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Alexandroff Spaces

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“Alexandroff Spaces”

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Abstract

An Alexandroff space is a topological space where the arbitrary intersections of open sets are also open. These are named after the Russian mathematician Pavel Sergeyevich Alexandroff, characterized by the minimal open neighborhoods existing for every point of the space, all the while they stay in the realm of low-level separation axioms. This thesis explores Alexandroff spaces from their foundational definitions to their intricate connections with quasiorders. It navigates through the province of functional Alexandroff spaces, also known as primal topologies, by studying the significance of their topological properties. Finally, it delves into their application in tackling the famous Collatz Conjecture of number theory, as shown by a surprising series of recent papers. Although the statement of the conjecture is very simple, mathematicians from different areas have studied it for years without being able to prove or refute it. In this instance, the problem is addressed from a topological point of view, so that equivalent propositions based on the properties of the set of natural numbers endowed with the primal topology induced by the Collatz function are obtained. This journey unveils the beauty and profound implications of Alexandroff spaces within the broader tapestry of topology and mathematics as a whole.

Keywords Alexandroff space; quasiorder; primal spaces; Collatz conjecture.

Resumen

Un espacio de Alexandroff es un espacio topológico donde se verifica que la intersección arbitraria de conjuntos abiertos es también abierto. Deben su nombre al matemático ruso Pavel Sergeyevich Alexandroff, y se caracterizan por las vecindades abiertas minimales existentes para cada punto del espacio; todo mientras se mantienen en el dominio de los axiomas de separación de bajo nivel. Esta tesis explora los espacios de Alexandroff desde sus definiciones fundamentales hasta sus intrincadas conexiones con los preórdenes. Examina también el ámbito de los espacios de Alexandroff funcionales, también conocidos como topologías primales, estudiando la importancia de sus propiedades topológicas. Finalmente, se profundiza en su aplicación al abordar la famosa conjetura de Collatz de la teoría de números, somo se muestra en una sorprendente serie de artículos recientes. Aunque el enunciado de la conjetura es muy sencillo, matemáticos de distintas áreas la han estudiado durante años sin haber logrado probarla o refutarla. En este caso, se aborda el problema desde un punto de vista topológico, de forma que se obtienen proposiciones equivalentes basadas en las propiedades del conjunto de los números naturales dotado con la topología primal inducida por la función de Collatz. Este viaje revela la belleza y las profundas implicaciones de los espacios de Alexandroff dentro del entramado más amplio de la topología y las matemáticas en su conjunto.

Palabras clave Topología de Alexandroff; preorden; espacios primales; Conjetura de Collatz.

Contents

1 Introduction

Alexandroff spaces, named after the Russian mathematician Pavel Sergeyevich Alexandroff (1896-1982), constitute a stimulating topic of topology that has increasingly attracted the attention of scholars since their introduction in 1937 [1]. These spaces, characterized by minimal open neighborhoods, possess a rich structure while staying in the realm of low-level separation axioms. The purpose of the present work is to explore the world of Alexandroff spaces, describing their defining characteristics and attributes.

The survey starts with the foundational concepts in Chapter 2, *i.e.*, the definition and characterization via the existence of minimal open neighborhoods. In due course, the notions of separation, connectedness, compactness, and countability are examined. The chapter is closed with a consideration of how new Alexandroff spaces can be derived from previous ones through operations like intersections, products, quotients, and identification maps. In Chapter 3, the profound connection between Alexandroff spaces and quasiorders is presented, shedding light on the deep interplay between topology and order theory. Chapter 4 leads us into the domain of the primal topology, by navigating through its notable classes of sets, namely, the minimal neighborhoods, invariant sets, and orbits. Separation properties, compactness, and connectedness continue to be examined within the context of primal spaces. Lastly, Chapter 5 ventures into the application of Alexandroff spaces as the Collatz Conjecture is investigated, showcasing the relevance of these topological structures on the noted open problem. A primal topology is induced over the set $\mathbb{N}$ by the Collatz function, which has led to inquiries from a novel perspective. Several equivalences have been proposed in connection with the conjecture [23], hence being added to the several approaches attempted to solve it.

While navigating this mathematical landscape, let us uncover not only the beauty of Alexandroff spaces but also their connection to the broader tapestry of topology and the potential impact on other branches of Mathematics.

2 Alexandroff spaces

It is well known that a topological space is a pair (X, τ) where X is a nonempty set and τ is a family of subsets containing X and $\varnothing$ which is closed under arbitrary unions and finite intersections. However, there exist well-known topologies, like the discrete topology, that not only are closed under finite intersections but arbitrary intersections.

Nowadays, topological spaces satisfying this property are known as Alexandroff spaces and they were introduced in [1] under the name *discrete spaces* (*diskreten Räumen* in German). We devote this chapter to study these topologies in detail. Our basic references for this chapter are [2, 13, 19, 22]. Let us start with its definition.

Definition 1 (Alexandroff space, [2])**.** *A topological space (X, τ) is called an Alexandroff space if the arbitrary intersections of open sets are open.*

Example 1. *If X is a finite space and τ is a topology on X then (X, τ) is clearly an Alexandroff space.*

Example 2. *Sets endowed with the discrete topology are Alexandroff spaces since every subset is open in the discrete topology, that includes any arbitrary intersection of open sets.*

Example 3. *The real line with the usual topology is not an Alexandroff space. For example, $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a family of open sets but its intersection $\{0\}$ is not open, as singletons are not open.*

A straightforward consequence of the definition of an Alexandroff space is that arbitrary unions of closed sets are closed.

Proposition 1. *Let $\mathcal{F}$ be an arbitrary family of closed sets in an Alexandroff space (X, τ). Then $\bigcup \mathcal{F}$ is closed.*

Proof. Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a family of closed sets and let $C = \bigcup_{\lambda \in \Lambda} F_\lambda$. Then $X \backslash C = \bigcap_{\lambda \in \Lambda} X \backslash F_\lambda$. Since $X \backslash F_\lambda$ is open for all $\lambda \in \Lambda$ and (X, τ) is an Alexandroff space, then $X \backslash C$ is open, hence C is closed. □

Proposition 2. *Let (X, τ) be an Alexandroff space, then for any subset $S \subseteq X$ the closure is $\bar{S} = \bigcup \{\bar{x} : x \in S\}$.*

Proof. Since (X, τ_f) is also an Alexandroff space, then any union of closed sets is also closed. Therefore, the union $\bigcup \{\bar{x} : x \in S\}$ is a closed set containing S, and so $\bar{S} \subseteq \bigcup \{\bar{x} : x \in S\}$. On the other hand, $x \in S$ implies that $\bar{x} \subseteq \bar{S}$, with $\bigcup \{\bar{x} : x \in S\} \subseteq \bar{S}$ as a result. Hence, $\bar{S} = \bigcup \{\bar{x} : x \in S\}$. □

The following is an important characterization of Alexandroff spaces as those topological spaces having minimal open neighborhoods.

Theorem 1 ([19]). *A topological space (X, τ) is Alexandroff if and only if each point $x \in X$ has a minimal open neighborhood $\mathcal{N}_x$.*

Proof. ($\rightarrow$) Let (X, τ) be an Alexandroff space, and $x \in X$. Take the collection $\{O \in \tau : x \in O\}$, and let $\mathcal{N}_x = \bigcap\{O \in \tau : x \in O\}$. Since (X, τ) is Alexandroff and $\mathcal{N}_x$ is an intersection of open sets, then it is open. Moreover, it is obviously the minimal neighborhood of x in the ordering by inclusion since U is a neighborhood of x, thus we can find an open set G such that $x \in G \subseteq U$; therefore, $\mathcal{N}_x \subseteq G \subseteq U$.

($\leftarrow$) Suppose that for each $x \in X$, there exists a minimal open neighborhood $\mathcal{N}_x$. Let $\{O_i : i \in I\}$ be an arbitrary family of open sets. If $\cap_{i \in I} O_i = \varnothing$ then the intersection is obviously open. So let us suppose that there exists $x \in \cap_{i \in I} O_i$. Since O_i is a neighborhood of x and $\mathcal{N}_x$ is the minimal neighborhood of x, then $\mathcal{N}_x \subseteq O_i$ for all $i \in I$, that is, $\mathcal{N}_x \subseteq \cap_{i \in I} O_i$. Hence $\cap_{i \in I} O_i$ is a neighborhood of x. Due to the arbitrariness of x we deduce that $\cap_{i \in I} O_i$ is open. Consequently, (X, τ) is Alexandroff. □

From the previous result we immediately deduce that if (X, τ) is an Alexandorff space then $\{\mathcal{N}_x\}$ is a basis for the neighborhood system of x for every $x \in X$. Therefore, we can prove the following.

Proposition 3. *The collection of minimal open neighborhoods constitutes a basis for the open sets in an Alexandroff space.*

Proof. Let (X, τ) be an Alexandroff space, and $U \in \tau$ be a nonempty set. Then, for every $x \in U$ there exists a minimal neighborhood $\mathcal{N}_x \subseteq U$, so that $U = \bigcup \mathcal{N}_x$ for all $x \in U$. □

Hence, any open set of an Alexandroff space is a union of minimal neighborhoods.

We next characterize the properties that a family of subsets must satisfy to be a base of an Alexandroff topology.

Proposition 4. *Let X be a nonempty set. A family $\mathcal{B}$ of subsets of X is a base for an Alexandroff topology on X if and only if*

1. $X = \bigcup_{B \in \mathcal{B}} B$;
2. *if* $\{B_i : i \in I\} \subseteq \mathcal{B}$ *then* $\bigcap_{i \in I} B_i \in \mathcal{B}$.

Proof. Let X be a nonempty set.

($\rightarrow$) Suppose that $\mathcal{B}$ is a base for an Alexandroff topology τ on X. This obviously implies condition 1 since it is a base. Moreover, $\mathcal{B} \subseteq \tau$ is closed under arbitrary intersection so 2 holds.

($\leftarrow$) Conversely, suppose that $\mathcal{B}$ satisfies properties 1 and 2. Then if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ we have that $x \in B_3 \subseteq B_1 \cap B_2$ where $B_3 = \cap\{B \in \mathcal{B} : x \in B\}$. Since $B_3 \in \mathcal{B}$ by 2, then $\mathcal{B}$ is a base for a topology $\tau(\mathcal{B})$ on X.

Moreover, let $\{O_i : i \in I\} \subseteq \tau$. If $\cap_{i \in I} O_i = \varnothing$ then the intersection is open. Otherwise, let $x \in \cap_{i \in I} O_i$. Then, for each $i \in I$ there exists $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq O_i$. Then $B = \cap_{i \in I} B_i \in \mathcal{B}$ and $x \in B \subseteq \cap_{i \in I} O_i$. Since x is arbitrary, we deduce that $\cap_{i \in I} O_i$ is also open, hence $\tau(\mathcal{B})$ is Alexandroff. □

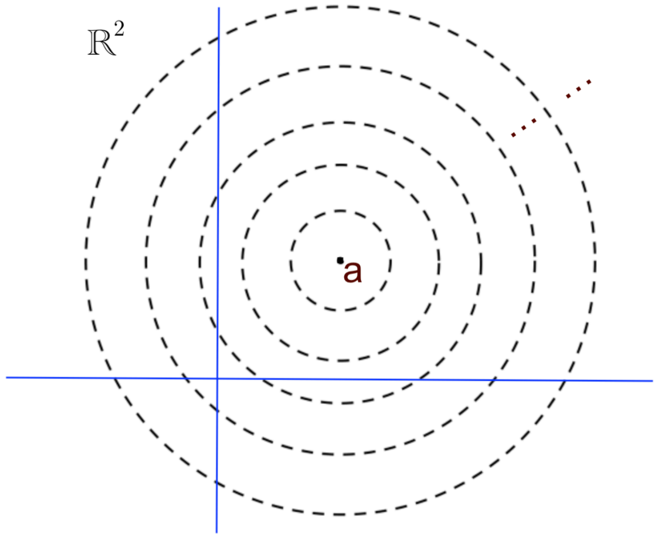

Figure 2.1: Example 4. A numerable family of concentric *balls* can be a basis for an Alexandroff topology on $\mathbb{R}^2$.

Consequently, a family $\mathcal{B}$ satisfying the requirements of the previous result produces an Alexandroff topology since it ensures that every point has a minimal neighborhood, that is, the intersection of all the basic open sets containing the point.

We next provide two examples of the construction of an Alexandroff topology.

Example 4. *Let X be $\mathbb{R}^2$, and $d : X \times X \to \mathbb{R}$ be the Euclidean distance. Given an element $a \in X$, fixed, it is possible to define a set $B(a, n)$, a 'ball', just as follows:*

$$B(a, n) = \{x \in \mathbb{R}^2 : d(a, x) \leq n \textit{ with } n \in \mathbb{N}\}$$

Then the collection $\mathcal{B} = \{B(a, n)\}_{n \in \mathbb{N}}$ is a basis for an Alexandroff topology on $\mathbb{R}^2$; since $\forall x \in \mathbb{R}^2$, there exists a set acting as the minimal open neighborhood in the topology generated by $\mathcal{B}$.

Example 5. *Let $X = \mathbb{R}$. The collection $\mathcal{B} = \{[n, n+1) : n \in \mathbb{Z}\}$ is a basis for an Alexandroff topology on $\mathbb{R}$.*

A short yet very useful result is the following lemma:

Lemma 1. *Let (X, τ) be an Alexandroff space, and $a, b \in X$. Then,*

$$a \in \mathcal{N}_b \textit{ if and only if } b \in \bar{a},$$

where $\bar{a}$ denotes the closure of the singleton $\{a\}$.

Proof. $(\leftarrow)$ If $b \in \bar{a}$, then every open neighborhood of b contains a, that includes the minimal open neighborhood $\mathcal{N}_b$; hence $a \in \mathcal{N}_b$.

$(\rightarrow)$ If $a \in \mathcal{N}_b$, straightforwardly, $b \in \bar{a}$, for $\mathcal{N}_b$ is already the minimal open neighborhood of b. □

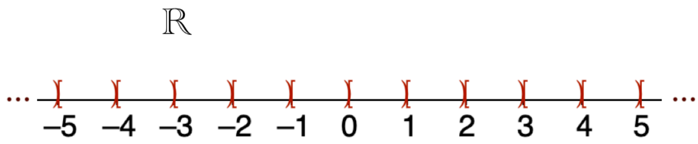

Figure 2.2: Example 5. A numerable family of intervals on $\mathbb{R}$.

