The Envelopes of Consonant Intervals and Chords in Just Intonation and Equal Temperament

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Abstract. Musical consonances are most times assumed to be associated with the absence of beats or roughness, which can only be achieved, in a strict sense, in just intonation. In this paper, signals representing consonant intervals and chords are analyzed, both in just intonation and in an arbitrary slight deviation from it. Analytical approximate formulas for their envelopes are obtained and then applied to the particular case of equal temperament. It was found that, in just intonation, the envelopes are flat but, in the other cases, the envelopes have a ripple which corresponds to beats or roughness, thus indicating a loss of consonance. Both the amplitudes and periodicities of the ripples are obtained for all types of consonances.

Keywords: consonance · beat · roughness · envelope · just intonation · equal temperament.

1 Introduction

Searching for the basis of musical consonances has grabbed the attention of many musicians, musical instrument manufactures, and all kind of scientists for centuries. . . and is still under discussion. Relevant answers to this phenomenon have been given from different areas of knowledge, such as: mathematics, physics, biology, psychology or sociology [1–8]. Apart from the theoretical models, also experimental results, based on tests and surveys, have been obtained. Unfortunately, most of the studies have been carried out on intervals and not on chords.

In spite of the difficulty to give an answer covering every point of view, it is generally accepted that there are 6 basic types of consonances: 4 related to the intervals and 2 to the chords. In the first case, they are the octave, perfect fifth, major and minor thirds (the unison is not here considered, because it is a trivial consonance); and, in the second case, the major and minor chords (which are, in fact, combinations of consonant intervals). As well, the inversions both of the intervals and chords are considered consonant, too.

This study has been carried out for pure tones, that is, simple sinusoids, in contrast to complex tones, which are formed by a series of harmonics that affect the sensation of consonance differently. Then, in our case, all types of consonances will be combinations of (pure) tones whose frequencies are small multiples of a fundamental one, that is, tones belonging to a harmonic series, particularly the harmonics 2 (octave), 3 (perfect fifth) and 5 (major third). But, additionally, there is another important characteristic of the consonances that is pointed out in many studies: the absence of beats or roughness, which are rhythmical fluctuations of the amplitude as a result of constructive and destructive interferences.

In parallel to those discussions there is another ancient problem: the tuning of musical instruments. Thus, in [9], more than 180 systems are described, which fortunately can be reduced to about 20 basic ones. As well, [10] is a good reference on this matter in Spanish. Among those systems, the Pythagorean tuning perfectly matches the harmonics 2 and 3, while the just intonation also matches the harmonic 5. Other systems, called temperaments, slightly deviate from those harmonics in order to meet other requirements. For example, the meantone temperament reduces the Pythagorean fifths in a quarter of syntonic comma in order to match the major thirds, that is, the harmonic 5, but at the expense of deviating from harmonic 3. Another important example is the 12-tone equal temperament (12-TET) or, simply, equal temperament, which divides the octave into 12 equal parts (in a logarithmic frequency scale), thus resulting in a uniform and closed system. As a matter of fact, most tuning systems tend to approach the just intonation.

Then, on the one hand, the consonances are combinations of harmonics 2, 3 and 5 from a harmonic series and, on the other hand, those harmonics are perfectly matched by the just intonation. Therefore, the following question arises: how consonant is a consonance with respect to its deviation from just intonation? This paper is devoted to answer that question. To do that, signals containing 2 or 3 pure tones corresponding to a consonance are analyzed, both in just intonation and in an arbitrary slight deviation from it. It was found that, in just intonation, the envelopes of those signals are flat, but when there is a slight deviation from it, the envelopes have a ripple which corresponds to beats or roughness. This fact is closely related to the above-mentioned characteristic of the consonances, which indicates that the ripple is a measure of the loss of consonance. The ripple resembles a vibrato in amplitude and is characterized by its amplitude and its "periodicity", which may include one or more periods.

The simplest type of beat arises from the combination of two tones with similar frequencies, and is well-known. However, in this study the beats correspond to the combination of two or three tones whose frequencies are "almost multiples" of a fundamental one, and obtaining analytical expressions of their envelopes is not an easy task. Thus, in sections 2 and 3, approximate formulas for those envelopes are obtained, which represents the main contribution of this paper. In section 4, these formulas are applied to just intonation and equal temperament, although they are general formulas that can be applied to most tuning systems. To keep this study under a reasonable extension, the inversions of the consonances were not analyzed. As well, the octave was not considered here, because it is perfectly matched by practically all tuning systems.

