp} = \theta_i(\bar{s}_i)^{\sharp} = \theta_i(\bar{s}_i^{\sharp})$ and $t_i \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_{i+1})$, we can further say that $\bar{s}_i \in \mathcal{NHT}_{\mathcal{Q}}$.

Since $\pi(u_i)\ (\gtrsim \cup \sqsupset)\ \pi(v_i)$ for all $u_i \to v_i \in \mathcal{Q} \subseteq \mathcal{P}$, by stability of $\gtrsim$ and $\sqsupset$, we have $\sigma_{\pi}(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_{\pi}(\pi(v_i))$ for all $i \geq 1$.

No pair $u \to v \in \mathcal{Q}$ satisfies that $\pi(u) \sqsupset \pi(v)$. Otherwise, we get a contradiction by considering the following two cases:

(1) If $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$, then $t_i = \sigma(v_i) \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_{i+1})$ and by Proposition 12, $\pi(t_i) \gtrsim \pi(\sigma(u_{i+1}))$. By Lemma 7, $\pi(t_i) \gtrsim \sigma_{\pi}(\pi(u_{i+1}))$. Since we have $\sigma_{\pi}(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_{\pi}(\pi(v_i)) = \pi(\sigma(v_i)) = \pi(t_i)$ (using Lemma 7), by using transitivity of $\gtrsim$ and compatibility between $\gtrsim$ and $\sqsupset$, we conclude that $\sigma_{\pi}(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_{\pi}(\pi(u_{i+1}))$.

(2) If $u_i \to v_i = u_i \to x_i \in \mathcal{Q}_{\mathcal{X}}$, then $\sigma(v_i) = \sigma(x_i) \trianglerighteq_{\mu} s_i$. Since $i \in \mu(f)$ implies that $i \in \pi(f)$, we can say that $\pi(\sigma(x)) = \sigma_{\pi}(x) \trianglerighteq_{\mu_{\pi}} \pi(s_i)$. Since $\trianglerighteq_{\mu_{\pi}} \subseteq \gtrsim$ we have $\sigma_{\pi}(\pi(v_i)) = \sigma_{\pi}(x_i) \gtrsim \pi(s_i)$. Furthermore, we are assuming that $\pi(t)\ (\gtrsim \cup \sqsupset)\ \pi(t^{\sharp})$ for all $t \in \mathcal{NHT}_{\mathcal{Q}} \subseteq \mathcal{NHT}_{\mathcal{P}}$. Since $s_i = \theta_i(\bar{s}_i)$, we have that $\pi(s_i) = \pi(\theta_i(\bar{s}_i)) = \theta_{i,\pi}(\pi(\bar{s}_i))$ (using Lemma 7 again) and, similarly, $\pi(s_i^{\sharp}) = \theta_{i,\pi}(\pi(\bar{s}_i^{\sharp}))$. By stability we have that $\pi(s_i)\ (\gtrsim \cup \sqsupset)\ \pi(s_i^{\sharp})$. Hence, by transitivity of $\gtrsim$ (and compatibility of $\gtrsim$ and $\sqsupset$), we have $\sigma_{\pi}(\pi(v_i)) = \sigma_{\pi}(x_i)\ (\gtrsim \cup \sqsupset)\ \pi(s_i^{\sharp})$. Finally, since $\pi(s_i^{\sharp}) = \pi(t_i)$ and $t_i \hookrightarrow^{*}_{\mathcal{R},\mu} \sigma(u_{i+1})$ for all $i \geq 1$, by Proposition 12 and Lemma 7, $\pi(t_i) \gtrsim \sigma_{\pi}(\pi(u_{i+1}))$. Therefore, again by transitivity of $\gtrsim$ and compatibility of $\gtrsim$ and $\sqsupset$, we conclude that $\sigma_{\pi}(\pi(u_i))(\gtrsim \cup \sqsupset)\sigma_{\pi}(\pi(u_{i+1}))$.

Since $u \to v$ occurs infinitely often in B, there is an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that $\sigma_\pi(\pi(u_i)) \sqsupset \sigma_\pi(\pi(u_{i+1}))$ for all $i \in \mathcal{I}$. And we have $\sigma_\pi(\pi(u_i)) (\gtrsim \cup \sqsupset) \sigma_\pi(\pi(u_{i+1}))$ for all other $u_i \to v_i \in \mathcal{Q}$. Thus, by using the compatibility conditions of the μ_π-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$.

Therefore, $\mathcal{Q} \subseteq (\mathcal{P} - \mathcal{P}_\sqsupset)$, which means that B is an infinite minimal $(\mathcal{P} - \mathcal{P}_\sqsupset, \mathcal{R}, \mu)$-chain, thus leading to a contradiction. □

The following example shows that the 'compatibility' between the replacement map μ and the argument filtering π which is required when collapsing pairs are present is necessary in Theorem 7.

Example 17 *Consider the following TRS:*

$$\begin{aligned} \mathtt{a} &\to \mathtt{c}(\mathtt{h}(\mathtt{f}(\mathtt{a}), \mathtt{b})) \\ \mathtt{f}(\mathtt{c}(x)) &\to x \end{aligned}$$

together with the replacement map μ given by $\mu(\mathtt{f}) = \mu(\mathtt{h}) = \{1\}$ and $\mu(\mathtt{c}) = \varnothing$. Then, $\mathsf{DP}(\mathcal{R}, \mu)$ consists of a single (collapsing) CSDP:

$$\mathtt{F}(\mathtt{c}(x)) \to x$$

and $\mathcal{NHT}_{\mathsf{DP}(R,\mu)} = \{\mathtt{f}(\mathtt{a}), \mathtt{a}\}$. Note that $\mathcal{R}$ is not *μ-terminating:*

$$\mathtt{f}(\underline{\mathtt{a}}) \hookrightarrow \underline{\mathtt{f}(\mathtt{c}(\mathtt{h}(\mathtt{f}(\mathtt{a}), \mathtt{b})))} \hookrightarrow \mathtt{h}(\mathtt{f}(\underline{\mathtt{a}}), \mathtt{b}) \hookrightarrow \cdots$$

However, by using the argument filtering π given by $\pi(\mathtt{h}) = []$, $\pi(\mathtt{F}) = \pi(\mathtt{f}) = [1]$ and $\pi(\mathtt{c}) = 1$, we would get the constraints:

$$\begin{aligned} \pi(\mathtt{a}) = {} & \mathtt{a} \gtrsim \mathtt{h} = \pi(\mathtt{c}(\mathtt{h}(\mathtt{f}(\mathtt{a}), \mathtt{b}))) \\ \pi(\mathtt{f}(\mathtt{c}(x))) = {} & \mathtt{f}(x) \gtrsim x = \pi(x) \\ \pi(\mathtt{F}(\mathtt{c}(x))) = {} & \mathtt{F}(x) \sqsupset x = \pi(x) \end{aligned}$$

which are easily satisfiable (by an RPO with precedence $\mathtt{a} \succ \mathtt{h}$, for instance). Thus, we would wrongly conclude μ-termination of $\mathcal{R}$. Note that $\pi(\mathtt{c}) = 1$ but $\mu(\mathtt{c}) = \varnothing$ and that $\pi(\mathtt{h}) = []$ but $\mu(\mathtt{h}) = \{1\}$. Note also that $\mu_\pi(\mathtt{f}) = \mu_\pi(\mathtt{F}) = \{1\}$ and $\mu_\pi(\mathtt{a}) = \mu_\pi(\mathtt{h}) = \varnothing$.

Example 18 *Consider the TRS $\mathcal{R}$ [Zan97, Example 5]:*

$$\begin{aligned} \mathtt{if}(\mathtt{true}, x, y) &\to x & \qquad \mathtt{f}(x) &\to \mathtt{if}(x, \mathtt{c}, \mathtt{f}(\mathtt{true})) \\ \mathtt{if}(\mathtt{false}, x, y) &\to y \end{aligned}$$

with $\mu(\mathtt{if}) = \{1, 2\}$. Then, $\mathsf{DP}(\mathcal{R}, \mu)$ consists of a CSDP in $\mathsf{DP}_\mathcal{F}(\mathcal{R}, \mu)$ and another one in $\mathsf{DP}_\mathcal{X}(\mathcal{R}, \mu)$:

$$\mathtt{F}(x) \to \mathtt{IF}(x, \mathtt{c}, \mathtt{f}(\mathtt{true})) \qquad\qquad \mathtt{IF}(\mathtt{false}, x, y) \to y$$

with $\mu^\sharp(\mathtt{F}) = \{1\}$ *and* $\mu(\mathtt{IF}) = \{1, 2\}$*. The* μ*-reduction pair* $(\geq, >)$ *induced by the polynomial interpretation*

$$\begin{array}{rcc} [\mathtt{c}] = [\mathtt{true}] = 0 & [\mathtt{f}](x) = x & [\mathtt{F}](x) = x \\ [\mathtt{false}] = 1 & [\mathtt{if}](x, y, z) = x + y + z & [\mathtt{IF}](x, y, z) = x + z \end{array}$$

can be used to prove the μ*-termination of* $\mathcal{R}$*. Consider* $\mathcal{P} = \mathsf{DP}(\mathcal{R}, \mu)$*. We have* $\mathcal{NHT}_\mathcal{P} = \{\mathtt{f}(\mathtt{true})\}$*. First, as required by Theorem 7, we can see that the quasi-ordering includes the* μ*-subterm property for symbols in* $\mathcal{F}$*:*

$$\begin{aligned} [\mathtt{f}(x)] &= \quad x \quad \geq x = [x] \\ [\mathtt{if}(x, y, z)] &= x + y + z \geq x = [x] \\ [\mathtt{if}(x, y, z)] &= x + y + z \geq y = [y] \end{aligned}$$

Now we can see that the condition on the only hidden term in $\mathcal{NHT}_\mathcal{P}$ *is also fulfilled:*

$$[\mathtt{f}(\mathtt{true})] = 0 \geq 0 = [\mathtt{F}(\mathtt{true})]$$

Finally, for the three rules in $\mathcal{R}$ *and the two pairs in* $\mathcal{P}$*, we have:*

$$\begin{aligned} [\mathtt{f}(x)] &= \quad x \quad \geq x = [\mathtt{if}(x, \mathtt{c}, \mathtt{f}(\mathtt{true}))] \\ [\mathtt{if}(\mathtt{true}, x, y)] &= x + y \geq x = [x] \\ [\mathtt{if}(\mathtt{false}, x, y)] &= x + y \geq y = [y] \\ [\mathtt{F}(x)] &= \quad x \quad \geq x = [\mathtt{IF}(x, \mathtt{c}, \mathtt{f}(\mathtt{true}))] \\ [\mathtt{IF}(\mathtt{false}, x, y)] &= y + 1 > y = [y] \end{aligned}$$

So, we remove the pair $\mathtt{IF}(\mathtt{false}, x, y) \to y$ *from* $\mathcal{P}$*. With the remaining pair* $\mathtt{F}(x) \to \mathtt{IF}(x, \mathtt{c}, \mathtt{f}(\mathtt{true}))$ *no infinite chain is possible. Thus, the* μ*-termination of* $\mathcal{R}$ *is proved.*

The next processor is useful when all terms in $\mathcal{NHT}_\mathcal{P}$ are *ground*. The advantage is that the quasi-ordering $\gtrsim$ of the μ-reduction pair does not need to have any μ-subterm property.

