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Parsimonious Graphs for the Most Common Trichords and Tetrachords

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Parsimonious transformations are common patterns in different musical styles and eras. In some cases, they can be represented on the Tonnetz, Cube Dance, Power Towers, or the central region of an orbifold, mainly when they only include the most even trichords and tetrachords. In this paper, two novel graphs, called Cyclopes, are presented, which include more than double the number of chord types in previously published graphs, thus allowing to represent a larger musical repertoire in a practical way. Apart from parsimonious transformations, they are also especially suitable for representing trichords a major third apart, tetrachords a minor third apart, and the cadences V7–I(m) and II$^{\varnothing}$–V7–I(m) with major or minor tonic chords. Therefore, they allow to clearly visualize the relationship among the corresponding chords and better understand those composition patterns, as well as being efficient mnemonic resources, all of which make them useful tools both for music analysis and composition.

Keywords: *Tonnetz*; neo-Riemannian theory; chord class; chord type; voice-leading; parsimonious transformations; orbifold; Cube Dance; Power Towers; Cyclops

2010 Mathematics Subject Classification: 00A65; 97M80

1. Introduction

Among the recurrent and repeated structures in musical compositions are the parsimonious transformations. They have been widely used in such different musical styles and eras as, for example, Classical period, Romanticism, Latin music or Jazz, thus being a well-established pattern in music. Their analysis can be carried out with the *neo-Riemannian theory*, which arose in the 1980s for analyzing some chromatic passages by nineteenth-century composers and is still evolving with the contributions of algebra and geometry. According to Gollin (2005), it is characterized by three elements: mathemat-

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ical groups of transformations, voice-leading parsimony, and graphical representations. The paradigmatic example corresponds to the *PLR-group*[1] and the *Tonnetz*, although they are limited to major and minor triads.

As a starting point, a primary rule in harmony for connecting chords is the "law of the shortest way" (Schönberg 1983, page 39, quoting Bruckner). This means to sustain the common notes and move the others by the smallest possible intervals. In this respect, Douthett and Steinbach (1998) state that two chords with the same cardinality are $P_{m,n}$*-related* if one of them can be transformed into the other by sustaining the common notes and, for the rest of them, moving m by a semitone and n by a whole tone. Then, *parsimony* is a $P_{m,n}$ relation with low values for m and n, normally $m + 2n \leq 2$. The simplest case is $P_{1,0}$, which we will call *single-semitonal* (after Tymoczko 2011). In that paper, the authors also provide several remarkable parsimonious graphs, particularly the *Chicken-Wire Torus* (the dual of the *Tonnetz*) and *Cube Dance* for nearly and most even trichords, respectively, and the *Towers Torus* and *Power Towers* for nearly and most even tetrachords, respectively. Twenty years before, however, Waller (1978) published a torus equivalent to the Chicken-Wire, but which clearly shows its full hexagonal tessellation, as well as all PL, PR, and – although a bit harder to visualize – LR cycles. These and other PLR compound operations were later studied extensively by Cohn (1996, 1997, 1998, and, particularly, 2012). A different approach is given by Tymoczko (2006), who provides the full theory for representing all n-note pitch-class sets in the orbifold $\mathbb{T}^n/S_n$, here abbreviated *n-orbifold*, which is a kind of generalized Möbius strip. As well, he represents the 2-orbifold in 2D and part of the 3-orbifold in 3D, before twisting and bending the figures to obtain the real orbifolds. Callender, Quinn, and Tymoczko (2008) provide further representations in this sense. In practice, however, due to the complexity of the spaces, only the central regions of the orbifolds are normally represented.

In this paper, I present a novel chord pattern representation, based on cyclic circular graphs called "Cyclopes", which show broader groups of trichords and tetrachords related by single-semitonal transformations. As well, they provide a wider view around the centre of the corresponding orbifolds. Therefore, they allow to represent a greater number of musical works in a practical way and can be used both for music analysis and composition.

The reader is assumed to be familiar with *Forte names* and *set classes* (Forte 1973), also called *chord classes*. Here, the *non-inversionally-symmetrical* ones are split into two *chord types* related by *inversion*, named "a" and "b", in accordance with Nuño (2020). As well, large parts of this study deal with *chord geometry* (Tymoczko 2011) and *most even chord transformations* (Cohn 2012), although the main concepts are explained here.

2. Dyads

Tymoczko (2006) represents the *unordered pairs of pitch classes*, or simply *two-note chords* (Tymoczko 2011), in a 2-orbifold or Möbius strip. Now, we will obtain the same result by a procedure and with a notation more suitable for developing our final graphs.

There are 6 different 2-note chord classes, interval classes or dyads, all of them being inversionally symmetrical. They are represented in Figure 1 (left), where they are assigned interval names (m2, M2, m3, M3, P4, and Tr or tritone). The chord class 1-1 with two equal notes or unison (in fact, a multiset) is also included and represented by "X" (this uncommon notation is used instead of P1 for consistency with next sections). The arrows show how to transform the dyads by raising one note by a semitone (or, in the opposite

[1] P, L and R stand for the basic operations *Parallel*, *Leading-tone exchange*, and *Relative*, which respectively map, for example, C major to C minor, C major to E minor, and C major to A minor; and vice versa.

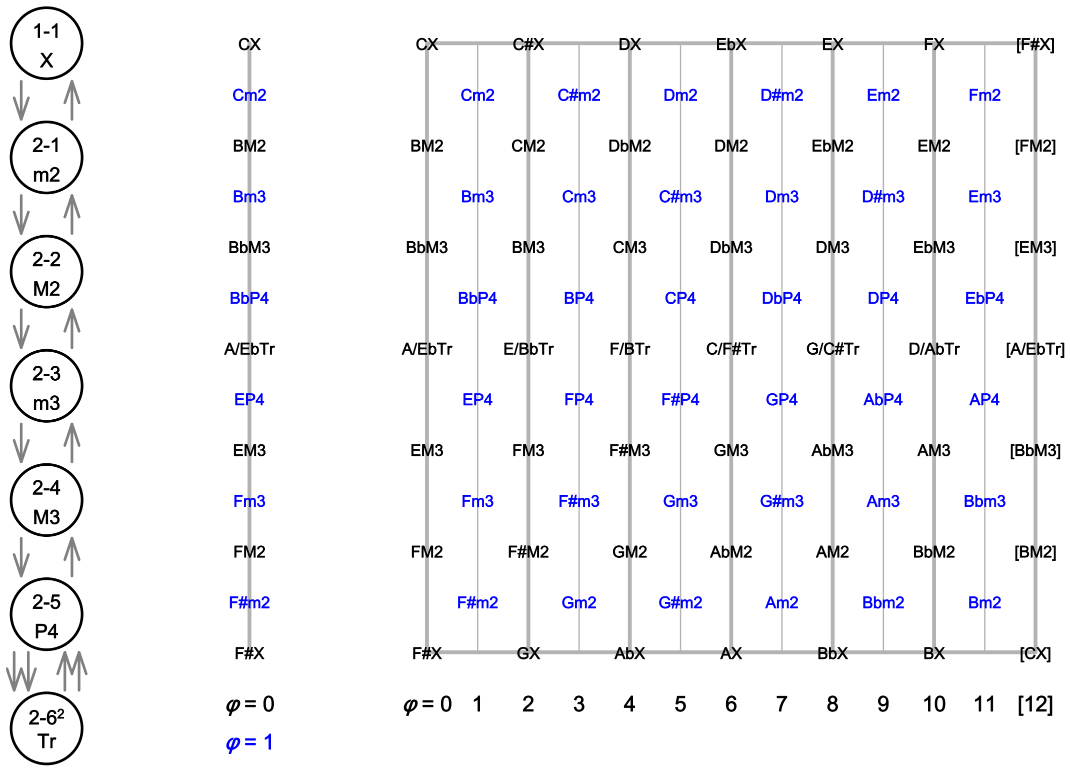

Figure 1. (Left) The 2-note chord classes plus the 2-note unison (multiset) with their single-semitonal transformations. (Centre) Zones 0 and 1 of the 2-orbifold. (Right) The 2-orbifold.

direction, by lowering one note by a semitone). The superscript 2 in 2-6 is its *degree of transpositional symmetry*, which doubles the arrows connecting this dyad. This diagram does not include the chord roots and represents the "local relationships" in the 2-orbifold.