2.1 Separation

In this section, some separation properties of Alexandroff spaces will be studied. These concepts deal with the possibility of distinguishing points, *i.e.*, singletons, in a topological space, via their corresponding neighborhood systems. This motivates the introduction of the following definitions [6, 18, 24].

Definition 2 (Distinguishable points). *In a topological space (X, τ), two distinct points $x, y \in X$ are called topologically distinguishable if they do not have the same neighborhood system.*

This means that there exist at least a neighborhood containing one of the points but not the other. It can be said that one of the points is separated from the other, but not vice versa [18]. On the other hand, two points become indistinguishable if they both belong to the closure of the other.

Definition 3 (Separated points). *Let (X, τ) be a topological space, and $x, y \in X$. The points x and y are called separated if for each point there exists a neighborhood that is not a neighborhood of the other.*

Definition 4 (Separation by neighborhoods). *Let (X, τ) be a topological space, and $x, y \in X$. It is said that the points x and y are separated by neighborhoods if and only if they both have disjoint neighborhoods.*

Clearly, separation by neighborhoods imply that points are separated which, in turn, implies topological distinguishability. The following definitions are due to [24, 4].

Definition 5 (T_0). *A topological space (X, τ) is T_0 or Kolmogorov if every two distinct points $x, y \in X$ are topologically distinguishable.*

The separation axiom R_0 was first introduced by Shanin in 1943, for the proceedings of the Russian Academy of sciences [20]. The R_0 separation axiom has been proved useful for some types of Alexandroff spaces.

Definition 6 (R_0). *A topological space (X, τ) is R_0 if all pairs of topologically distinguishable points are separated.*

Definition 7 (T_1). *A topological space (X, τ) is T_1 or Fréchet if every two distinct points $x, y \in X$ are both separated.*

Notice that a topological space is T_1 if and only if it is T_0 and R_0.

In Example 4, there are topologically indistinguishable points; for instance, the elements sharing the same radius n. In Example 5 the topologically indistinguishable points are the ones being in the same interval $[n, n+1)$; thus those are not Kolmogorov spaces. However, there are topologically distinguishable points in both of them. In Example 4, points with radii differing in value higher than 1 are distinguishable. Points lying in different intervals are distinguishable in Example 5. The latter is an instance of an R_0 space, for every pair of topologically distinguishable points are separated.

The following result shows that separation axioms stronger than or equal to T_1 are not useful in Alexandroff spaces, for the discrete topology is obtained.

Theorem 2. *Let X be a nonempty set. Then the only T_1 Alexandroff topology over X is the discrete topology.*

Proof. Let τ be a T_1 Alexandroff topology over the set X and let $x \in X$. Thus $\forall y \in X$ with $x \neq y$, y is separated from x, hence there exists a neighborhood of x not containing y. Let $\mathcal{N}_x$ denote the intersections of all neighborhoods of x, thus $\mathcal{N}_x$ does not contain y, and so for every other y, therefore $\mathcal{N}_x = \{x\}$ and τ is the discrete topology. □

There is a minor result as well, namely: if X is a T_1 Alexandroff space, and $x \in X$, then $\mathcal{N}_x = \{x\}$.

Lemma 2. *Let (X, τ) be an Alexandroff space. Then $\mathcal{N}_x = \mathcal{N}_y$ if and only if $\bar{x} = \bar{y}$.*

Proof. If $\mathcal{N}_x = \mathcal{N}_y$, then $x \in \mathcal{N}_y$ and $y \in \mathcal{N}_x$, and so $\mathcal{N}_x \cap \{y\} \neq \varnothing$ and $\mathcal{N}_y \cap \{x\} \neq \varnothing$; hence $x \in \bar{y}$ and $y \in \bar{x}$, therefore $\bar{x} \subseteq \bar{y}$ and $\bar{y} \subseteq \bar{x}$. Consequently, $\bar{x} = \bar{y}$.

It is easy to follow the converse path to prove the logic equivalence. □

Proposition 5 (Arenas, F.G. [2])**.** *Let (X, τ) be an Alexandroff space. (X, τ) is T_0 if and only if $\mathcal{N}_x = \mathcal{N}_y$ implies that $x = y$*

Proof. Let (X, τ) be an Alexandroff space.

($\rightarrow$) Let (X, τ) be T_0. Take the contrapositive, and suppose $x \neq y$, then they are topologically distinguishable since X is T_0, and thus $\bar{x} \neq \bar{y}$. Therefore $\mathcal{N}_x \neq \mathcal{N}_y$, as a consequence of Lemma 2.

($\leftarrow$) By the contrapositive, if $x \neq y$ implies $\mathcal{N}_x \neq \mathcal{N}_y$, it follows directly that they do not have the same neighborhood system, therefore the space is T_0. For it being Alexandroff, the neighborhood systems differ at least in the minimal open neighborhood. □

Theorem 3 (Characterization of R_0 in Alexandroff spaces)**.** *An Alexandroff space (X, τ) is R_0 if and only if $\bar{x} \neq \bar{y}$ implies that $\bar{x} \cap \bar{y} = \varnothing$ for all $x, y \in X$.*

Proof. Let (X, τ) be an Alexandroff space.

($\rightarrow$) Suppose that (X, τ) is R_0 and let $x, y \in X$ such that $\bar{x} \neq \bar{y}$. By Lemma 2, $\mathcal{N}_x \neq \mathcal{N}_y$ so x and y are topologically distinguishable. Since the space is R_0, then they are also separated, that is, $\mathcal{N}_x \not\subseteq \mathcal{N}_y$ and $\mathcal{N}_y \not\subseteq \mathcal{N}_x$.

Let $z \in \bar{x}$. Then $x \in \mathcal{N}_z$ so $\mathcal{N}_x \subseteq \mathcal{N}_z$ which implies that $\mathcal{N}_z \not\subseteq \mathcal{N}_y$. Since the space is R_0, then $\mathcal{N}_y \not\subseteq \mathcal{N}_z$. Hence $z \notin \bar{y}$ and $\bar{x} \cap \bar{y} = \varnothing$.

($\leftarrow$) Suppose $x, y \in X$ are topologically distinguishable, that is, $\mathcal{N}_x \neq \mathcal{N}_y$. By Lemma 2, $\bar{x} \neq \bar{y}$ so $\bar{x} \cap \bar{y} = \varnothing$ by hypothesis. Hence $y \notin \mathcal{N}_x$ and $x \notin \mathcal{N}_y$, that is, $\mathcal{N}_x$ is not a neighborhood of y and $\mathcal{N}_y$ is not a neighborhood of x. Consequently x, y are separated and the topological space is R_0. □

Substantially, all interesting properties of Alexandroff spaces are to be sought in the realm of separability axioms weaker than T_1. The only Hausdorff Alexandroff space is the discrete space, and so on. Several authors put emphasis in Alexandroff spaces being T_0, *i.e.,* Kolmogorov [2].

2.2 Connectedness

This section is aimed to study one of the fundamental properties of a topological space within the framework of Alexandroff spaces: connectedness. Let us recall the definition.

Definition 8. *A topological space (X, τ) is disconnected if there exist disjoint nonempty open sets O_1, O_2 such that $X = O_1 \cup O_2$.*

If (X, τ) is not disconnected we say that it is connected. Furthermore, the topological space is locally connected if each point has a neighborhood base of open connected sets.

The following result characterizes the disconnection of a subspace of a topological space.

Lemma 3 (Mendelson, B.[16])**.** *Let (X, τ) be a topological space, and $A \subseteq X$, then A is disconnected if and only if there exist $O_1, O_2 \in \tau$, such that:*

- $A \subseteq O_1 \cup O_2$,
- $O_1 \cap O_2 \subseteq X \backslash A$, *and*
- $A \cap O_1 \neq \varnothing$ *and* $A \cap O_2 \neq \varnothing$.

Using the previous result, we next prove that the minimal neighborhood of a point is always connected in an Alexandroff space.

Theorem 4. *Let (X, τ) be an Alexandroff space, then the minimal neighborhoods are connected.*

Proof. Let (X, τ) be an Alexandroff space and $a \in X$. Let $\mathcal{N}_a$ be the minimal neighborhood of a. If there exist open sets O_1, O_2 such that $\mathcal{N}_a \subseteq O_1 \cup O_2$, then $a \in O_1$ or $a \in O_2$. Suppose that $a \in O_1$, then $a \in \mathcal{N}_a \subseteq O_1$, so that $\mathcal{N}_a \cap O_1 = \mathcal{N}_a \neq \varnothing$, then it is impossible that $\varnothing \neq \mathcal{N}_a \cap O_2 \subseteq O_1 \cap O_2$ be contained in $X \backslash \mathcal{N}_a$. Therefore, $\mathcal{N}_a$ is connected. □

Corollary. *Alexandroff spaces are locally connected.*

Since Alexandroff spaces are locally connected, it can be readily seen that any connected component is clopen. Several connectedness properties for T_0-Alexandroff spaces are now covered. Let us first recall a few definitions regarding chain connectedness [24].

Definition 9 (Chain connectedness)**.** *Let (X, τ) be a topological space, and $\mathcal{V}$ be a family of open sets in X. Let $x, y \in X$. A $\mathcal{V}$-chain connecting x and y is a finite sequence $V_1, V_2, ...V_k$ of open sets in $\mathcal{V}$ such that $x \in V_1$, $y \in V_k$, and $V_i \cap V_j \neq \varnothing$ for any $|i - j| \leq 1$.*

A subset $C \subseteq X$ is chain connected if for every open covering $\mathcal{V}$ of X, and any $x, y \in C$, there exists a $\mathcal{V}$-chain connecting x and y. Moreover, if this property holds for any $x, y \in X$, then X is said to be a chain connected space.

The following lemma is taken from Willard, [24].

Lemma 4. *Let (X, τ) be a topological space. If X is connected, then X is chain connected.*

Proof. Let (X, τ) be a connected topological space, $\mathcal{V}$ be an open cover of X, and $a \in X$. Let P_a be the set:

$$P_a = \{p \in X : \text{ there is a } \mathcal{V}\text{-chain connecting } p \text{ with } a \ \}.$$

Then $P_a \neq \varnothing$ for $a \in P_a$. Take $b \in P_a$, then there exists a finite sequence $V_1, V_2, ...V_k$ of elements of $\mathcal{V}$ connecting a and b, hence $b \in V_k$ with V_k open. Then for any other element c of V_k the finite sequence $V_1, V_2, ...V_k$ is a $\mathcal{V}$-chain connecting a with c. Hence $V_k \subseteq P_a$. Therefore P_a is open, for it contains an open set containing b, for any $b \in P_a$.

We next show that P_a is closed. Take a point in the closure, say $z \in \overline{P_a}$. Since $\mathcal{V}$ is an open covering of X, there exists $V \in \mathcal{V}$ such that $z \in V$. Then it holds $V \cap P_a \neq \varnothing$; hence there exists a $w \in V \cap P_a$, and so $w \in P_a$. Let $V_1, V_2, ...V_k$ be a $\mathcal{V}$-chain connecting w with a, with $a \in V_1$ and $w \in V_k$. Then $V_1, V_2, ...V_k, V$ is a $\mathcal{V}$-chain connecting a with z, hence $z \in P_a$. As a consequence, $\overline{P_a} \subseteq P_a$ and P_a is closed.

Since X is connected and P_a is clopen, then $P_a = X$, and the whole space is chain connected. □

Theorem 5 (Arenas, F.G.[2])**.** *Let (X, τ) be a T_0-Alexandroff space. The following statements are equivalent.*

1. *X is path connected.*
2. *X is connected.*
3. *X is chain connected.*
4. *For every $x, y \in X$, there exist $a_0, a_1, ...a_n$ such that $a_0 = x$, $a_n = y$, with $\mathcal{N}_{a_i} \cap \mathcal{N}_{a_j} \neq \varnothing$ whenever $|i - j| \leq 1$.*
5. *For every $x, y \in X$, there exist $a_0, a_1, ...a_k$ such that $a_0 = x$, $a_k = y$, with $\overline{\mathcal{N}}_{a_i} \cap \overline{\mathcal{N}}_{a_j} \neq \varnothing$ whenever $|i - j| \leq 1$.*
6. *For every $x, y \in X$, there exist $a_0, a_1, ...a_m$ such that $a_0 = x$, $a_m = y$, with $\overline{\{a_i\}} \cap \overline{\{a_j\}} \neq \varnothing$ whenever $|i - j| \leq 1$.*

Proof. $(1 \to 2)$ As in any topological space, path connectedness is sufficient for connectedness.

$(2 \to 3)$ Likewise, this follows from Lemma 4, just as in any topological space.

$(3 \to 4)$ Let $\mathcal{V}$ be the family of all minimal open neighborhoods of elements of X. Since X is chain connected and $x, y \in X$, we can find a $\mathcal{V}$-chain $V_0, \dots, V_n$ connecting x and y, so the conclusion follows.

$(4 \to 5)$ If 4. holds for minimal neighborhoods, then it holds for their respective closures.

$(5 \to 6)$ If 5. holds, then for every pair of consecutive indices, say i, j, with $j = i+1$, $\overline{\mathcal{N}}_{a_i} \cap \overline{\mathcal{N}}_{a_j} \neq \varnothing$; hence there exists $p_i \in \overline{\mathcal{N}}_{a_i}$ and $p_i \in \overline{\mathcal{N}}_{a_j}$, so that $\mathcal{N}_{p_i} \cap \mathcal{N}_{a_i} \neq \varnothing$ and $\mathcal{N}_{p_i} \cap \mathcal{N}_{a_j} \neq \varnothing$. This guarantees the existence of at least two points, name them q_i, q_j such that $q_i \in \mathcal{N}_{a_i}$ and $q_i \in \mathcal{N}_{p_i}$; while $q_j \in \mathcal{N}_{p_i}$ and $q_j \in \mathcal{N}_{a_j}$. By applying the Lemma 1, $a_i \in \overline{\{q_i\}}$, $p_i \in \overline{\{q_i\}} \cap \overline{\{q_j\}}$, and $a_j \in \overline{\{q_j\}}$. By executing the same procedure to all the other consecutive pairs, it is shown that there exists a finite sequence $\{q_m\}$, such that $\overline{\{q_i\}} \cap \overline{\{q_j\}} \neq \varnothing$, whenever $|i - j| \leq 1$.

