Parsimonious Graphs for Selected Heptatonic and Pentatonic Scales

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Abstract. Tonal music is based on major, melodic and harmonic minor scales. In some cases, the harmonic major scale is also used. In this paper, four additional heptatonic scale types, derived from them, are considered. The harmonic characteristics of these eight scale types are analyzed by the trichord- and tetrachord-type vectors, which list, respectively, the number of times each trichord and tetrachord type is contained in a set type. Then, a novel parsimonious graph is provided, called 7-Cyclops, which relate those scales by single-semitonal transformations. On the other hand, their complements are eight pentatonic scales, whose harmonic characteristics are also analyzed and the corresponding parsimonious graph, called 5-Cyclops, is given. These graphs highlight the cycles of fifths and fourths, which are the only possible circumferences linking the same scale types in these graphs. Other parsimonious transformations, like moving one note by a whole tone, are easily found in these graphs, too. The acoustical relationship between those heptatonic and pentatonic scale types is analyzed by the pentachord-type vector, which lists the number of times each pentachord type is contained in a set type. With the inclusion of a musical example, all this information is intended mainly for theorists and composers.

Keywords: Parsimonious Transformation, Heptatonic Scale, Pentatonic Scale, Cyclops, Trichord-Type Vector, Tetrachord-Type Vector, Pentachord-Type Vector, Cycle of Fifths, Cycle of Fourths

1 Introduction

The major scale is the basis of Western music. Although perfectly well known, it is worth reviewing now some of its main characteristics. It consists of seven notes showing great acoustical affinity among them, to the extent that they constitute a “complete and versatile” set. Thus, most popular songs – and not so popular – are composed in a major key. Our musical notation, based on the staff and the key signatures, is ideal for writing music in major keys. The piano, a crucial musical instrument, is especially suitable for playing in the C major key. The names of the notes are seven – instead of twelve –, precisely those in the C major scale. The term octave (a Latin word for “eighth”) indicates its extension (including an ending tonic), whereas the terms whole tone and semitone describe the types of intervals between two consecutive notes in it. As well, the quality of the intervals (perfect, major, minor, …) are defined with respect to a major scale. In summary, the major scale is a cornerstone of music theory and composition.

Other kinds of scales can be directly derived from the major one. Thus, by choosing any of its notes as the tonic, we obtain seven *modes*, from which the Aeolian one constitutes the minor scale, which is also prevalent in Western music. In this case, its sixth and seventh degrees can be natural or altered (raised by a semitone), thus giving rise to the natural, melodic, and harmonic minor scales.

On the other hand, the complement of a major scale is a major pentatonic one. In general, pentatonic scales – not only the major pentatonic – have been used since ancient times by many different cultures. Although they predominate in Eastern countries (China, Japan, India, Java, etc.), they are also used in several Western styles, such as Classical, Scottish, Andean, Jazz, etc.

This study is based on the standard twelve-tone chromatic system ($\mathbb{Z}_{12}$) and uses the nomenclature of *Forte names* and *set classes* [1], which here will be called *scale classes*. Additionally, the *non-inversionally-symmetrical* ones are split into two *scale types* related by *inversion*, named “a” and “b”, following [2]. Under these premises, the major and major pentatonic scales are the most even set types with seven and five notes, respectively, both possessing an exclusive property: apart from the set types with one and eleven notes, they are the only set types that can be transformed into the same set types by a *single-semitonal transformation*.[1] For example, by raising in the C major scale the note F by a semitone, we obtain the G major scale. This property, in the case of major scales, gives rise to the order of sharps and flats, as well as the cycle of fifths [3, 4], which is essential in the theory of *modulation*, that is, the change from one key to another. In this respect, given a key, its “nearest” keys are those having one sharp or flat more or less in their key signatures [5, 6]. This also applies to the minor keys, since they have the same key signatures as their relative major ones.

A *parsimonious transformation* is a more general concept, where one or more notes move by a semitone or a whole tone (in practice, normally no more than two semitones in total), while sustaining the rest of them [7]. Thus, the *Tonnetz* is a first representation of them for major and minor triads, while [7] provides more complex and interesting graphs. In the nineteenth century, several composers made extensive use of this kind of transformations and a large number of their works are analyzed in [8]. A different approach is given in [9], where pitch-class sets are represented in special spaces called *orbifolds*. In [10], the “most common” trichords and tetrachords are represented in cyclic circular graphs called *Cyclopes*, which allow us to analyze a great number of such musical works in a practical way. In this paper, two novel parsimonious graphs of this kind are developed for heptatonic and pentatonic scales. In each case, eight scale types are chosen following specific criteria. As well, an aforementioned result is shown graphically: the cycle of fifths for the major scales, together with the cycle of fourths for the major pentatonic ones, are the only possible circumferences connecting pitch-class sets of the same type in this kind of graphs [3, 4].

[1] A transformation of a pitch-class set (in our case, a scale) consisting in raising or lowering one pitch-class (note) by a semitone, while sustaining the rest of them.