2 Consonant Intervals

The combination of two pure tones whose frequencies are almost multiples of a fundamental one, f_0, gives rise to the signal

$$\begin{aligned} s(t) &= \cos k\omega_0 t + \cos(l\omega_0 t + \alpha) \\ \alpha &= at + \theta \end{aligned} \tag{1}$$

$\omega_0 = 2\pi f_0$ being the fundamental angular frequency and θ the initial angle or phase. In just intonation, $a = 0$ and the overall period of $s(t)$ is $T_0 = 2\pi/\omega_0 = 1/f_0$ divided by the greatest common divisor of (k, l), which will be 1 in all cases here considered. In other tuning systems, the angular frequency a represents a slight deviation from it, which gives rise to the period $T_a = 2\pi/a$. Throughout this paper, it will be assumed that $T_a \gg T_0$. This condition assures that $T_a \gg T_0/k$, T_0/l, as well.

Excluding the unison and the octave, there are 3 types of consonant intervals (apart from their inversions): the perfect fifth, where the pair $(k, l) = (2, 3)$, the major third, where $(k, l) = (4, 5)$, and the minor third, where $(k, l) = (5, 6)$.

As explained in [11], the "force" (loudness) of a sound is related to the amplitude of the oscillations. In our case, except in just intonation, the amplitude of $s(t)$ varies with time, giving rise to an envelope formed by the maximum and minimum values of $s(t)$ in each period T_0. For the perfect fifth, those maximum and minimum values will be represented by functions $E_1^5(\alpha)$ and $E_2^5(\alpha)$, respectively. The solid line in Fig. 1 named $(2, 3)$ shows the upper envelope $E_1^5(\alpha)$ of $s(t)$ as a function of $\alpha/\pi \in [0, 2]$. The graph was obtained with 200 points for α/π; and, for each of them, 2000 values of t were used to obtain the maximum of $s(t)$.

Due to the shape of this graph, the function $|\cos x\,|$ will be used to approximate $E_1^5(\alpha)$, in the following form

$$F_1(\alpha) = A + B|\cos\alpha\,| \tag{2}$$

To obtain the coefficients A, B, the function $F_1(\alpha)$ is forced to have the same maximum and minimum values as $E_1^5(\alpha)$. The result is

$$A = 1.6342, \qquad B = 0.3658 \tag{3}$$

The same procedure was followed to approximate the lower envelope $E_2^5(\alpha)$ of $s(t)$, the final formulas being

$$\begin{aligned} F_1(\alpha) &= 1.6342 + 0.3658|\cos\alpha\,| \\ F_2(\alpha) &= -1.6342 - 0.3658|\sin\alpha\,| \end{aligned} \tag{4}$$

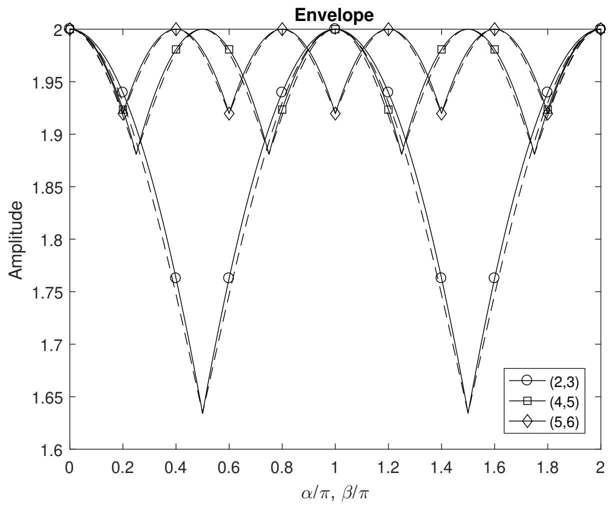

Fig. 1. Upper envelopes of $s(t)$ for consonant intervals (solid lines), along with their approximating functions (dashed lines). Curve $(2,3)$ corresponds to the perfect fifth, $(4,5)$ to the major third, and $(5,6)$ to the minor third.