Let us now represent the "global relationships" among all 2-note chords (with their roots). To do this, we group them into *voice-leading zones* (Cohn 2012, page 102) or, simply, *zones* $\varphi \in [0, \ldots, 11]$. First called *sum classes* (Cohn 1998), they are the equivalence classes defined by the sum of the notes in a chord, modulo 12. For example, Bm3 = (B, D) is in the zone $\varphi = 11 + 2 = 1 \pmod{12}$. This way, given a chord in the zone φ, the one obtained from it by raising one note by a semitone will be in $\varphi + 1$. And chords related by *pure contrary motion*, such as FM2 = (F, G) and EM3 = (E, G♯), will be in the same zone (in this case, $\varphi = 0$).

Figure 1 (centre) is a compact diagram showing the 2-note chords in the zones $\varphi = 0$ and $\varphi = 1$, while the chords in $\varphi = k$ will be those in $\varphi = k - 2$ but raising the two notes by a semitone. Note that the dyads of the same class whose roots are 6 semitones apart are in the same zone. And Figure 1 (right) shows all 2-note chords as given by Tymoczko (2006), but with a different notation.[2] In this diagram, each chord is transformed into the nearest ones (in oblique directions) by raising or lowering one note by a semitone, as indicated by the arrows in Figure 1 (left). Note that the tritones (2-6) are at the central horizontal axis, the perfect fourths (2-5) are one semitone apart from them, and the remaining chords are two or more semitones apart. By twisting 180° the right side of this figure and connecting it to the left one, we obtain the 2-orbifold (a Möbius strip).

[2] Tymoczko (2006) represents the 2-note chords by their actual notes, in integer notation (e.g. 48 for EM3).

3. Selection of trichords and tetrachords

There are 12 different 3-note chord classes, the trichords, 5 of them being inversionally symmetrical, while the remaining 7 can be split into two chord types related by inversion, which makes a total of 19 chord types. And there are 29 different 4-note chord classes, the tetrachords, 15 of them being inversionally symmetrical; and, by splitting the remaining 14, we obtain a total of 43 chord types. In both cases, the number of chord types is too high to obtain practical and visually simple graphs relating them. Therefore, we will just focus on the "most common" trichords and tetrachords. Let us see how to select them.

In the common practice period (around 1650 to 1900) the harmonies are mainly built by superimposing thirds on the 7 degrees of the major, harmonic and melodic minor scales (see, for example, Schönberg 1983 or Piston 1988). This leads to the 4 basic triads and the 7 basic seventh chords, which are 3-10, 3-11a, 3-11b, 3-12, and 4-19a, 4-19b, 4-20, 4-26, 4-27a, 4-27b, 4-28, respectively. In addition, the augmented sixth chords add the 3-8a (Italian) and 4-25 (French). All these set types are, consequently, prevalent in western music. On the other hand, for set classes with the same cardinality, the Forte ordinals are assigned so that the corresponding *interval-class vectors*[3] are arranged in decreasing lexicographic order.[4] This means that the number of smaller interval classes is progressively reduced, which arranges the set classes from the chromatic to the maximally even ones. Thus, the criterion here taken is to select "full series of chord types", from the ones in the above groups having the lexicographically greatest interval-class vectors (3-8 and 4-19) to the corresponding maximally even ones (3-12 and 4-28).

Table 1 shows those trichord and tetrachord types with the symbols here used to represent them, their *intervallic forms*[5] (Nuño 2020) starting from the root, and their interval-class vectors. The added chord types are 3-8b, 3-9, 4-21, 4-22a, 4-22b, 4-23, and 4-24, which are sometimes interpreted as chromatic, incomplete or passing chords. In other musical styles, such as Pop, Latin or Jazz, all chord types in the table are frequently used (see, for example, the list of chords given by Sher 1991, page iv). Therefore, in order to keep the selected chord types to a reasonable and manageable number, as well as retaining the most relevant ones, just those in the table will be considered here.

4. Parsimonious graphs

Straus (2003) gives two diagrams showing all 3- and 4-note chord classes, linked by single-semitonal transformations. Figures 2 and 3 are reduced versions of them, which only include the chord classes here considered, but splitting those being non-inversionally-symmetrical into two chord types related by inversion. These figures are analogous to Figure 1 (left), but now the arrows in opposite directions forming a pair correspond to different chord types of the same class. Similarly, multiple arrows show the different ways to move between two chord types. Among other things, splitting the two types of a set class allows to show the relations between them, when they exist. This is the case for P and L operations between major and minor triads (Figure 2).

In these diagrams, Arabic numerals indicate the initial and final notes referring to the chord roots, where 1, 3, 4, and 5 stand for perfect or major intervals, which may

[3]The vector listing the number of times each of the 6 dyads is contained in a given set class or set type.

[4]Except in the case of *Z-related* pairs (two different set classes with the same interval-class vector), where one member of each pair is placed at the end of the corresponding group.

[5]The sequence of intervals, in semitones, between every two adjacent pitch classes in a set type, including the interval between the last and the first ones. Any of its circular shifts.

Table 1. Trichord and tetrachord types here considered. A superscript on the Forte ordinal indicates the degree of transpositional symmetry, when greater than 1. An asterisk (*) means "omit 5" and a double asterisk (**) "omit ♭3". A major chord (3-11b) is normally represented by the root without any symbol. Symbol "(9)" means "add 9", whereas symbol "9" adds both the minor seventh and the major ninth. The intervallic forms start from the root.

Trichord	Symbol	Int. Form	Int.-Class Vect.
3-8a	7*	462	010101
3-8b	Ø**	642	010101
3-9	sus4	525	010020
3-10	dim	336	002001
3-11a	m	345	001110
3-11b	M	435	001110
$3\text{-}12^3$	+	444	000300

Tetrachord	Symbol	Int. Form	Int.-Class Vect.
4-19a	mΔ	3441	101310
4-19b	Δ♯5	4431	101310
4-20	Δ	4341	101220
4-21	9*	2262	030201
4-22a	(9)	2235	021120
4-22b	m4	3225	021120
4-23	7sus	5232	021030
4-24	7♯5	4422	020301
$4\text{-}25^2$	7♭5	4242	020202
4-26	m7	3432	012120
4-27a	Ø	3342	012111
4-27b	7	4332	012111
$4\text{-}28^4$	O	3333	004002

be altered with ♯ or ♭, whereas major, minor, and diminished sevenths are denoted by Δ, 7, and d7, respectively; and Roman numerals at the middle of the arrows indicate the difference between the two chord roots, in semitones (letter "O" means zero). For example, Cm consists of notes (C, E♭, G) and, by raising the root (1) by a semitone, the new note is the minor seventh (7) of the new chord, a "7*" with root C + III, that is, E♭7* = (E♭, G, D♭). Or, by lowering the minor third (♭3) by a semitone, it turns into the perfect fifth (5) of the "sus4" chord with root C − V, that is, Gsus4 = (G, C, D).