2 Selection of Heptatonic Scales

Tonal music is based on major, melodic and harmonic minor scales [5, 6], whose "extended" Forte names are 7-35, 7-34, and 7-32a, respectively. To complete the numerical series, we can also consider 7-32b (harmonic major) and 7-33 (which we will call "Neapolitan major", following [11]). These are the five most even heptatonic scale types, as they have the least *interval-class vectors*[2] [1], with respect to the lexicographic order. The most even one is, obviously, the major scale (7-35).

In order to obtain other heptatonic scales related to them, we can simply combine two groups of four consecutive notes or *tetrachords*.[3] An example of this procedure is provided by the traditional Indian music, where up to 72 heptatonic scales, called "Melakarta ragas", are obtained by combining different types of tetrachords [12]. However, since the total number of heptatonic scale types is 66, some of those ragas are, in fact, modes of other ragas, the number of different scale types being 36. In any case, both 66 and 36 are too many scale types to develop practical and visually simple graphs relating them.

Another option is to start with a major scale and raise or lower one or more notes by a semitone, as done with the natural minor scale to obtain the melodic and harmonic minor ones. In this case, it seems appropriate to choose the altered notes from the nearest key signatures.

These two procedures are now used to obtain a "reasonable" number of heptatonic scale types, which can be of interest for theorists and composers.

2.1 Combinations of two Tetrachords

Let us consider the C major scale. It is composed of the tetrachords C – D – E – F and G – A – B – C, whose "intervallic structures", in semitones, are the same: 221; and the interval between the two tetrachords is 2 semitones. So, we can write the full intervallic structure of C major as 221 2 221; and the tetrachord 221 can be called "major". As well, starting from notes D or A in the C major scale, the first four notes give the tetrachord 212, which we will call "minor". Similarly, starting from E or B, we obtain the tetrachord 122, which we will call "Phrygian". And starting from F, we obtain the tetrachord 222, which we will call "Lydian".

Thus, we can obtain different heptatonic scales by combining any two of those tetrachords. But, to obtain the harmonic minor or major scales, we need another tetrachord: the 131, which we will call "harmonic". Table 1 shows the 25 combinations of these 5 tetrachords, with the resulting intervallic structures and the names and symbols given here to the corresponding scale types. The less common names are taken from [11] and all modes of a scale type are given the same name. Note that, in all cases, the interval between the two tetrachords is such that the starting and ending

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[2] The vector listing the number of times each of the 6 dyads (intervals from 1 to 6 semitones) is contained in a given set type or set class (in our case, scale type or scale class). It characterizes, to a great extent, the sonority of a set class. In [1], it was called *interval vector*.

[3] The term tetrachord also means "4-note chord". However, throughout this paper, its right meaning will easily be determined by the context.

notes in the scale be the same; or, in other words, the sum of the semitones in every intervallic structure be 12.

Table 1. Heptatonic scale types obtained by combining two out of five tetrachords, with their intervallic structures. Scale symbols: M: major, mm: melodic minor, hm: harmonic minor, hM: harmonic major, NpM: Neapolitan major, Npm: Neapolitan minor, hL: harmonic Lydian, hh: double harmonic or Hungarian, WT: whole tone.

1st ↓ \ 2nd →	Major	minor	Phrygian	Lydian	harmonic
Major	221 2 221 M	221 2 212 M	221 2 122 mm	221 1 222 NpM	221 2 131 hM
minor	212 2 221 mm	212 2 212 M	212 2 122 M	212 1 222 mm	212 2 131 hm
Phrygian	122 2 221 NpM	122 2 212 mm	122 2 122 M	122 1 222 M	122 2 131 Npm
Lydian	222 1 221 M	222 1 212 mm	222 1 122 NpM	222 222 WT	222 1 131 hL
harmonic	131 2 221 hL	131 2 212 hM	131 2 122 hm	131 1 222 Npm	131 2 131 hh

The combination of two Lydian tetrachords gives rise to the whole-tone scale (WT), which only has six notes, thus being excluded from this study. The rest of the combinations give rise to eight different scale types, which include the five most even and is a suitable number for developing our graphs. Table 2 shows those scale types with their extended Forte names, the symbols here used to represent them, their *intervallic forms*[4] [2] starting from the tonic, and their interval-class vectors.

Table 2. Heptatonic scale types considered here. The intervallic forms start from the tonic.

Heptatonic Scale	Symbol	Intervallic Form	Interval-Class Vector
7-22	hh	1312131	424542
7-30a	Npm	1222131	343542
7-30b	hL	2221131	343542
7-32a	hm	2122131	335442
7-32b	hM	2212131	335442
7-33	NpM	1222221	262623
7-34	mm	2122221	254442
7-35	M	2212221	254361

2.2 Combinations of the Altered Notes from the Nearest Key Signatures

Let us consider again the C major scale. Its two nearest key signatures, both in the order of the sharps and the flats, introduce the altered notes F♯, C♯, B♭, and E♭. Thus, using the C major scale with any of those notes either natural or altered, gives rise to 16 combinations, which are shown in Table 3, together with the resulting scales. As can be seen, we obtain the same eight scale types as with the previous procedure. In fact, according to [13], they are the ones with a "span" ≤ 10. Therefore, they will be the heptatonic scale types considered in this study, which are those listed in Table 2.