Theorem 8 (μ-reduction pair processor for ground hidden terms) *Let* $\mathcal{R} = (\mathcal{F}, R)$ *and* $\mathcal{P} = (\mathcal{G}, P)$ *be TRSs and* $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$*. Let* π *be an argument filtering for* $\mathcal{F} \cup \mathcal{G}$ *such that, for all* $t \in \mathcal{NHT}_\mathcal{P}$*,* $\pi(t)$ *is* ground*. Let* $(\gtrsim, \sqsupset)$ *be a* μ_π*-reduction pair such that*

(1) $\pi(\mathcal{R}) \subseteq \gtrsim$*,* $\pi(\mathcal{P}_\mathcal{G}) \subseteq \gtrsim \cup \sqsupset$*, and*

(2) for all $u \to v \in \mathcal{P}_\mathcal{X}$ and all $t \in \mathcal{NHT}_\mathcal{P}$, $\pi(u)\ (\gtrsim \cup \sqsupset)\ \pi(t^\sharp)$

Let $\mathcal{P}_\sqsupset = \{u \to v \in \mathcal{P}_\mathcal{G} \mid \pi(u) \sqsupset \pi(v)\} \cup \{u \to v \in \mathcal{P}_\mathcal{X} \mid \forall t \in \mathcal{NHT}_\mathcal{P}, \pi(u) \sqsupset \pi(t^\sharp)\}$. Then, the processor Proc_{RPg} given by

$$\mathsf{Proc}_{RPg}(\mathcal{P},\mathcal{R},\mu) = \begin{cases} \{(\mathcal{P}-\mathcal{P}_\sqsupset,\mathcal{R},\mu)\} & \text{if (1) and (2) hold} \\ \{(\mathcal{P},\mathcal{R},\mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

Proof. The proof is analogous to that of Theorem 7. Assume the facts and notation in the first paragraph of such a proof. Again, we proceed by contradiction and assume that a pair $u \to v \in \mathcal{Q}$ is in $\mathcal{P}_\sqsupset$. Again, we have $\sigma_\pi(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_\pi(\pi(u_{i+1}))$ for all pairs $u_i \to v_i \in \mathcal{Q}_\mathcal{G}$.

Now, if $u_i \to v_i = u_i \to x_i \in \mathcal{Q}_\mathcal{X}$, then since $\pi(u_i)\ (\gtrsim \cup \sqsupset)\ \pi(t^\sharp)$ for all $t \in \mathcal{NHT}_\mathcal{Q} \subseteq \mathcal{NHT}_\mathcal{P}$, by stability we have that $\sigma_\pi(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_\pi(\pi(t^\sharp))$. Since $\pi(t)$ is ground, we have $\sigma_\pi(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \pi(t^\sharp)$. Therefore, since $s_i \in \mathcal{NHT}_\mathcal{Q}$ and $t_i = s_i^\sharp$, we have $\sigma_\pi(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \pi(t_i)$. Finally, since $s_i^\sharp = t_i$ and $t_i \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_{i+1})$ for all $i \geq 1$, by Proposition 12 and Lemma 7, we have that $\pi(t_i) \gtrsim \sigma_\pi(\pi(u_{i+1}))$. Thus, we also have $\sigma_\pi(\pi(u_i))\ (\gtrsim \cup \sqsupset)\ \sigma_\pi(\pi(u_{i+1}))$.

Since $u \to v$ occurs infinitely often in B, by using the compatibility conditions of the μ_π-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$. In particular, if $u \to v \in \mathcal{Q}_\mathcal{X} \cap \mathcal{P}_\sqsupset$, then $\pi(u) \sqsupset \pi(t^\sharp)$ for all $t \in \mathcal{NHT}_\mathcal{Q}$, so each time that $u \to v$ is used, a strict decrease occurs. $\square$

The following example shows that Theorem 8 can succeed when Theorem 7 fails.

Example 19 *Consider the TRS $\mathcal{R}$:*

$$\mathtt{a} \to \mathtt{f}(\mathtt{d}(\mathtt{c}(\mathtt{a}))) \tag{16}$$
$$\mathtt{f}(\mathtt{c}(x)) \to x \tag{17}$$
$$\mathtt{d}(\mathtt{c}(x)) \to \mathtt{b} \tag{18}$$

together with the replacement map μ given by $\mu(\mathtt{c}) = \varnothing$ and $\mu(\mathtt{f}) = \mu(\mathtt{d}) = \{1\}$. This TRS has three CSDPs:

$$\mathtt{A} \to \mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a}))) \tag{19}$$
$$\mathtt{A} \to \mathtt{D}(\mathtt{c}(\mathtt{a})) \tag{20}$$
$$\mathtt{F}(\mathtt{c}(x)) \to x \tag{21}$$

Take $\mathcal{P} = \{(19),(21)\}$. Then, since $\mathcal{NHT}_\mathcal{P} = \{\mathtt{a}\} \neq \varnothing$ and $\mathtt{F}(\mathtt{c}(x)) \to x$ is a collapsing CSDP, according to Theorem 7 we would require that any μ-

reduction ordering used in the theorem satisfies $\trianglerighteq_\mu \subseteq \gtrsim$ *(assume the trivial filtering* $\pi_\top$ *here) and that* $\mathtt{a}\,(\gtrsim \cup \sqsupset)\,\mathtt{A}$*. In this case, though, since* $\mathtt{d}(\mathtt{c}(\mathtt{a})) \trianglerighteq_\mu \mathtt{c}(\mathtt{a})$*, we must have* $\mathtt{d}(\mathtt{c}(\mathtt{a})) \gtrsim \mathtt{c}(\mathtt{a})$*; by* μ*-monotonicity of* $\gtrsim$*,* $\mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a}))) \gtrsim \mathtt{F}(\mathtt{c}(\mathtt{a}))$*. Now, one of the following two cases must hold:*

(1) $\mathtt{A} \sqsupset \mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a})))$ *and* $\mathtt{F}(\mathtt{c}(x))\,(\gtrsim \cup \sqsupset)\,x$*. Then, by stability of* $\gtrsim$ *and* $\sqsupset$*, we have* $\mathtt{F}(\mathtt{c}(\mathtt{a}))\,(\gtrsim \cup \sqsupset)\,\mathtt{a}$*. Hence,*

$$\mathtt{A} \sqsupset \mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a}))) \gtrsim \mathtt{F}(\mathtt{c}(\mathtt{a}))\,(\gtrsim \cup \sqsupset)\,\mathtt{a}\,(\gtrsim \cup \sqsupset)\,\mathtt{A}.$$

By compatibility of $\gtrsim$ *and* $\sqsupset$*, we have* $\mathtt{A} \sqsupset \cdots \sqsupset \mathtt{A}$*, contradicting the well-foundedness of* $\sqsupset$*.*

(2) $\mathtt{A}\,(\gtrsim \cup \sqsupset)\,\mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a})))$ *and* $\mathtt{F}(\mathtt{c}(x)) \sqsupset x$*. Hence,*

$$\mathtt{A}\,(\gtrsim \cup \sqsupset)\,\mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a}))) \gtrsim \mathtt{F}(\mathtt{c}(\mathtt{a})) \sqsupset \mathtt{a}\,(\gtrsim \cup \sqsupset)\,\mathtt{A}.$$

Again, by compatibility of $\gtrsim$ *and* $\sqsupset$*, we have* $\mathtt{A} \sqsupset \cdots \sqsupset \mathtt{A}$*.*

Thus, Theorem 7 cannot be used with this example.

Since $\mathcal{NHT}_\mathcal{P} \subseteq \mathcal{T}(\mathcal{F})$*, Theorem 8 is applicable here. The* μ*-reduction pair* $(\geq, >)$ *induced by the following polynomial interpretation*[3] *:*

$$\begin{array}{llll} [\mathtt{a}] = 1 & [\mathtt{b}] = 0 & [\mathtt{c}](x) = x+1 & [\mathtt{d}](x) = \frac{1}{4}x \\ [\mathtt{f}](x) = x & [\mathtt{A}] = 1 & [\mathtt{F}](x) = 0 & \end{array}$$

can be used to remove (19) *from* $\mathcal{P}$*. For the three rules in* $\mathcal{R}$ *and pair* (19)*, we have:*

$$\begin{aligned} [\mathtt{a}] &= 1 \geq \tfrac{1}{2} = [\mathtt{f}(\mathtt{d}(\mathtt{c}(\mathtt{a})))] \\ [\mathtt{f}(\mathtt{c}(x))] &= x+1 \geq x = [x] \\ [\mathtt{d}(\mathtt{c}(x))] &= \tfrac{1}{4}x + \tfrac{1}{4} \geq 0 = [\mathtt{b}] \\ [\mathtt{A}] &= 1 > \tfrac{1}{2} = [\mathtt{F}(\mathtt{d}(\mathtt{c}(\mathtt{a})))] \end{aligned}$$

The collapsing pair (21) *generates a constraint* $\mathtt{F}(\mathtt{c}(x))\,(\gtrsim \cup \sqsupset)\,\mathtt{A}$ *which is also satisfied by the previous interpretation:*

$$[\mathtt{F}(\mathtt{c}(x))] = x+1 \geq 1 = [\mathtt{A}]$$

So, we remove (19) *from* $\mathcal{P}$ *to obtain* P'*. With* $\mathcal{P}' = \{(21)\}$*, no infinite chain is possible because* $\mathcal{NHT}_{\mathcal{P}'} = \varnothing$*. Thus, the* μ*-termination of* $\mathcal{R}$ *is proved.*

On the other hand, even when $\mathcal{NHT}_\mathcal{P} \subseteq \mathcal{T}(\mathcal{F})$, Theorem 7 can be helpful when Theorem 8 fails.

[3] See [Luc05,Luc07] for details about the use of this kind of polynomial intepretations with rational coefficients.

Example 20 *Consider $\mathcal{R}$ and μ as in Example 16. Theorem 8 cannot be used here because, reasoning as in Example 16, we would obtain constraints which are incompatible with the well-foundedness of $\sqsupset$. However, the μ-termination of $\mathcal{R}$ can be easily proved with Theorem 7. The μ-reduction pair $(\geq, >)$ generated by the following polynomial interpretation:*

$$[\mathtt{b}] = 1 \qquad [\mathtt{c}](x) = 0 \qquad [\mathtt{f}](x) = x$$
$$[\mathtt{B}] = 2 \qquad [\mathtt{F}](x) = x + 1$$

satisfies the requirements of Theorem 8 and can be used to show a weak decrease of the rules and a strict decrease of the two CSDPs which can both be removed.

Our last result establishes that if we are able to provide a strict comparison between unmarked and marked versions of the (filtered) hidden terms in $\mathcal{NHT}_{\mathcal{P}}$, then we can remove *all* collapsing pairs at the same time.