This notation also allows to easily find other parsimonious transformations, particularly $P_{0,1}$, which corresponds to two consecutive arrows where the ending note on the first matches the starting note on the second one. For example, if in Cm we raise the perfect fifth (5) by a semitone, it turns into the root (1) of a major chord; and by raising again this note by a semitone, it turns into the root (1) of a "dim" chord whose root is C − IV + I, that is, Adim = (A, C, E♭). As well, if in Cm we lower the root (1) by a semitone, it turns into the 1, 3, or ♯5 of an augmented triad; and by lowering again *the same note*

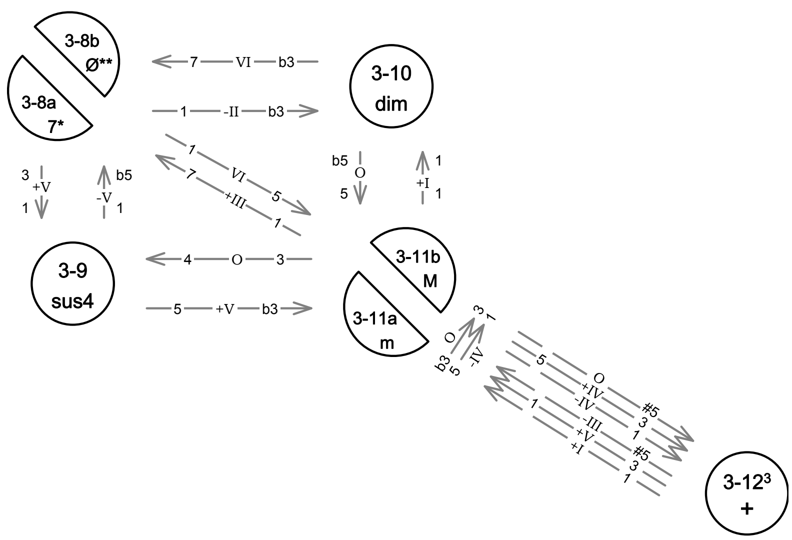

Figure 2. The 3-note chord types included in Table 1 with their single-semitonal transformations.

by a semitone, it turns into the perfect fifth (5) of a major chord whose root is C + III, that is, E♭M = (E♭, G, B♭). This is, precisely, the R operation.

It was found that there are, respectively, 18 and 36 $P_{1,0}$, and 13 and 18 $P_{0,1}$ relations among the trichords and tetrachords here considered, and all but one of the $P_{0,1}$ relations can be derived with Figures 2 and 3 as explained above. However, the exception must be derived differently, as it corresponds (or may correspond) to a voice crossing between the tetrachord types "Δ♯5" and "mΔ". For example, transforming CΔ♯5 = (C, E, G♯, B), into C♯mΔ = (C♯, E, G♯, B♯) or (C♯, E, G♯, C), can be achieved by raising B by a whole tone, which crosses C. And in Figure 3, this is found by first raising the root (C) by a semitone, giving C♯m7, and then raising its minor seventh (B) by a semitone, thus avoiding the voice crossing. Cannas (2018) gives two diagrams showing both the $P_{1,0}$ and $P_{0,1}$ relations among the 4 trichord types 3-10, 3-11a, 3-11b, 3-12, and among the 5 tetrachord types 4-20, 4-26, 4-27a, 4-27b, 4-28. She also extended the analysis of tetrachords to include 4-19a, 4-19b, 4-24, and 4-25, but without providing the corresponding diagram.

After representing the local relationships among the trichords and tetrachords here considered (without indicating the roots), we will obtain the corresponding global ones (with all roots). Following the theory by Tymoczko (2011, §3.8), we developed the Figure 4, whose left diagram is analogous to Figure 1 (centre). It shows the 3-note chords in the zones $\varphi = 0$, $\varphi = 1$, and $\varphi = 2$, while the chords in $\varphi = k$ will be those in $\varphi = k - 3$ but raising the three notes by a semitone. Chords at the vertices are of class 1-1, but with three equal notes, and are represented by "XX". The remaining chords at the edges are

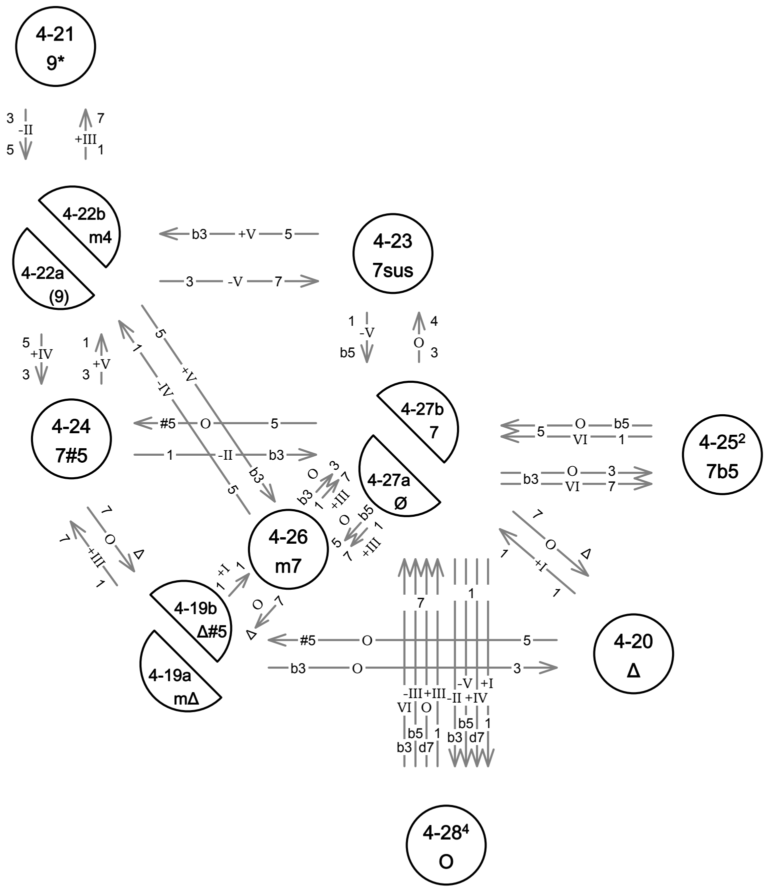

Figure 3. The 4-note chord types included in Table 1 with their single-semitonal transformations.