[4] The intervallic form is the sequence of intervals, in semitones, between every two adjacent pitch classes in a set type (in our case, a scale type), including the interval between the last and the first ones, or any of its circular shifts. If it starts from a scale tonic, then it matches the "intervallic structure" previously used in this section.

Table 3. Heptatonic scales obtained from CM by combining the two nearest altered notes.

Altered Notes	-	F → F♯	C → C♯	F → F♯, C → C♯
-	CM	GM	Dmm	DM
B → B♭	FM	Gmm	Dhm	DhM
E → E♭	Cmm	GhM	DNpM	GhL
B → B♭, E → E♭	B♭M	Ghm	DNpm	Dhh

2.3 Harmonic Characteristics of the Selected Heptatonic Scales

To evaluate the harmonic characteristics of the heptatonic scale types considered here, we can use two generalizations of the interval-class vector: the *trichord-* and *tetrachord-type vectors*, which list, respectively, the number of times each trichord and tetrachord type is contained in a set type. Table 4 shows these vectors for those scale types, where each digit corresponds to a chord type in the order established in [2]. Thus, for example, the first digits from right to left in the trichord-type vector correspond to the augmented, major, minor, and diminished triads. And the first digits from right to left in the tetrachord-type vector correspond to the diminished, dominant, half-diminished, and minor seventh chords. Digits in bold correspond to the trichord and tetrachord types considered in [10].

Table 4. Trichord- and Tetrachord-Type Vectors of the Heptatonic scale types considered here.

Scale	Trichord-Type Vector	Tetrachord-Type Vector
7-22	111333322-011**2212331**	000111110220-0001100111111111222**222-0000011110**
7-30a	111212321-312**3321321**	010001110110-0110100111120121010**202-2120211110**
7-30b	111123212-321**3321231**	001010110110-0111000111102211100**022-2210211110**
7-32a	022222221-122**1224321**	000100000110-1112121120110111122**211-0011101211**
7-32b	022222212-122**2124231**	000100000110-1111212211001111211**121-0101101121**
7-33	111111111-611**6611112**	011000110000-0111100001111110001**110-6110630110**
7-34	022111111-333**3332221**	000100000000-2111111111111110001**110-2222211220**
7-35	022002211-344**1151330**	000000000010-2220011221100110000**002-1334003110**

3 Parsimonious Graphs for the Selected Heptatonic Scales

Figure 1 is a diagram showing the eight heptatonic scale types considered here, linked by single-semitonal transformations, and where link crossings are avoided. The arrows show how to transform the scale types by raising one note by a semitone (or, in the opposite direction, by lowering one note by a semitone). Note that the major scale is the only scale type that can be self-transformed, which will give rise to the cycle of fifths. When a scale class consists of two scale types related by inversion, they are placed next to each other (7-30a next to 7-30b and 7-32a next to 7-32b). This allows us to clearly see the relations between them, when they exist, as is the case for 7-32a and 7-32b. As well, the links connecting two such scale types with others always consist of pairs of arrows in opposite directions, one for each scale type.

Arabic numerals indicate the initial and final notes referring to the scale tonics, where 1 to 6 stand for perfect or major intervals, which may be altered with ♯ or ♭, and

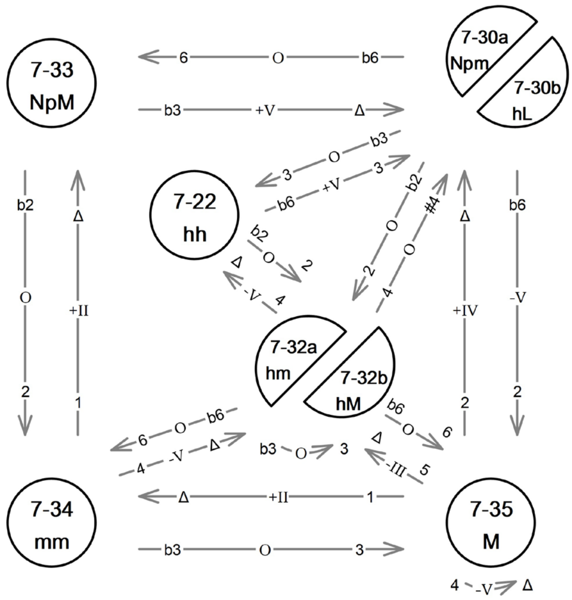

Fig. 1. The heptatonic scale types included in Table 2 with their single-semitonal transformations.

major and minor sevenths are denoted by Δ and 7, respectively. And the Roman numerals at the middle of the arrows indicate the interval between the scale tonics, *in semitones* (letter "O" means zero). For example, Cmm consists of notes (C, D, E♭, F, G, A, B) and, by raising the perfect fourth (4) by a semitone, the new note is the major seventh (Δ) of the new scale, a "hM" with tonic C – V semitones, that is, GhM = (G, A, B, C, D, E♭, F♯). Or, by lowering in Cmm the major seventh (Δ) by a semitone, it turns into the tonic (1) of the "M" scale with tonic C – II, that is, B♭M = (B♭, C, D, E♭, F, G, A). Other parsimonious transformations can be found in this diagram, particularly those obtained by moving one note by a whole tone. But this will be explained in Section 4.2, together with the transformations of pentatonic scale types.