The only question left is about the first and last elements in the sequence. It is contended that the first element can be x, and y the last. In the first case, the previous result guarantees the existence of q_0, p_0, q_1, such that $q_0 \in \mathcal{N}_{p_0}$ and $q_0 \in \mathcal{N}_{a_0}$, with $a_0 = x$. Therefore, by Lemma 1, $x \in \overline{\{q_0\}}$; hence $\overline{\{x\}} \cap \overline{\{q_0\}} \neq \varnothing$. The new finite sequence starts with x, and similar reasoning can show that the last element is y.

$(6 \to 1)$ If 6. holds, then for all $x, y \in X$, there exist $a_0, a_1, ...a_m$ such that $a_0 = x$, $a_m = y$, with $\overline{\{a_i\}} \cap \overline{\{a_j\}} \neq \varnothing$ whenever $|i - j| \leq 1$. Take a pair of consecutive indices, say $i, i+1$, then there exists $b_i \in \overline{\{a_i\}} \cap \overline{\{a_{i+1}\}}$. By Lemma 1, $a_i \in \mathcal{N}_{b_i}$ and $a_{i+1} \in \mathcal{N}_{b_i}$; thus $\mathcal{N}_{a_i} \subseteq \mathcal{N}_{b_i}$ and $\mathcal{N}_{a_{i+1}} \subseteq \mathcal{N}_{b_i}$. It will be shown that a_i and a_{i+1} are path connected via the following function $f_i : [0,1] \to X$:

$$f_i(t) = \begin{cases} a_i & \text{if } t \in [0, \frac{1}{2}) \\ b_i & \text{if } t = \frac{1}{2} \\ a_{i+1} & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

The correspondence f_i is continuous: Let O be an open set; if O does not include a_i, b_i nor a_{i+1}, then has an empty preimage, $f_i^{-1}(O) = \varnothing$. If O contains a_i only, then $f_i^{-1}(O) = [0, \frac{1}{2})$, which is open with the relative topology on the domain. If O contains only a_{i+1}, then $f_i^{-1}(O) = (\frac{1}{2}, 1]$, which is also open in $[0,1]$. If O contains both a_i and a_{i+1}, then $f_i^{-1}(O) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ open in $[0,1]$. Any open set containing b_i will containing also a_i and a_{i+1}, therefore its preimage will be $[0,1]$, open as well. Hence, f_i is continuous, for the preimages of open sets are open.

A path connecting a_i and a_{i+1} was constructed. The same constructive process can be carried out for all the other consecutive pairs of elements in the finite sequence, by taking $b_i \in \overline{\{a_i\}} \cap \overline{\{a_{i+1}\}}$ for every $i \in \{0, 1, \dots, m-1\}$. Hence, the linking of all the paths can be done with just one function, $\alpha : [0,1] \to X$, as follows:

$$\alpha(t) = \begin{cases} a_k & \text{if } \frac{k}{m+1} < t < \frac{k+1}{m+1}, k \in \{0, \dots, m-1\}, \\ b_k & \text{if } t = \frac{k+1}{m+1}, k \in \{0, \dots, m-1\}, \\ x & \text{if } t = 0, \\ y & \text{if } t = 1. \end{cases}$$

By construction, $\alpha(0) = x$ and $\alpha(1) = y$, and it can be readily checked that α is continuous in the context of an Alexandroff space.

□

Corollary. *Every T_0-Alexandroff space is locally path-connected.*

2.3 Compactness and countability

Several topological properties such as the Lindelöf property and countability axioms are addressed in this section, within the framework of Alexandroff spaces.

Proposition 6 (Arenas, F.G.[2])**.** *Let (X, τ) be an Alexandroff space, then X satisfies the first axiom of countability.*

Proof. By Theorem 1, every point $x \in X$ has a minimal neighborhood $\mathcal{N}_x$. Then $\{\mathcal{N}_x\}$ is a countable neighborhood base at x so the space is first countable. □

Bear in mind that a topological space (X, τ) is said to be orthocompact if every open cover has an interior-preserving open refinement.

Proposition 7 (Arenas, F.G.[2])**.** *Let (X, τ) be an Alexandroff space, then X is orthocompact.*

Proof. In an Alexandroff space, every open cover is interior-preserving, for arbitrary intersections of open sets are always open. Therefore, it is obviously orthocompact. □

The following definition is given by McCord in [13].

Definition 10. *A topological space (X, τ) is said to be locally finite if every point has a finite neighborhood.*

Proposition 8 (Arenas, F.G.[2])**.** *Let (X, τ) be a T_0-Alexandroff space. If X is paracompact, then X is locally finite.*

Proof. If X is paracompact, then for every open cover of X there exists a refinement, namely the minimal open neighborhoods refinement, such that every $\mathcal{N}_x$ satisfies $\mathcal{N}_x \cap \mathcal{N}_y \neq \varnothing$ for a finite number of $\mathcal{N}_y$. Proceeding by contradiction, suppose X is not locally finite, then there exists $x \in X$ such that $\mathcal{N}_x$ is infinite. In that case, $\mathcal{N}_y \subseteq \mathcal{N}_x$ for every $y \in \mathcal{N}_x$, $\Rightarrow\Leftarrow$. Hence X is locally finite. □

Proposition 9. *Let (X, τ) be a T_0-Alexandroff space. Then X is second countable if and only if X is countable.*

Proof. $(\leftarrow)$ If X is countable and X is also first countable by Proposition 6, then it is also second countable as with any topological space.

$(\rightarrow)$ Since the second countable axiom implies that the minimal neighborhood basis is countable, then the space is countable. □

Theorem 6. *Let (X, τ) be a T_0-Alexandroff space, X is Lindelöf if and only if $X = \bigcup_{n=1}^{\infty} \mathcal{N}_{x_n}$ for some countable set $\{x_n : n \in \mathbb{N}\} \subseteq X$.*

Proof. $(\rightarrow)$ If X is Lindelöf, then given the open cover $X = \bigcup\{\mathcal{N}_x : x \in X\}$ constituted by the minimal neighborhoods, there exists a countable subcover, *i.e.*, $X = \bigcup_{n=1}^{\infty} \mathcal{N}_{x_n}$ for some countable set $\{x_n : n \in \mathbb{N}\} \subseteq X$.

($\leftarrow$) Suppose that $X = \bigcup_{n=1}^{\infty} \mathcal{N}_{x_n}$ for some countable set $\{x_n : n \in \mathbb{N}\} \subseteq X$. Given an arbitrary open cover $X = \bigcup\{\mathcal{U}_i : i \in I\}$, then $\forall n \in \mathbb{N}$, by the Axiom of Choice, it is possible to pick an element $x_n \in \mathcal{N}_{x_n} \subseteq \mathcal{U}_i$ for some $i \in I$. Rename the selected $\mathcal{U}_i$ as $\mathcal{U}_n$ and make a countable subcover of X. Hence X is Lindelöf. □

The next definition can be found in [17].

Definition 11. *A topological space (X, τ) is said to be locally countable if for every $x \in X$, there exists a countable open set in τ containing x.*

Proposition 10. *Let (X, τ) be a T_0-Alexandroff space. X is countable if and only if X is locally countable and Lindelöf.*

Proof. ($\rightarrow$) If X is countable, then there is a countable cover consisting of minimal neighborhoods, hence it is Lindelöf and locally countable.

($\leftarrow$) If X is Lindelöf and locally countable, then X is a countable union of locally countable sets, hence X is countable. □

Proposition 11. *Let (X, τ) be a locally finite T_0-Alexandroff space. X is compact if and only if X is finite.*

Proof. ($\leftarrow$) Straightforward, as with any finite space.

($\rightarrow$) Let X be a locally finite T_0-Alexandroff space. If X is compact, then the cover comprised by minimal neighborhoods has a finite subcover, $X = \bigcup_{k=1}^{N} \mathcal{N}_{x_k}$. Since X is locally finite, then X is finite. □

2.4 Obtaining Alexandroff spaces out of others

In this section, several rules for obtaining new Alexandroff spaces from old ones are examined, starting with the called opposite topology which is distinguishing topology associated with Alexandroff spaces.

2.4.1 The opposite topology

Alexandroff spaces arise in pairs. Since arbitrary intersections of open sets are also open, then the arbitrary unions of their complementary closed sets are closed as well. This feature allows for the appearance of the labeled opposite topology.

Theorem 7. *If τ is an Alexandroff topology over X, then the collection τ_o described as:*

$$\tau_o = \{X \backslash V : V \in \tau\}$$

is also an Alexandroff topology on X.

Proof. X and $\varnothing$ belong to τ_o for $\varnothing, X \in \tau$, respectively.

- Let $U_1, U_2 \in \tau_o$, thus $U_i = X \backslash V_i$ with $V_i \in \tau$. Then $U_1 \cap U_2 = X \backslash V_1 \cap X \backslash V_2 = X \backslash (V_1 \cup V_2)$. Hence $U_1 \cap U_2 \in \tau_o$, for $V_1 \cup V_2 \in \tau$.

- Let $\{U_i\}_{i\in I}$ be an arbitrary family of elements of τ_o, so that $U_i = X\backslash V_i$ with $V_i \in \tau$ for all $i \in I$. Then the union belong to τ_o, for:

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} X\backslash V_i = X\backslash \bigcap_{i\in I} V_i$$

  and $\bigcap_{i\in I} V_i \in \tau$ because of τ being an Alexandroff topology.

Therefore, τ_o is also a topology on X. Moreover, the intersection $\bigcap_{i\in I} U_i$ of an arbitrary collection of open sets $\{U_i\}_{i\in I}$ in τ_o is open as well, for:

$$\bigcap_{i\in I} U_i = \bigcap_{i\in I} X\backslash V_i = X\backslash \bigcup_{i\in I} V_i$$

and $\bigcup_{i\in I} V_i$ is closed in τ. Therefore, τ_o is an Alexandroff topology. □

2.4.2 Subspaces

Given a topological space (X, τ), and $Y \subset X$, it is possible to define a topology on Y, by restricting the open subsets O of X to Y via $A = O \cap Y$. This produces a family τ_A of subsets of Y which is a topology on Y called the *relative topology* on A. In this case, we say that Y is a subspace of X.

Theorem 8. *Let (X, τ) be an Alexandroff space and $Y \subset X$, with $Y \neq \varnothing$. Then (Y, τ_Y) is an Alexandroff space.*

Proof. Let $y \in Y \subset X$, and let $\{\mathcal{U} \in \tau : y \in \mathcal{U}\}$ be the collection of its open neighborhoods. Then, each $\mathcal{V} = \mathcal{U} \cap Y$, is an open neighborhood of y as well, regarding the relative topology on Y. Therefore, the intersection:

$$\mathcal{N}_y = \bigcap\{\mathcal{V} \in \tau_Y : y \in \mathcal{V}\} = \bigcap\{\mathcal{U} \cap Y \in \tau_Y : y \in \mathcal{U}\}$$

is open, for $\bigcap \mathcal{V} = Y \cap (\bigcap \mathcal{U})$ with $\bigcap \mathcal{U}$ being open on X. Moreover, $\mathcal{N}_y \neq \varnothing$, for $y \in \mathcal{N}_y$, and it is minimal with respect to the inclusion order. As a consequence, (Y, τ_Y) is an Alexandroff subspace of X. □

It can be readily shown that the collection of minimal neighborhoods $\{\mathcal{N}_y : y \in Y\}$, with $\mathcal{N}_y$ as defined above, constitutes a basis for the relative Alexandroff topology on Y.

2.4.3 Intersections

Given a nonempty set X and two topologies on it, namely τ_1 and τ_2, it is a well-known result that the intersection $\tau_1 \cap \tau_2$ is a topology as well. It holds for the arbitrary intersection as well. Let $\{\tau_i\}_{i\in I}$ be a collection of topologies over X, and $\tau = \bigcap \tau_i$. Then X and $\varnothing$ are in τ for they belong to every τ_i in the collection. If $\{A_j\}_j$ is an arbitrary family of open sets in τ, then each one of them belong to each one of the topologies in the family. Therefore, the union $\bigcup_j A_j$ is open in every τ_i, hence it is open in τ. Finally, if $A_1, A_2 \in \tau$, then $A_1, A_2 \in \tau_i$ for all $i \in I$; thus $A_1 \cap A_2 \in \tau_i$, for all $i \in I$, hence $A_1 \cap A_2 \in \tau$.

Theorem 9 (Intersection of Alexandroff spaces). *Let $\{\tau_i\}_{i\in I}$ be a collection of Alexandroff topologies over X, then (X,τ), with $\tau = \bigcap \tau_i$, is an Alexandroff space.*

Proof. Let $\{\tau_i\}_{i\in I}$ be an arbitrary family of Alexandroff topologies over X; it is already known that $\tau = \bigcap \tau_i$ is also a topology over X.

Let $x \in X$ and $\{U_j : x \in U_j \in \tau\}$ the collection of all open neighborhoods containing x. Then $\mathcal{N}_x = \bigcap\{U_j : x \in U_j\}$ is open and a neighborhood of x, being also minimal in the sense of the inclusion order. If V is any other open neighborhood of x in τ, then $V \in \tau_i$ for all $i \in I$. This means that $V \in \{U_j : x \in U_j\}$ and $\mathcal{N}_x \subseteq V$. Therefore, (X,τ) is Alexandroff. □

Corollary. *Finite intersection of Alexandroff topologies is also Alexandroff.*

2.4.4 Products

Let $(X_1,\tau_1),(X_2,\tau_2),...,(X_n,\tau_n)$ be n topological spaces, and let $X = \prod_{i=1}^n X_i$ be the cartesian product $X_1 \times X_2 \times ...X_n$. It has been shown that it is possible to construct a topology on X as a result of the topologies on each factor X_i [16]. For finite products, it has been proved that the *box* topology coincides with the *Tychonoff* topology; both being generated by a basis of the form:

$$\mathcal{B} = \{O = O_1 \times O_2 \times ...O_n : O_i \text{ is open in } X_i\}.$$

Likewise, given a point $x \in X$, and a neighborhood V of x, then $x = (x_1, x_2, ...x_n) \in V$ and there exists a set of the form $V_1 \times V_2 \times ... \times V_n$ contained in V, with every V_i being a neighborhood of $x_i \in X_i$. Then there exist open sets $O_i \subseteq V_i$ containing x_i, for all $i = 1...n$.