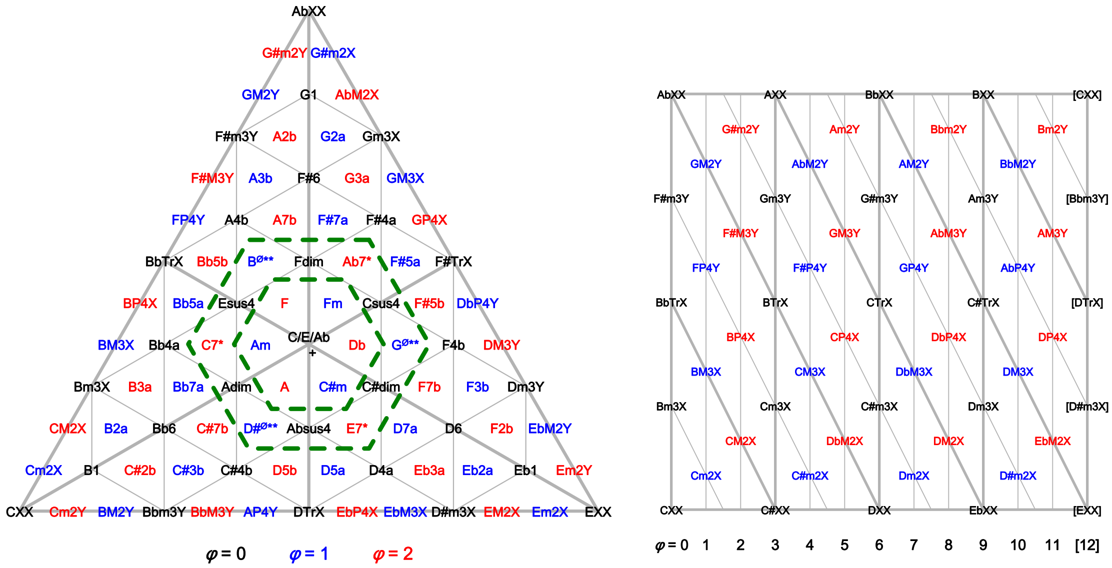

Figure 4. (Left) Zones 0, 1, and 2 of the 3-orbifold. The two central regions (dashed hexagons) contain the chord types included in Table 1. (Right) A side face of the 3-orbifold.

dyads with one note duplicated, either the root (symbol "X") or the other note (symbol "Y"). Chords in the central regions defined by the dashed hexagons are assigned the symbols in Table 1, whereas the remaining chords are represented by the root, according to the *normal intervallic form*[6] (Nuño 2020), followed by the Forte ordinal and the letter "a" or "b" when appropriate. Note that the trichords of the same type whose roots are 4 semitones apart are in the same zone.

Superimposing all zones $\varphi \in [0, \ldots, 11]$ gives rise to a triangular prism, one of its side faces is shown in Figure 4 (right), where the oblique lines correspond to the vertical lines in Figure 1 (right) (remember that now the dyads have one note duplicated). In that prism, each chord is transformed into the nearest ones (in oblique directions with respect to the current axes) by raising or lowering one note by a semitone. The dashed hexagons also give rise to two prisms, the smaller one including the axis with the augmented triads (3-12) plus the minor (3-11a) and major (3-11b) triads, which are 1 semitone apart from them. And the greater hexagonal prism adds the chords being 2 semitones apart (chord types 3-8 to 3-10). The remaining chords are 3 or more semitones apart from the prism axis. Now, by twisting 120° one of the bases of the triangular prism and connecting it to the other one, we obtain the 3-orbifold, which is a "triangular Möbius strip". The result for just the trichords here considered is represented in Figure 5 in a circular graph, here called *3-Cyclops*, where φ is actually an angular position starting from "twelve o'clock" ($\varphi = 0$ for C+) and increasing clockwise. The arrows in Figure 2 are now substituted by lines whose directions are assumed to be clockwise and no Roman numerals are used, since the roots are directly given.

The Cube Dance by Douthett and Steinbach (1998) shows the single-semitonal transformations among the augmented, major and minor triads, that is, those in the smaller hexagonal prism. Or, with respect to the 3-Cyclops, it just includes 1 chord type per zone. Tymoczko (2011, page 105) gives an alternative representation of those chords on a cube, in the 3-orbifold. For its part, the *Tonnetz* is an earlier representation of major and minor triads, connected by *PLR* operations. On the other hand, the 3-Cyclops can

[6] The least of all possible circular shifts of an intervallic form, with respect to the lexicographic order.

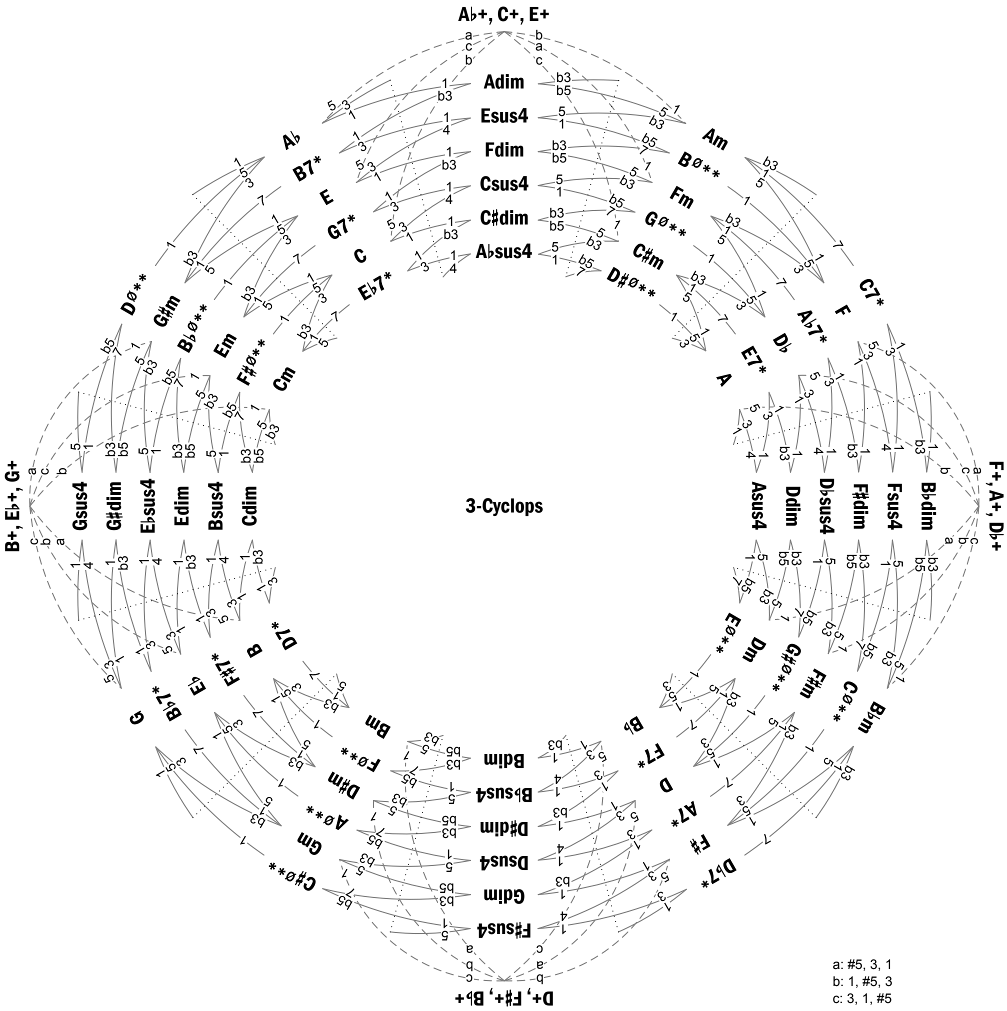

Figure 5. The 3-Cyclops, with the 3-note chords considered in Table 1.

be viewed as a "second-order" Cube Dance or *Tonnetz*, since it also includes the chords being 2 semitones apart from the prism axis. Thus, it contains a total of 7 chord types versus 3 in the Cube Dance or 2 in the *Tonnetz*. As well, the basic operations in the *Tonnetz* are easily visualized on it: P and L are lines oblique to a circumference centred with the graph, and R goes through an augmented triad entering and exiting by the same letter ("a", "b", or "c"). Symbolically, $P = /$, $L = \backslash$, and $R = \wedge$. As well, it clearly shows the *Weitzmann*[7] and *hexatonic*[8] *regions* (Cohn 2012), which correspond to the zones (11,1), (2,4), (5,7), (8,10), and (1,2), (4,5), (7,8), (10,11), respectively.