Theorem 9 (μ-reduction pair processor for collapsing pairs) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let π be an argument filtering for $\mathcal{F} \cup \mathcal{G}$ and $(\gtrsim, \sqsupset)$ be a μ_π-reduction pair such that*

(1) $\pi(\mathcal{R}) \subseteq \gtrsim$, $\pi(\mathcal{P}) \subseteq \gtrsim \cup \sqsupset$, and
(2) $\pi(t) \sqsupset \pi(t^\sharp)$ for all $t \in \mathcal{NHT}_{\mathcal{P}}$ and
  (a) for all $f \in \mathcal{F}$, either $\pi(f) = [i_1, \ldots, i_m]$ and $\mu(f) \subseteq \pi(f)$, or $\pi(f) = i$ and $\mu(f) = \{i\}$,
  (b) $\trianglerighteq_{\mu_\pi} \subseteq \gtrsim$, where $\trianglerighteq_{\mu_\pi}$ is the μ_π-replacing subterm relation on $\mathcal{T}(\mathcal{F}_\pi, \mathcal{X})$.

Then, the processor Proc_{RPc} given by

$$\mathsf{Proc}_{RPc}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P}_{\mathcal{G}}, \mathcal{R}, \mu)\} & \text{if (1) and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

Proof. As in the proof of Theorem 7, we proceed by contradiction. We assume that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain A, but that there is no infinite minimal $(\mathcal{P}_{\mathcal{G}}, \mathcal{R}, \mu)$-chain. Thus, there is $\mathcal{Q} \subseteq \mathcal{P}$ such that $\mathcal{Q} \cap \mathcal{P}_{\mathcal{X}} \neq \varnothing$ and A has a tail B as in the proof of Theorem 7. Now, we assume the notation as in the first paragraph of such a proof.

We have $\sigma_\pi(\pi(u_i))\,(\gtrsim \cup \sqsupset)\,\pi(t_i)$ and $\pi(t_i) \gtrsim \sigma_\pi(\pi(u_{i+1}))$ for all pairs $u_i \to v_i \in \mathcal{P}_{\mathcal{G}}$. If $u_i \to v_i = u_i \to x_i \in \mathcal{Q}_{\mathcal{X}}$, then by applying the considerations in the corresponding item of the proof of Theorem 7 and taking into account that $\pi(t) \sqsupset \pi(t^\sharp)$ for *all* $t \in \mathcal{NHT}_{\mathcal{P}}$, we have now that $\sigma_\pi(\pi(u_i))\,(\gtrsim \cup \sqsupset)\,\sigma_\pi(x_i) \sqsupset \pi(t_i) \gtrsim \sigma_\pi(\pi(u_{i+1}))$. Since pairs $u_i \to v_i \in \mathcal{Q}_{\mathcal{X}}$ occur infinitely often in B, by using the compatibility conditions of the μ_π-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$. $\Box$

13 Subterm criterion

In [HM04,HM07], Hirokawa and Middeldorp introduce a very interesting *subterm criterion* which permits to ignore certain cycles of the dependency graph *without paying attention to the rules of the TRS*. Hirokawa and Middeldorp's result applies to *cycles* in the *dependency graph*. Recently, Thiemann has adapted it to the DP-framework [Thi07, Section 4.6]. In our adaptation to *CSR*, we take ideas from both works. Our first definition is inspired by Thiemann's *head symbols* [Thi07, Definition 4.36].

Definition 11 (Root symbols of a TRS) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS. The set of* root symbols *associated to $\mathcal{R}$ is:*

$$Root(\mathcal{R}) = \{root(l) \mid l \to r \in R\} \cup \{root(r) \mid l \to r \in R, r \notin \mathcal{X}\}$$

The following result relates $Root(\mathcal{P})$ and the set $\mathcal{H}_\mathcal{P}$ of hidden symbols occurring *at the root* of terms in $\mathcal{NHT}_\mathcal{P}(\mathcal{R}, \mu)$. It is silently used in the statements of some theorems below.

Lemma 8 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs such that $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$, and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. For all $f \in \mathcal{H}_\mathcal{P}$, we have $f^\sharp \in Root(\mathcal{P})$.*

Proof. If $f \in \mathcal{H}_\mathcal{P}$, then there is $t \in \mathcal{NHT}_\mathcal{P}$ such that $f = root(t)$. Therefore, there are substitutions θ and θ' such that $\theta(t^\sharp) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(u)$ for some $u \to v \in \mathcal{P}$. Since $f^\sharp \notin \mathcal{F}$, μ-rewritings on $\theta(t^\sharp)$ using $\mathcal{R}$ do not remove it. Thus, $root(u) = f^\sharp$ and $f^\sharp \in Root(\mathcal{P})$. $\Box$

Thiemann uses argument filterings (see Section 12.1) instead of Hirokawa and Middeldorp's *simple projections* (see [HM04, Definition 10]). We find more convenient to follow Hirokawa and Middeldorp's style, so we generalize their definition to be used with TRSs rather than cycles in the dependency graph.

Definition 12 (Simple projection) *Let $\mathcal{R}$ be a TRS. A* simple projection *for $\mathcal{R}$ is a mapping π that assigns to every k-ary symbol $f \in Root(\mathcal{R})$ an argument position $i \in \{1, \ldots, k\}$. The mapping that assigns to every term $t = f(t_1, \ldots, t_k)$ with $f \in Root(\mathcal{R})$ its subterm $\pi(t) = t|_{\pi(f)}$ is also denoted by π; we also let $\pi(x) = x$ if $x \in \mathcal{X}$.*

Given a simple projection π for a TRS $\mathcal{R}$, we let $\pi(\mathcal{R}) = \{\pi(l) \to \pi(r) \mid l \to r \in \mathcal{R}\}$.

Theorem 10 (Subterm processor for noncollapsing pairs) *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs such that $\mathcal{P}$ contains no collapsing rule, i.e., for all $u \to v \in \mathcal{P}$, $v \notin \mathcal{X}$, and $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$. Let $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$ and let π be a simple projection for $\mathcal{P}$. Let $\mathcal{P}_{\pi,\rhd_\mu} = \{u \to v \in \mathcal{P} \mid \pi(u) \rhd_\mu \pi(v)\}$. Then, the processor $\mathsf{Proc}_{subNColl}$ given by*

$$\mathsf{Proc}_{subNColl}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_{\pi,\rhd_\mu}, \mathcal{R}, \mu)\} & \text{if } \pi(\mathcal{P}) \subseteq \unrhd_\mu \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

Proof. We have to prove that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain if and only if there is an infinite minimal $(\mathcal{P} - \mathcal{P}_{\pi,\rhd_\mu}, \mathcal{R}, \mu)$-chain. The *if* part is obvious. For the *only if* part, we proceed by contradiction. Assume that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain A but there is no infinite minimal $(\mathcal{P} - \mathcal{P}_{\pi,\rhd_\mu}, \mathcal{R}, \mu)$-chain. Since $\mathcal{P}$ is finite, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B which is an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu)$-chain where all pairs in $\mathcal{Q}$ are infinitely often used. Assume that B is as follows (since $\mathcal{Q}_\mathcal{X} = \varnothing$, we use a simpler notation):

$$t_0 \hookrightarrow^*_{\mathcal{R},\mu} s_1 \stackrel{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} t_1 \hookrightarrow^*_{\mathcal{R},\mu} s_2 \stackrel{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} t_2 \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

where, there is a substitution σ such that, for all $i \geq 1$, $s_i = \sigma(u_i)$ and $t_i = \sigma(v_i)$ for some $u_i \to v_i \in \mathcal{Q}$. Furthermore, w.l.o.g. we also assume that $t_0 = \sigma(v_0)$ for some $u_0 \to v_0 \in \mathcal{P}$.

Note that, for all $i \geq 1$, $root(s_i) \in Root(\mathcal{P})$ because $root(u_i) \in Root(\mathcal{P})$ and for all $i \geq 0$, $root(t_i) \in Root(\mathcal{P})$ due to $root(v_i) \in Root(\mathcal{P})$, which holds because $root(v_i) \notin \mathcal{X}$. Therefore, we can apply π to s_{i+1} and t_i for all $i \geq 0$. Moreover, since $t_i \hookrightarrow^*_{\mathcal{R},\mu} s_{i+1}$ for all $i \geq 0$ and $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$, we can actually write $t_i \stackrel{>\Lambda}{\hookrightarrow}^*_{\mathcal{R},\mu} s_{i+1}$ because μ-rewritings with $\mathcal{R}$ cannot change $root(t_i)$. Hence $\pi(t_i) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_{i+1})$ and also $root(t_i) = root(s_{i+1})$ for all $i \geq 0$. Finally, since $\pi(u_i) \unrhd_\mu \pi(v_i)$ for all $i \geq 0$, by stability of $\unrhd_\mu$, we have

$$\pi(s_i) = \pi(\sigma(u_i)) = \sigma(\pi(u_i)) \unrhd_\mu \sigma(\pi(v_i)) = \pi(\sigma(v_i)) = \pi(t_i)$$

for all $i \geq 1$.

No pair $u \to v \in \mathcal{Q}$ satisfies that $\pi(u) \rhd_\mu \pi(v)$. Otherwise, we get a contradiction in both of the following two complementary cases:

(1) if $\pi(f) \notin \mu(f)$ for all $f \in Root(\mathcal{Q})$, then, for all $i \geq 0$, $\pi(t_i) = \pi(s_{i+1})$, because no μ-rewritings are possible on the $\pi(root(t_i))$-th immediate subterm $\pi(t_i)$ of t_i. Since $\pi(s_{i+1}) \unrhd_\mu \pi(t_{i+1})$, we have that $\pi(t_i) \unrhd_\mu \pi(t_{i+1})$ for all $i \geq 0$. Furthermore, since we assume $\pi(u) \rhd_\mu \pi(v)$ for some $u \to v \in \mathcal{Q}$

which occurs infinitely often in B, and by stability of $\rhd_\mu$, there is a maximal infinite set $J = \{j_1, j_2, \ldots\} \subseteq \mathbb{N}$ such that $\pi(t_{j_i}) \rhd_\mu \pi(t_{j_i+1})$ for all $i \geq 1$. Thus, we obtain an infinite sequence $\pi(t_{j_1}) \rhd_\mu \pi(t_{j_2}) \rhd_\mu \cdots$ which contradicts the well-foundedness of $\rhd_\mu$.

(2) if $\pi(f) \in \mu(f)$ for some $f \in \mathit{Root}(\mathcal{Q})$, then, since $\mathit{root}(t_i) = \mathit{root}(s_{i+1})$ and all pairs in $\mathcal{Q}$ occur infinitely often in B, we can assume that $\mathit{root}(t_0) = f$. Furthermore, since A is minimal, we can assume that t_0 is μ-terminating (w.r.t. $\mathcal{R}$). Since $\pi(t_i) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_{i+1})$ and $\pi(s_{i+1}) \unrhd_\mu \pi(t_{i+1})$ for all $i \geq 0$, the sequence B is transformed into an infinite $\hookrightarrow_{\mathcal{R},\mu} \cup \unrhd_\mu$-sequence

$$\pi(t_0) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_1) \unrhd_\mu \pi(t_1) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_2) \unrhd_\mu \pi(t_2) \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

containing infinitely many $\rhd_\mu$-steps, due to $\pi(u) \rhd_\mu \pi(v)$ for some $u \to v \in \mathcal{Q}$ which occurs infinitely often in B. Since $\rhd_\mu$ is well-founded, the infinite sequence must also contain infinitely many $\hookrightarrow_{\mathcal{R},\mu}$-steps. By making repeated use of the fact that $\rhd_\mu \circ \hookrightarrow_{\mathcal{R},\mu} \subseteq \hookrightarrow_{\mathcal{R},\mu} \circ \rhd_\mu$, we obtain an infinite $\hookrightarrow_{\mathcal{R},\mu}$-sequence starting from $\pi(t_0)$. Thus, $\pi(t_0)$ is not μ-terminating with respect to $\mathcal{R}$. Since $\pi(f) \in \mu(f)$ and hence $t_0 \rhd_\mu \pi(t_0)$, this implies that t_0 is not μ-terminating (use Lemma 1(1)). This contradicts μ-termination of t_0.