Regarding the major and minor thirds, and in order to distinguish their formulas from those of the perfect fifth, the following expression for $s(t)$ will be used:

$$\begin{gathered} s(t) = \cos k\omega_0 t + \cos(m\omega_0 t + \beta) \\ \beta = bt + \varphi \end{gathered} \tag{5}$$

Let us first consider the major third. The upper and lower envelopes will be represented by functions $E_1^3(\beta)$ and $E_2^3(\beta)$, respectively. The solid line in Fig. 1 named $(4,5)$ shows the upper envelope $E_1^3(\beta)$ of $s(t)$ as a function of $\beta/\pi \in [0,2]$, obtained as in the previous case. Following the above procedure, the approximating function for the upper envelope now takes the form

$$G_1(\beta) = A + B|\cos 2\beta\,| \tag{6}$$

The coefficients A, B are obtained by imposing the same conditions as in the previous case, the result being

$$A = 1.8809, \qquad B = 0.1191 \tag{7}$$

and the final formulas for approximating the upper and lower envelopes are, respectively,

$$\begin{gathered} G_1(\beta) = 1.8809 + 0.1191|\cos 2\beta\,| \\ G_2(\beta) = -1.8809 - 0.1191|\sin 2\beta\,| \end{gathered} \tag{8}$$

With respect to the minor third, the upper and lower envelopes will be represented by functions $E_1^{b3}(\beta)$ and $E_2^{b3}(\beta)$, respectively. The solid line in Fig. 1

named $(5, 6)$ shows the upper envelope $E_1^{b3}(\beta)$ of $s(t)$ as a function of $\beta/\pi \in [0, 2]$, obtained as in the previous cases. Following the same procedure, the approximating function for the upper envelope now takes the form

$$H_1(\beta) = A + B|\cos 2.5\beta\,| \tag{9}$$

The coefficients A, B are obtained by imposing the same conditions as in the previous cases, the result being

$$A = 1.9197, \qquad B = 0.0803 \tag{10}$$

and the final formulas for approximating the upper and lower envelopes are, respectively,

$$\begin{aligned} H_1(\beta) &= 1.9197 + 0.0803|\cos 2.5\beta\,| \\ H_2(\beta) &= -1.9197 - 0.0803|\sin 2.5\beta\,| \end{aligned} \tag{11}$$

In order to assess the accuracy of this approximation, Fig. 1 also includes the graphs of $F_1(\alpha)$, $G_1(\beta)$ and $H_1(\beta)$ (dashed lines), and good agreement between every two paired curves is observed. In the three types of intervals, the envelope has a ripple whose amplitude equals the coefficient B, and its period is defined by α or β.

3 Consonant Chords

The combination of three pure tones whose frequencies are almost multiples of a fundamental one, f_0, gives rise to the signal

$$\begin{aligned} s(t) = \cos k\omega_0 t + \cos(m\omega_0 t + \beta) + \cos(l\omega_0 t + \alpha) \\ \alpha = at + \theta, \qquad \beta = bt + \varphi \end{aligned} \tag{12}$$

$\omega_0 = 2\pi f_0$ being the fundamental angular frequency and θ, φ the initial angles or phases. Letters k, l, m, as well as α, β, are assigned following the order of harmonics, while the addends in (12) follow the order of notes in the chord. In just intonation, $a = b = 0$ and the overall period of $s(t)$ is $T_0 = 2\pi/\omega_0 = 1/f_0$ divided by the greatest common divisor of (k, m, l), which will be 1 in all cases here considered. In other tuning systems, the angular frequencies a and b represent slight deviations from it, which give rise to the periods $T_a = 2\pi/a$ and $T_b = 2\pi/b$, respectively. Throughout this paper, it will be assumed that T_a, $T_b \gg T_0$. This condition assures that $T_a, T_b \gg T_0/k$, T_0/l, T_0/m, as well.

There are 2 types of consonant chords: the major chord, where the triplet $(k, m, l) = (4, 5, 6)$, and the minor chord, where $(k, m, l) = (10, 12, 15)$.

