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Optimal extensions of Lipschitz maps on metric spaces of measurable functions

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ABSTRACT

We prove a factorization theorem for Lipschitz operators acting on certain subsets of metric spaces of measurable functions and with values on general metric spaces. Our results show how a Lipschitz operator can be extended to a subset of other metric space of measurable functions that satisfies the following optimality condition: it is the biggest metric space, formed by measurable functions, to which the operator can be extended preserving the Lipschitz constant. As an application, we show the coarsest metric that can be given for a metric space in which an order bounded lattice-valued-Lipschitz map is defined. Concrete examples involving the relevant space $L^0(\mu)$ are given.

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1. Introduction and basic definitions: metric function spaces

Optimal extension of linear operators between Banach function spaces is a classical topic of Functional Analysis. Once one has a relevant operator defined on an L^p-space —for example, the Fourier transform, convolution maps or other operators of interest in Harmonic Analysis—, the next step to solve some important problems is to analyze if there is a bigger domain for the same operator. Some developments have been introduced in recent years to provide a new methodology to systematically construct such a maximal space, and it has been successfully used for more advanced theoretical results and applications (see [11–14,16,25] and the references therein). Further research has been also done to extend these results in the locally convex context, for example for the case of Fréchet function spaces (see [6], in particular, see Section 3.3, Theorem 3.3.1), and even for the non-locally convex setting (quasi-Banach function spaces, see [26]).

Rather than extending the results to more general classes of linear domain spaces, in this paper we are interested in the analysis of optimal extensions of a relevant class of non-linear operators. We are

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concerned with Lipschitz operators acting in what we call metric function spaces, that form a weaker version —topologically and algebraically— of the notion of Banach function space, including for example some particular subsets of function spaces of measurable functions endowed with a Fréchet topology (see [10]). Moreover, our results are somewhat connected with the analysis of the Lipschitz structure of Banach and quasi-Banach spaces, which is a topic of current interest; let us mention here the relevant works [1–4,17,22]. On the other hand, the specific study of Lipschitz operators on certain function spaces has also been carried out in recent years; we should refer here to a series of profound papers published by Kalton already in the present century (see [19–21]). The focus of attention in almost all of these papers is on ∞-norms: in fact, the cases of c_0, ℓ^∞ and $C(K)$ have proved to be of central importance for the study of Lipschitz isomorphisms on Banach spaces and, in general, of extensions of Lipschitz maps. Looking at the arguments presented there, the underlying reason for this seems to be that in these cases uniform pointwise domination and norm domination are the same thing. However, this property does not hold for general Banach function spaces, that are the main references for our framework, so we have to work with different ideas. Anyway, some of our arguments are inspired in the previously mentioned papers —mainly in [20]—, although we do not explicitly use them in our results.

Motivated also by some classical problems of the function theory, the aim of the present paper is to obtain the main optimality results for operators on function spaces in the metric context. That is, if (Ω, Σ, μ) is a measure space, we will consider the natural elements for this study: metric spaces whose elements are classes of μ-a.e. equal measurable functions, that is, subsets of $L^0(\mu)$. Our original motivation is given by the classical research project of improvement —in the sense of the function spaces involved—, of the results of Carleson on almost everywhere convergence of norm convergent series [8]. One of the results that is needed for this aim is related to the need of finding a (quasi) Banach function space continuously included in $L^0(\mu)$ as big as possible and still preserving continuity of a given (linear) operator (see for example [5,9] and the references therein). The inclusion is of course a linear map, but we can consider weaker structures: Lipschitz injective maps and subsets of measurable functions endowed with a metric preserving a minimal structure as function space. Thus, emulating the method for linear operators, we are interested in this paper in providing a constructive method to obtain the largest "structured" metric space to which a Lipschitz map can be extended.

In the second part of this introductory section, we provide some definitions and notions that will be central in the paper. Throughout the paper (Ω, Σ, μ) will be a finite measure space and (E, d) a metric space. As usual, we write $L^0(\mu)$ for the space of (classes of μ-a.e. equal) μ-measurable functions endowed with the μ-a.e. convergence. If ν is another measure on Σ, we will write $\nu \ll \mu$ if ν is absolutely continuous with respect to μ.