A similar procedure can be carried out for the 4-note chords (Tymoczko 2011, §3.9), but it leads to a 4D prism whose bases are tetrahedra, which complicates the study. Additionally, the 4-note chords here considered do not correspond to a simple central region in that prism, whose axis contains the diminished seventh chords, 4-28. For example, the chords 4-21 are 4 semitones apart from them, while 4-18a and 4-18b are only 2 semitones apart and are not considered here (nor are other chords being 3 semitones apart). Therefore, only the final circular graph for the tetrachords here considered is

[7] 3 major and 3 minor triads adjacent to the same augmented triad.

[8] 3 major and 3 minor triads lying between two consecutive augmented triads (3 zones apart).

given in Figure 6, which we will call *4-Cyclops*. Note that the tetrachords of the same type whose roots are 3 semitones apart are in the same zone.

The Power Towers by Douthett and Steinbach (1998) show the single-semitonal transformations among the diminished (4-28), half-diminished (4-27a), dominant (4-27b), and minor seventh (4-26) chords, which correspond to 1 chord type per zone in the 4-Cyclops. Cannas (2018) adds the major seventh chords (4-20), obtaining the so-called *Clover graph*. In contrast, the Douthett's *4-Cube Trio* (Cohn 2012, page 158), as well as the representation by Tymoczko (2011, page 106) in the 4-orbifold, add the French sixth chords (4-25), which complete a 4D cube or tesseract (chord types 4-25 to 4-28). On the other hand, the 4-Cyclops can be viewed as a higher-order 4-Cube Trio, since it also includes 4-19 to 4-24. Thus, it contains a total of 13 chord types versus 5 in the 4-Cube Trio or the

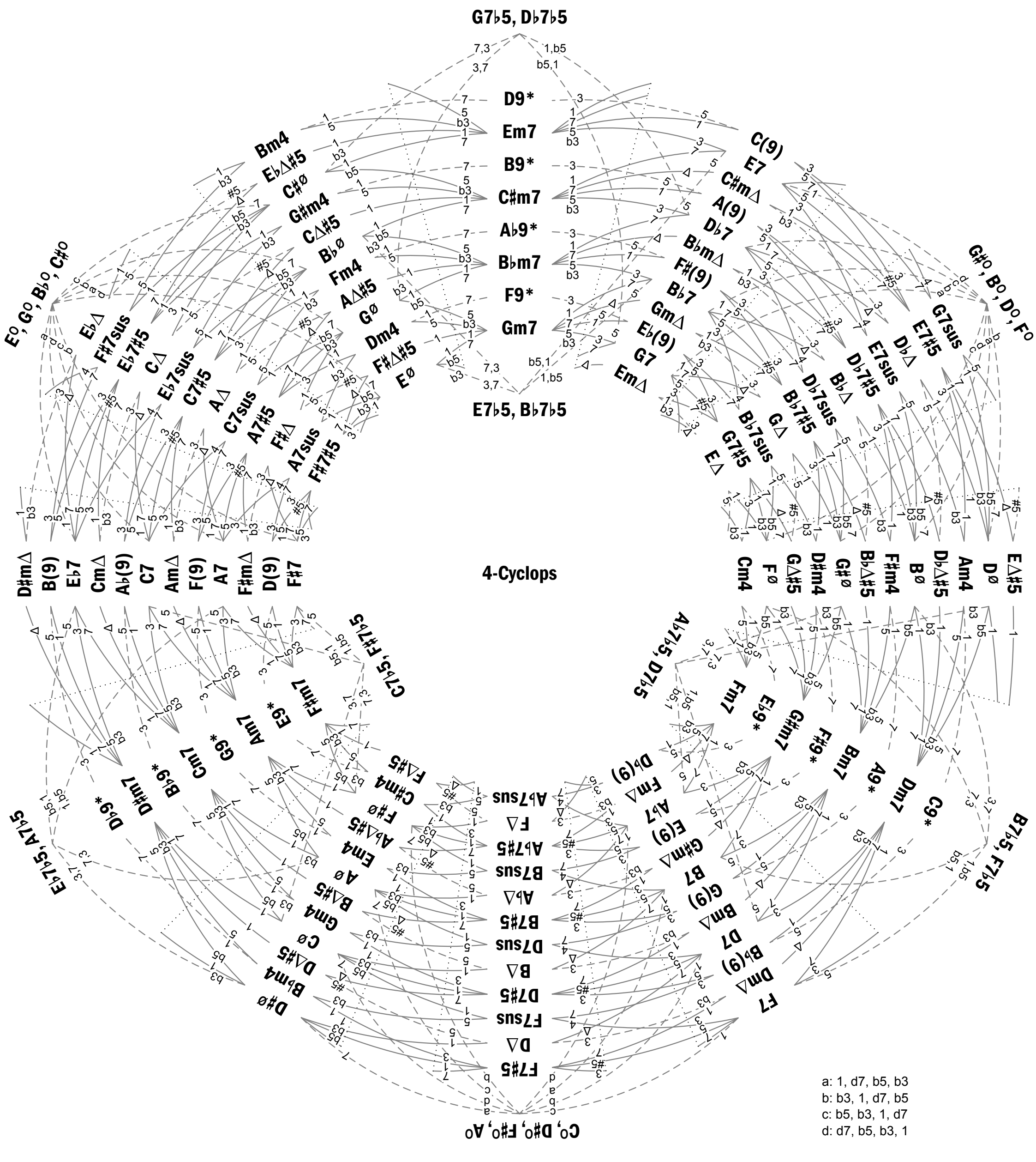

Figure 6. The 4-Cyclops, with the 4-note chords considered in Table 1.

Clover graph, which is a high number and makes this graph more complex than the 3-Cyclops. As well, it clearly shows the *Boretz*[9] and *octatonic*[10] *regions* (Cohn 2012), which correspond to the zones (1,3), (5,7), (9,11), and (11,1), (3,5), (7,9), respectively.

5. Chord patterns

The 3- and 4-Cyclops are especially suitable for representing some particular chord patterns used in musical compositions, which are given in Table 2. These patterns can also be represented on the *Tonnetz*, but only to a limited extent, since it just deals with minor (3-11a) and major (3-11b) triads; and, when seventh chords of class 4-27 are involved, normally the "*Tonnetz* reduction" consists in omitting the seventh in the "7" chords and the root in the "Ø" chords. Cohn (2012) and Tymoczko (2011) analyze many examples of these kinds, but also including the augmented triads (3-12); and, regarding the tetrachords, they consider the five most even chord types (4-25 to 4-28). On the other hand, the 3- and 4-Cyclops include more than double the number of chord types in both cases (3-8 to 3-12 and 4-19 to 4-28, respectively), thus allowing to analyze a greater number of musical works, as well as to obtain simpler and more compact representations.