As in last section, except in just intonation, the amplitude of $s(t)$ varies with time, giving rise to an envelope formed by the maximum and minimum values of $s(t)$ in each period T_0. For major chords, those maximum and minimum values will be represented by functions $E_1^M(\alpha, \beta)$ and $E_2^M(\alpha, \beta)$, respectively. Solid lines in Fig. 2 show the upper envelope $E_1^M(\alpha, \beta)$ of $s(t)$ as a function of $\beta/\pi \in [0, 2]$ for different values of $\alpha/\pi \in [0, 0.5]$. Each graph was obtained with 200 points for

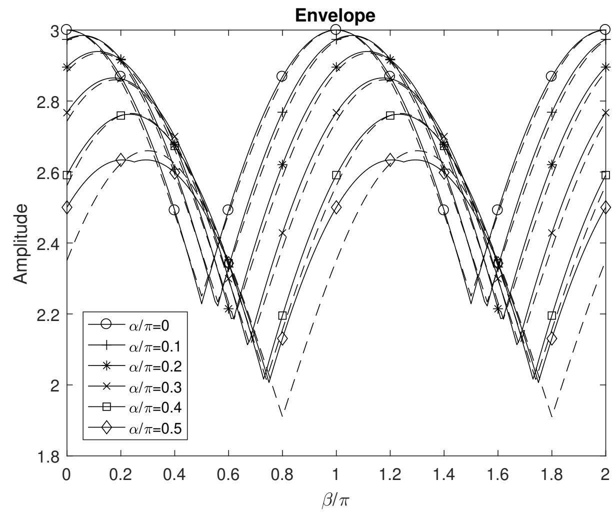

Fig. 2. Upper envelope $E_1^M(\alpha, \beta)$ of $s(t)$ as a function of β/π for $\alpha/\pi \in [0, 0.5]$ (solid lines), along with its approximating function $P_1(\alpha, \beta)$ (dashed lines)

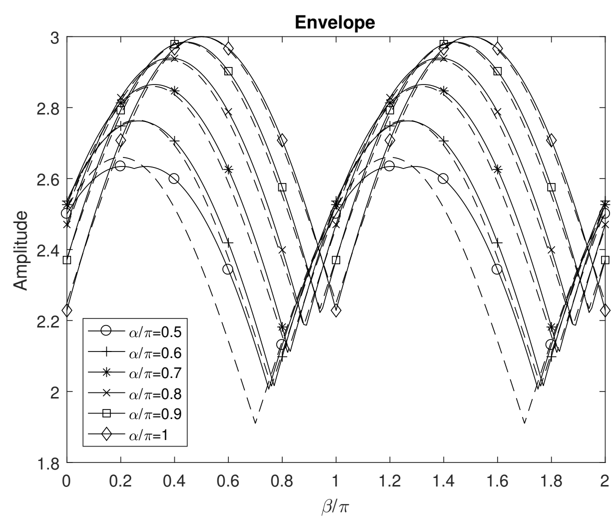

Fig. 3. Upper envelope $E_1^M(\alpha, \beta)$ of $s(t)$ as a function of β/π for $\alpha/\pi \in [0.5, 1]$ (solid lines), along with its approximating function $P_1(\alpha, \beta)$ (dashed lines)

β/π; and, for each of them, 2000 values of t were used to obtain the maximum of $s(t)$. The corresponding graphs for $\alpha/\pi \in [0.5, 1]$ are shown in Fig. 3.

As in the previous section, the function $|\cos x\,|$ will be used for approximating $E_1^M(\alpha,\beta)$, which now takes the form

$$\begin{aligned} P_{11}(\alpha,\beta) &= A + B|\cos(\beta - 0.6\alpha)\,| + C|\cos\alpha\,|, \quad 0 \le \alpha \le \pi/2 \\ P_{12}(\alpha,\beta) &= A + B|\cos(\beta - 0.6\alpha + 0.1\pi)\,| + C|\cos\alpha\,|, \quad \pi/2 \le \alpha \le \pi \end{aligned} \tag{13}$$

The phase shift between $P_{11}(\alpha,\beta)$ and $P_{12}(\alpha,\beta)$, which occurs at $\alpha = \pi/2$, can be expressed by the sawtooth function $\arctan(\tan x)$, thus allowing to combine the two formulas (13) in just one:

$$P_1(\alpha,\beta) = A + B|\cos[\beta - 0.5\alpha - 0.1\arctan(\tan\alpha)]\,| + C|\cos\alpha\,| \tag{14}$$

Note that Figs. 2 and 3 correspond to half a period in the variable α. For the other half, however, the graphs of $E_1^M(\alpha,\beta)$ are the same, but moved $\beta/\pi = 0.5$ to the right. Therefore, the formula (14) is valid for all α,β.