Definition 1.1. We say that a subset $I(\mu) \subseteq L^0(\mu)$ is a *function ideal set* if for every $A \in \Sigma$, $\chi_A \cdot I(\mu) \subseteq I(\mu)$. A function ideal set is a *metric function space* if there is a metric $\rho : I(\mu) \times I(\mu) \to \mathbb{R}$ such that $\rho(f\chi_A, g\chi_A) \leq \rho(f, g)$ for all $A \in \Sigma$ and $f, g \in I(\mu)$. We call such a metric ρ a function metric.

Example 1.2. Let us provide some examples.

1) Let us show first a standard example. Consider a Banach function space $X(\mu)$ over μ (see the definition at the end of the section). Then it is a metric function space endowed with the distance provided by the norm, i.e. $\rho(f, g) = \|f - g\|_{X(\mu)}$. However, note that the linearity is not required in the definition of metric function spaces, and so any set as $\{f\chi_A : A \in \Sigma\}$, $f \in X(\mu)$, is a metric function space with the metric given by the norm.
2) On the other hand, there are relevant distances other than normed metrics that we want to consider as function metrics. For instance, the canonical distance d_0 in $L^0(\mu)$ is a function metric, since for $f, g \in L^0(\mu)$,

$$d_0(f,g) = \int\limits_{\Omega} \frac{|f-g|}{|f-g|+1}\, d\mu \geq \int\limits_{A} \frac{|f-g|}{|f-g|+1}\, d\mu$$

$$= \int\limits_{\Omega} \frac{|f\chi_A - g\chi_A|}{|f\chi_A - g\chi_A|+1}\, d\mu = d_0(f\chi_A, g\chi_A), \quad A \in \Sigma.$$

Recall that this metric is not defined by any norm.

We are interested in *not assuming* the translation invariance of the metric in some of our results, and so we work in a more general framework than the one that is usually given by norms and seminorms in Banach and Fréchet function spaces.

The norms defined on classical Banach spaces of integrable functions (e.g., the $L^p(\mu)$ spaces for $1 \leq p \leq \infty$) provide metrics that are well suited to the measure used to support the function space. Indeed, the norm of a Banach function space $X(\mu)$ fits with the measure μ, since $\|\chi_A\|_{X(\mu)} = 0$ if and only if $\mu(A) = 0$. However, we do not restrict our attention to the case of Banach function spaces: we want our results to work for operators on the metric space $(L^0(\mu), d_0)$.

A recent attempt to relate metrics to measures in the context of Lipschitz functions is given by the so called intrinsic measures on metric spaces, appeared in the setting of the Dirichlet and Wiener spaces. A complete study of this concept can be found in [18] and the references therein. However, although this theory is interesting for applications in, for example, evolution equations, it does not fit with our aim. The reason is that the notion of measurable pseudometric is central in that development, and our main natural example —that is given by the distance defined by a norm in a Banach function space— does not satisfy the axioms of a measurable pseudometric. Concretely, Axiom 4 in [18, Definition 2.1] is not satisfied by metrics like $d(f,g) = \|f-g\|_{X(\mu)}$, where $X(\mu)$ is a Banach function space and $f, g \in X(\mu)$.

As we already said, along the paper we will compare our construction with the case of linear operators defined on Banach function spaces. Recall that $X(\mu)$ is a Banach function space over μ if it is a Banach space defined by equivalence classes of μ-a.e. equal measurable functions, with a lattice norm (that is, if $|f| \leq |g|$, then $\|f\| \leq \|g\|$). A strictly positive function $0 < h \in X(\mu)$ is what is called a weak unit for $X(\mu)$. The space $X(\mu)$ is order continuous if each decreasing sequence $f_n \downarrow 0$ satisfies that $\lim \|f_n\| = 0$. A continuous linear operator $T : X(\mu) \to E$, where E is a Banach space, always defines a vector measure $m_T : \Sigma \to E$ given by the formula