This diagram does not include the scale tonics, so it represents the "local relationships" in a more general scale space. Thus, let us now represent the "global relationships" among all the heptatonic scales considered here (with their tonics). To do this, we group them into *voice-leading zones* [8] or, simply, *zones* $\varphi \in [0, \dots, 11]$, which are the equivalence classes defined by the sum of the notes in a scale, modulo 12. For example, CM = (C, D, E, F, G, A, B) is in the zone $\varphi = 0 + 2 + 4 + 5 + 7 + 9 + 11 = 2$ (mod 12). This way, given a scale in the zone φ, the one obtained from it by raising one note by a semitone will be in $\varphi + 1$. And scales related by *pure contrary motion*, as CM and Dhm = (D, E, F, G, A, B♭, C♯) will be in the same zone (in this case, $\varphi = 2$). The final result is given in Figure 2 in a cyclic circular graph, here called *7-Cyclops*, where φ is actually an angular position starting from "12 o'clock" ($\varphi = 0$ for B♭M) and increasing clockwise. The arrows in Figure 1 are now substituted by lines whose directions are assumed to be clockwise and no Roman numerals are used, since the tonics are directly given. Because 7 and 12 are coprime integers, in each zone of the 7-Cyclops there is exactly one scale of each type. The links between major scales make up the *cycle of fifths*, which corresponds to the only possible circumference in this graph (the bold line).

A different circular diagram is given in [9, p. 136], which includes the major (there called *diatonic*), melodic minor (there called *acoustic* and whose tonic is the perfect fourth of the corresponding melodic minor scale), harmonic minor, and harmonic major scales, plus three non-heptatonic scales with *transpositional symmetry*: the whole-tone (6-35), *hexatonic* (6-20, also called *augmented*), and *octatonic* (8-28, also called *half-step/whole-step diminished*), whose intervallic forms starting from the tonic are, respectively, 222222, 131313, and 12121212. To properly allocate all these scales, 36 angular positions are used, which obviously cannot coincide with the zones considered here, and the notes that change from one scale to the other are not shown.

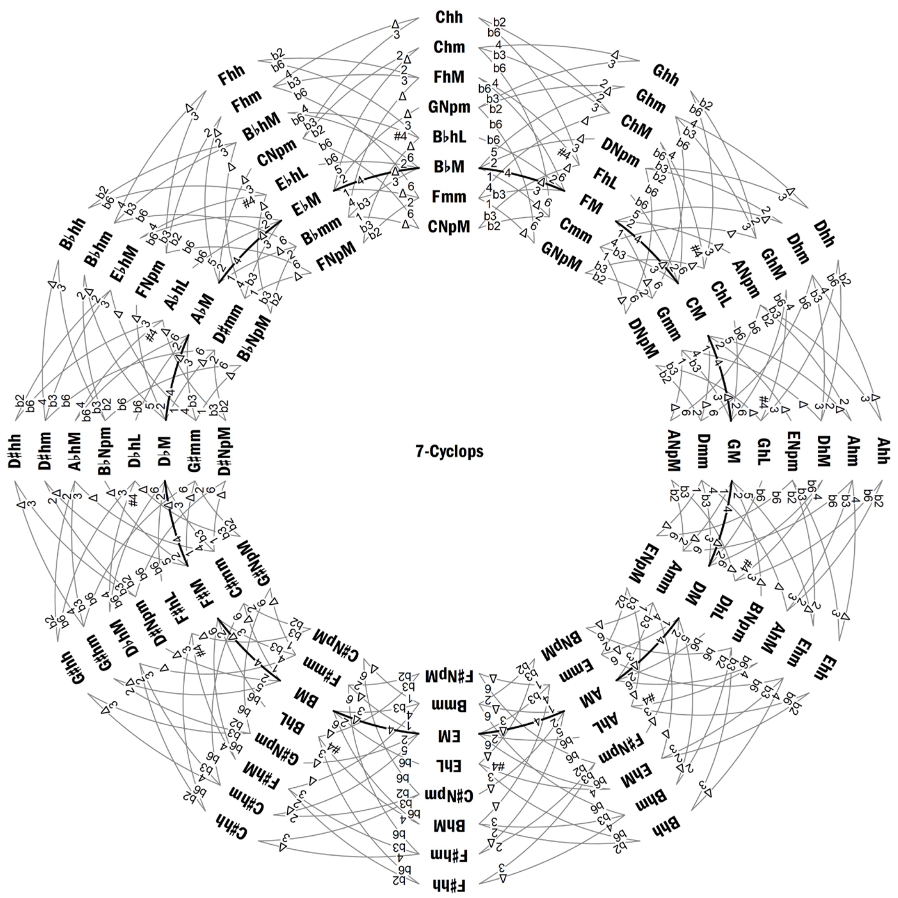

Fig. 2. The 7-Cyclops, with the heptatonic scales considered in Table 2.