Theorem 10 (Finite products of Alexandroff spaces). *Let $(X_1,\tau_1),(X_2,\tau_2),...,(X_n,\tau_n)$ be n Alexandroff topological spaces, then the product $X = \prod_{i=1}^n X_i$ is an Alexandroff space as well.*

Proof. Let $x = (x_1, x_2, \dots, x_n) \in X = \prod_{i=1}^n X_i$. For each $i \in \{1,\dots,n\}$, (X_i,τ_i) is an Alexandroff space so x_i has a minimal open neighborhood $\mathcal{N}_{x_i}$ which constitutes a neighborhood base at x_i. Since the cartesian product of neighborhood bases at x_i gives a neighborhood base at $(x_1,\dots,x_n)$, then $\{\mathcal{N}_{x_1} \times \dots \times \mathcal{N}_{x_n}\}$ is a open neighborhood base at x. Since this neighborhood is also minimal, then X is also Alexandroff. □

In the product space, it is possible to define the canonical projection $\pi_i : X \to X_i$ such that, given $a = (a_1, a_2, ...a_n) \in X$, then $\pi_i(a) = a_i$. Hence, given an open set $U_i \in \tau_i$ with $a_i \in U_i$, then the preimage is:

$$\pi_i^{-1}(U_i) = X_1 \times X_2 \times ...X_{i-1} \times U_i \times X_{i+1} \times ...X_n.$$

Since $\pi_i^{-1}(U_i)$ is open, then the projection maps are continuous. Therefore, it is possible to write an open set in X as:

$$O = O_1 \times O_2 \times ...O_n = \pi_1^{-1}(O_1) \cap \pi_2^{-1}(O_2) \cap ... \cap \pi_n^{-1}(O_n)$$

Hence the Tychonoff's generalization for an arbitrary product is obtained.

Definition 12 (Tychonoff's product topology)**.** *Let $\{(X_i, \tau_i)\}_{i\in I}$ be an arbitrary collection of topological spaces. Then the product space $X = \prod_{i\in I} X_i$ is a topological space, having as a basis for the topology open sets of the form:*

$$\bigcap_{k=1}^{K} \pi_{i_k}^{-1}(O_{i_k})$$

with $K \in \mathbb{N}$ and $O_{i_k} \in \tau_{i_k}$.

Addressing the question whether an arbitrary product of Alexandroff spaces is also Alexandroff, the following is a counterexample to this idea.

Example 6. *Let $S = \{a, b\}$ and (S, τ_S) be a Sierpiński space, namely, $\tau_S = \{\varnothing, \{a\}, S\}$. Then it is also an Alexandroff space. Let us construct the following product:*

$$\prod_{n\in\mathbb{N}} S_n$$

with $S_n = S$. The Tychonoff's product topology requires that only a finite number of components in the product of open set are not the whole space S. Therefore, it is possible to construct the following family of open sets, $\{O_n\}_{n\in\mathbb{N}}$:

$$\begin{aligned} O_1 &= \{a\} \times S \times S \times ... \\ O_2 &= S \times \{a\} \times S \times ... \\ O_3 &= S \times S \times \{a\} \times ... \\ &\vdots \\ O_n &= S \times ... \times \{a\} ... \times S \times ... \end{aligned}$$

where the only open set different from S is $\{a\}$ in the n^{th}-position, for every $n \in \mathbb{N}$. Hence, each O_n is an open set in the Tychonoff's product topology, and $\{O_n\}_{n\in\mathbb{N}}$ is an arbitrary family of open sets. However, the intersection

$$\bigcap_{n\in\mathbb{N}} O_n = \{a\} \times \{a\} \times \{a\} \times ...$$

is not an open set, as it does not satisfies the Definition 12. In consequence, the aforementioned product is not an Alexandroff space.

2.4.5 Quotients

Given a topological space (X, τ) and an equivalence relation $\sim$ on X, then the quotient set denoted, by $X/_\sim$, is the set of all the equivalence classes regarding $\sim$. If $x \in X$, the symbol $[x]$ denotes the class of equivalence of x, *i.e.*, the set $[x] = \{a \in X : x \sim a\}$. A surjective map called the canonical quotient map is naturally associated with the equivalence relation, namely $\mathfrak{q} : X \to X/_\sim$, where $\mathfrak{q}(x) = [x]$. In similar fashion, there is a natural topology defined for the quotient $X/_\sim$, namely a set $U \in X/_\sim$ is open if and only if $\mathfrak{q}^{-1}(U)$ is open in X.

Let $\tau_\mathfrak{q}$ denote the collection $\{U \subseteq X/_\sim : \mathfrak{q}^{-1}(U)$ is open in X$\}$. Then $\varnothing$ and $X/_\sim$ are in $\tau_\mathfrak{q}$, for $\mathfrak{q}^{-1}(\varnothing) = \varnothing$ and $\mathfrak{q}^{-1}(X/_\sim) = X$, both open in X. If $\{U_i\}_{i\in I}$ is an arbitrary collection of elements of $\tau_\mathfrak{q}$, then $\mathfrak{q}^{-1}(U_i)$ is open in X for each $i \in I$. Therefore $\bigcup_i U_i$ is in $\tau_\mathfrak{q}$, for $\mathfrak{q}^{-1}(\bigcup_i U_i) = \bigcup_i \mathfrak{q}^{-1}(U_i)$ is open in (X, τ). Finally, if $U_1, U_2 \in \tau_\mathfrak{q}$, then $\mathfrak{q}^{-1}(U_1)$ and $\mathfrak{q}^{-1}(U_2)$ are open in X; hence $U_1 \cap U_2 \in \tau_\mathfrak{q}$, for $\mathfrak{q}^{-1}(U_1 \cap U_2) = \mathfrak{q}^{-1}(U_1) \cap \mathfrak{q}^{-1}(U_2)$ is open in X. As a result, $\tau_\mathfrak{q}$ is a topology on $X/_\sim$, and it is called the quotient topology.

Theorem 11. *Let (X, τ) be an Alexandroff space, and $\sim$ an equivalence relation on X; then $X/_\sim$ is an Alexandroff space with the corresponding quotient topology $\tau_\mathfrak{q}$.*

Proof. Let $\{U_i\}_{i\in I}$ be an arbitrary collection of open sets in $\tau_\mathfrak{q}$; then $\mathfrak{q}^{-1}(U_i)$ is open in X for each $i \in I$. Therefore $\bigcap_i U_i$ is also open in $\tau_\mathfrak{q}$, on the grounds of $\mathfrak{q}^{-1}(\bigcap_i U_i) = \bigcap_i \mathfrak{q}^{-1}(U_i)$ being open in τ, for X is Alexandroff. In consequence, $(X/_\sim, \tau_\mathfrak{q})$ is Alexandroff. □

2.4.6 Identification topologies

Let X and Y be two topological spaces, and $p : X \to Y$ be a continuous map; then p is called an *identification* if and only if for each set $V \subseteq Y$, $p^{-1}(V)$ is open in X implies that V is open in Y, as described by Mendelson in [16]. The idea of identification is the rationale for a method to equip a set with a topology, via surjective functions.

Definition 13 (Identification topology, Mendelson [16])**.** *Let (X, τ_X) be a topological space, and $p : X \to Y$ be a surjective function mapping X onto a set Y. The identification topology on Y comprises all sets $V \subseteq Y$ such that $p^{-1}(V)$ is open in X.*

Hence, p becomes an identification mapping X onto Y. Let τ_Y be the collection of all sets $V \subseteq Y$ such that $p^{-1}(V)$ is open in X. Then it can be seen that $Y \in \tau_Y$, for $p^{-1}(Y) = X \in \tau_X$; as much as $\varnothing \in \tau_Y$ too. If $\{V_i\}_{i\in I}$ is an arbitrary family of elements in τ_Y, then it means that $p^{-1}(V_i)$ is open in X for each $i \in I$. Therefore $\bigcup_i V_i$ is also in τ_Y since $p^{-1}(\bigcup_i V_i) = \bigcup_i p^{-1}(V_i)$ is open in X. Last but not least, given two sets $V_1, V_2 \in \tau_Y$, then the intersection $V_1 \cap V_2$ belongs to τ_Y for the preimage $p^{-1}(V_1 \cap V_2) = p^{-1}(V_1) \cap p^{-1}(V_2)$ is open in X. Therefore, the collection τ_Y is a topology on Y, *i.e.*, a identification topology.

A few instances of identification have been already shown in the present work, *exempli gratia*: the quotient map $\mathfrak{q}$ from a space X onto the quotient $X/_\sim$, and the canonical projection π_k mapping a product space $\prod_i X_i$ onto one of the factors X_k.

Theorem 12. *Let (X, τ_X) be a topological space, and $p : X \to Y$ be a surjective map inducing an identification topology τ_Y on Y. Therefore, if (X, τ_X) is an Alexandroff space, then (Y, τ_Y) is an Alexandroff space.*

Proof. Let $\{V_i\}_{i\in I}$ be an arbitrary family of sets in τ_Y, then $\bigcap_{i\in I} V_i$ is open in τ_Y, for

$$p^{-1}(\bigcap_{i\in I} V_i) = \bigcap_{i\in I} p^{-1}(V_i)$$

is open in X on account of X being an Alexandroff space. □

The main idea of identification can be generalized for a family of surjections mapping a family of spaces onto a set, in order to obtain the denominated *final topology*. Given a set Y, a family of topological spaces (X_i, τ_{X_i}), with a corresponding family of functions $\mathcal{F} = \{f_i : X_i \to Y\}$, then the final topology on Y, induced by the family $\mathcal{F}$, is the finest topology $\tau_{\mathcal{F}}$ on Y such that each f_i is continuous for each $i \in I$. This unequivocally means that a set U is open in Y, *i.e.* $U \in \tau_{\mathcal{F}}$, if and only if each $f_i^{-1}(U)$ is open in (X_i, τ_{X_i}) for each $i \in I$. Therefore, the final topology becomes an indexed intersection of identification topologies, hence if $(X_i, \tau_{X_i})_{i \in I}$ is a family of Alexandroff spaces, then $(Y, \tau_{\mathcal{F}})$ is also an Alexandroff space.

3 Equivalence between Alexandroff spaces and quasiordered spaces

3.1 Introduction

There is a close relationship between topology and order. The neighborhood systems in a topological space allow to define the commonly designated *specialization quasiorder* in an arbitrary topological space by considering that a point x is less than or equal to a point y if the neighborhood system at x is included in the neighborhood system at y. When the topological space is T_1 then the specialization quasiorder becomes equality and it is useless. Therefore, in this connection between topological and ordered spaces, non-T_1 spaces play a significant role more than T_1 spaces.

In general, several different topological spaces can induce the same specialization quasiorder. This makes impossible to recover the topology from its specialization quasiorder. However, in case that one considers an Alexandroff topological space, the recovery is possible. In fact, there exists an equivalence between the categories of Alexandroff spaces and quasiordered spaces.

This chapter is devoted to develop this equivalence and it is mainly inspired in the references [8, 9, 19].

3.2 Order theory

Given a set X, let us recall that an equivalence relation $\sim$ on X is a reflexive, symmetric and transitive relation; while, on the other hand, a non-strict partial order is a relation which is reflexive, antisymmetric and transitive. Hence, the essential difference comes down to the symmetric–antisymmetric dichotomy. By removing this condition altogether, a relation denominated *quasiorder* is obtained.

Definition 14 (Quasiorder)**.** *Let X be a nonempty set. A quasiorder on X is a reflexive and transitive relation $\lesssim$ on X. That is, $\lesssim$ is a quasiorder on X if and only if for all $x, y, z \in X$:*

- $x \lesssim x$
- $x \lesssim y$ *and* $y \lesssim z$ *implies* $x \lesssim z$

In this case, we say that the pair $(X, \lesssim)$ is a quasiordered set.

If a quasiorder $\lesssim$ on X is also antisymmetric then it is called a partial order and $(X, \lesssim)$ is a partially ordered set (a poset for short).

Definition 15. *Given a quasiordered set $(X, \lesssim)$, the dual quasiorder $\gtrsim$ on X is defined as $x \gtrsim y$ if and only if $y \lesssim x$.*

Example 7. *If* $\mathbb{R}$ *is endowed with the usual topology, the binary relation* $\lesssim$ *on* $\mathcal{P}(\mathbb{R})$ *given by*

$$A \lesssim B \text{ if and only if } A \subseteq \overline{B}$$

for any $A, B \in \mathcal{P}(\mathbb{R})$*, is a quasiorder which does not satisfy the antisymmetry. For example, if* $A = [0, 1)$ *and* $B = (0, 1]$*, then* $A \lesssim B$ *and* $B \lesssim A$ *but* $A \neq B$*.*

However, a quasiorder is at short distance to be an equivalence relation as the next result shows [19].

Theorem 13 ([19, Theorem 8.2.2])**.** *Let* $(X, \lesssim)$ *be a quasiordered set. Then:*

1. *The relation* $\sim$ *defined on* X *as* $x \sim y$ *if and only if* $x \lesssim y$ *and* $y \lesssim x$*, for all* $x, y \in X$ *is an equivalence relation.*
2. *Given an equivalence relation defined as indicated above, the quotient set* $X/_\sim$ *has a natural partial order defined as* $[x] \leq [y]$ *if and only if* $x \lesssim y$*.*
3. *Moreover, given a partition of a set* C*, such that there is a partial order* $\leq$ *between the blocks of the partition, then there exists a quasiorder* $\lesssim$ *on* C *such that* $x \lesssim y$ *if and only if* $[x] \leq [y]$*.*

Proof. Let $(X, \lesssim)$ be a quasiordered set.

1. Since $\lesssim$ is reflexive, $x \lesssim x$; therefore $x \sim x$ and thus $\sim$ is reflexive.

   Let $x, y, z \in X$ be such that $x \sim y$ and $y \sim z$; that is, $x \lesssim y$, $y \lesssim z$, $y \lesssim x$ and $z \lesssim y$. Since $\lesssim$ is transitive, then $x \lesssim z$ and $z \lesssim x$; hence, $x \sim z$. Therefore, $\sim$ is transitive.

   Finally, given $x, y \in X$ such that $x \sim y$, then we have $x \lesssim y$ and $y \lesssim x$. That is exactly the same as $y \sim x$, for the conjunction *and* is commutative. Thus, $\sim$ is symmetric.

   Therefore, $\sim$ is an equivalence relation.

2. Given $(X, \lesssim)$ and the aforementioned equivalence relation $\sim$, define $\leq$ in $X/_\sim$ as $[x] \leq [y]$ if and only if $x \lesssim y$, for all $x, y \in X$. First, let us show that it is well-defined.