Therefore, $\mathcal{Q} \subseteq \mathcal{P} - \mathcal{P}_{\pi,\rhd}$. Hence, B is an infinite minimal $(\mathcal{P} - \mathcal{P}_{\pi,\rhd_\mu}, \mathcal{R}, \mu)$-chain. This contradicts our initial argument. □

Example 21 *Consider the CS-TRS $(\mathcal{R}, \mu)$ in Example 14. We can apply Theorem 10 to the two cycles $\{(13)\}$ and $\{(14)\}$ in the CSDG (see Figure 2).*

(1) Taking $\pi(\mathtt{G}) = 1$, we have that $\pi(\mathtt{G}(\mathtt{s}(x))) = \mathtt{s}(x) \rhd_\mu x = \pi(\mathtt{G}(x))$ and we conclude (by using Proposition 8(1)), that there is no infinite minimal $(\{(13)\}, \mathcal{R}, \mu^\sharp)$-chain.

(2) Taking $\pi(\mathtt{SEL}) = 1$, we have that $\pi(\mathtt{SEL}(\mathtt{s}(x), \mathtt{cons}(y, z))) = \mathtt{s}(x) \rhd_\mu x = \pi(\mathtt{SEL}(x, z))$, and similarly conclude that there is no infinite minimal $(\{(14)\}, \mathcal{R}, \mu^\sharp)$-chain.

Thus, the μ-termination of $\mathcal{R}$ is proved.

The following examples shows that if we allow collapsing rules in $\mathcal{P}$, then Theorem 10 does not hold.

Example 22 *Consider the TRS $\mathcal{R}$ consisting of a rule*

$$\mathtt{h}(x) \to \mathtt{f}(\mathtt{g}(\mathtt{h}(x)))$$

and $\mathcal{P}$ containing a single collapsing rule

$$\mathtt{f}(\mathtt{g}(x)) \to x.$$

Let μ be given by $\mu(f) = \{1, \ldots, k\}$ for all symbols f. Note that, as required in Theorem 10, Root($\mathcal{P}$) = $\{\mathtt{f}\}$ and $\mathcal{D} = \{\mathtt{h}\}$ are disjoint. By using the projection $\pi(\mathtt{f}) = 1$, we get $\pi(\mathtt{f}(\mathtt{g}(x))) = \mathtt{g}(x) \rhd_\mu x$. After removing the pair in $\mathcal{P}$, a finite CS-termination problem $(\varnothing, \mathcal{R}, \mu)$ is obtained. However, $(\mathcal{P}, \mathcal{R}, \mu)$ is not *finite:*

$$\underline{\mathtt{f}(\mathtt{g}(\mathtt{h}(x)))} \hookrightarrow_{\mathcal{P},\mu} \underline{\mathtt{h}(x)} \hookrightarrow_{\mathcal{R},\mu} \underline{\mathtt{f}(\mathtt{g}(\mathtt{h}(x)))} \hookrightarrow_{\mathcal{P},\mu} \cdots$$

In the following theorem, we show how to use the subterm criterion to *remove* all collapsing pairs from $\mathcal{P}$. The interesting point is that, in contrast with non-collapsing pairs, we do *not*need to have $u \rhd_\mu v$ to be able to remove collapsing pairs $u \to v$.

Theorem 11 (Subterm processor for collapsing pairs) *Let $\mathcal{R} = (\mathcal{F}, R)$ $= (\mathcal{C} \uplus \mathcal{D}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs such that $\mathcal{P}_\mathcal{G}$ contains no collapsing rule, Root($\mathcal{P}$) $\cap\, \mathcal{D} = \varnothing$, and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let π be a simple projection for $\mathcal{P}$ such that*

(1) $\pi(\mathcal{P}) \subseteq \unrhd_\mu$, and
(2) whenever $\mathcal{P}_\mathcal{X} \neq \varnothing$, we have $\pi(f^\sharp) \in \mu(f^\sharp) \cap \mu(f)$ for all $f \in \mathcal{H}_\mathcal{P}$.

Then, the processor $\mathsf{Proc}_{subColl}$ given by

$$\mathsf{Proc}_{subColl}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_\mathcal{X}, \mathcal{R}, \mu)\} & \text{if (1) and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

Proof. We have to prove that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain if and only if there is an infinite minimal $(\mathcal{P} - \mathcal{P}_\mathcal{X}, \mathcal{R}, \mu)$-chain. The *if* part is obvious. For the *only if* part, we proceed by contradiction. Assume that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain A but there is no infinite minimal $(\mathcal{P} - \mathcal{P}_\mathcal{X}, \mathcal{R}, \mu)$-chain. Since $\mathcal{P}$ is finite, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B which is an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu)$-chain where all pairs in $\mathcal{Q}$ are infinitely often used and $\mathcal{Q}$ contains some collapsing pair $u \to x \in \mathcal{Q}_\mathcal{X}$. Assume that B is

$$t_0 \hookrightarrow^*_{\mathcal{R},\mu} s_1 \overset{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} \circ \unrhd^\sharp_\mu t_1 \hookrightarrow^*_{\mathcal{R},\mu} s_2 \overset{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} \circ \unrhd^\sharp_\mu t_2 \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

where there is a substitution σ such that, for all $i \geq 1$, $s_i = \sigma(u_i)$ for some $u_i \to v_i \in \mathcal{P}$, and

(1) if $v_i \notin \mathcal{X}$, then $t_i = \sigma(v_i)$, and
(2) if $v_i = x_i \in \mathcal{X}$, then $x_i \notin \mathcal{V}ar^\mu(u_i)$ and $t_i = r_i^\sharp$ for some $r_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(x_i) \unrhd_\mu r_i$, and $r_i = \theta_i(\bar{r}_i)$ for some $\bar{r}_i \in \mathcal{NHT}_\mathcal{Q}$ and substitution

θ_i.

W.l.o.g. (because we can freely choose the starting term of B) we assume that t_0 is a particular case of the second alternative above, i.e., there is a collapsing pair $u_0 \to x_0$ such that $\sigma(x_0) \trianglerighteq_\mu r_0$ and $t_0 = r_0^\sharp$. Note that, for all $i \geq 1$, $root(s_i) \in Root(\mathcal{P})$ because $root(u_i) \in Root(\mathcal{P})$. Furtermore, for all $i \geq 0$, $root(t_i) \in Root(\mathcal{P})$ because:

(1) If $u_i \to v_i \in \mathcal{Q}_\mathcal{G}$, then $root(v_i) \in Root(\mathcal{P})$ and $t_i = \sigma(v_i)$.
(2) If $u_i \to v_i \in \mathcal{Q}_\mathcal{X}$, then $root(t_i) \in \mathcal{F}^\sharp$; since $t_i \hookrightarrow^*_{\mathcal{R},\mu} s_{i+1}$ and $\mathcal{F}^\sharp \cap \mathcal{F} = \varnothing$, rewritings with $\mathcal{R}$ cannot remove the marked root symbol in t_i; hence, we can further conclude $root(t_i) = root(s_{i+1}) \in Root(\mathcal{P})$.

Therefore, we can apply π to s_{i+1} and t_i for all $i \geq 0$. Moreover, since $t_i \hookrightarrow^*_{\mathcal{R},\mu} s_{i+1}$ for all $i \geq 0$ and $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$, we can actually write $t_i \stackrel{>\Lambda}{\hookrightarrow}{}^*_{\mathcal{R},\mu} s_{i+1}$; hence $\pi(t_i) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_{i+1})$ and also $root(t_i) = root(s_{i+1})$ for all $i \geq 0$.

Since $u \to x \in \mathcal{Q}_\mathcal{X}$ and B is infinite, it must be $\mathcal{H}_\mathcal{Q} \neq \varnothing$ (hence $\mathcal{H}_\mathcal{P} \neq \varnothing$). Thus, we have $\pi(f^\sharp) \in \mu(f)$ for all $f \in \mathcal{H}_\mathcal{Q} \subseteq \mathcal{H}_\mathcal{P}$. Then, since $root(t_i) = root(s_{i+1})$ and all pairs in $\mathcal{Q}$ occur infinitely often in B, we can assume that $root(t_0) = f$. Furthermore, since A is minimal, we can assume that t_0 is μ-terminating. We have that $\pi(u_i) \trianglerighteq_\mu \pi(v_i)$ for all $u_i \to v_i \in \mathcal{Q}$. Now we distinguish two cases:

(1) If $u_i \to v_i \in \mathcal{Q}_\mathcal{G}$, then $s_i = \sigma(u_i)$ and $t_{i+1} = \sigma(v_i)$. By stability of $\trianglerighteq_\mu$ we have $\pi(s_i) \trianglerighteq_\mu \pi(t_{i+1})$.
(2) If $u_i \to v_i = u_i \to x_i \in \mathcal{Q}_\mathcal{X}$, then $s_i = \sigma(u_i)$ and there is a term r_i, such that $\sigma(x_i) \trianglerighteq_\mu r_i$ and $r_i^\sharp = t_{i+1}$. Since $\pi(u_i) \trianglerighteq_\mu x_i$, by stability of $\trianglerighteq_\mu$ we have
$$\pi(s_i) = \pi(\sigma(u_i)) = \sigma(\pi(u_i)) \trianglerighteq_\mu \sigma(x_i) \trianglerighteq_\mu r_i.$$
Note that $f_i = root(r_i) = root(\bar{r}_i) \in \mathcal{H}_\mathcal{P}$. Since $\pi(t_{i+1}) = t_{i+1}|_{\pi(f_i^\sharp)} = r_i^\sharp|_{\pi(f_i^\sharp)} = r_i|_{\pi(f_i^\sharp)}$ and $\pi(f_i^\sharp) \in \mu(f_i)$, we have that $r_i \triangleright_\mu \pi(t_{i+1})$ and thus $\pi(s_i) \triangleright_\mu \pi(t_{i+1})$.

Therefore, by applying the simple projection π, the sequence B is transformed into an infinite $\hookrightarrow_{\mathcal{R},\mu} \cup \triangleright_\mu$-sequence B'

$$\pi(t_0) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_1) \trianglerighteq_\mu \pi(t_1) \hookrightarrow^*_{\mathcal{R},\mu} \pi(s_2) \trianglerighteq_\mu \pi(t_2) \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

Since $u \to x$ occurs infinitely often in B, and by the second case above, B' contains infinitely many $\triangleright_\mu$ steps, starting from $\pi(t_0)$. Since $\triangleright_\mu$ is well-founded, the infinite sequence must also contain infinitely many $\hookrightarrow_{\mathcal{R},\mu}$-steps. By making repeated use of the fact that $\triangleright_\mu \circ \hookrightarrow_{\mathcal{R},\mu} \subseteq \hookrightarrow_{\mathcal{R},\mu} \circ \triangleright_\mu$, we obtain an infinite $\hookrightarrow_{\mathcal{R},\mu}$-sequence starting from $\pi(t_0)$. Thus, $\pi(t_0)$ is not μ-terminating with respect to $\mathcal{R}$. Since $\pi(f^\sharp) \in \mu(f^\sharp)$ and hence $t_0 \triangleright_\mu \pi(t_0)$, this implies that

t_0 is not μ-terminating (use Lemma 1(1)). This contradicts μ-termination of t_0.