Another relevant work is [14]. Based on just intonation and "commatic transition series", three groups of heptatonic scales are obtained:

- Hiatal: 7-35, 7-32a, 7-32b, 7-30a, 7-30b, 7-22
- Octatonic: 7-35, 7-34, 7-32a, 7-32b, 7-31a, 7-31b
- Whole-tone: 7-35, 7-34, 7-33, pseudo-whole-tone

where "pseudo-whole-tone" is a whole-tone scale plus one enharmonic note. So, in [14], this scale and the pair 7-31a/7-31b are added to those in Table 2. As well, the nomenclature used there for some scale names differs from the one used in this paper.

For each group, the corresponding diagrams for both the local and global relationships are given, although the notes that change from one scale to the other are not indicated. Finally, the diagrams of the three groups are superimposed, both those with the local and the global relationships, the latter resulting in a really complex diagram, so that only the links are shown, but not the scale names. As well, the links between 7-33 and the pair 7-30a/7-30b are not included, since they belong to different groups.

4 Pentatonic Scales

A similar process is now followed for the pentatonic scales.

4.1 Selection of Pentatonic Scales

For consistency with previously selected heptatonic scales, we will select the pentatonic scales that are their complements. Table 5 shows these scale types with their extended Forte names, the symbols here used to represent them, their intervallic forms starting from the tonic, and their interval-class vectors. The scales whose symbols are of the form "m…P" and "7…P" are derived, respectively, from the minor (mP) and "dominant" (7P) pentatonic scales, whose intervallic forms, starting from the tonic, are 32232 and 22332. Note that "mP" is a mode of the major pentatonic scale (MP).

Table 5. Pentatonic scale types considered here. The intervallic forms start from the tonic.

Pentatonic Scale	Symbol	Intervallic Form	Interval-Class Vector
5-22	mΔ♯4P	33141	202321
5-30a	mΔP	32241	121321
5-30b	Δ♯5P	42231	121321
5-32a	m♯4P	33132	113221
5-32b	7♯9P	31332	113221
5-33	7♯5P	22422	040402
5-34	7P	22332	032221
5-35	MP	22323	032140

We can also obtain these scale types in a similar way as we did for the heptatonic scales by using the nearest altered notes. However, because now we cannot use the concept of key signature as before, we will talk of the nearest *modified* notes. For example, starting from the CMP scale, we search for the notes that, by moving by a semitone, generate other major pentatonic scales. There are just two possibilities:

raising E to F to obtain FMP or lowering C to B to obtain GMP. Thus, considering the two nearest modified notes, both raising and lowering, we obtain the Table 6, which is analogous to Table 3 but for the pentatonic scales. As can be seen, again eight different scale types are obtained, which are precisely those listed in Table 5.

Table 6. Pentatonic scales obtained from CMP by combining the two nearest modified notes.

Modified Notes	-	E → F	A → B♭	E → F, A → B♭
-	CMP	FMP	C7P	B♭MP
C → B	GMP	G7P	Em♯4P	G7♯9P
G → F♯	D7P	D7♯9P	D7♯5P	F♯Δ♯5P
C → B, G → F♯	DMP	Bm♯4P	BmΔP	BmΔ♯4P

The harmonic characteristics of these scale types are shown in Table 7, which includes their trichord- and tetrachord-type vectors. This is analogous to Table 4 but for the pentatonic scales, the same conventions being used here.

Table 7. Trichord- and Tetrachord-Type Vectors of the Pentatonic scale types considered here.

Scale	Trichord-Type Vector	Tetrachord-Type Vector
5-22	000111111-000**0001111**	000000000010-000000000000000011**110-0000000000**
5-30a	000100110-101**1110101**	000000000000-000000000001001000**100-0010100000**
5-30b	000011001-110**1110011**	000000000000-000000000000110000**010-0100100000**
5-32a	000110010-011**0101210**	000000000000-000000000001000101**000-0000001100**
5-32b	000110001-011**1001120**	000000000000-000000000000100110**000-0000001010**
5-33	000000000-300**3300001**	000000000000-000000000000000000**000-2000210000**
5-34	000000000-211**1111110**	000000000000-000000000000000000**000-1110000110**
5-35	000000000-122**0030110**	000000000000-000000000000000000**000-0112001000**

4.2 Parsimonious Graphs for the Selected Pentatonic Scales

Figure 3 is a diagram showing the eight scale types considered here, linked by single-semitonal transformations. Since if two scale types are related by a single-semitonal transformation, then so are their complements, this figure is completely analogous to Figure 1, the same conventions being used here. Note that the major pentatonic scale is the only scale type that can be self-transformed, which, in this case, will give rise to the cycle of fourths.