To obtain the coefficients A, B, C, a first condition is $A + B + C = 3$, so that the maximum value $P_{1,\max}(\alpha,\beta) = 3$. Then, in order to achieve a good approximation for most values of α, the function $P_1(\alpha,\beta)$ is forced to have the same maximum and minimum values as $E_1^M(\alpha,\beta)$ for $\alpha = 0.4\pi$. The result is

$$A = 1.91, \qquad B = 0.75, \qquad C = 0.34 \tag{15}$$

Since this is a different kind of approximation compared to the one used in last section, in this case it was considered exact enough to use coefficients to two decimal places. The same procedure was followed to approximate the lower envelope $E_2^M(\alpha,\beta)$ of $s(t)$, the final formulas being

$$\begin{aligned} P_1(\alpha,\beta) &= 1.91 + 0.75|\cos[\beta - 0.5\alpha - 0.1\arctan(\tan\alpha)]\,| + 0.34|\cos\alpha\,| \\ P_2(\alpha,\beta) &= -1.91 - 0.75|\cos[\beta - 0.5\alpha + 0.1\arctan(\cot\alpha)]\,| - 0.34|\sin\alpha\,| \end{aligned} \tag{16}$$

In order to assess the accuracy of this approximation, Figs. 2 and 3 also include the graphs of $P_1(\alpha,\beta)$ (dashed lines), and good agreement between every two paired curves is observed. The greatest error occurs for $\alpha/\pi = 0.5$ at the minimum values, which is less than 5%. The value of $\arctan(\tan\alpha)$ for $\alpha/\pi = 0.5$ in Fig. 2 was obtained from the left (0.5^-) and in Fig. 3 from the right (0.5^+).

Regarding the minor chords, the upper and lower envelopes will be represented by functions $E_1^m(\alpha,\beta)$ and $E_2^m(\alpha,\beta)$, respectively. Following the above procedure, which again requires the use of the sawtooth function, the approximating function for the upper envelope now takes the form

$$Q_1(\alpha,\beta) = A + B|\cos[2.5\beta - 1.4\arctan(\tan\alpha)]\,| + C|\cos\alpha\,| \tag{17}$$

which is valid for all α,β. The coefficients A, B, C are obtained by imposing the same conditions as in the previous case, the result being

$$A = 2.53, \qquad B = 0.13, \qquad C = 0.34 \tag{18}$$

and the final formulas for approximating the upper and lower envelopes are, respectively,

$$\begin{aligned} Q_1(\alpha,\beta) &= 2.53 + 0.13|\cos[2.5\beta - 1.4\arctan(\tan\alpha)]\,| + 0.34|\cos\alpha\,| \\ Q_2(\alpha,\beta) &= -2.53 - 0.13|\sin[2.5\beta + 1.4\arctan(\cot\alpha)]\,| - 0.34|\sin\alpha\,| \end{aligned} \quad (19)$$

Now, the greatest error also occurs for $\alpha/\pi = 0.5$ at the minimum values, which in this case is less than 2.5%.

In both types of chords, the envelope has a ripple whose amplitude is the sum of the coefficients B and C, and its periodicity is defined by α and β.

4 Results

Formulas obtained in sections 2 and 3 are valid for any kind of tuning or temperament, with the only condition that $T_a, T_b \gg T_0$. In this section, results are given for just intonation and equal temperament. In the first case, $a = b = 0$, while in the second one, $a = k\omega_0(2^{7/12} - 3/2) = -1.693 \cdot 10^{-3} k\omega_0$ (perfect fifth) and $b = k\omega_0(2^{4/12} - 5/4) = 9.921 \cdot 10^{-3} k\omega_0$ (major third) or $b = k\omega_0(2^{3/12} - 6/5) = -10.793 \cdot 10^{-3} k\omega_0$ (minor third). Then, the periods defined by a and b range from $T_a = 295T_0$ (perfect fifth) to $T_b = 9.27T_0$ (minor chord). Therefore, in all cases, the required condition is fulfilled. Additionally, we can obtain the corresponding values for other tuning systems. For example, in Pythagorean tuning, $a = 0$ and $b = k\omega_0(81/64 - 5/4) = 15.625 \cdot 10^{-3} k\omega_0$ (major third) or $b = k\omega_0(32/27 - 6/5) = -14.815 \cdot 10^{-3} k\omega_0$ (minor third). And in meantone temperament, $a = k\omega_0[(3/2)(81/80)^{-1/4} - 3/2] = -4.651 \cdot 10^{-3} k\omega_0$ (perfect fifth) and $b = 0$ (major third) or $b = k\omega_0[(32/27)(81/80)^{3/4} - 6/5] = -3.721 \cdot 10^{-3} k\omega_0$ (minor third). Therefore, the only period being less than previous ones is obtained for the Pythagorean tuning, minor chord, where $T_b = 6.75T_0$, the approximation being still acceptable.