Table 2. Particular Chord Patterns especially fitting the 3- and 4-Cyclops.

3-Cyclops	4-Cyclops
Parsimonious progressions of Trichords Same Trichord types a *major* third apart	Parsimonious progressions of Tetrachords Same Tetrachord types a *minor* third apart

First, we will consider some examples based on trichords a major third apart, thus lying on the same zone on the 3-Cyclops, which also include parsimonious progressions. With respect to the "7" and "Ø" chords, we will use their incomplete forms, "7*" and "Ø**", which are better approximations to the real chords than those used with the *Tonnetz* and, what is very advantageous, they lead to more compact representations.

Let us start with Beethoven's Sonata for Violin and Piano in F major, Op. 24. The harmonies in the 2nd mvt., mm. 38–54, are the following:

$$\{\mathrm{B\flat m}\ \ \text{%}\}\ \ \{\mathrm{G\flat}\ \ \mathrm{D\flat 7}\ \ \text{%}\ \ \mathrm{G\flat{-}C\flat}\ \ \mathrm{G\flat{-}D\flat 7}\ \ \mathrm{G\flat}\}\ \ \{\mathrm{F\sharp m}\}$$
$$\{\mathrm{D{-}G}\ \ \mathrm{D{-}A7}\ \ \mathrm{D}\}\ \ \{\mathrm{Dm}\}\ \ \{\mathrm{F7}\ \ \mathrm{B\flat{-}E\flat}\ \ \mathrm{B\flat{-}F7}\ \ \mathrm{B\flat}\}$$

where each chord or each pair linked by a dash lasts one measure and symbol "%" means to repeat the previous measure. Chords related to the same consonant triad are grouped in curly brackets. This chord progression is represented in Figure 7 on the 3-Cyclops, where the initial chord is specially marked. The 3 minor chords (B♭m, F♯m, Dm) are a major third apart in descending order, as are the 3 major chords related to them by L and P operations (G♭, D, B♭). The latter are affirmed by cadences including the dominant seventh and subdominant chords, each group lying in one zone. Since we used the incomplete form of the "7" chords, the result is very compact, only requiring 3 nearby zones: 4, 5, and 8. If we had used the "7" chords with the seventh omitted, as is usual with the *Tonnetz*, they would have lain in the zone 2 of Figure 5. And regarding their 4-note forms, they lie in different zones (1, 5, 9) of Figure 6 and are not grouped together.

[9] 4 dominant and 4 half-diminished seventh chords adjacent to the same diminished seventh chord.

[10] 4 dominant and 4 half-diminished seventh chords lying between two diminished seventh chords. These groups are 2 semitones apart, but are connected by single-semitonal transformations by means of the minor seventh or the French sixth chords.

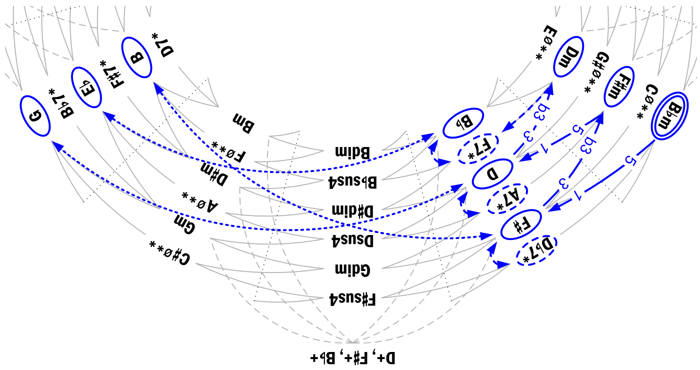

Figure 7. Beethoven, Sonata for Violin and Piano in F major, Op. 24, 2nd mvt., mm. 38–54.

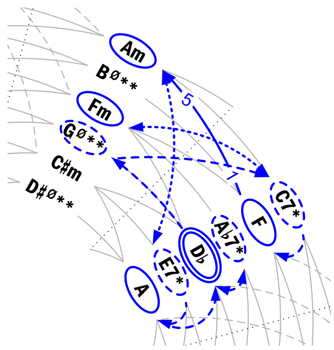

Figure 8. Liszt, Consolation No. 3, Op. 102, mm. 23–43.

We will now analyze the mm. 23–43 of the Consolation in D♭ major, Op. 102, No. 3 by Liszt, whose harmonies are

$$\{\mathrm{D}\flat\}\quad \{\mathrm{G}^{\varnothing}\quad \mathrm{G}^{\varnothing}{-}\mathrm{C7}\quad \mathrm{Fm}\quad \%\quad \mathrm{C7/F}\quad \mathrm{Fm}\}\quad \{\mathrm{C7/F}\quad \mathrm{F}\quad \%\}$$

$$\{\mathrm{Am}\quad \mathrm{Am}{-}\mathrm{E7}\quad \mathrm{Am}\quad \mathrm{E7}\quad \mathrm{Am}\}\quad \{\mathrm{E7/A}\quad \mathrm{A}\quad \%\}\quad \{\mathrm{D}\flat\quad \mathrm{A}\flat 7\quad \mathrm{D}\flat\}$$

where some chords are played over a pedal note, here represented by a slash followed by the pedal. This chord progression is represented in Figure 8 on the 3-Cyclops (without the pedals) and can be compared with Cohn (2012, page 187), who also provides a Web animation. Now the 3 major chords (D♭, F, A) are a major third apart but in ascending order, and there are only 2 minor chords (Fm, Am), related to them by L and P operations, which are affirmed by longer cadences. A "∅" chord is now included, whose incomplete form, together with those of the "7" chords, make the representation really compact, just requiring 2 consecutive zones (1 and 2). In fact, the 3-Cyclops is especially suitable for representing the cadences V7–I(m) and II$^{\varnothing}$–V7–I(m) with major or minor tonic chords. The Jazz tune Giant Steps by Coltrane (Sher 1991) is closely related to this, as it just consists of cadences V7–IΔ and IIm7–V7–IΔ a major third apart.

Regarding examples with the 4-Cyclops, let us start with the Piano Concert No. 2 in C minor, Op. 18, by Rachmaninoff. In the 1st mvt., mm. 1–8, there is a pure single-semitonal progression, represented in Figure 9 on the 4-Cyclops with a simple line:

$$[\mathrm{Fm(5)}]\quad \mathrm{D}\flat\Delta\quad \mathrm{D}^{\varnothing}\quad \mathrm{Fm7}\quad \mathrm{F7}\quad \mathrm{Fm7}\quad \mathrm{D}^{\varnothing}\quad \mathrm{D}\flat\Delta$$

Here, a note in parentheses means to add that note to the chord. Thus, Fm(5) is Fm with two C. This chord is written in brackets because it does not appear in the 4-Cyclops, but was included in the figure to illustrate the example. They are precisely those two C who first raise and then lower semitone by semitone to change the chords, except F7. A pedal F−C (in three octaves), which belongs to all the harmonies, gives consistency to the full chord progression. There is another pedal A♭ (in two octaves), except with F7. The first chord moves to D$^{\varnothing}$ through D♭Δ instead of D$^{\mathrm{O}}$, possibly because the latter does not contain the pedal C and, additionally, it has two tritones and the former none.