This graph also allows us to easily find other parsimonious transformations, particularly those obtained by moving one note by a whole tone. They correspond to two consecutive arrows, where the ending note on the first matches the starting note on the second one. For example, if in C7♯9P = (C, D♯, E, G, B♭) we raise the major third (3) by a semitone, it turns into the major second (2) of an "MP" scale; and by raising again this note by a semitone, it turns into the augmented fourth (♯4) of an "m♯4P" scale whose tonic is C + III – III, that is, Cm♯4P = (C, E♭, F♯, G, B♭). There are 5 parsimonious transformations of this kind, the rest of them being: from "7♯5P" to the same scale type through "7P", from "7P" to "mΔP" and "Δ♯5P" through "MP", and from "Δ♯5P" to "mΔP" through "MP". The latter is harder to see because there is, or there may be, a voice crossing. For example, transforming CΔ♯5P = (C, E, F♯, G♯, B) into C♯mΔP = (C♯, E, F♯, G♯, B♯) or (C♯, E, F♯, G♯, C) can be done by raising B by a whole tone, which crosses C. But, in Figure 3, this must be done by first raising the tonic (C) by a semitone, giving EMP, and then raising its perfect fifth (B) by a semitone, thus avoiding the voice crossing. There is, however, a simpler way to find this parsimonious transformation. To this end, we have to use Figure 1 and take into account that: 1) the complements of 5-30a, 5-30b, and 5-35 are, respectively, 7-30b, 7-30a, and 7-35; 2) raising/lowering a note by a semitone in a scale type corresponds to lowering/raising a note by a semitone in its complement; and 3) if a note of a scale type, by moving by a whole tone, crosses a voice, it does not produce any voice crossing in its complement. Therefore, the pentatonic transformation from "Δ♯5P" to "mΔP" through "MP", by raising one note by a whole tone, corresponds to the heptatonic transformation from "Npm" to "hL" through "M", by lowering one note by a whole tone, which is clearly seen in Figure 1 (note Δ lowers to 2, which lowers to b6). Consequently, using the two diagrams, we easily find all transformations of this kind.

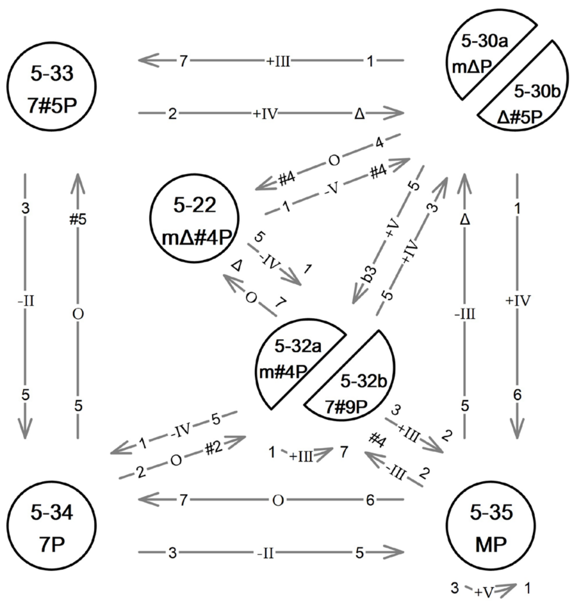

Fig. 3. The pentatonic scale types included in Table 5 with their single-semitonal transformations.

From the local relationships among the pentatonic scales (without the tonics), we can obtain the global ones (with all the tonics). They are shown in Figure 4 in a cyclic circular graph, here called *5-Cyclops*. This is analogous to Figure 2 but for pentatonic scales, the same conventions being used here. As well, in each zone of the 5-Cyclops there is exactly one scale of each type. But now, the links between major pentatonic scales make up the *cycle of fourths*, which also corresponds to the only possible circumference in this graph (the bold line).