Therefore, $\mathcal{Q}$ cannot contain any collapsing pair. This contradicts our initial assumption $u \to x \in \mathcal{Q}$. $\square$

Remark 8 *The use of Theorem 11 only makes sense if $\mathcal{P} \subseteq \mathcal{P}_{\mathcal{G}} \cup \mathcal{P}^1_{\mathcal{X}}$. If $u \to x \in \mathcal{P}_{\mathcal{X}} - \mathcal{P}^1_{\mathcal{X}}$ for some $u = f(u_1, \ldots, u_k)$, then for all $i \in \{1, \ldots, k\}$, whenever $x \in \mathcal{V}ar(u_i)$ we have $i \in \mu(f)$ and $u_i \not\rhd_\mu x$. Thus, there is no simple projection π such that $\pi(u) \trianglerighteq_\mu x$.*

Example 23 *Consider the following TRS $\mathcal{R}$:*

$$\begin{aligned} \mathtt{g}(x,y) &\to \mathtt{f}(x,y) \\ \mathtt{f}(\mathtt{c}(x),y) &\to \mathtt{g}(x,\mathtt{g}(y,y)) \end{aligned}$$

together with the replacement map μ given by $\mu(\mathtt{c}) = \mu(\mathtt{g}) = \{1\}$ and $\mu(\mathtt{f}) = \varnothing$. The CSDPs are:

$$\mathtt{G}(x,y) \to \mathtt{F}(x,y) \tag{22}$$

$$\mathtt{F}(\mathtt{c}(x),y) \to \mathtt{G}(x,\mathtt{g}(y,y)) \tag{23}$$

$$\mathtt{F}(\mathtt{c}(x),y) \to x \tag{24}$$

and all of them are part of the only SCC $\mathcal{P} = \{(22),(23),(24)\}$ in the CSDG of $(\mathcal{R},\mu)$. Note that $\mathcal{NHT}_{\mathcal{P}} = \{\mathtt{g}(y,y)\}$, hence $\mathcal{H}_{\mathcal{P}} = \{\mathtt{g}\}$. Consider the simple projection π given by $\pi(\mathtt{F}) = \pi(\mathtt{G}) = 1$. Note that $\pi(\mathtt{G}) \in \mu(\mathtt{G}) \cap \mu(\mathtt{g})$ as required by Theorem 11. Since

- $\pi(\mathtt{G}(x,y)) = x \trianglerighteq_\mu x = \pi(\mathtt{F}(x,y))$
- $\pi(\mathtt{F}(\mathtt{c}(x),y)) = \mathtt{c}(x) \rhd_\mu x = \pi(\mathtt{G}(x,\mathtt{g}(y,y)))$, *and*
- $\pi(\mathtt{F}(\mathtt{c}(x),y)) = \mathtt{c}(x) \rhd_\mu x = \pi(x)$

we can first use Theorem 11 to remove the CSDP (24) from the SCC $\mathcal{P}$ to obtain a new problem $(\{(22),(23)\},\mathcal{R},\mu^\sharp)$ to which Theorem 10 applies to finally obtain $(\{(22)\},\mathcal{R},\mu^\sharp)$ for which $\mathsf{G}(\{(22)\},\mathcal{R},\mu^\sharp)$ contains no cycle, thus proving the μ-termination of $\mathcal{R}$.

The following result provides a kind of generalization of the subterm criterion to simple projections which only take *non-μ-replacing* arguments.

Theorem 12 (Non-μ-replacing projection processor) *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs such that $\mathcal{P}_{\mathcal{G}}$ contains no collapsing rule, $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$, and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$ Let $\gtrsim$ be a stable quasi-ordering on terms whose strict and stable part $>$ is well-founded and π be a simple projection for $\mathcal{P}$ such that*

(1) for all $f \in Root(\mathcal{P})$, $\pi(f) \notin \mu(f)$,
(2) $\pi(\mathcal{P}) \subseteq \gtrsim$, and,
(3) whenever $\mathcal{NHT}_\mathcal{P} \neq \varnothing$ and $\mathcal{P}_\mathcal{X} \neq \varnothing$, we have that $\trianglerighteq_\mu \subseteq \gtrsim$, where $\trianglerighteq_\mu$ is the μ-replacing subterm relation on $\mathcal{T}(\mathcal{F},\mathcal{X})$, and $t \gtrsim t|_{\pi(root(t)^\sharp)}$ for all $t \in \mathcal{NHT}_\mathcal{P}$.

Let $\mathcal{P}_> = \{u \to v \in \mathcal{P} \mid \pi(u) > \pi(v)\}$. Then, the processor Proc_{NRP} *given by*

$$\mathsf{Proc}_{NRP}(\mathcal{P},\mathcal{R},\mu) = \begin{cases} \{(\mathcal{P}-\mathcal{P}_>,\mathcal{R},\mu)\} & \textit{if (1), (2), and (3) hold} \\ \{(\mathcal{P},\mathcal{R},\mu)\} & \textit{otherwise} \end{cases}$$

is sound and complete.

Proof. We have to prove that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain if and only if there is an infinite minimal $(\mathcal{P}-\mathcal{P}_>,\mathcal{R},\mu)$-chain. The *if* part is obvious. For the *only if* part, we proceed by contradiction. Assume that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain A but there is no infinite minimal $(\mathcal{P}-\mathcal{P}_>,\mathcal{R},\mu)$-chain. Since $\mathcal{P}$ is finite, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B

$$\sigma(u_1) \overset{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} \circ \trianglerighteq^\sharp_\mu \; t_1 \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2) \overset{\Lambda}{\hookrightarrow}_{\mathcal{Q},\mu} \circ \trianglerighteq^\sharp_\mu \; t_2 \hookrightarrow^*_{\mathcal{R},\mu} \cdots$$

for some substitution σ and pairs $u_i \to v_i \in \mathcal{Q}$, and

(1) if $v_i \notin \mathcal{X}$, then $t_i = \sigma(v_i)$, and
(2) if $v_i = x_i \in \mathcal{X}$, then $x_i \notin \mathcal{V}ar^\mu(u_i)$ and $t_i = s_i^\sharp$ for some s_i such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i = \theta_i(\bar{s}_i)$ for some $\bar{s}_i \in \mathcal{NHT}_\mathcal{P}$ and substitution θ_i.

Furthermore, *all pairs* in $\mathcal{Q}$ are used infinitely often in B. As discussed in the proof of Theorem 10, for all $i \geq 1$, $root(t_i) \in Root(\mathcal{P})$, $\pi(t_i) \hookrightarrow^*_{\mathcal{R},\mu} \pi(\sigma(u_{i+1}))$ and also $root(t_i) = root(u_{i+1})$ for all $i \geq 1$.

No pair $u \to v \in \mathcal{Q}$ satisfies that $\pi(u) > \pi(v)$. Otherwise, by applying the simple projection π to the sequence B, for all $i \geq 1$ we get a contradiction as follows:

(1) Since $\pi(f) \notin \mu(f)$ for all $f \in Root(\mathcal{Q})$, for all $i \geq 1$, $\pi(t_i) = \pi(\sigma(u_{i+1})) = \sigma(\pi(u_{i+1}))$, because no μ-rewritings are possible on the $\pi(root(t_i))$-th immediate subterm $\pi(t_i)$ of t_i, and
(2) Due to $\pi(u_i) \gtrsim \pi(v_i)$ and by stability of $\gtrsim$, we have that $\pi(\sigma(u_i)) = \sigma(\pi(u_i)) \gtrsim \sigma(\pi(v_i))$. Now, we distinguish two cases:
   (a) If $u_i \to v_i \in \mathcal{Q}_\mathcal{G}$, then $\pi(t_i) = \pi(\sigma(v_i)) = \sigma(\pi(v_i))$. Thus, $\pi(\sigma(u_i)) \gtrsim \pi(t_i)$.
   (b) If $u_i \to v_i \in \mathcal{Q}_\mathcal{X}$, then $\sigma(\pi(v_i)) = \sigma(x_i)$. Since $\sigma(x_i) \trianglerighteq_\mu s_i$, we have that $\sigma(x_i) \gtrsim s_i$ (because $\trianglerighteq_\mu \subseteq \gtrsim$). Let $f = root(u_{i+1}) = root(t_i) = root(\bar{s}_i^\sharp)$. Since $t \gtrsim t|_{\pi(root(t)^\sharp)}$ for all $t \in \mathcal{NHT}_\mathcal{P}$, by stability, we

have $s_i = \theta_i(\bar{s}_i) \gtrsim \theta_i(\bar{s}_i|_{\pi(f)}) = \theta_i(\bar{s}_i)|_{\pi(f)} = s_i|_{\pi(f)}$. Since $s_i|_{\pi(f^\sharp)} = t_i|_{\pi(f^\sharp)} = \pi(t_i)$, we have $s_i \gtrsim \pi(t_i)$. Hence, $\pi(\sigma(u_i)) \gtrsim \pi(t_i)$.

Thus, we always have $\pi(\sigma(u_i)) \gtrsim \pi(t_i)$. Therefore, we obtain an infinite $\gtrsim$ sequence

$$\pi(\sigma(u_1)) \gtrsim \pi(t_1) = \pi(\sigma(u_2)) \gtrsim \pi(t_2) \cdots$$

Since the dependency pairs in $\mathcal{Q}$ occur infinitely many, this sequence contains infinitely many $>$ steps starting from $\pi(\sigma(u_1))$. This contradicts the well-foundedness of $>$.

Therefore, $\mathcal{Q} \subseteq \mathcal{P} - \mathcal{P}_>$, which means that B is an infinite minimal $(\mathcal{P} - \mathcal{P}_>, \mathcal{R}, \mu)$-chain, thus leading to a contradiction with our initial assumption. $\square$

Example 24 *Consider the CS-TRS* $(\mathcal{R}, \mu)$ *in Example 9.* $\mathsf{DP}(\mathcal{R}, \mu)$ *is:*

$$\mathtt{G}(x) \to \mathtt{H}(x) \qquad\qquad \mathtt{H}(\mathtt{d}) \to \mathtt{G}(\mathtt{c})$$

where $\mu^\sharp(\mathtt{G}) = \mu^\sharp(\mathtt{H}) = \varnothing$. *The dependency graph contains a single cycle including both of them. The only simple projection is* $\pi(\mathtt{G}) = \pi(\mathtt{H}) = 1$. *Since* $\pi(\mathtt{G}(x)) = \pi(\mathtt{H}(x))$, *we only need to guarantee that* $\pi(\mathtt{H}(\mathtt{d})) = \mathtt{d} > \mathtt{c} = \pi(\mathtt{G}(\mathtt{c}))$ *holds for a stable and well-founded ordering* $>$. *This is easily fulfilled by, e.g., a polynomial ordering.*

Theorem 13 (Non-μ-replacing projection processor II) *Let* $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ *and* $\mathcal{P} = (\mathcal{G}, P)$ *be TRSs such that* $\mathcal{P}_\mathcal{G}$ *contains no collapsing rule,* $Root(\mathcal{P}) \cap \mathcal{D} = \varnothing$, *and* $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$ *Let* $\gtrsim$ *be a stable quasi-ordering on terms whose strict and stable part* $>$ *is well-founded and* π *be a simple projection for* $\mathcal{P}$ *such that*

(1) for all $f \in Root(\mathcal{P})$, $\pi(f) \notin \mu(f)$,
(2) $\pi(\mathcal{P}) \subseteq \gtrsim$, *and,*
(3) whenever $\mathcal{NHT}_\mathcal{P} \neq \varnothing$ *and* $\mathcal{P}_\mathcal{X} \neq \varnothing$, *we have that* $\trianglerighteq_\mu \subseteq \gtrsim$, *where* $\trianglerighteq_\mu$ *is the* μ*-replacing subterm relation on* $\mathcal{T}(\mathcal{F}, \mathcal{X})$, *and* $t > t|_{\pi(root(t)^\sharp)}$ *for all* $t \in \mathcal{NHT}_\mathcal{P}$.