Our next example is *Indudable* (Bossa Nova) by Nuño (2012), whose mm. 19–27 consist of the following chords (actually, some of them include additional tensions):

$$\mathrm{G}\sharp\mathrm{m7}\quad \mathrm{C}\sharp\Delta\quad \mathrm{Fm7}\quad \mathrm{B}\flat\Delta\quad \mathrm{Dm7}\quad \mathrm{G6}\quad \mathrm{Bm7}\quad \mathrm{E7sus}\quad \mathrm{G}\sharp\mathrm{m7}$$

This chord progression is represented in Figure 10 on the 4-Cyclops. The 4 minor seventh chords (G♯m7, Fm7, Dm7, Bm7) are a minor third apart, thus lying on the

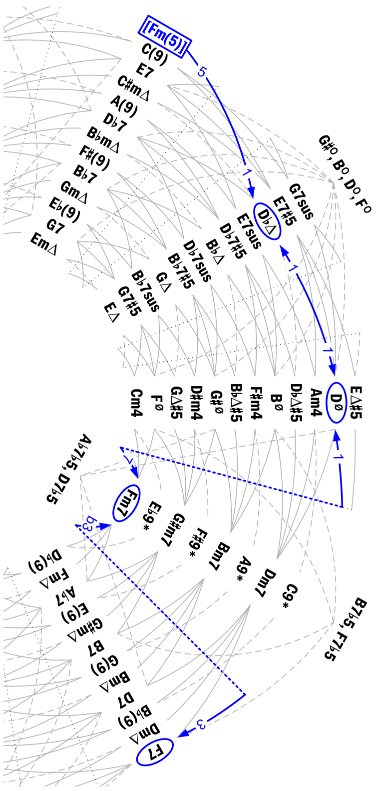

Figure 9. Rachmaninoff, Piano Concert No. 2, Op. 18, 1st mvt., mm. 1–8.

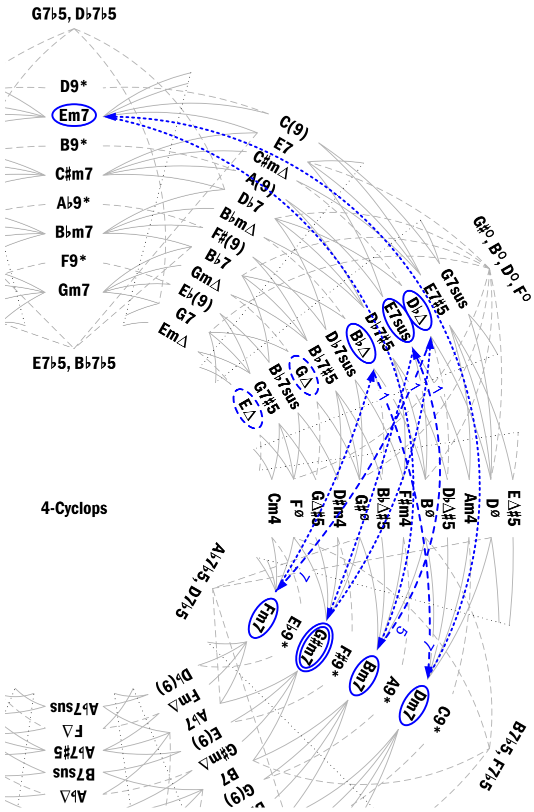

Figure 10. Nuño, *Indudable*, mm. 19–27.

same zone. With respect to the other chords, their roots are also a minor third apart, but instead of having the uniform sequence C♯Δ, B♭Δ, GΔ, EΔ, the last two chords (marked with dashed lines in the figure) are replaced by G6 (enharmonic to Em7) and E7sus, respectively. Even so, the representation is again simple and compact.

The last example is Chopin's Prelude in E minor, Op. 28, No. 4, one of the most interesting pieces analyzed by Tymoczko (2011, pages 287–293) and Cohn (2012, pages 160–166), both providing Web animations. Figure 11 is a simplified score with mm. 1–12. As will be seen, this composition is best understood by first analyzing the harmonies in the three lower voices, represented in Figure 12 on the 3-Cyclops. They pass through all the trichord types considered in this graph, except the augmented triads (perhaps too dissonant?). As well, Chopin also included the chord types "m7*" (3-7a) and "Δ*" (3-4a), defined by the intevallic forms {372} and {471}, which are the incomplete tonic seventh chords in natural minor and major keys, respectively. From the second chord (F♯m7*), the three lower voices strictly follow a descending single-semitonal ($P_{1,0}$) line, covering more than one full turn on the graph. Then, other parsimonious transformations are employed to finish the phrase, as indicated in the score.

On the other hand, the austere melody also draws a descending line, B–A–G♯–F♯, which completes the harmonies and leads to a more complex representation on the 4-Cyclops (Figure 13). Apart from the chord types considered in this graph, Chopin also included "(♭9)" (4-18a) and "Δ♭5" (4-16a), defined by {1335} and {4251}, respectively.

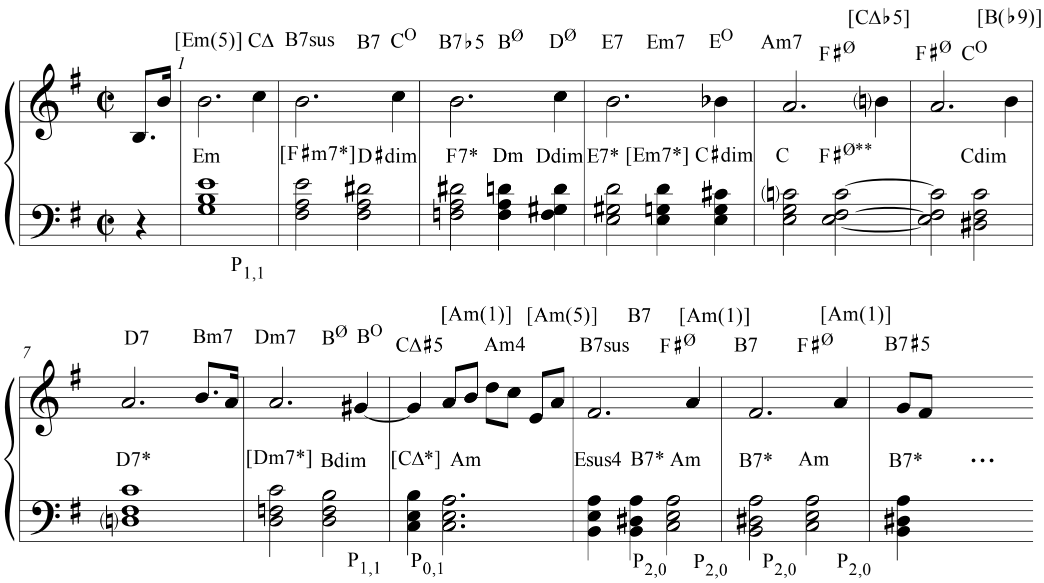

Figure 11. Chopin, Prelude in E minor, Op. 28, No. 4, mm. 1–12. Melody and harmonic structure.