All signals in the following examples last 4 s and their graphs were obtained by sampling them at 44,100 Hz, as in the WAV audio format. In fact, the corresponding audio files were generated, too. For simplicity, all the examples will start with note C4, whose frequency is 261.626 Hz and, therefore, $k\omega_0 = 1643.842$ rad/s.

Regarding the consonant intervals, Figs. 4 and 5 show $s(t)$ and its envelopes $F_1(\alpha)$ and $F_2(\alpha)$ for the perfect fifth C4G4, in just intonation and equal temperament, for $\theta = 0$, obtained with (1) for $(k,l) = (2,3)$ and (4). Fig. 6 shows a detail of $s(t)$ and $F_1(\alpha)$. In the cases of major and minor thirds, the corresponding graphs are similar to Figs. 4 and 5, but with different ripples. Fig. 7 shows a detail of $s(t)$ and its upper envelope $G_1(\alpha)$ for the major third C4E4 in equal temperament, for $\varphi = -0.7\pi$, obtained with (5) for $(k,m) = (4,5)$ and (8), while Fig. 8 shows a detail of $s(t)$ and its lower envelope $H_2(\alpha)$ for the minor third C4Eb4 in equal temperament, for $\varphi = 0.3\pi$, obtained with (5) for $(k,m) = (5,6)$ and (11).

With respect to consonant chords, Figs. 9–11 show $s(t)$ and its envelopes $P_1(\alpha,\beta)$ and $P_2(\alpha,\beta)$ for the major chord C4E4G4, in just intonation and equal temperament, for $\theta = 0.3\pi$, $\varphi = -0.7\pi$, obtained with (12) for $(k, m, l) = (4, 5, 6)$ and (16). And Figs. 12–14 show $s(t)$ and its envelopes $Q_1(\alpha, \beta)$ and $Q_2(\alpha, \beta)$ for the minor chord C4Eb4G4, in just intonation and equal temperament, for $\theta = -0.2\pi$, $\varphi = 0.6\pi$, obtained with (12) for $(k, m, l) = (10, 12, 15)$ and (19). Since in equal temperament $a \ll b$, the ripples in these graphs include a fast variation with a short period, defined by b, superimposed to a slow variation with a large period, defined by a.