Then, the processor Proc_{NRP2} *given by*

$$\mathsf{Proc}_{NRP2}(\mathcal{P}, \mathcal{R}, \mu) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_\mathcal{X}, \mathcal{R}, \mu)\} & \text{if (1), (2), and (3) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu)\} & \text{otherwise} \end{cases}$$

is sound and complete.

14 Narrowing Transformation

The starting point of a proof of termination is the computation of the *estimated* dependency graph (see Definition 9) followed by the use of the SCC processor (Theorem 4). The estimation of the graph can lead to *overestimate* the arcs that connect two dependency pairs.

Example 25 *Consider the following example [Luc06, Proposition 7]:*

$$\begin{array}{rlrl} \mathtt{f}(\mathtt{0}) & \to \mathtt{cons}(\mathtt{0}, \mathtt{f}(\mathtt{s}(\mathtt{0}))) & \qquad\qquad \mathtt{p}(\mathtt{s}(x)) & \to x \\ \mathtt{f}(\mathtt{s}(\mathtt{0})) & \to \mathtt{f}(\mathtt{p}(\mathtt{s}(\mathtt{0}))) & & \end{array}$$

together with $\mu(\mathtt{f}) = \mu(\mathtt{p}) = \mu(\mathtt{s}) = \mu(\mathtt{cons}) = \{1\}$ *and* $\mu(\mathtt{0}) = \varnothing$. *Then,* $\mathsf{DP}(\mathcal{R}, \mu)$ *consists of the following pairs:*

$$\mathtt{F}(\mathtt{s}(\mathtt{0})) \to \mathtt{F}(\mathtt{p}(\mathtt{s}(\mathtt{0}))) \tag{25}$$
$$\mathtt{F}(\mathtt{s}(\mathtt{0})) \to \mathtt{P}(\mathtt{s}(\mathtt{0})) \tag{26}$$

The estimated *CS-dependency graph contains one cycle:* $\{(25)\}$. *Note, however, that this cycle does* not *belong to the CS-dependency graph because there is no way to* μ*-rewrite* $\mathtt{F}(\mathtt{p}(\mathtt{s}(\mathtt{0})))$ *into* $\mathtt{F}(\mathtt{s}(\mathtt{0}))$*!*

As already observed by Arts and Giesl for the standard case [AG00], in our case the overestimation comes when a (noncollapsing) pair $u_i \to v_i$ is followed in a chain by a second one $u_{i+1} \to v_{i+1}$ and v_i and u_{i+1} are not directly unifiable, i.e., at least one μ-rewriting step is needed to μ-reduce $\sigma(v_i)$ to $\sigma(u_{i+1})$. Then, the μ-reduction from $\sigma(v_i)$ to $\sigma(u_{i+1})$ requires at least one step, i.e., we always have $\sigma(v_i) \hookrightarrow_{\mathcal{R},\mu^\sharp} \sigma(v_i') \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$. Then, v_i' is a one-step μ-narrowing of v_i and we could require $u_i \sqsupset v_i'$ (which could be easier to prove) instead of $u_i \sqsupset v_i$. Furthermore, we could discover that v_i has *no* μ*-narrowings*. In this case, we know that no chain starts from $\sigma(v_i)$.

According to the discussion above, we can be more precise when connecting two pairs $u \to v$ and $u' \to v'$ in a chain, if we perform all possible one-step μ-narrowings on v in order to develop the possible reductions from $\sigma(v)$ to $\sigma(u')$. Then, we obtain new terms $v_1, \ldots, v_n$ which are one-step μ-narrowings of v using unifiers θ_i (i.e., $v \rightsquigarrow_{\mathcal{R},\mu,\theta_i} v_i$) for $i \in \{1, \ldots, n\}$, respectively. These unifiers are also applied to the left-hand side u of the pair $u \to v$. Therefore, we can replace a pair $u \to v$ by all its (one-step) μ*-narrowed pairs* $\theta_1(u) \to v_1, \ldots, \theta_n(u) \to v_n$.

As in [AG00,GTSF06], a pair $u \to v \in \mathcal{P}$ may only be replaced by its narrowings if the right-hand side v does not unify with any left-hand side u' of a (possibly renamed) pair $u' \to v' \in \mathcal{P}$ (note that this excludes pairs $u \to v$ with $v \in \mathcal{X}$). Moreover, the term v must be *linear*. We need to demand linearity instead of (the apparently more natural) μ-linearity (i.e., something like "no multiple *μ-replacing* occurrences of the same variable are allowed").

Example 26 *Consider the following TRS which is used in [AG00] to motivate the requirement of linearity.*

$$\begin{aligned} \mathtt{f}(\mathtt{s}(x)) &\to \mathtt{f}(\mathtt{g}(x,x)) \\ \mathtt{g}(\mathtt{0},\mathtt{1})) &\to \mathtt{s}(\mathtt{0}) \\ \mathtt{0} &\to \mathtt{1} \end{aligned}$$

We make it a CS-TRS by adding a replacement map μ given by $\mu(\mathtt{f}) = \mu(\mathtt{s}) = \{1\}$, $\mu(\mathtt{g}) = \{2\}$. The only cycle in the CSDG consists of the CSDP

$$\mathtt{F}(\mathtt{s}(x)) \to \mathtt{F}(\mathtt{g}(x,x)).$$

If linearity of the right-hand sides is not required for narrowing CSDPs, then it will be removed since $\mathtt{F}(\mathtt{g}(x,x))$ and the (renamed version of) the left-hand side $\mathtt{F}(\mathtt{s}(x'))$ do not unify, thus, there are no μ-narrowings. However the system is not μ-terminating*:*

$$\underline{\mathtt{f}(\mathtt{s}(\mathtt{0}))} \hookrightarrow \mathtt{f}(\mathtt{g}(\mathtt{0},\underline{\mathtt{0}})) \hookrightarrow \mathtt{f}(\underline{\mathtt{g}(\mathtt{0},\mathtt{1})}) \hookrightarrow \underline{\mathtt{f}(\mathtt{s}(\mathtt{0}))} \ldots$$

The problem is that the μ-reduction from $\sigma(\mathtt{F}(\mathtt{g}(x,x)))$ to $\sigma(\mathtt{F}(\mathtt{s}(x')))$ takes place 'in σ' and therefore it cannot be captured by μ-narrowing. Note that $\mathtt{F}(\mathtt{g}(x,x))$ is "μ-linear".

Another restriction to take into account when μ-narrowing a noncollapsing pair $u \to v$ is that the μ-replacing variables in v have to be μ-replacing in u as well (this corresponds with the notion of conservativeness). Furthermore, they cannot be both μ-replacing and non-μ-replacing at the same time. This corresponds to the following definition.

Definition 13 (Strongly Conservative [GLU08]**)** *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. A rule $l \to r$ is strongly μ-conservative if it is μ-conservative and $\mathcal{V}ar^{\mu}(l) \cap \mathcal{V}ar^{\not\mu}(l) = \mathcal{V}ar^{\mu}(r) \cap \mathcal{V}ar^{\not\mu}(r) = \varnothing$.*

The following result shows that, under these conditions, the set of CSDPs can be safely replaced by their μ-narrowings.

Theorem 14 (Narrowing processor) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $u \to v \in \mathcal{P}$ be such that*

(1) $u \to v$ is strongly conservative,

(2) v linear, and
(3) for all $u' \to v' \in \mathcal{P}$ (with possibly renamed variables), v and u' do not unify.

Let $\mathcal{Q} = (\mathcal{P} - \{u \to v\}) \cup \{u' \to v' \mid u' \to v'$ is a μ-narrowing of $u \to v\}$. Then, the processor Proc_{narr} given by

$$\mathsf{Proc}_{narr}(\mathcal{P},\mathcal{R},\mu) = \begin{cases} \{(\mathcal{Q},\mathcal{R},\mu)\} & \textit{if (1), (2), and (3) hold} \\ \{(\mathcal{P},\mathcal{R},\mu)\} & \textit{otherwise} \end{cases}$$

is sound and complete.

Proof. We have to prove that there is an infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain iff there is an infinite minimal $(\mathcal{Q},\mathcal{R},\mu)$-chain. The proof of this theorem is analogous to the proof of [GTSF06, Theorem 31], which we adapt here. For the first direction, we prove that given a minimal $(\mathcal{P},\mathcal{R},\mu)$-chain "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$", there is a μ-narrowing v' of v with the mgu θ such that "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is also a minimal $(\mathcal{Q},\mathcal{R},\mu)$-chain. Hence, every infinite minimal $(\mathcal{P},\mathcal{R},\mu)$-chain yields an infinite minimal $(\mathcal{Q},\mathcal{R},\mu)$-chain.

If "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is a minimal $(\mathcal{P},\mathcal{R},\mu)$-chain, then there is an substitution σ such that for all pairs $s \to t$ in the chain,

(1) if $s \to t \in \mathcal{P}_\mathcal{G}$, then $\sigma(t)$ is μ-terminating and it μ-reduces to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain
(2) if $s \to t = s \to x \in \mathcal{P}_\mathcal{X}$ then, $\sigma(x)$ has a μ-replacing subterm s_0, $\sigma(x) \trianglerighteq_\mu s_0$ such that $s_0^\sharp$ is μ-terminating and it μ-reduces to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain; furthermore, there is $\bar{s}_0 \in \mathcal{NHT}(\mathcal{R},\mu)$ such that $s_0 = \theta_0(\bar{s}_0)$ for some substitution θ_0.

Assume that σ is a substitution satisfying the above requirements and such that the length of the sequence $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$ is *minimal*.