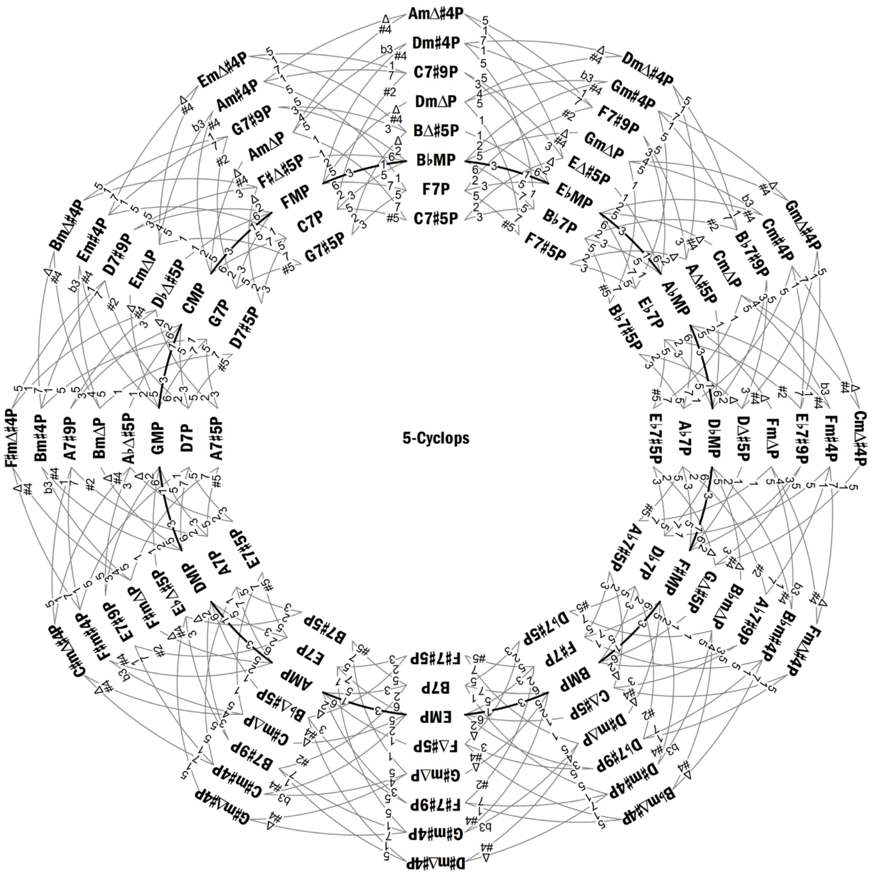

Fig. 4. The 5-Cyclops, with the pentatonic scales considered in Table 5.

4.3 Relation between the Selected Heptatonic and Pentatonic Scales

Apart from the complementary relationship between heptatonic and pentatonic scales, a more acoustical relationship can be found by using a further generalization of the interval-class vector: the pentachord-type vector, which lists the number of times each pentachord type (or pentatonic scale type) is contained in a given scale type. Thus, Table 8 shows the pentachord-type vectors of the heptatonic scale types included in Table 2 plus the pair 7-31a/7-31b, considered in [14]. Now, each digit corresponds to a pentatonic scale type in the order established in [2] and those in bold correspond to the ones considered here. Thus, from left to right, the first digit in bold corresponds to 5-22 and the last nine digits (those after the space) correspond to 5-30a, 5-30b, 5-31a, 5-31b, 5-32a, 5-32b, 5-33, 5-34, and 5-35.

Table 8. Pentachord-Type Vector of several Heptatonic scales.

Scale	Pentachord-Type Vector
7-22	000000001100-00000000000000111111110011112-0000000001100 **000011000**
7-30a	0000000000100-0100010000100010000010100111000-0011000111010 **200010110**
7-30b	000000001000-00100010000100100001010001101**0**-0011001011100 **020001110**
7-31a	0000000000000-0002100000000000120000002100000**0**-0000210000210 **003021000**
7-31b	0000000000000-0001200000000000210000001200000**0**-0000120000120 **000312000**
7-32a	0000000000000-000100000100000010100110001101**1**-1000101101001 **102110000**
7-32b	0000000000000-000010000100000100100110010011**1**-0100011110001 **011201000**
7-33	0000000000000-1110000000110010000000000000000**0**-0011001100110 **110000610**
7-34	0000000000000-0001100000000000001000000000000**0**-1111111100111 **110000121**
7-35	0000000000000-0000000001000000000000000011000**0**-2211110022001 **000000013**

As can be seen, 7-35 contains three 5-35 scale types. This is the maximum number of 5-35 contained in any heptatonic scale type (the rest of the heptatonic scale types – not only those in the table – contain no more than two). As well, 7-34, 7-33, and 7-22 contain, respectively, the maximum number of 5-34, 5-33, and 5-22, which are 2, 6 and 2 (in all other cases, they contain no more than one of each of them). Regarding the pairs of scale types forming a scale class, we must take into account that the complement of an a-type is a b-type and vice versa [2]. Then, 7-30a and 7-30b contain, respectively, the maximum number of 5-30a and 5-30b (that is, the inversions of their complements), which is two (in all other cases, they contain no more than one of each of them); and they do not contain their corresponding complements. As well, 7-31a and 7-31b contain, respectively, the maximum number of 5-31a and 5-31b, which is three (in all other cases, they contain no more than two) and do not contain their corresponding complements. And, with respect to 5-32a and 5-32b, it turns out that the heptatonic scale types containing the maximum number of them are, respectively, 7-31a and 7-31b, which is two (in all other cases, they contain no more than one). At least, 7-32a and 7-32b contain, respectively, one 5-32a and one 5-32b (again the inversions of their complements) and do not contain their corresponding complements. Therefore, in all these cases there is, to a greater or lesser extent, a clear acoustical relationship between each heptatonic scale type and the inversion of its complement.