   Let $[x_1] = [x_2]$ and $[y_1] = [y_2]$, *i.e.*, different representatives from the same equivalence classes, then it is true that $[x_1] \leq [y_1]$ if and only if $[x_2] \leq [y_2]$. Since $[x_1] = [x_2]$ and $[y_1] = [y_2]$, then $x_1 \lesssim x_2$, $x_2 \lesssim x_1$, $y_1 \lesssim y_2$ and $y_2 \lesssim y_1$. Hence, if $[x_1] \leq [y_1]$, then $x_2 \lesssim x_1 \lesssim y_1 \lesssim y_2$, thus $x_2 \lesssim y_2$ by transitivity of $\lesssim$; therefore $[x_2] \leq [y_2]$. In similar fashion, $[x_2] \leq [y_2]$ implies $[x_1] \leq [y_1]$, hence $\leq$ is well-defined.

   Now, $\leq$ is reflexive, for $x \lesssim x$, hence $[x] \leq [x]$. If $[x] \leq [y]$ and $[y] \leq [z]$, then $x \lesssim y$ and $y \lesssim z$, then $x \lesssim z$ for $\lesssim$ is transitive, therefore $[x] \leq [z]$ so $\leq$ is transitive. Finally, if $[x] \leq [y]$ and $[y] \leq [x]$, then $x \lesssim y$ and $y \lesssim x$, thus $x \sim y$, hence, they are in the same equivalence class: $[x] = [y]$ and $\leq$ is antisymmetric.

   As a consequence, $\leq$ is a partial order on $X/_\sim$.

3. Let C be a set, and $\{[x] : x \in C\}$ be the blocks of a partition of C. Let $\leq$ be a partial order between the blocks of partitions. Then, define $x \lesssim y$ if and only if $[x] \leq [y]$ for all $x, y \in C$. Hence, $\lesssim$ is reflexive, for $\leq$ is reflexive, thus $[x] \leq [x]$ implies that $x \lesssim x$. $\lesssim$ is transitive, for $\leq$ is transitive, *i.e.*, if $[x] \leq [y]$ and $[y] \leq [z]$, then $[x] \leq [z]$; therefore, if $x \lesssim y$ and $y \lesssim z$, it follows that $x \lesssim z$. As a result, $\lesssim$ is a quasiorder on C.

□

Corollary ([19, Corollary 8.2.3])**.** *There is a one-to-one correspondence between quasiorders on X and partial orders on blocks of partitions of X.*

Given a quasiordered set $(X, \lesssim)$, and $C \subseteq X$, the *increasing hull* of C is the set $\uparrow C = \{x \in X : c \lesssim x \text{ for some } c \in C\}$. Likewise, the *decreasing hull* of C is defined as $\downarrow C = \{x \in X : x \lesssim c \text{ for some } c \in C\}$. Since $\lesssim$ is reflexive, then $C \subseteq \uparrow C$ and $C \subseteq \downarrow C$. When equality holds, the set C is called an *increasing set* or *upper set*, or a *decreasing set* or *lower set*, respectively [8, 19].

Definition 16 (Upper set, lower set)**.** *Let $(X, \lesssim)$ be a quasiordered set, let $C \subseteq X$. Then C is said to be an increasing set or upper set if $C = \uparrow C = \{x \in X : c \lesssim x \text{ for some } c \in C\}$.*

Likewise, C is a decreasing set or lower set if $C = \downarrow C = \{x \in X : x \lesssim c \text{ for some } c \in C\}$.

Moreover, the term *increasing hull* of a set, *i.e.*, $\uparrow C$, is also know as the *upward closure* of C, while the *decreasing hull* $\downarrow C$ is also called *downward closure* of C. For the singleton $\{x\}$, the downward closure is denoted $\downarrow x$, and the upward closure is denoted by $\uparrow x$ [11].

Proposition 12 ([19, Theorem 8.2.10])**.** *Let $(X, \lesssim)$ be a quasiordered set, and C a subset of X. Then:*

1. *$\uparrow C$ is the smallest increasing set containing C.*

2. *C is increasing if and only if $C = \bigcup\{\uparrow x : x \in C\}$*

Proof. Let C be a subset of $(X, \lesssim)$.

1. Let A be another increasing different from $\uparrow C$ such that $C \subseteq A \subseteq \uparrow C$. Then there exists $z \in \uparrow C$ but $z \notin A$ and $c \lesssim z$ for some $c \in C$. However, $A = \uparrow A$, hence $A = \{x : a \lesssim x \text{ for some } a \in A\}$, therefore any element in $\uparrow C$ actually pertains to $\uparrow A = A$. Thus $A = \uparrow C$.

2. $(\rightarrow)$ Let $C = \uparrow C$ and let $x \in C$. Then straightforward $c \lesssim x$ for some $c \in C$, as $\lesssim$ is reflexive so $x \lesssim x$. Thus $x \in \bigcup\{\uparrow x : x \in C\}$ and $C \subseteq \bigcup\{\uparrow x : x \in C\}$.

   Let $z \in \bigcup\{\uparrow x : z \in C\}$, then $z \in \uparrow C$, but $C = \uparrow C$, therefore $\bigcup\{\uparrow x : x \in C\} \subseteq C$.

   $(\leftarrow)$ If $C = \bigcup\{\uparrow x : x \in C\}$, then $C = \uparrow C$, hence C is increasing.

□

Similar arguments show that $\downarrow C$ is the smallest decreasing set containing C; and C is decreasing if and only if $C = \bigcup\{\downarrow x : x \in C\}$.

3.2.1 Quasiordered spaces as topological spaces

Several ways to endow a quasiordered space $(X, \lesssim)$ with a topology will be shown next.

Theorem 14 ([9, Proposition 4.2.11],[19, Theorem 8.3.1])**.** *Let $(X, \lesssim)$ be a quasiordered set. Then the family of all upper sets*

$$\tau_{\lesssim} := \{A \subseteq X : A = \uparrow A\}$$

*is an Alexandroff topology on X called the Alexandroff topology or the **specialization topology** of $\lesssim$.*

Proof. Let $(X, \lesssim)$ be a quasiordered set.

- Since $\lesssim$ is reflexive, then it is true that any $x \lesssim x$ for each $x \in X$, hence $x \in \uparrow X$, hence $X = \uparrow X$, thus X is an upper set. $\varnothing \subseteq X$, hence, it is (vacuously) an upper set.
- Let $\{A_i\}_{i \in I}$ be an arbitrary family of upper sets.

  ($\subseteq$) It is always true that $\bigcup_{i \in I} A_i \subseteq \uparrow \bigcup_{i \in I} A_i$, on grounds of proposition 12-1.

  ($\supseteq$) Let $x \in \uparrow \bigcup_{i \in I} A_i$, then $a \lesssim x$ for some $a \in \bigcup_{i \in I} A_i$; hence, there exists at least a $j \in I$ such that $a \in A_j$, thus $x \in \uparrow A_j = A_j$. Therefore, $x \in \bigcup_{i \in I} A_i$.

  Since $\bigcup_{i \in I} A_i = \uparrow \bigcup_{i \in I} A_i$, then $\bigcup_{i \in I} A_i$ is an upper set.
- Let $\{A_i\}_{i \in I}$ be an arbitrary family of upper sets.

  ($\subseteq$) $\bigcap_{i \in I} A_i \subseteq \uparrow \bigcap_{i \in I} A_i$, as shown in proposition 12-1.

  ($\supseteq$) Let $x \in \uparrow \bigcap_{i \in I} A_i$, then there exists $a \in \bigcap_{i \in I} A_i$, and thus $a \in A_i$ for each $i \in I$, such that $a \lesssim x$. Therefore, $x \in \uparrow A_i = A_i$ for each $i \in I$. As a result, $x \in \bigcap_{i \in I} A_i$, and so $\uparrow \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i$.

  Since $\bigcap_{i \in I} A_i = \uparrow \bigcap_{i \in I} A_i$, then $\bigcap_{i \in I} A_i$ is an upper set.

Therefore $\tau_{\lesssim}$ is an Alexandroff topology on X. □

It can be observed that a basis for the Alexandroff topology $\tau_{\lesssim}$ of a quasiordered set $(X, \lesssim)$ is the family of all the sets of the form $\uparrow x = \{y \in X : x \lesssim y\}$. Hence, the set $\uparrow x$ is the minimal open neighborhood of x.

It is important to notice that this Alexandroff topology is one of the topologies obtained from the quasiordered set $(X, \lesssim)$ yet not the only one. It is possible to define a topology, also known as the *upper topology*, via the complement of lower sets $X \setminus \downarrow F$, where $F \subseteq X$ is finite, by taking them as a basis for the open sets.

Theorem 15. *Let $(X, \lesssim)$ be a quasiordered set. Then the family:*

$$\mathcal{C} = \{X \setminus \downarrow F : \ F \subseteq X \textit{ is finite}\},$$

is the basis for a topology $\tau^u_{\lesssim}$ on X.

The topology $\tau^u_{\lesssim}$ is also called the upper topology induced by $\lesssim$.

Proof. Let us shows that $\mathcal{C}$ satisfies the properties for being a basis for a topology on X.

1. Since $\varnothing$ is finite and $\downarrow \varnothing = \varnothing$, then $X \setminus \varnothing = X \in \mathcal{C}$, hence $\mathcal{C}$ covers X.

2. Let $X\backslash\downarrow F_1$ and $X\backslash\downarrow F_2$ be two elements of $\mathcal{C}$, so F_1 and F_2 are finite. Then:

$$\begin{aligned} x \in (X\backslash\downarrow F_1) \cap (X\backslash\downarrow F_2) &\Leftrightarrow x \notin\downarrow F_1, x \notin\downarrow F_2 \\ &\Leftrightarrow x \notin (\downarrow F_1) \cup (\downarrow F_2) \\ &\Leftrightarrow x \notin\downarrow (F_1 \cup F_2) \\ &\Leftrightarrow x \in X\backslash\downarrow (F_1 \cup F_2) \end{aligned}$$

Therefore, $(X\backslash\downarrow F_1) \cap (X\backslash\downarrow F_2) = X\backslash\downarrow (F_1 \cup F_2)$. Since $F_1 \cup F_2$ is finite, then $X\backslash\downarrow (F_1 \cup F_2) \in \mathcal{C}$.

Hence, $\mathcal{C}$ is a basis for a topology on X.

□

The closure of a point x in the topological space $(X, \tau^u_{\lesssim})$ is $\downarrow x$, the smallest closed set containing the singleton; so the upper topology is the coarsest one being associated to a quasiordered space [9]. It can be seen that the upper topology is not Alexandroff in the manner the topology defined by Theorem 14 is; as illustrated by the following example.

Example 8. *Let us consider the partially ordered set* $(\mathbb{R}, =)$.

It is plain to see that every subset of $\mathbb{R}$ *is an upper set so the Alexandroff topology* $\tau_=$ *is the discrete topology.*

On the other hand, let us compute the upper $\tau^u_=$ *on* $\mathbb{R}$ *induced by* $=$ *. Given a finite subset* F *of* $\mathbb{R}$*, then* $\downarrow F = F$*, so a basis for the topology* $\tau^u_=$ *is the family:*

$$\{\mathbb{R}\backslash\downarrow F : F \text{ is finite}\} = \{\mathbb{R}\backslash F : F \text{ is finite}\}.$$

But this family is indeed a topology, i.e., the cofinite topology. However, this topology is not Alexandroff. For example, let $O_n = \mathbb{R}\backslash\{1, \ldots, n\}$ *for all* $n \in \mathbb{N}$*. It is clear that* O_n *is open in* $\tau^u_=$ *but*

$$\bigcap_{n\in\mathbb{N}} O_n = \bigcap_{n\in\mathbb{N}} \Big(\mathbb{R}\backslash\{1, \ldots, n\}\Big) = \mathbb{R}\backslash \left(\bigcup_{n\in\mathbb{N}}\{1, \ldots, n\}\right) = \mathbb{R}\backslash\mathbb{N}$$

is not open in $\tau^u_=$*. Hence the upper topology* $\tau^u_=$ *is not Alexandroff.*

There exist more topologies that can be obtained from a quasiordered space $(X, \lesssim)$. The *lower topology* is constructed in similar fashion as the upper topology by using the dual quasiorder instead. There are also the *interval topology*, the *Lawson topology*, the *Scott topology* —defined via the directed sets in a quasiordered space, *et cetera*; all of them being beyond the scope of this work. However, a question arises regarding the possibility of constructing a one-to-one correspondence between quasiordered spaces and Alexandroff spaces.

3.2.2 Topological spaces as quasiordered spaces

In the previous section, the endowing of a quasiordered set with a topology was examined. The reverse problem of constructing a quasiorder from a topological space, will be dealt with next. In order to accomplish this, the designated *specialization quasiorder* (the locution coming from algebraic geometry [12]) is introduced.

Although the following lemma is valid for any topological space, it is relevant within the present context.

Lemma 5 (Gierz et al., [8]). *Let (X, τ) be a topological space, then the following statements are equivalent:*

1. $\overline{\{x\}} \subseteq \overline{\{y\}}$
2. $x \in \overline{\{y\}}$
3. *$x \in U$ implies that $y \in U$, for all open set U.*

Proof. Let (X, τ) be a topological space.
(1 → 2) If $\overline{\{x\}} \subseteq \overline{\{y\}}$, then the conclusion follows suit since $x \in \overline{\{x\}}$ always.
(2 → 3) If $x \in \overline{\{y\}}$, then any open neighborhood U of x intersects $\{y\}$, hence $y \in U$.
(3 → 1) Let $a \in \overline{\{x\}}$, then any open neighborhood U of a intersects $\{x\}$, thus $x \in U$; however, this implies that $y \in U$, therefore $a \in \overline{\{y\}}$ and $\overline{\{x\}} \subseteq \overline{\{y\}}$. □

Theorem 16. *Let (X, τ) be a topological space, then the binary relation $\lesssim_\tau$ on X given by*

$$x \lesssim_\tau y \text{ if and only if } x \in \overline{\{y\}},$$

for all $x, y \in X$, is a quasiorder on X called the ***specialization quasiorder*** *induced by the topology τ.*

Proof.

- For each $x \in X$, it is true that $x \in \overline{\{x\}}$, therefore $x \lesssim_\tau x$ and the relation $\lesssim_\tau$ is reflexive.
- Let $x, y, z \in X$ such that $x \lesssim_\tau y$ and $y \lesssim_\tau z$. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$. As a consequence of Lemma 5, all the neighborhoods of x contain also y, so they are neighborhoods of y as well. The same occurs between y and z, all neighborhoods of y contain z, thus they are neighborhoods of z, yet that includes the neighborhoods of x. Therefore $x \in \overline{\{z\}}$, that is $x \lesssim_\tau y$. The relation $\lesssim_\tau$ is transitive.