$$m_T(A) = T(\chi_A), \quad A \in \Sigma.$$

If $X(\mu)$ is order continuous, then m_T is countably additive. The semivariation of this vector measure is given by

$$\|m_T\|(A) = \sup_{B \in \Sigma} \|T(\chi_{A \cap B})\|, \quad A \in \Sigma.$$

For the definition of what a Banach function spaces is and its main properties, we refer to [24, p.28]. The space $L^1(m_T)$ of integrable functions with respect to the vector measure m_T plays a fundamental role in the general theory of operators acting in Banach function spaces, since it provides the description of the so-called optimal domain of a given operator. Indeed, if $X(\mu)$ is an order continuous Banach function space with a weak unit and $T : X(\mu) \to E$ is a linear and continuous operator, there is a factorization of T through $L^1(m_T)$

$$
\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, "i"'] & & E, \\
& L^1(m_T) \arrow[ur, "I_{m_T}"'] &
\end{tikzcd}
$$

where I_{m_T} is the integration operator associated to the vector measure m_T. This factorization is maximal (or optimal); in the sense that $L^1(m_T)$ is the biggest order continuous Banach function space with weak unit to which T can be extended. In other words, if there is another factorization like that through a space $Y(\mu)$ satisfying these requirements, then $Y(\mu) \subseteq L^1(m_T.)$ The reader can find all the information needed on these spaces in [25, Ch.3], and on the optimal factorization in [25, Ch.4] (see Th.4.14).

2. Extension of Lipschitz functions to maximal metric domains

The extension of Lipschitz functions defined on a subset of a metric space to the whole space is a central topic in mathematical analysis. Let us recall here some important facts on the topic. For the real-valued case, the McShane-Whitney theorem establishes that, given a subset U of a metric space (M, ρ) and a Lipschitz function $T : U \to \mathbb{R}$ with Lipschitz constant k, there always exist Lipschitz functions $M \to \mathbb{R}$ extending T and with the same Lipschitz constant k; for example, the functions

$$T_1(x) := \inf_{u \in U} \{T(u) + k\, \rho(x,u)\}, \quad x \in M,$$

and

$$T_2(x) := \sup_{u \in U} \{T(u) + k\, \rho(x,u)\}, \quad x \in M,$$

provide two extensions of T that keep as Lipschitz constant k.

In the vector valued case, the celebrated Kirszbraun theorem gives extensions of Lipschitz operators that take values in Hilbert spaces. It states that if H and K are Hilbert spaces, U is a subset of H and $T : U \to K$ is a Lipschitz operator, then there is another Lipschitz operator $\tilde{T} : H \to K$ that extends T and has the same Lipschitz constant as T (see [23], [27, p.21]). However, the result is not true for general Banach spaces, even in the finite dimensional case.

As we said in the introduction, there are many recent results on the extension of Lipschitz maps from subsets of Banach spaces to the whole space. Special attention must be paid to the case of Lipschitz maps on $C(K)$ spaces; see [19] and the references already given. Our results are of a different nature, and they deal with extensions of Lipschitz maps on metric (non-linear) subspaces $I(\mu)$ of measurable functions in $L^0(\mu)$ to a bigger space. Technically, they must be probably called “factorizations” rather than “extensions”, as they give factorization theorems preserving Lipschitz maps.