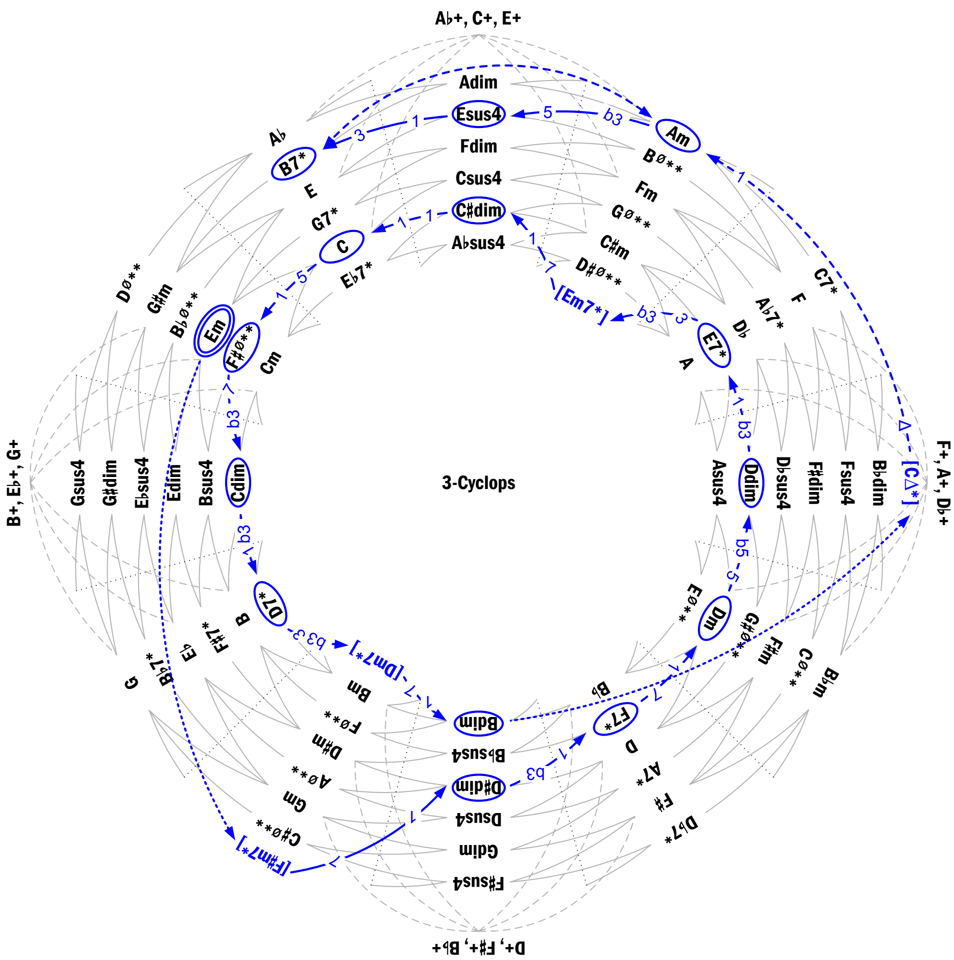

Figure 12. Chopin, Prelude in E minor, Op. 28, No. 4, mm. 1–12. Harmonies in the three lower voices.

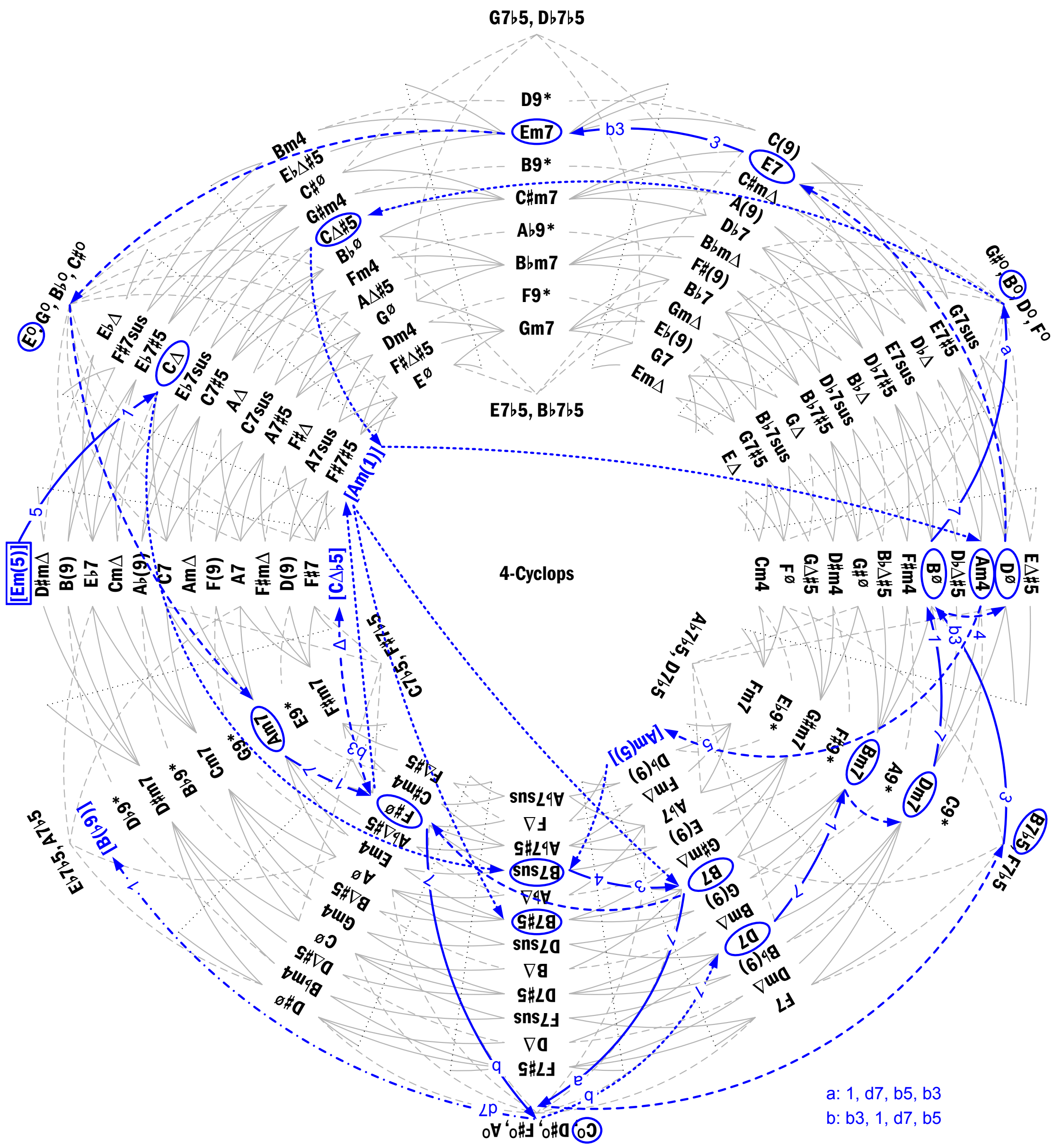

Figure 13. Chopin, Prelude in E minor, Op. 28, No. 4, mm. 1–12. Full harmonies.

6. Conclusions

Two novel graphs, called Cyclopes, are presented, which relate the most common trichords and tetrachords by single-semitonal transformations. They include more than double the number of chord types in previously published graphs, thus allowing to analyze a larger repertoire in a practical way. They are especially suitable for representing parsimonious chord progressions, trichords a major third apart, tetrachords a minor third apart, and the cadences V7–I(m) and II$^{\varnothing}$–V7–I(m) with major or minor tonic chords. In all those cases, the results are simple and compact, thus allowing to clearly visualize the relationship among the corresponding chords and better understand those composition patterns, as well as being efficient mnemonic resources. Consequently, they proved to be practical tools that can be used both for music analysis and composition.

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