As a consequence, $\lesssim_\tau$ is a quasiorder. □

Corollary. *If (X, τ) is a topological space, then two points $x, y \in X$ verify that $x \lesssim_\tau y$ if and only if they satisfy any of the following conditions:*

- $\overline{\{x\}} \subseteq \overline{\{y\}}$;
- *if U is a neighborhood of x then U is also a neighborhood of y.*

Proposition 13 ([9], Proposition 4.2.3). *Let (X, τ) be a topological space, and $\lesssim_\tau$ be the corresponding specialization quasiorder. Then:*

1. *X is T_0 if and only if $\lesssim_\tau$ is a partial ordering.*
2. *X is T_1 if and only if $\lesssim_\tau$ is the equality.*

Proof. If (X, τ) is a topological space and $\lesssim_\tau$ its specialization quasiorder, then:

1. ($\rightarrow$) If X is T_0, let us suppose the specialization quasiorder lacks antisymmetry, *i.e.*, there exist $x, y \in X$ such that $x \lesssim_\tau y$ and $y \lesssim_\tau x$, but $x \neq y$. Since X is T_0, then there exists a neighborhood containing x but not y, thus contradicting that $x \lesssim_\tau y$ as per Lemma 5; or a neighborhood containing y but not x, contradicting that $y \lesssim_\tau x$ on similar reasoning. Therefore, $\lesssim_\tau$ is antisymmetric, and a partial ordering.

   ($\leftarrow$) If $\lesssim_\tau$ is a partial ordering, and given $x \neq y$, then either $x \not\lesssim_\tau y$, hence there is an open set containing x but not y; or $y \not\lesssim_\tau x$, thus there is an open set containing y but not x. Therefore, X is T_0.

2. ($\rightarrow$) If X is T_1, then X is T_0 and the quasiorder $\lesssim_\tau$ is already a partial order. However, T_1 also means that, whenever $x \neq y$, there is an open set containing x but not y, *and* an open set containing y but not x; hence $x \not\lesssim_\tau y$, *and* $y \not\lesssim_\tau x$, as per Lemma 5. This means that pairs (x, y) with $x \neq y$ do not pertain to the relationship $\lesssim_\tau$ (*i.e.*, are not comparable), hence $\lesssim_\tau$ is an equality.

   ($\leftarrow$) An equality is a partial order, *i.e.*, reflexive, transitive and antisymmetric. In addition, whenever $x \neq y$, $x \not\lesssim_\tau y$ and $y \not\lesssim_\tau x$; that is $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$, hence there exists an open set containing x but not y, and an open set containing y but not x, thus making X a T_1 space.

□

Proposition 14. *Let (X, τ) be a topological space. Then $\tau \subseteq \tau_{\lesssim_\tau}$, that is, the topology τ is coarser than the Alexandroff topology induced by the specialization quasiorder of τ.*

Proof. Let O be an open set in τ. Let $x \in \uparrow O$. Then we can find $o \in O$ such that $o \lesssim_\tau x$. If $o \lesssim_\tau x$, then $o \in \overline{\{x\}}$, hence $x \in O$. Therefore, $\uparrow O \subseteq O$; and $O \subseteq \uparrow O$.

Since $\uparrow O = O$, then O is open in the Alexandroff topology $\tau_{\lesssim_\tau}$ induced by the specialization quasiorder of τ. □

Notice that from the above result we can deduce the following:

Corollary. *Let $(X, \lesssim)$ be a quasiordered space. Then the Alexandroff topology $\tau_\lesssim$ is the finest topology on X whose specialization quasiorder is $\lesssim$.*

Proof. Let τ' be a topology on X such that its specialization quasiorder is $\lesssim$, that is $\lesssim_{\tau'} = \lesssim$. By the previous result $\tau' \subseteq \tau_{\lesssim_{\tau'}} = \tau_\lesssim$ so the Alexandroff topology $\tau_\lesssim$ induced by $\lesssim$ is finer than τ'. □

3.3 The one-to-one link

We have seen that a quasiordered set yields its Alexandroff topology and that from any topology we can construct a quasiorder. At this point, it is natural to wonder if these two constructions are inverse. The answer is positive as we show next.

Theorem 17. *Let* $\mathsf{QOr}, \mathsf{ATop}$ *be the families of quasiordered spaces and Alexandroff topological spaces, respectively. Then the function* $\mathcal{A} : \mathsf{QOr} \to \mathsf{ATop}$ *given by*

$$\mathcal{A}(X, \lesssim) = (X, \tau_\lesssim)$$

is bijective and its inverse is given by

$$\mathcal{A}^{-1}(X,\tau) = (X, \lesssim_\tau).$$

Proof. By Theorem 14, we deduce that $\mathcal{A}$ is well-defined.

Suppose that $\mathcal{A}(X_1, \lesssim_1) = \mathcal{A}(X_2, \lesssim_2)$, that is, $(X_1, \tau_{\lesssim_1}) = (X_2, \tau_{\lesssim_2})$. Then $X_1 = X_2$ and $\tau_{\lesssim_1} = \tau_{\lesssim_2}$. Moreover, let $x \in X_1$. Since $x \in \{y \in X : x \lesssim_1 y\} \in \tau_{\lesssim_1}$ and $\tau_{\lesssim_1} = \tau_{\lesssim_2}$, then there exists an upper set G with respect to $\lesssim_2$ such that $x \in G \subseteq \{y \in X : x \lesssim_1 y\}$. Hence $\{y \in X : x \lesssim_2 y\} \subseteq G \subseteq \{y \in X : x \lesssim_1 y\}$. So if $x \lesssim_1 y$ then $x \lesssim_2 y$. A similar argument also shows that $x \lesssim_2 y$ implies $x \lesssim_1 y$. Consequently $\lesssim_1 = \lesssim_2$ so $\mathcal{A}$ is injective.

Furthermore, given an Alexandroff topological space (X,τ) then $(X, \lesssim_\tau)$ is a quasiordered space. Let us show that $\mathcal{A}(X, \lesssim_\tau) = (X, \tau_{\lesssim_\tau}) = (X,\tau)$. By Proposition 14, we know that $\tau \subseteq \tau_{\lesssim_\tau}$. For the converse inclusion, let O be an open set in $\tau_{\lesssim_\tau}$. Then O is an upper set in $(X, \lesssim_\tau)$. Therefore

$$O = \bigcup_{o\in O} \{o\} = \bigcup_{o \in O} \uparrow \{o\} = \bigcup_{o\in O} \left(\bigcap_{o \in G \in \tau} G \right).$$

Since τ is Alexandroff, then $\bigcap_{o\in G\in\tau} G$ is open in τ so O is also open in τ. This proves that $\tau = \tau_{\lesssim_\tau}$.

Therefore, $\mathcal{A}$ is surjective and thus bijective. □

This review of the important connection between Alexandroff topologies and quasiorders can not be complete without mentioning that the dual quasiorder $\gtrsim$ is in correspondence with the opposite Alexandroff topology [19].

4 Primal topology

An interesting typology of Alexandroff spaces are the denominated primal spaces, first introduced under the name of *functional Alexandroff spaces* in 2011, in an article by Shirazi and Golestani [21]. The designation *primal space* was introduced a year later, independently, by Echi in [6].

Proposition 15. *Given a nonempty set X and a function $f : X \to X$, define the family τ_f of subsets of X as:*

$$\tau_f = \{C \subseteq X : f^{-1}(C) \subseteq C\}.$$

Then τ_f is an Alexandroff topology on X, also called functional Alexandroff topology or primal topology induced by f.

Proof. Let $X \neq \varnothing$ and $f : X \to X$; let $\tau_f = \{C \subseteq X : f^{-1}(C) \subseteq C\}$. Then:

- $f^{-1}(\varnothing) = \varnothing$, thus $\varnothing \in \tau_f$
- $f^{-1}(X) = X$, so $X \in \tau_f$
- If the collection $\{C_i\}_{i\in I} \in \tau_f$, then $f^{-1}(\bigcup C_i) = \bigcup f^{-1}(C_i) \subseteq \bigcup C_i$; therefore $\bigcup C_i \in \tau_f$
- If $\{C_i\}_{i\in I} \in \tau_f$, then $f^{-1}(\bigcap C_i) = \bigcap f^{-1}(C_i) \subseteq \bigcap C_i$, consequently $\bigcap C_i \in \tau_f$.

Therefore, τ_f is a topology on X. □

Since the property $f^{-1}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f^{-1}(A_i)$ holds for arbitrary functions and indexing sets, then it follows that primal spaces are indeed Alexandroff spaces [21]. The symbol τ_f will henceforth denote the primal topology induced by the map f and we will say that (X, τ_f) is a primal space.

In the following, the expression f^n will denote the iterative composition $f^n = f \circ f \circ ... \circ f$ (n times), for $n \in \mathbb{N} \cup \{0\}$, taking f^0 to represent the identity map [6]. Likewise, f^{-n} denotes the iterative application of the preimage f^{-1}.

Example 9. *Given a nonempty set X, and the identity map $i_d : X \to X$, then each $x \in X$ is a fixed point, i.e., $i_d(x) = x$. Therefore, singletons are open sets in τ_{i_d} for $i_d^{-1}(\{x\}) \subseteq \{x\}$, and τ_{i_d} corresponds to the discrete topology over X.*

An interesting type of primal spaces is given in [15], classified as linear operators on $\mathbb{R}^n$ as illustrated by the following example.

Example 10. *Let $X = \mathbb{R}^2$, $\theta \in \mathbb{R}$ and let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of rotation given by:*

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Take for instance $\theta = \frac{\pi}{3}$. R_θ *has only one fixed point, namely the zero vector:* $R_\theta(\vec{0}) = \vec{0}$*; hence* $R_\theta^{-1}(\vec{0}) \subseteq \{\vec{0}\}$*, thus* $\{\vec{0}\}$ *is an open set in the topology* τ_{R_θ}*. No other single vector does hold this property. However, given a vector* $v \in \mathbb{R}^2$*, it is possible to construct open sets by reiterative union of its preimages:*

$$C = \bigcup \{R_\theta^{-n}(v) : n \geq 0\}$$

C *is an open set in the topology* τ_{R_θ}*. In this example, the function satisfies the property* $R_\theta \circ R_\theta = R_{2\theta}$*, thus with* $\theta = \frac{\pi}{3}$*, the set* $\{R_\theta^n(v) : n \geq 0\}$ *has actually 6 elements:*

$$\{v, R_\theta^1(v), R_\theta^2(v), R_\theta^3(v), R_\theta^4(v), R_\theta^5(v)\},$$

for every nonzero vector in the space. It can be shown that each nonzero element in $\mathbb{R}^2$ *has an open set containing it, with the above specified form.*

Example 11. *Given* $X = \mathbb{R}$*, and the function* $f : \mathbb{R} \to \mathbb{R}$ *defined as* $f(x) = 1$*, for all* $x \in \mathbb{R}$*; yields the trivial topology on* $\mathbb{R}$*. For any set* $A \subseteq \mathbb{R}$*:*

$$f^{-1}(A) = \begin{cases} \mathbb{R}, & A \supseteq \{1\} \\ \varnothing, & A \not\supseteq \{1\} \end{cases}$$

Therefore, it is only true $f^{-1}(\mathbb{R}) \subseteq \mathbb{R}$ *and* $f^{-1}(A) \subseteq \varnothing$*, whenever* $1 \notin A$*. Hence,*

$$\tau_f = \{A \subseteq \mathbb{R} : A \not\supseteq \{1\}\} \cup \{\mathbb{R}\},$$

i.e., the excluded point topology.

4.1 Notable classes of sets

As with any Alexandroff space, the minimal open neighborhood is the most distinctive set in a primal space. However, other equally relevant ones are the closed sets, characterized by their invariance; and the orbits, appearing after repeated iterations of the map inducing the topology.

4.1.1 Minimal neighborhoods

Let (X, τ_f) be a primal space, the minimal neighborhood of an element $x \in X$, is the intersection of all open sets containing x. If $\mathcal{N}_x$ denotes this minimal neighborhood, then the following properties are true [21]:

Proposition 16. *Given a primal space* (X, τ_f) *then:*

1. *The minimal open neighborhood of a point* $x \in X$*, is* $\mathcal{N}_x = \bigcup \{f^{-n}(x) : n \geq 0\}$
2. *Given a subset* $A \subseteq X$*, the smallest open set containing it is* $\mathcal{N}_A = \bigcup \{\mathcal{N}_x : x \in A\}$*.*

Proof. 1. Let us see that $\mathcal{N}_x$ is open. Let $y \in f^{-1}(\mathcal{N}_x)$, thus $f(y) \in \mathcal{N}_x$, *i.e.*, there exists $n \geq 0$ such that $f^n(f(y)) = x$. Thus, $f^{n+1}(y) = x$ and $y \in \mathcal{N}_x$, meaning that $f^{-1}(\mathcal{N}_x) \subseteq \mathcal{N}_x$, hence $\mathcal{N}_x$ is open.

Now, let us see $\mathcal{N}_x$ is minimal in the inclusion order. Let U be another open set containing x, so $f^{-1}(U) \subseteq U$, for it is open. We will proceed by mathematical induction: First, $x \in U$, so $f^{-1}(x) \subseteq f^{-1}(U) \subseteq U$. Let $k \in \mathbb{N}$ such that $f^{-k}(x) \in U$; therefore $f^{-1}(f^{-k}(x)) \subseteq f^{-1}(U)$; that is $f^{-(k+1)}(x) \subseteq f^{-1}(U) \subseteq U$; implying that the set $\bigcup\{f^{-n}(x) : n \geq 0\} \subseteq U$, thus $\mathcal{N}_x$ is minimal.