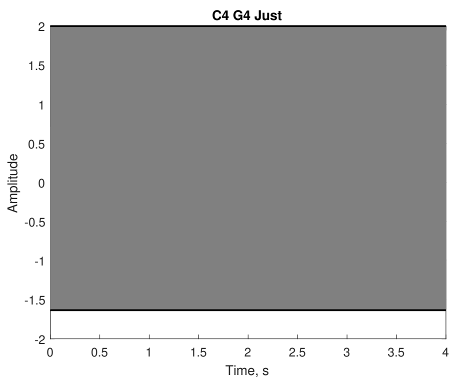

Fig. 4. Perfect fifth C4G4 in just intonation for $\theta = 0$

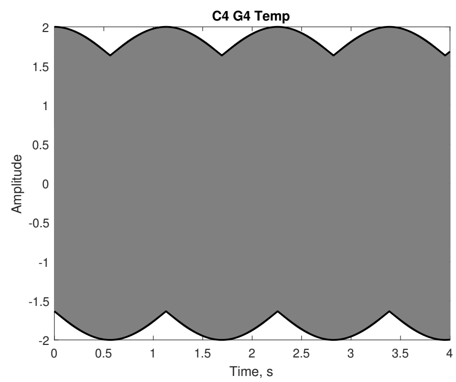

Fig. 5. Perfect fifth C4G4 in equal temperament for $\theta = 0$

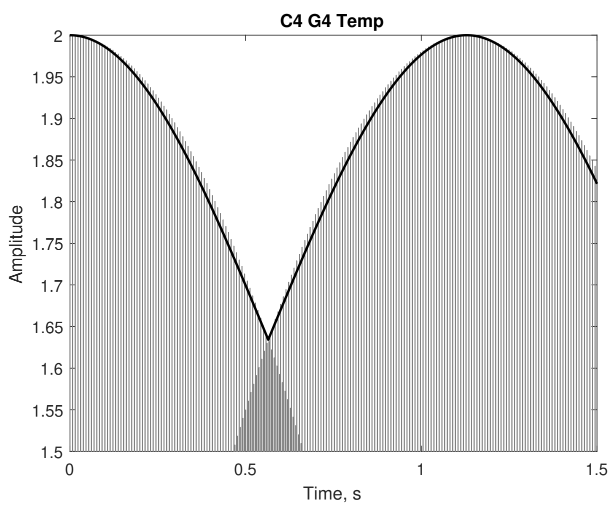

Fig. 6. Perfect fifth C4G4 in equal temperament for $\theta = 0$ (detail)

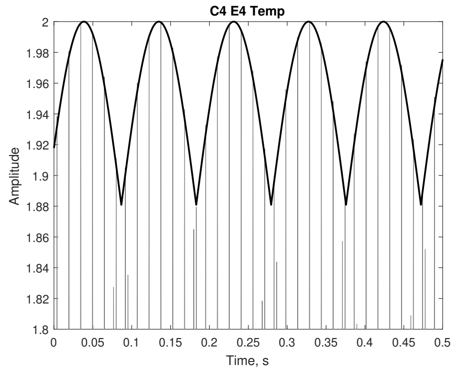

Fig. 7. Major third C4E4 in equal temperament for $\varphi = -0.7\pi$ (detail)

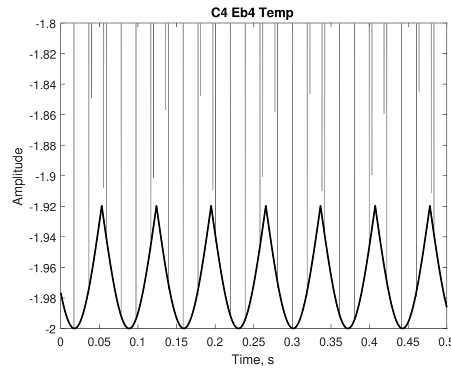

Fig. 8. Minor third C4Eb4 in equal temperament for $\varphi = 0.3\pi$ (detail)

5 Conclusions

Signals corresponding to consonant intervals and chords have been analyzed. They were combinations of 2 or 3 pure tones whose frequencies are almost multiples of a fundamental one. The procedure for obtaining approximate formulas for the envelopes of those signals has been developed. In just intonation, the envelopes are flat, but when there is a slight deviation from it, they have a ripple which corresponds to beats or roughness, thus indicating a loss of consonance. Both the amplitudes and periodicities of the ripples have been obtained for all types of consonances. It has been found that the amplitude of a ripple is determined by the type of consonance itself, while its periodicity is defined by the frequency deviations from just intonation. In the case of intervals (2 tones), the ripple includes one period, while in the case of chords (3 tones) the ripple includes two periods superimposed.

Conducting a survey to evaluate the human perception of ripples in consonant intervals and chords is now under consideration, but preliminary results indicate that ripples with small amplitudes (as in minor third intervals) or large periods (as in perfect fifth intervals) are hardly perceived, while ripples with greater amplitudes and lesser periods (as in major chords) are clearly perceived.

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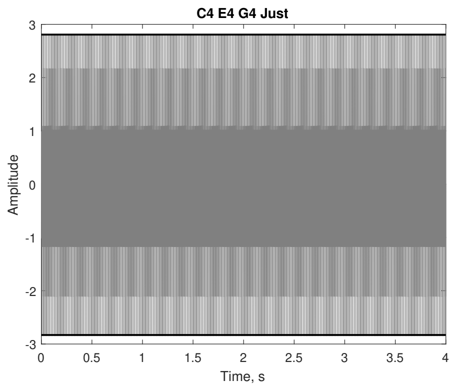

Fig. 9. Major chord C4E4G4 in just intonation for $\theta = 0.3\pi$, $\varphi = -0.7\pi$