Note that the length of this μ-reduction sequence cannot be zero because v and u_2 do not unify, that is, $\sigma(v) \neq \sigma(u_2)$. Hence, there is a term q such that $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$. We consider two possible cases:

(1) The reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ takes place within a binding of σ, i.e., there is a term r, a μ-replacing variable position $p \in \mathcal{P}os^\mu_\mathcal{X}(v)$, and a μ-replacing variable $x \in \mathcal{V}ar^\mu(v)$ such that $v|_p = x$, $q = \sigma(v[r]_p)$ and $\sigma(x) \hookrightarrow_{\mathcal{R},\mu} r$. Since v is linear, x occurs only once in v. Thus, $q = \sigma'(v)$ for the substitution σ' with $\sigma'(x) = r$ and $\sigma'(y) = \sigma(y)$ for all variables $y \neq x$. As we assume that all occurrences of pairs in the chain are variable disjoint, $\sigma'(x)$ behaves like σ for all pairs except $u \to v$. We have $\sigma(z) \hookrightarrow^*_{\mathcal{R},\mu}$

$\sigma'(z)$ for all $z \in \mathcal{X}$. Since $u \to v$ is strongly conservative we also have $\sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$ because all occurrences of x in u must be μ-replacing. Hence, if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}$ we have

$$\sigma'(v_1) = \sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$$

and if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$, then there is $s_1 \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that

$$\sigma'(v_1) = \sigma(v_1) \trianglerighteq_\mu s_1 \textit{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$$

and, in both cases,

$$\sigma'(v) = q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2) = \sigma'(u_2).$$

Note that, by minimality and because $u \to v \in \mathcal{P}_{\mathcal{G}}$, $\sigma(v)$ is $(\mathcal{R},\mu)$-terminating and, since $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$, the term q is $(\mathcal{R},\mu)$-terminating as well. Therefore, $\sigma'(x) = q$ is $(\mathcal{R},\mu)$-terminating and σ' satisfies the two conditions above. Since the length of the sequence $\sigma'(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u_2)$ is shorter than the sequence $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$, we obtain a contradiction and we conclude that the μ-reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ cannot take place in a binding of σ.

(2) The reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ 'touches' v, i.e., there is a nonvariable position $p \in \mathcal{P}os^\mu_{\mathcal{F}}(v)$, and a rewrite rule $l \to r \in \mathcal{R}$ such that $\sigma(v|_p) = \rho(l)$, for some substitution ρ and

$$\sigma(v) = \sigma(v)[\sigma(v|_p)]_p = \sigma(v)[\rho(l)]_p \hookrightarrow_{\mathcal{R},\mu} \sigma(v)[\rho(r)]_p = q$$

Since we can assume that variables in l are fresh, we can extend σ to behave like ρ on variables in l. Thus, $\sigma(l) = \sigma(v|_p)$, i.e, l and $v|_p$ unify and there is a mgu θ and an substitution τ satisfying $\sigma(x) = \tau(\theta(x))$ for all variables x. We have that v μ-narrows to $\theta(v)[\theta(r)]_p = v'$ with unifier θ. Again, we can extend σ to behave like τ on the variables of $\theta(u)$ and v'. Therefore, if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}$ we have

$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) = \tau(\theta(u)) = \sigma(\theta(u))$$

and if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$, then there is $s_1 \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that

$$\sigma(v_1) = \sigma(x) \trianglerighteq_\mu s_1 \textit{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) = \tau(\theta(u)) = \sigma(\theta(u))$$

and

$$\sigma(v') = \tau(v') = \tau(\theta(v))[\tau(\theta(r))]_p = \sigma(v)[\sigma(r)]_p = \sigma(v)[\rho(r)]_p = q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$$

Hence, "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is also a minimal chain.

The other side is also analogous to the 'completeness' part of [GTSF06, Theorem 31]. If "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is an infinite minimal

$(\mathcal{Q}, \mathcal{R}, \mu)$-chain where v' is a one-step μ-narrowing of v using the mgu θ, then "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain. There is a substitution σ such that

$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(\theta(u)) \text{ if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}, \text{ and}$$

$$\sigma(v_1) = \sigma(x) \trianglerighteq_\mu s_1 \text{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(\theta(u)) \quad \text{if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$$

Finally, we also have

$$\sigma(v') \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2).$$

Since the variables in the pairs are pairwise disjoint, we may extend σ to behave like $\sigma(\theta(x))$ on $x \in \mathcal{V}ar(u)$ then $\sigma(u) = \sigma(\theta(u))$ and therefore

$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \text{ if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}, \text{ and}$$
$$\sigma(v_1) \trianglerighteq_\mu s_1 \text{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \quad \text{if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$$

Moreover, by definition of μ-narrowing, we have $\theta(v) \hookrightarrow_{\mathcal{R},\mu} v'$. This implies that $\sigma(\theta(v)) \hookrightarrow_{\mathcal{R},\mu} \sigma(v')$ and since $\sigma(v) = \sigma(\theta(v))$, we obtain

$$\sigma(v) \hookrightarrow_{\mathcal{R},\mu} \sigma(v') \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2).$$

Hence, "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is a minimal $(\mathcal{P}, \mathcal{R}, \mu)$-chain as well.

□

Example 27 *(Continuing Example 25) Since the right-hand side of pair* (25) *in Example 25 does not unify with any (renamed) left-hand side of a CSDP (including itself) and it can be μ-narrowed at position 1 (notice that $\mu(\mathtt{f})=\{1\}$) by using the rule $\mathtt{p}(\mathtt{s}(x)) \to x$, we can replace it by its μ-narrowed pair:*

$$\mathtt{F}(\mathtt{s}(\mathtt{0})) \to \mathtt{F}(\mathtt{0})$$

*The μ-*narrowed *pair does not form any cycle in the estimated narrowed CS-dependency graph and μ-termination is easily proved now.*

The following example shows that strongly conservativeness cannot be dropped for the pair $u \to v$ to be μ-narrowed[4]. This requirement was not taken into account in [AGL07, Theorem 5.3].

Example 28 *Consider the following TRS $\mathcal{R}$:*

$$\mathtt{c}(\mathtt{e}(x)) \to \mathtt{d}(x, x)$$
$$\mathtt{a} \to \mathtt{e}(\mathtt{a})$$

[4] We thank Fabian Emmes for providing this example.

and $\mathcal{P}$ consisting of the following pair:

$$\mathtt{F}(\mathtt{d}(x,x)) \to \mathtt{F}(\mathtt{c}(x))$$

together with $\mu(\mathtt{c}) = \mu(\mathtt{d}) = \mu(\mathtt{F}) = \{1\}$ and $\mu(\mathtt{e}) = \varnothing$. There is an infinite $(\mathcal{P},\mathcal{R},\mu)$-chain as follows:

$$\mathtt{F}(\mathtt{c}(\underline{\mathtt{a}})) \hookrightarrow_{\mathcal{R},\mu} \mathtt{F}(\underline{\mathtt{c}(\mathtt{e}(\mathtt{a}))}) \hookrightarrow_{\mathcal{R},\mu} \underline{\mathtt{F}(\mathtt{d}(\mathtt{a},\mathtt{a}))} \hookrightarrow_{\mathcal{P},\mu} \mathtt{F}(\mathtt{c}(\underline{\mathtt{a}})) \hookrightarrow_{\mathcal{R},\mu} \cdots$$

Since $\mathtt{F}(\mathtt{c}(x))$ does not unify with any left-hand side of another pair, we can μ-narrow the pair in $\mathcal{P}$. We obtain $\mathcal{P}'$ consisting of the μ-narrowed pair

$$\mathtt{F}(\mathtt{d}(\mathtt{e}(x),\mathtt{e}(x))) \to \mathtt{F}(\mathtt{d}(x,x))$$

No infinite $(\mathcal{P}',\mathcal{R},\mu)$-chain is possible now.

Note that $\mathcal{P}$ is μ-conservative, but it is not strongly μ-conservative (the variable x is both μ-replacing and non-μ-replacing in $\mathtt{F}(\mathtt{d}(x,x))$).

15 Experiments

The processors described in the previous sections have been implemented as part of the tool MU-TERM [AGIL07,Luc04a]. We have tested the impact of the CSDP-framework in practice on the 90 examples in the Context-Sensitive Rewriting subcategory of the 2007 Termination Competition:

`http://www.lri.fr/~marche/termination-competition/2007`

which are part of the Termination Problem Data Base (TPDB, version 4.0):

`http://www.lri.fr/~marche/tpdb`

We have addressed this task in three different ways:

(1) We have compared CSDPs with previously existing techniques for proving termination of *CSR*: transformations, CSRPO, and polynomial orderings.
(2) We have compared the improvements introduced by the different CS-processors which have been defined in this paper.
(3) We have participated in the *CSR* subcategory of the 2007 International Termination Competition.

Tool Version	Proved	Total Time	Average Time
CSDPs	65/90	0.31 sec.	0.00 sec.
CSRPO	37/90	0.21 sec.	0.00 sec.
Polynomial Orderings	27/90	0.06 sec.	0.00 sec.
Transformations	56/90	5.59 sec.	0.10 sec.

Table 1
Comparison among *CSR* Termination Techniques

15.1 CSDPs vs. other techniques for proving termination of CSR

Several methods have been developed to prove termination of *CSR* for a given CS-TRS $(\mathcal{R}, \mu)$. Two main approaches have been investigated so far:

(1) *Direct proofs*, which are based on using μ-reduction orderings (see [Zan97]) such as the (context-sensitive) recursive path orderings [BLR02] and polynomial orderings [GL02,Luc04b,Luc05]. These are orderings $>$ on terms which can be used to directly compare the left- and right-hand sides of the rules in order to conclude the μ-termination of the TRS.
(2) *Indirect proofs* which obtain a proof of the μ-termination of $\mathcal{R}$ as a proof of termination of a transformed TRS $\mathcal{R}^\mu_\Theta$ (where Θ represents the transformation). If we are able to prove *termination* of $\mathcal{R}^\mu_\Theta$ (using the standard methods), then the μ-termination of $\mathcal{R}$ is ensured.

We have used MU-TERM to compare all these techniques with respect to the aforementioned benchmark examples. The results of this comparison are summarized in Table 1.

Remark 9 *A number of transformations Θ from TRSs $\mathcal{R}$ and replacement maps μ that produce TRSs $\mathcal{R}^\mu_\Theta$ have been investigated by Lucas (transformation L [Luc96]), Zantema (transformation Z [Zan97]), Ferreira and Ribeiro (transformation FR [FR99]), and Giesl and Middeldorp (transformations*[5] *GM, sGM, and C [GM99,GM04]), see [GM04,Luc06] for recent surveys about these transformations which also include a thorough analysis about their relative power. All these transformations were considered in our experiments, so item* "Transformations" *in Table 1 concentrate the* joint *impact of all of them.*

From the benchmarks summarized in Table 1, we clearly conclude that the CSDP-framework is the most powerful and fastest technique for proving termination of *CSR*. Actually, all examples which were solved by using CSRPO or polynomial orderings were also solved using CSDPs. Regarding transformations, there is only one example (namely, `Ex9_Luc06`, which can be solved by using transformation GM) that could not be solved with our current implementation of the CS-processors.

[5] The labels for these transformations correspond to the ones introduced in [Luc06].

Tool Version	Narrowing	Non-μ-Replacing Projection	Subterm	Proved	Total Time	Average Time
1.	No	No	No	54/90	3.00 sec.	0.05 sec.
2.	No	No	Yes	62/90	0.55 sec.	0.01 sec.
3.	No	Yes	No	57/90	0.82 sec.	0.01 sec.
4.	No	Yes	Yes	65/90	0.49 sec.	0.01 sec.
5.	Yes	No	No	54/90	3.22 sec.	0.06 sec.
6.	Yes	No	Yes	62/90	2.64 sec.	0.04 sec.
7.	Yes	Yes	No	57/90	1.27 sec.	0.02 sec.
8.	Yes	Yes	Yes	65/90	0.31 sec.	0.00 sec.