2.- Since $\forall x \in A$, the set $\mathcal{N}_x$ is open, then the set $\mathcal{N}_A$ is open, for it is a union of open sets. Let $\mathcal{V}$ be another open set containing A. Then, for every $x \in A$, the minimal neighborhood $\mathcal{N}_x \subseteq \mathcal{V}$, hence the union $\bigcup\{\mathcal{N}_x : x \in A\}$ is contained in $\mathcal{V}$, then $\mathcal{N}_A$ is minimal. ☐

Corollary. *In a straightforward result, $a \in \mathcal{N}_x$ if and only if there exists $n \geq 0$ such that $f^n(a) = x$.*

Correspondingly with Alexandroff spaces, the collection $\{\mathcal{N}_x : x \in X\}$ is a basis for the primal topology. The following property was first presented by Shirazi and Golestani as a trichotomy, *i.e.*, $\mathcal{N}_x \cap \mathcal{N}_y = \varnothing$ or $\mathcal{N}_x \subseteq \mathcal{N}_y$ or $\mathcal{N}_y \subseteq \mathcal{N}_x$ [21]:

Theorem 18. *Let (X, τ_f) be a primal space, and $x, y \in X$. If $\mathcal{N}_x \cap \mathcal{N}_y \neq \varnothing$, then $\mathcal{N}_x \subseteq \mathcal{N}_y$ or $\mathcal{N}_y \subseteq \mathcal{N}_x$.*

Proof. Let a be in $\mathcal{N}_x \cap \mathcal{N}_y$, then there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $f^n(a) = x$ and $f^m(a) = y$. If $m \geq n$, then $f^{m-n}(x) = y$ and $x \in \mathcal{N}_y$; conversely, if $n \geq m$, then $f^{n-m}(y) = x$ and $y \in \mathcal{N}_x$. Hence $\mathcal{N}_x \subseteq \mathcal{N}_y$ or $\mathcal{N}_y \subseteq \mathcal{N}_x$. ☐

4.1.2 Invariant sets

Given a space X, and a function $f : X \to X$, a set $C \subseteq X$ is said to be invariant under f, or f-invariant, if and only if $f(C) \subseteq C$. Closed sets are characterized by being invariant in primal spaces, as was first shown by Echi in [6].

Theorem 19 ([6, Proposition 1.2])**.** *Let (X, τ_f) be a primal space, then it follows that:*

1. *$C \subseteq X$ is closed if and only if C is invariant.*
2. *The closure of a point $x \in X$ is given by $\bar{x} = \{f^n(x) : n \geq 0\}$.*

Proof. Let (X, τ_f) be a primal space, then:

1. $(\rightarrow)$ If C is closed, then $X \backslash C$ is open and thus $f^{-1}(X \backslash C) \subseteq X \backslash C$; however $f^{-1}(X \backslash C) = X \backslash f^{-1}(C) \subseteq X \backslash C$, therefore $C \subseteq f^{-1}(C)$, hence $f(C) \subseteq C$ and C is invariant.

$(\leftarrow)$ If C is invariant, then $f(C) \subseteq C$, thus $X \backslash C \subseteq X \backslash f(C)$. Therefore, $f^{-1}(X \backslash C) \subseteq f^{-1}(X \backslash f(C)) = X \backslash f^{-1}(C) \subseteq X \backslash C$. Consequently, $X \backslash C$ is open, and C closed.

2. It is evident that $\{f^n(x) : n \geq 0\}$ is invariant under f, hence being a closed set containing x, and so $\bar{x} \subseteq \{f^n(x) : n \geq 0\}$. On the other hand, let $y \in \{f^n(x) : n \geq 0\}$, thus there exists $m \geq 0$ so that $f^m(x) = y$. Let $\mathcal{N}_y$ the minimal open set containing y, then $x \in f^{-m}(y) \subseteq \mathcal{N}_y$, which implies that $\mathcal{N}_y \cap \bar{x} \neq \varnothing$ and $y \in \bar{x}$. Since this holds for every $y \in \{f^n(x) : n \geq 0\}$, therefore $\{f^n(x) : n \geq 0\} \subseteq \bar{x}$. Consequently, $\{f^n(x) : n \geq 0\} = \bar{x}$. ☐

4.1.3 Orbits

Given a function $f : X \to X$, the orbit of a point $x \in X$ is defined as the set $\Omega_f(x) = \{x, f(x), f^2(x), ...\}$. Furthermore, a point $x \in X$ is defined periodic if and only if $f^m(x) = x$ for some integer $m \geq 1$. The smallest m holding this property is named the period of x. A periodic point thus has a finite orbit.

Following Theorem 19, the closure of a point is exactly its orbit, $\bar{x} = \Omega_f(x)$. On top of that, while orbits are always countable, periodic orbits are finite. Moreover, Echi proved in [6] that, in a primal space (X, τ_f), finite orbits are exactly the non-empty, closed f-invariant sets which are minimal in the inclusion order.

Proposition 17 ([6, Lemma 1.4])**.** *Let (X, τ_f) be a primal space, then the orbits of its periodic points are the minimal non-empty closed f-invariant sets.*

Proof. Let M be a non-empty closed f-invariant set, minimal in the inclusion order. For every $x \in X$, the orbit $\Omega_f(x)$ is a closed, f-invariant set. Hence, if $x \in M$, then $M = \Omega_f(x)$. Likewise, it can be argued that $M = \Omega_f f(x)$. As a consequence, x is in the orbit of $f(x)$ indicating that x is periodic.

Reciprocally, if x is periodic, then $\Omega_f(x) = \overline{x}$, which is closed and invariant under f. Since the closure of x is the minimal closed set containing x, then $\Omega_f(x)$ is a minimal non-empty f-invariant closed set. □

4.2 Separation

As a special case of Alexandroff spaces, primal spaces are also restricted to T_1 in terms of separation axioms. As a matter of fact, the only T_1-Alexandroff space, namely the discrete space, has a primal topology induced by the identity map $f(x) = x$ as shown in [21] (see Example 9).

Following Proposition 17 and Lemma 2, it can be readily observed that in a primal space, all the points in the same periodic orbit have the same minimal open neighborhood; *i.e.*, if x is a periodic element, then $\mathcal{N}_x = \mathcal{N}_y$ for all $y \in \Omega_f(x)$. The next theorem show the conditions for a primal space to be T_0 [21].

Theorem 20 ([21, Lemma 2.6])**.** *A primal space (X, τ_f) is T_0 if and only if the only periodic points of f are its fixed points.*

Proof. ($\to$) Let X be T_0. Let us proceed by *Reductio ad absurdum*: Let $x \in X$ be a periodic point but not a fixed point. Then $f(x) \neq x$, but $\bar{x} = \overline{f(x)}$. By Lemma 2, then $\mathcal{N}_x = \mathcal{N}_{f(x)}$. However, X is T_0, therefore this implies that $f(x) = x$, *contradictio* $\Rightarrow\Leftarrow$.

($\leftarrow$) Suppose that every periodic points for f is a fixed point. Let $x, y \in X$ such that $\mathcal{N}_x = \mathcal{N}_y$, then $\bar{x} = \bar{y}$, hence x and y are periodic. By assumption, x, y are fixed points so $x = y$. Hence X is T_0. □

4.2.1 R_0 and weakly-R_0 primal spaces

The notions relative to R_0 are now reviewed within the framework of primal spaces.

Theorem 21. *A primal space (X, τ_f) is R_0 if and only if every point is periodic.*

Proof. ($\rightarrow$) Suppose that (X, τ_f) is R_0 and let $x \in X$. If x is a fixed point, then it is periodic. If x is not a fixed point, then $x \neq f(x)$. If $x \notin \overline{f(x)}$, then $\bar{x} \neq \overline{f(x)}$; hence $\bar{x} \cap \overline{f(x)} = \varnothing$ which is not possible in a primal space. Therefore $x \in \overline{f(x)}$. Consequently, there exists $k > 0$ so that $f^{k+1}(x) = x$ and x is periodic.

($\leftarrow$) If every point is periodic in a primal space (X, τ_f), then for a point $a \in X$, all the elements in the orbit $\Omega_f(a)$ are non distinguishable. Let $a, b \in X$ be distinguishable points, then they are not in the same orbit. However, every point is periodic, then $\Omega_f(a)$ and $\Omega_f(b)$ are disjoint, therefore, the sets $\{f^{-n}(a) : n \geq 0\}$ and $\{f^{-n}(b) : n \geq 0\}$ are disjoint, for f is a function. Therefore a and b are separated, hence X is R_0. □

Proposition 18. *A primal space (X, τ_f) is T_1 if and only if every point is a fixed point, equivalently, τ_f is the discrete topology.*

Proof. Let (X, τ_f) be a primal space. A point being fixed is equivalent to being periodic with period 1, that is $f^1(x) = x$. Hence, every point is periodic is equivalent to (X, τ_f) being R_0, according to Theorem 21. Moreover, that all the points are fixed is equivalent to (X, τ_f) being T_0. Therefore, it is equivalent to (X, τ_f) being T_0 and R_0, which is equivalent to $T1$. Hence, (X, τ_f) has the discrete topology. □

An associated concept, introduced by Di Maio [5], is that of weakly R_0 spaces, denoted by $w - R_0$.

Definition 17 (Weakly R_0 spaces)**.** *A topological space, X, is said to be weakly R_0, or $w - R_0$, if the intersection of the closures of all its points is empty:*

$$\bigcap \{\bar{x} : x \in X\} = \varnothing.$$

The following result characterizing the weakly R_0 primal spaces was first provided in [23].

Theorem 22 ([23, Theorem 3.1])**.** *A primal space (X, τ_f) is **not** $w - R_0$ if and only if it has exactly one periodic orbit Ω, and $\forall x \in X \backslash \Omega$ there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Omega$.*

Proof. ($\leftarrow$) Let (X, τ_f) be a primal space, with exactly one periodic orbit Ω, and for every $x \in X \backslash \Omega$ there exists $n \in \mathbb{N}$ such that $f^n(x) \in \Omega$; then $\Omega \subseteq \bar{x}$ for every $x \in X \backslash \Omega$, therefore:

$$\bigcap_{x \in X} \bar{x} = \Omega \neq \varnothing,$$

hence X is not $w - R_0$.

($\rightarrow$) If (X, τ_f) is not $w - R_0$, then $\Omega = \bigcap_{x \in X} \bar{x} \neq \varnothing$. As an intersection of closures of every point, Ω is a minimal closed set in the inclusion order, hence it is a periodic orbit, following proposition 17. Ω is the only periodic orbit, for any other periodic orbit Ψ would be disjoint, and then $\Omega \cap \Psi = \varnothing$ and X would be $w - R_0 \Rightarrow\Leftarrow$. Moreover, for any $x \in X \backslash \Omega$, it follows that $\Omega \subseteq \bar{x}$; hence for any $a \in \Omega \subseteq \bar{x}$ and there exists $n \in \mathbb{N}$ such that $f^n(x) = a$, thus $f^n(x) \in \Omega$. □

4.3 Compactness

Given a function $f : X \to X$, the set of all periodic points of f is denoted by $Pe(f)$. The compactness of the primal space induced by f on X is characterized by $Pe(f)$. Let us observe that the minimal neighborhood of a point in a primal space is always compact, for any open cover has such a minimal neighborhood as a finite subcover. The following result, due to Echi and Turki [7], shows a characterization of compact primal spaces.

Theorem 23 ([7, Theorem 2.2])**.** *Let (X, τ_f) be a primal space. Then, the following statements are equivalent:*

1. *X is compact*

2. *$Pe(f)$ is finite, and $\forall x \in X$ there exists $n \geq 0$ such that $f^n(x) \in Pe(f)$*

Proof. $(1 \to 2)$ If X is compact, then there exists a finite set $\{a_1, a_2, ...a_k\}$ such that $X = \bigcup \mathcal{N}a_i$. Without loss of generality, suppose that $\{\mathcal{N}a_1, \mathcal{N}a_2, ...\mathcal{N}a_k\}$ are not comparable with the inclusion order, hence $\mathcal{N}a_i \cap \mathcal{N}a_j = \varnothing$ for all $i \neq j$. For each i, $f(a_i) \in \mathcal{N}a_j$, for some $j = 1, 2, ...k$. However, $a_i \in \mathcal{N}_{f(a_i)}$, therefore $\mathcal{N}a_i \subseteq \mathcal{N}a_j$, hence $i = j$, for the minimal neighborhoods were assumed disjoint for $i \neq j$. Consequently, $f(a_i) \in \mathcal{N}a_i$, indicating that a_i is periodic and $Pe(f) \supseteq \bigcup_{i=1}^{k}\{f^n(a_i) : n \geq 0\}$.

Let $b \in Pe(f)$, then $b \in \mathcal{N}a_i$ for some $i = 1, 2, ...k$. Let T be the period of b, *i.e.*, $f^T(b) = b$. Since $b \in \mathcal{N}a_i$, then $f^n(b) = a_i$ for some $n \leq T$, hence $f^{T-n}(a_i) = b$, *i.e.*, the periodic points are in finite orbits, thus $Pe(f) \subseteq \bigcup_{i=1}^{k}\{f^n(a_i) : n \geq 0\}$. Therefore, $Pe(f) = \bigcup_{i=1}^{k}\{f^n(a_i) : n \geq 0\}$. Since each a_i is periodic, then its respective orbit is finite; hence $Pe(f)$ is finite, for it is a finite union of finite sets.

Let $x \in X$; if x is periodic, then obviously $f^n(x) \in Pe(f)$, for any $n \geq 0$. If x is not periodic, it is still being covered by some $\mathcal{N}_{a_i}$, for X is compact. Suppose that $x \in \mathcal{N}_{a_j}$, then, as a direct application of the corollary to Proposition 16, there exists $n \in \mathbb{N}$ such that $f^n(x) = a_j$, thus $f^n(x) \in Pe(f)$.

$(2 \to 1)$ If $Pe(f)$ is finite and for all $x \in X$ there is some $n \geq 0$ such that $f^n(x) \in Pe(f)$, then $X = \bigcup\{\mathcal{N}_a : a \in Pe(f)\}$. Since $\mathcal{N}_a$ is compact, then X is compact, for it is a finite union of compact sets. ☐

4.4 Connectedness

Since primal spaces are Alexandroff, then their basic properties regarding minimal open neighborhoods follow suit, namely, minimal neighborhoods are connected, as per Lemma 3; primal spaces are locally connected and, consequently, connected components are clopen, as first shown by Shirazi and Golestani in [21].

Proposition 19. *Orbits are connected in primal spaces.*

Proof. In any topological space, points are connected. If X is a primal space, any singleton $\{x\}$ is connected. Then, its closure $\bar{x}$ is connected, being $\bar{x}$ the orbit of x. ☐

Proposition 20 ([21, Theorem 2.1.(8)])**.** *Let (X, τ_f) be a primal space, and $x, y \in X$. Then, x and y are in the same connected component if and only if there exist $n, m \in \mathbb{N}$ such that $f^n(x) = f^m(y)$.*

Proof. ($\leftarrow$) It can be argued in this sense that, if $c \in X$ is such that there exist $n, m \in \mathbb{N}$ so that $f^n(x) = c = f^m(y)$, then $\mathcal{N}_c$ is the smallest connected neighborhood containing x, y and c.