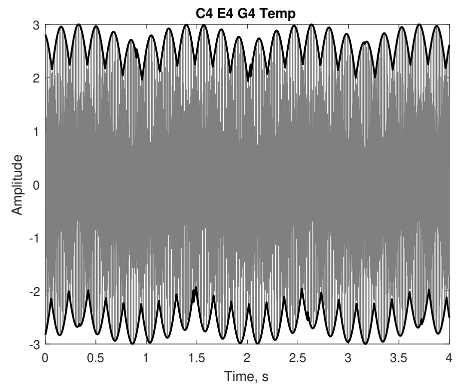

Fig. 10. Major chord C4E4G4 in equal temperament for $\theta = 0.3\pi$, $\varphi = -0.7\pi$

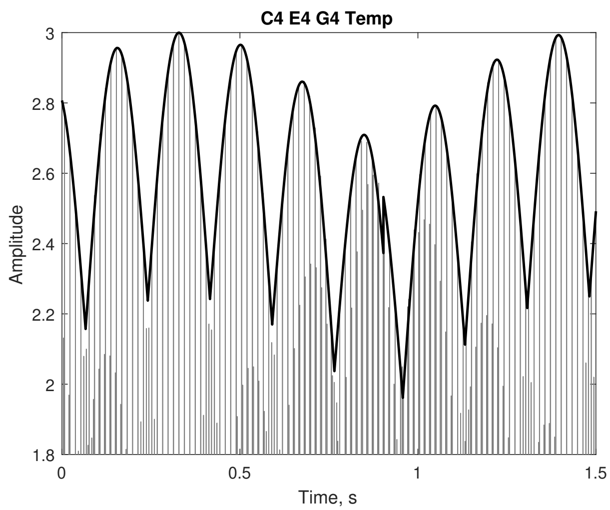

Fig. 11. Major chord C4E4G4 in equal temperament for $\theta = 0.3\pi$, $\varphi = -0.7\pi$ (detail)

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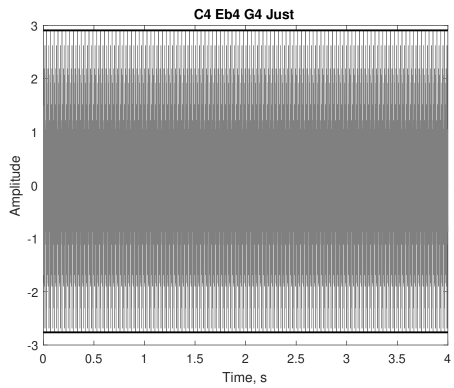

Fig. 12. Minor chord C4Eb4G4 in just intonation for $\theta = -0.2\pi$, $\varphi = 0.6\pi$

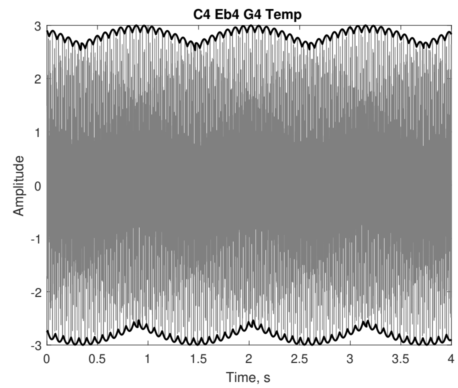

Fig. 13. Minor chord C4Eb4G4 in equal temperament for $\theta = -0.2\pi$, $\varphi = 0.6\pi$

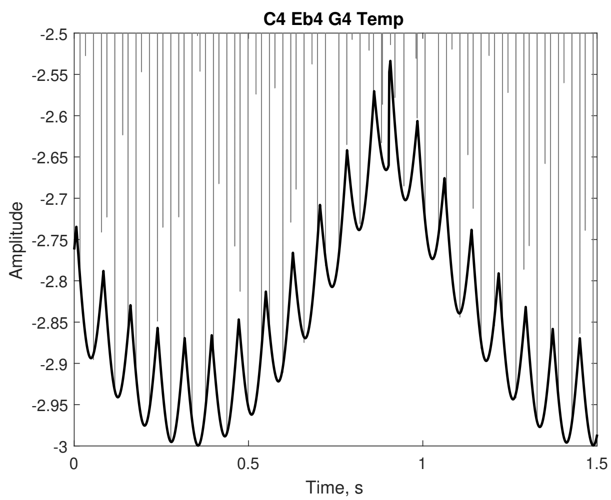

Fig. 14. Minor chord C4Eb4G4 in equal temperament for $\theta = -0.2\pi$, $\varphi = 0.6\pi$ (detail)

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