The technique we use is inspired in a series of results that are sometimes called optimal domain theorems for Banach function spaces, which give descriptions of the biggest Banach function spaces satisfying a certain property to which a given linear operator can be extended. We have already given some indications in this regard in the introduction. The reader can find information about the fundamental structure of the theory in [25, Ch.4] and the references therein; see also the references given in the introduction. The same type of technique has been recently applied in more general contexts, for example in the case of continuous operators acting in quasi-Banach spaces ([26]), or for operators on Banach function spaces that satisfy a certain domination inequality ([7]). The papers which are directly connected with the present one are [10,11]. In [11] an optimal domain theorem for operators on $L^0(\mu)$ is given. Conceptually, this is the starting point of our analysis, since it provides a description of the maximal domain for operators having values in the space $L^0(\mu)$, in which the topology is not given by a (quasi) norm but by a metric. The main difference of our study is that we are interested in working with (non necessarily linear) function spaces preserving some lattice properties in the context of the metric spaces, which is the natural one when dealing with Lipschitz functions. In [10] it is treated the situation of factoring Lipschitz maps on metric spaces with values in a Banach function space through Lipschitz maps that are maximal in the sense that any other similar factorization scheme relates the extension to the former one. In the present paper the point of view is completely different: instead of considering spaces of measurable functions as the range of the extendable Lipschitz map, we locate the measurable functions in the domain, where we provide a structure of what we call metric function space, which can be considered an extension of the notion of Banach function space to the metric setting, with no linear structure.

Lemma 2.1. *Let $(I(\mu), \rho)$ be a metric function space, $S \subseteq I(\mu)$ a subset, and consider a Lipschitz map $T : I(\mu) \to E$ into the complete metric space (E, d). Then the restriction $T|_S$ of T to S can be factored through a map i with Lipschitz constant $Lip(i) \leq Lip(T)$ and a Lipschitz operator $\overline{T} : S_T \to E$ with $Lip(\overline{T}) \leq 1$ as*

$$\begin{tikzcd} S \arrow[rr, "T|_S"] \arrow[dr, "i"'] & & E, \\ & S_T \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

where $(S_T, \overline{d}_T)$ is a complete metric space in which $i(S)$ is dense, and $\overline{d}_T$ is the quotient metric associated to the pseudometric

$$d_T(f, g) := \sup_{A \in \Sigma} d(T(f\chi_A), T(g\chi_A)), \quad f, g \in S.$$

Besides, if we assume that $S = S(\mu) \subseteq I(\mu)$ is a metric function space (that is, $\chi_A \cdot S \subset S$ for all $A \in \Sigma$) then $Lip(\overline{T}) = 1$, and if we assume that $S = I(\mu)$ then $Lip(i) = Lip(T)$.

Proof. Consider the function $d_T : S \times S \to \mathbb{R}$ defined by

$$d_T(f, g) := \sup_{A \in \Sigma} d(T(f\chi_A), T(g\chi_A)), \quad f, g \in S.$$

Note that it can only take a finite value, since for $f, g \in S$ and $A \in \Sigma$ we have

$$d(T(f\chi_A), T(g\chi_A)) \leq Lip(T)\, \rho(f\chi_A, g\chi_A) \leq Lip(T)\, \rho(f, g) < \infty, \tag{1}$$

due to the fact that ρ is a function metric. Let us show that d_T is a pseudo metric on $I(\mu)$. Clearly, d_T is a symmetric function, for each $f \in S$, $d_T(f, f) = 0$ and for $f, g, h \in S$, $d_T(f, h) \leq d_T(f, g) + d_T(g, h)$.

Consider the quotient S/d_T endowed with the quotient metric $\overline{d}_T$.

Let $i : S \to S/d_T$ be the quotient map and denote S_T the completion of S/d_T keeping the notation $\overline{d}_T$ for the extended metric. We have that $(S_T, \overline{d}_T)$ is the desired factorization space. Indeed, the map $i : S \to S_T$ given by $i(f) := [f]_{d_T}$, where $[\cdot]_{d_T}$ is the corresponding equivalence class, is the required map. It follows from (1) that i is a Lipschitz map with $Lip(i) \leq Lip(T)$.

Define $T_0 : i(S) \to E$ by $T_0(i(f)) := T(f)$. Let us show that T_0 is well defined. Take $f, g \in S$ so that $i(f) = i(g)$. Then, $d_T(f, g) = 0$. Hence, $d(T(f), T(g)) = 0$. Thus $T_0(i(f)) = T(f) = T(g) = T_0(i(g))$.