($\rightarrow$)Let $a \in X$ and K be the connected component of a. Let us remark that $\forall c \in K$, $\mathcal{N}_c \subseteq K$ for the minimal neighborhoods are connected and components are maximal, in the inclusion order. However, not every $\mathcal{N}_c$ would contain a. Let $V = \bigcup\{\mathcal{N}_c : a \in \mathcal{N}_c,\ c \in K\}$, *i.e.*, the union of all minimal neighborhoods in K containing a. Then V is open, for it is a union of open sets. Let us show that V is closed. Let $b \in K \backslash V$ so that $b \in \bar{V}$, that is $\mathcal{N}_b \cap V \neq \varnothing$, thus there exists $x \in \mathcal{N}_b \cap V$ and $\exists c \in V$ with $\mathcal{N}_c \supseteq \{x\}$. Then $\mathcal{N}_b \cap \mathcal{N}_c \neq \varnothing$ implies $\mathcal{N}_b \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{N}_b$, by virtue of Theorem 18. If $\mathcal{N}_b \subseteq \mathcal{N}_c$, then $b \in V$, *contradictio*. If $\mathcal{N}_c \subseteq \mathcal{N}_b$, then $a \in \mathcal{N}_b$ and $b \in V$, *contradictio* $\Rightarrow\Leftarrow$. Therefore, $\mathcal{N}_b \cap V = \varnothing$, $b \in int\ K \backslash V$, hence V is closed. Since K is connected, then V being clopen implies $V = K$.

Therefore, given any $x, y \in X$ within the same connected component K, there always exists $c \in K$ such that $x, y \in \mathcal{N}_c \subseteq K$, *i.e.*, $\exists m, n \in \mathbb{N}$ such that $f^m(x) = c = f^n(y)$. □

Theorem 24 ([10, Theorem 3.2])**.** *Let (X, τ_f) be a connected primal space. If there exists $b \in X$ such that $\bar{b}$ is finite, then the set $\bar{x}$ is finite, $\forall x \in X$.*

Proof. If $\bar{b}$ is finite, then there exists a point $p = f^k(b)$, such that p is periodic, $p = f^k(p)$. Since X is connected and by proposition 20, then for all $x \in X$ there exist m, n such that $f^n(x) = f^m(p)$ and this implies that $\bar{x}$ is finite. □

It can be shown that two points form a connected set when they are in the same minimal neighborhood, due to the configuration of the open sets in primal spaces. This is better examined via path connectedness, as shown by the next two statements by Guale *et al.*, [10].

Proposition 21 ([10, Lemma 4.1])**.** *Let (X, τ_f) be a primal space, with $a, b \in X$ such that $a \in \mathcal{N}_b$. Then, there exists a continuous function $g : [0, 1] \to X$, namely, a path, such that $g(0) = a$ and $g(1) = b$.*

Proof. This can be proved by considering the function $g : [0, 1] \to X$ with:

$$g(t) = \begin{cases} a & ; for\ t \in [0, 1) \\ b & ; for\ t = 1 \end{cases} .$$

Then, with the minimal neighborhoods $\mathcal{N}_a$ y $\mathcal{N}_b$ it can be shown that $g^{-1}(\mathcal{N}_a) = [0, 1)$, $g^{-1}(\mathcal{N}_b) = [0, 1]$, and g is continuous. □

Theorem 25. *Every connected primal space is path-connected.*

Proof. Let (X, τ_f) be a connected primal space. Let $a, b \in X$; then there exists $q \in X$ such that $f^n(a) = q = f^m(b)$ for some $m, n \in \mathbb{N}$, due to the connectedness of X. Since $q \in \bar{a} \cap \bar{b}$, then it is possible to construct a path g connecting a with q, then a second path h from b to q; thus finally concatenate g with h^{-1} to form a path from a to b. □

5 Case in point: The Collatz Conjecture

The Collatz conjecture, also noted as the 3n+1 problem, the Ulam problem, the Hasse's algorithm, the Kakutani problem, the Thwaites conjecture, or the Syracuse problem; is a notorious unsolved problem in mathematics. The conjecture presents a simple yet delusively difficult issue: take any positive integer n and, if it is even, divide it by 2; if odd, multiply it by 3 and add 1. By iterating this process with the resulting numbers, the sequence eventually arrives at 1. The Collatz conjecture persists as an open problem, despite being proposed over 70 years ago [3]. However, it has attracted the attention of countless mathematicians and enthusiasts the same, becoming a fascinating topic of study.

Let us start defining the function $C : \mathbb{N} \to \mathbb{N}$, the Collatz conjecture is based upon:

$$C(n) = \begin{cases} n/2 & ; for\ n\ even \\ 3n+1 & ; for\ n\ odd \end{cases},$$

where the symbol $\mathbb{N}$ stands for the set $\mathbb{N} = \{1, 2, 3, ...\}$. Henceforth, the primal topology induced by the Collatz function is denoted by τ_C.

In Figure 5.1, a selection of orbits after iterating the Collatz function is presented. For instance, the number 5 is odd, then it maps into 16, *i.e.*, $3 \times 5 + 1$. However, 16 is even, hence it maps into 8, *i.e.*, $(16/2)$; and so on. The presented orbits eventually reach the final cycle $4 - 2 - 1 - 4$, and the question is to find out whether this is always the case or not.

Starting in 2019, several papers appeared tackling this problem from a topological perspective, by relating it to the primal topology [23]. In Figure 5.2, left panel, the graphic shows a representation of the minimal open neighborhoods of two given points $x = 3$ and $y = 32$. According to the theory, $\mathcal{N}_3 = \bigcup\{f^{-n}(3) : n \geq 0\}$ and $\mathcal{N}_{32} = \bigcup\{f^{-n}(32) : n \geq 0\}$, thus the graphic representation is not complete by any means. Likewise, in the right panel, it is shown the closure of two points, $x = 6$ and $y = 512$, that it: $\bar{6} = \bigcup\{f^{n}(6) : n \geq 0\} = \{6, 3, 10, 5, 16, 8, 4, 2, 1\}$ and so on.

The following theorem was first proposed by Vielma *et al.*, in 2019 [23]; later expanded by Guale *et al.* [10], and Mejías *et al.* [14]; and presents a list of equivalences such that, proving one of them would prove the Collatz conjecture to be true.

Theorem 26 (Vielma [23], Guale [10] and Mejías [14])**.** *Let $C : \mathbb{N} \to \mathbb{N}$ be the Collatz function, and $(\mathbb{N}, \tau_C)$ the primal space induced by C. Therefore, the following statements are equivalent:*

1. *The Collatz conjecture is true;*
2. $(\mathbb{N}, \tau_C)$ *is not* $w{-}R_0$*;*
3. $(\mathbb{N}, \tau_C)$ *is compact with a single periodic orbit;*
4. $(\mathbb{N}, \tau_C)$ *is connected;*

5. $(\mathbb{N}, \tau_C)$ *is path-connected.*

Proof. Let $C : \mathbb{N} \to \mathbb{N}$ be the Collatz function, and $(\mathbb{N}, \tau_C)$ the primal space induced by C.

- $(1 \Rightarrow 2)$ If the Collatz conjecture is true, then, for all $n \in \mathbb{N}$ there exists $k \geq 0$ such that $f^k(n) = 1$, and the primal space has only one periodic orbit, namely, $\Omega_C(1) = \{1, 4, 2\}$; by the theorem 22, the primal space $(\mathbb{N}, \tau_C)$ is not R_0.
- $(2 \Rightarrow 3)$ If $(\mathbb{N}, \tau_C)$ is not $w - R_0$, by the theorem 22, there exists a finite orbit $\Omega_C(1) = \{1, 4, 2\}$ such that $\forall n \in \mathbb{N}$ there is $k \geq 0$ with $f^k(n) = 1$; hence, under the theorem 23, $(\mathbb{N}, \tau_C)$ is compact and its only periodic orbit is $\Omega_C(1)$.
- $(3 \Rightarrow 4)$ If $(\mathbb{N}, \tau_C)$ is compact and with single periodic orbit, then that orbit must be the one already known: $\Omega_C(1) = \{1, 4, 2\}$. Given $a, b \in \mathbb{N}$, there exist $n, m \geq 0$ such that $C^n(a) = 1 = C^m(b)$. Following proposition 20, we have that a and b are in the same connected component. Since this is valid for any $a, b \in \mathbb{N}$, then $(\mathbb{N}, \tau_C)$ is connected.
- $(4 \Rightarrow 5)$ If $(\mathbb{N}, \tau_C)$ is connected, then it is path-connected, by virtue of theorem 25.
- $(5 \Rightarrow 1)$ If $(\mathbb{N}, \tau_C)$ is path-connected, then it is connected. Since $\bar{1}$ is finite and by theorem 24, then we have that the point closure $\bar{x}$ is finite for every $x \in \mathbb{N}$. Since they all are in the same connected component as 1 and $\bar{1} = \Omega_C(1)$, then $\forall x \in \mathbb{N}$ there exist $k, l \geq 0$ such that $C^k(x) = C^l(1)$. However, $\Omega_C(1) = \{1, 4, 2\}$ is finite, therefore $\forall x \in \mathbb{N}$ there exists $m \geq 0$ such that $C^m(x) = 1$ and the Collatz conjecture is true.

□

The presented theorem does not solve the Collatz conjecture, yet gives a novel approach enabling further investigation.

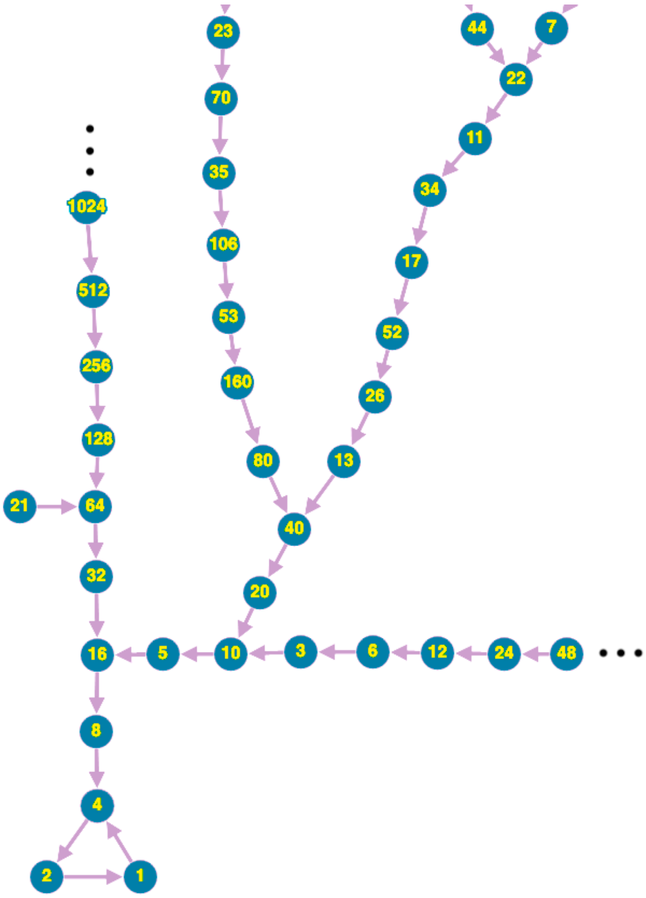

Figure 5.1: A tree-shaped schematic representation of selected orbits from Collatz function.

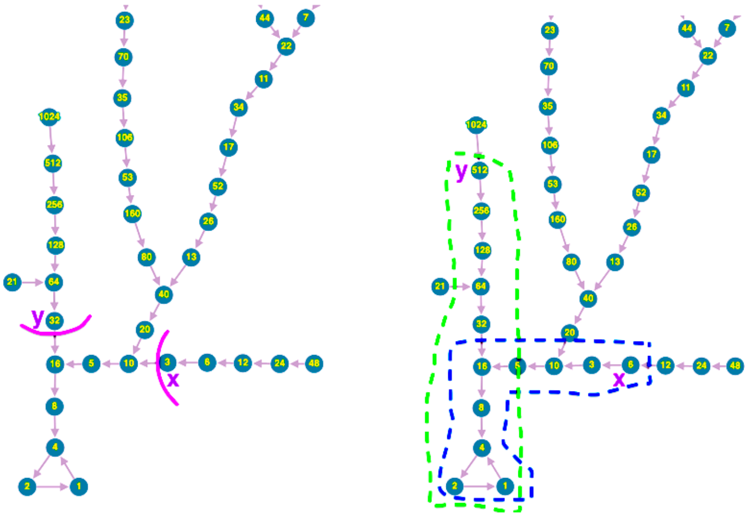

Figure 5.2: Schematic representation of minimal open neighborhoods (left panel–under no circumstance should be considered complete), and closures of points (right panel)

6 Concluding remarks

The present work has explored the landscape of Alexandroff spaces, delving into their attributes and interrelations.

In Chapter 2, their defining characteristics, namely, the closure under arbitrary intersections of open sets, and the existence of minimal neighborhoods were introduced, together with their interplay with characteristics such as separation, connectedness, compactness and countability.

Examples of construction of Alexandroff spaces were given, by using a nested family of basis sets (Example 4), and another with basis partitioning the space (Example 5).

Lemma 1 is very useful for proofs found in the related literature, yet it is rarely made explicit as in this work.

Regarding the separation axioms, the focus is on low level axioms, T_1 or lesser; for any T_1-Alexandroff space has already the discrete topology.

Some authors put emphasis on Alexandroff spaces being T_0 [2]; however several propositions and theorems had this condition removed, for it was not required during the proving process.

With respect to connectedness, it turned up that Alexandroff spaces are locally connected since their minimal open neighborhoods are connected.

Chapter 3 offers a perspective about quasiordered spaces and topological spaces: how a topology can produce a quasiorder relation, and how a quasiorder can give rise to a family of topologies. However, only the Alexandroff topologies are related to quasiordered spaces through a bijective correspondence.

Functional Alexandroff spaces, also known as primal spaces, are presented in Chapter 4. The distinct classes of sets and their interrelations are shown within the framework of this particular case. Finally, some attention is turned up on the Collatz Conjecture, under the light of primal spaces. A list of propositions are shown to be equivalent to the conjecture, so that proving one of them would solve this prominent open problem. Further investigation is required along this direction.

This work thus contributes to the comprehensive understanding of Alexandroff spaces but also underscores their relevance across various mathematical domains.

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