Table 2
Comparison among CS-processors

15.2 Contribution of the different CS-processors

In our implementation of the CSDP-framework, besides the CS-processor Proc_{SCC}, the μ-reduction-pair CS-processors described in Section 12 are the most basic ones: we use the CS-processors described in Sections 13 and 14 as much as possible; otherwise, the CS-processors in Section 12 are used.

The impact of the CS-processors in Sections 13 and 14 is summarized in Table 2. Our benchmarks show that the CS-processors described in Section 13 play an important role in our proofs. The subterm processors $\mathsf{Proc}_{subNColl}$ and $\mathsf{Proc}_{subColl}$ are quite efficient, but the ones which are based on simple projections for non-μ-replacing arguments (Proc_{NRP} and Proc_{NRP2}) also increase the power and the speed of the CSDPs technique. Furthermore, these two groups of CS-processors are complementary: the extra problems which are specifically solved by them are different. Narrowing is useful to simplify the graph, but it doesn't play an important role in the benchmarks, because it only applies to solve two examples (which can be solved without narrowing as well). Furthermore, we have to carefully use it because recomputing the graph can be expensive in that case.

Complete details of our experiments can be found here:

`http://zenon.dsic.upv.es/muterm/benchmarks/csdp`

15.3 CSDPs at the 2007 International Termination Competition

Nowadays, AProVE [GST06] is the only tool (besides MU-TERM) which implements specific methods for proving termination of *CSR*.

Both AProVE and MU-TERM participated in the *CSR* subcategory of the 2007

International Termination Competition. AProVE participated with a termination expert for *CSR* which, given a CS-TRS $(\mathcal{R},\mu)$, successively tries different transformations Θ for proving termination of *CSR* (those which are enumerated in Remark 9, i.e., $\Theta \in \{C, FR, GM, L, sGM, Z\}$) and then uses (on the obtained TRS $\mathcal{R}^{\mu}_{\Theta}$) a huge variety of different and complementary techniques for proving termination of rewriting (according to the DP-framework). Actually, AProVE is currently the most powerful tool for proving termination of TRSs and implements most existing results and techniques regarding DPs and related techniques.

However, MU-TERM's implementation of CSDPs was able to beat AProVE in the *CSR* category, thus witnessing that CSDPs are actually a very powerful technique for proving termination of *CSR*.

16 Related work

This paper is an extended and revised version of [AGL06,AGL07]. The first presentation of the context-sensitive dependency pairs was given in [AGL06].

Besides providing complete proofs for all results, discovering some bugs in previous results which are reported in the main text of this paper, and giving many examples about the use of the theory, the main conceptual differences between [AGL06,AGL07] and this paper are:

(1) In this paper, we have investigated and successfully combined two different notions of minimal non-μ-terminating terms: the so-called *strongly minimal terms* ($\mathcal{T}_{\infty,\mu}$, which have been introduced and investigated for the first time in this paper) and the *minimal terms* ($\mathcal{M}_{\infty,\mu}$), which were introduced in [AGL06] and further investigated in [AGL07]. The combined use of both of them leads us to a much better development of the theory which has brought new essential results, remarkably Theorem 1 which is the basis (at the level of pure context-sensitive rewriting) of the new notions of CSDP and minimal chain.

(2) Although most of the ideas in this first part of the paper (Sections 3, 4, and 5) were already present in [AGL07, Section 3], here we have made explicit some aspects which were only implicit there. For instance, the essential notion of *hidden term*, which is a consequence of Lemma 5 and further developed in Lemma 6 and Proposition 3 was implicit in [AGL07, Section 3], but only the notion of *hidden symbol* was made explicit. Actually, the proofs of the aforementioned results in this paper correspond (with minor changes) to that of Lemma 3.4, Lemma 3.5 and Proposition 3.6 in [AGL07], respectively.

(3) The notion of context-sensitive dependency pairs was first introduced in

[AGL06, Definition 1], but the narrowing condition that we have included now for the noncollapsing CSDPs is new. Although such a condition is inspired in the recent extension of the DP-method to Order-Sorted TRSs [LM08b], in this paper we have elaborated this in depth to show that it is a natural requirement, actually (see Section 5.1). In [LM08b] it is already showed that including 'narrowability' in the usual definition of dependency pair can be useful to automatically prove termination of rewriting. Similar considerations are valid for CSR.

(4) In [AGL06], a notion of minimal chain was introduced but not really used in the main results of the paper. Actually, the notion of minimal chain in this paper is completely different from the old one and is a consequence of the analysis of infinite μ-rewrite sequences developed in the first part of the paper. Furthermore, in this paper, the notion of minimal chain of pairs is essential for the definition of the context-sensitive dependency graph and the development of the CSDP-framework in the third part of the paper.

(5) The notion of context-sensitive dependency graph was first introduced in [AGL06] and further refined in [AGL07] thanks to the introduction of the hidden symbols. The definition in this paper introduces a new refinement through the notion of 'narrowable hidden term' and shows a nice symmetry between the arcs associated to noncollapsing and collapsing pairs.

(6) The definition of a CSDP-framework for the mechanization of proofs of termination of *CSR* using CSDPs is new. A number of processors introduced here had a kind of counterpart in [AGL06] (for instance, the use of μ-reduction orderings was formalized in [AGL06, Theorem 4] and the subterm criterion for noncollapsing pairs was formalized in [AGL06, Theorem 5]) or in [AGL07] (for instance, the narrowing transformation in [AGL07, Theorem 5.3]), but they were formulated in a DP-approach style.

(7) This paper introduces a number of new techniques which can be used for proving termination of *CSR* as new processors: the SCC processor [6], the processors for filtering or transforming collapsing pairs (see Section 11), the use of argument filterings [7], the use of the subterm criterion with collapsing pairs (Theorem 11), etc.

(8) Finally, for the first time, we have considered how to *disprove* termination of *CSR* within the CSDP framework.

[6] This is already mentioned in [AGL06, Section 4.2] but without any formal description.

[7] Again, this is very briefly mentioned at the end of [AGL06, Section 4.2] but never formalized.

Given a TRS $\mathcal{R}$ and a replacement map μ, if no replacement restrictions are imposed, i.e., $\mu(f) = \{1, \ldots, ar(f)\}$ for all $f \in \mathcal{F}$, then no collapsing pair is possible, and we would have $\trianglerighteq_\mu = \trianglerighteq$, and $\mathsf{DP}(\mathcal{R}, \mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$.

Regarding the CSDPs in $\mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$, Definition 3 differs from the standard definition of dependency pair (e.g., [AG00,GTSF06]) in two additional requirements:

(1) As in [HM04], which follows Dershowitz's proposal in [Der04], we require that subterms s of the right-hand sides r of the rules $l \to r$ which are considered to build the dependency pairs $l^\sharp \to s^\sharp$ are not subterms of the left-hand side (i.e., $l \not\trianglerighteq_\mu s$).
(2) As in [LM08b], we require 'narrowability' of the (appropriately renamed) term s: $\textsc{Narr}^\mu(\textsc{Ren}^\mu(s))$.

Except for these provisos, we could say that Definition 3 boils down to the definition of dependency pair when no replacement restrictions are imposed.

Regarding the definition of (minimal) chain (Definition 4), the correspondence is exact: if μ imposes no replacement restriction, then $\to_{\mathcal{R}} = \hookrightarrow_{\mathcal{R},\mu}$ and our definition coincides with the standard one (see, e.g., [GTSF06, Definition 3]): again, since all variables are μ-replacing now, item (2) in Definition 4 does not apply. Due to the absence of replacement restrictions, we have $\mathcal{V}ar^\mu(u) = \mathcal{V}ar(u)$, hence $\mathcal{V}ar(u) - \mathcal{V}ar^\mu(u) = \varnothing$ for all $u \to v \in \mathcal{P}$. Then, the condition $v \notin \mathcal{V}ar(u) - \mathcal{V}ar^\mu(u)$ vacuously holds and all pairs in $\mathcal{P}$ satisfy item (1) of Definition 4.

17 Conclusions

We have investigated the structure of infinite context-sensitive rewrite sequences starting from minimal non-μ-terminating terms (Theorem 1). This knowledge used to provide an appropriate definition of context-sensitive dependency pair (Definition 3), and the related notion of chain (Definition 4). in sharp contrast to the standard dependency pairs approach, where all dependency pairs have tuple symbols $f^\sharp$ both in the left- and right-hand sides, we have *collapsing* dependency pairs having a single *variable* in the right-hand side. These variables reflect the effect of the *migrating* variables into the termination behavior of *CSR*. At the level of *minimal chains*, though, this contrast is somehow recovered by a nice symmetry arising from the central notion of *hidden term* (Definition 2): a noncollapsing pair $u \to v$ is followed by a pair

$u' \to v'$ if $\sigma(v)$ μ-rewrites into $\sigma(u')$ for some substitution σ; a collapsing pair $u \to v$ is followed by a pair $u' \to v'$ if there is a *hidden term* t such that $\sigma(t)$ μ-rewrites into $\sigma(u')$ for some substitution σ. We have shown how to use the context-sensitive dependency pairs in proofs of termination of *CSR*. As in Arts and Giesl's approach, the presence or absence of infinite chains of dependency pairs from $\mathsf{DP}(\mathcal{R}, \mu)$ characterizes the μ-terminaton of $\mathcal{R}$ (Theorems 2 and 3).

We have provided a suitable adaptation of Giesl et al.'s *dependency pair framework* to *CSR* by defining appropriate notions of *CS-termination problem* (Definition 5) and *CS-processor* (Definition 6). In this setting we have described a number of sound and (most of them) complete CS-processors which can be used in any practical implementation of the CSDP-framework. In particular, we have introduced the notion of (estimated) *context-sensitive (dependency) graph* (Definitions 7 and 9) and the associated CS-processor (Theorem 4). We have also described some CS-processors for removing or transforming collapsing pairs from CS-termination problems in some particular cases (Theorems 5 and 6). We are also able to use μ-reduction pairs (Definition 10) and argument filterings to ensure the absence of infinite chains of pairs (Theorems 7, 8, and 9). We have also adapted Hirokawa and Middeldorp's *subterm criterion* which permits concluding the absence of infinite chains by paying attention only to the pairs in the corresponding CS-termination problem (Theorems 10 and 11). Following this appealing idea, we have also introduced two new processors which work in a similar way but use a very basic kind of orderings instead of the subterm relation (Theorems 12 and 13). Narrowing context-sensitive dependency pairs has also been investigated. It can also be helpful to simplify or restructure the dependency graph and eventually simplify the proof of termination (Theorem 14).

We have implemented these ideas as part of the termination tool MU-TERM [AGIL07,Luc04a]. The implementation and practical use of the developed techniques yield a novel and powerful framework which improves the current state-of-the-art of methods for proving termination of *CSR*. Actually, CSDPs were an essential ingredient for MU-TERM in winning the context-sensitive subcategory of the 2007 competition of termination tools.

As for future work, we plan to extend the basic CSDP-framework described in this paper with further CS-processors integrating the recently introduced *usable rules* for *CSR* [GLU08] as well as proofs of termination of *innermost CSR* using CSDPs [AL07].

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