We extend T_0 to $\overline{T} : (S_T, \overline{d}_T) \to E$ by continuity, providing the factorization $T|_S = \overline{T} \circ i$.

Note that

$$\begin{aligned} d(\overline{T}(i(f)), \overline{T}(i(g))) = d(T(f), T(g)) &\leq \sup_{A \in \Sigma} d(T(f\chi_A), T(g\chi_A)) \\ &= d_T(f,g) = \overline{d}_T(i(f), i(g)), \end{aligned}$$

and so $Lip(\overline{T}) \leq 1$.

Note also that

$$\begin{aligned} \overline{d}_T(i(f), i(g)) &= \sup_{A \in \Sigma} d(T(f\chi_A), T(g\chi_A)) \\ &\leq Lip(T) \sup_{A \in \Sigma} \rho(f\chi_A, g\chi_A) \leq Lip(T)\rho(f,g), \end{aligned}$$

and so $Lip(i) \leq Lip(T)$.

Let us assume now that $S = I(\mu)$ and let us prove that $Lip(i) = Lip(T)$. Note that given $\varepsilon > 0$ there are $f, g \in I(\mu)$ such that $d(T(f), T(g)) > (Lip(T) - \varepsilon)\rho(f,g)$. Thus,

$$\overline{d}_T(i(f), i(g)) = d_T(f,g) \geq d(T(f), T(g)) > (Lip(T) - \varepsilon)\rho(f,g),$$

what gives that indeed $Lip(i) = Lip(T)$.

We have already proved that $Lip(\overline{T}) \leq 1$. Let us see that $Lip(\overline{T}) = 1$ under the assumption that $\chi_A \cdot S \subset S$ for all $A \in \Sigma$. Assume that this is not the case, so there is a constant $0 < C < 1$ such that

$$d(\overline{T}(i(f)), \overline{T}(i(g))) \leq C\overline{d}_T(i(f), i(g))$$

for all $f, g \in S$. Fix $f, g \in S$, $f \neq g$. Let $\epsilon := \frac{1-C}{2} d_T(f,g) > 0$. By the definition of d_T, we can find $A \in \Sigma$ such that

$$d(T(f\chi_A), T(g\chi_A)) > \overline{d}_T(i(f), i(g)) - \epsilon.$$

By the definition of d_T it is easy to check that $d_T(f,g) \geq d_T(f\chi_A, g\chi_A)$. Then we have that

$$\begin{aligned} Cd_T(f,g) &\geq Cd_T(f\chi_A, g\chi_A) \\ &= C\overline{d}_T(i(f\chi_A), i(g\chi_A)) \\ &\geq d(\overline{T}(i(f\chi_A)), \overline{T}(i(g\chi_A))) \\ &= d(T(f\chi_A), T(g\chi_A)) \\ &> \overline{d}_T(i(f), i(g)) - \epsilon \\ &= d_T(f,g) - \epsilon \end{aligned}$$

Therefore, we get that $(1-C)d_T(f,g) < \epsilon = \frac{1-C}{2} d_T(f,g)$, which is a contradiction. We have then proved that $Lip(\overline{T}) = 1$.

This finishes the proof. $\square$

Remark 2.2. Note that an alternative definition to the one given in Lemma 2.1 would be given by a formula as

$$\tau(f,g) := \sup_{A,B \in \Sigma} d(T(f\chi_A), T(g\chi_B)).$$

However, this formula does not generalize our main examples. In fact, it is not a pseudometric in general. For instance, if the metric is given by a Banach function space norm $\|\cdot\|$ and T is the identity map, that is $d(f,g) := \|f-g\|$, we have that the expression

$$\tau(f,g) := \sup_{A,B\in\Sigma} d(T(f\chi_A), T(g\chi_B)) = \sup_{A,B\in\Sigma} \|f\chi_A - g\chi_B\|$$

does not satisfy that $\tau(f,f) = 0$.