5 Example of Musical Analysis

An interesting chromatic excerpt is analyzed in [14]: the Fantasy in C minor, K. 475, by Mozart, mm. 1–25. The involved scales are determined there, although some of them are incomplete. With the nomenclature used in this paper, they are

Ghh ⁒ FhM ⁒ D♭M E♭mm FNpm B♭hm B♭Npm BM ⁒ G♯Npm

D♭hL F♯Npm C♭hL BhM Bhm GM ⁒ ⁒ F♯Npm–F♯hh F♯hM ⁒ ⁒ ⁒

where each scale or a pair linked by a dash lasts one measure, and symbol “⁒” means to repeat the previous measure. These scales are represented on the diagram with the global relationships of the hiatal group [14, Example 27], as it contains most of the scale types of this excerpt. However, the E♭mm scale (measure 6) could not be represented there, as it does not belong to that group, but to the octatonic and whole-tone groups. Thus, the scales of mm. 3–7 are then represented on the complex diagram with the global relationships of the three groups superimposed [14, Example 30].

For comparison, Figure 5 shows the same excerpt on the 7-Cyclops, which includes all the required scales, the initial one being marked with a double line. As can be seen, although the composition is in C minor, none of the scales Chm, Cmm, or C natural minor (E♭M) are used. Nevertheless, the initial scale, Ghh, includes the C minor chord and is played starting with C (its fourth mode). Then, the scales move counterclockwise in the diagram, that is, in the direction of flats (with back and forth movements), until reaching GM, whose tonic (at the bass) is the dominant of C. However, it does not resolve to any C minor scale, but to several scales with tonic F♯, a tritone away from C, the last four measures being based on the F♯ major chord.

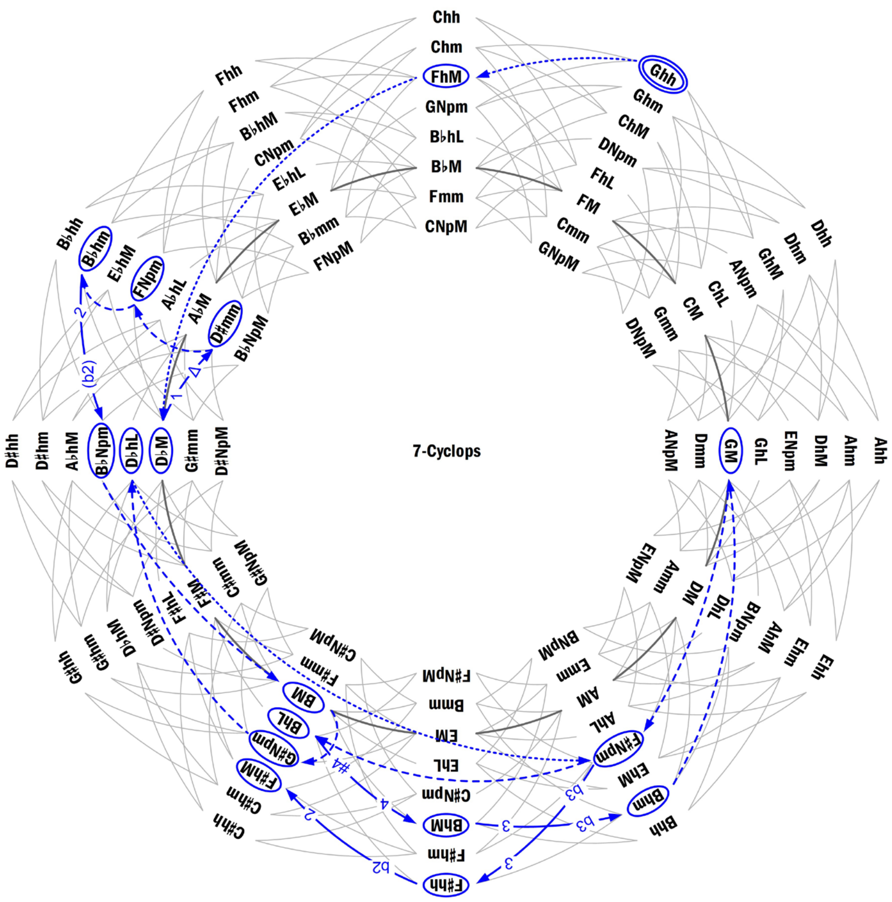

Fig. 5. Mozart, Fantasy in C minor, K. 475, mm. 1–25. Analysis with the 7-Cyclops.

6 Conclusions

Eight heptatonic scale types, together with their pentatonic complements, are selected following specific criteria. Their harmonic characteristics are analyzed and an acoustical relationship is found between each heptatonic scale type and the inversion of its pentatonic complement. Two novel parsimonious graphs, called 7-Cyclops and 5-Cyclops, are provided, which relate those heptatonic and pentatonic scales by single-semitonal transformations. Other parsimonious transformations, like moving one note by a whole tone, are easily found in them, too. As well, these graphs highlight the cycles of fifths and fourths, which are the only possible circumferences linking the same scale types in this kind of graphs (apart from the trivial cases of set types with one and eleven notes). Finally, an example of musical analysis is included, so all this information is expected to be of interest for theorists and composers.

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