Results on this kind of factorization for the case of linear operators acting on Banach function spaces —even for quasi-Banach spaces— are nowadays well-known, and can be compared with the previous lemma. In the linear case, the role that plays the space S_T in Lemma 2.1 is played by the space $L^1(m_T)$ of integrable functions with respect to a vector measure (see for example [12,15] and for a complete explanation of the underlying ideas and applications, see also [25, Ch.4]). Of particular interest for the present paper —due to the fact that non-normable topologies are explicitly considered—, are the papers [11,26]. In these cases, the factorization is also optimal; we will see in Lemma 2.3 below that this is also the case with our factorization.

If $I(\mu)$ and $J(\nu)$ are metric function spaces such that $\nu \ll \mu$, then any pair f,g of μ-measurable functions which are equal μ-a.e., are ν-measurable and coincide ν-a.e. We are interested in metric function spaces $I(\mu)$ and $J(\nu)$ such that the canonical mapping $i_0 : I(\mu) \to J(\nu)$ that sends a function $f \in I(\mu)$ to its class in $J(\nu)$ can be defined and is continuous. Note that for any $f \in I(\mu)$ and any $A \in \Sigma$ we have that $i_0(f\chi_A) = i_0(f)\chi_A$.

Lemma 2.3. *Let $(I(\mu),\rho)$ be a metric function space and consider a Lipschitz map $T : I(\mu) \to E$ into the complete metric space (E,d). Then the factorization through $I(\mu)_T$ given by Lemma 2.1 is maximal in the following sense. Assume there is a complete metric function space $(J(\nu),\overline{\rho})$ over a measure ν with $\nu \ll \mu$ such that the canonical map $i_0 : I(\mu) \to J(\nu)$ is well defined and continuous. Assume that:*

1. *the operator T factors as $T = \widetilde{T} \circ i_0$, with $\widetilde{T} : J(\nu) \to E$,*
2. *$\widetilde{T}$ is Lipschitz with $Lip(\widetilde{T}) \le 1$,*
3. *$i_0(I(\mu))$ is dense in $J(\nu)$.*

Then there is a Lipschitz map $k : J(\nu) \to I(\mu)_T$ such that $Lip(k) \le 1$, $k \circ i_0 = i$, and $\overline{T} \circ k = \widetilde{T}$. Besides, if $Lip(\widetilde{T}) = 1$ then $Lip(k) = 1$.

Proof. For the proof, note that we are in the setting of Lemma 2.1 for $S = I(\mu)$, so the factorization through $I(\mu)_T$ follows with $Lip(\overline{T}) = 1$. Assume that there are a complete metric function space $J(\nu)$ for an absolutely μ-continuous measure ν such that $i_0(I(\mu))$ is dense in $J(\nu)$ and $Lip(i_0) \le Lip(T)$, and a Lipschitz mapping $\widetilde{T} : J(\nu) \to E$ with $Lip(\widetilde{T}) \le 1$ such that $T = \widetilde{T} \circ i_0$.

We aim to define a map k so that the following factorization diagram holds,

$$\begin{tikzcd}
I(\mu) \arrow[rr, "T"] \arrow[dr, "i"] \arrow[ddr, "i_0"'] & & E, \\
& I(\mu)_T \arrow[ur, "\overline{T}"] & \\
& J(\nu) \arrow[u, "k"] \arrow[uur, "\widetilde{T}"'] &
\end{tikzcd}$$

We define $k : i_0(I(\mu)) \to I(\mu)_T$ as the mapping given by $k(i_0(f)) := i(f)$. First we see that k is well defined, that is, if $i_0(f) = i_0(g)$ for $f, g \in I(\mu)$, then $i(f) = i(g)$. Indeed, if $i_0(f) = i_0(g)$ then $T(f) = T(g)$. Hence $i(f) = i(g)$ by the construction in the proof of Lemma 2.1.

Taking into account that $\overline{\rho}$ is a metric function on $J(\nu)$ we get that

$$\overline{d}_T(k(i_0(f)), k(i_0(g))) = \overline{d}_T(i(f), i(g)) = d_T(f, g) = \sup_{A\in\Sigma} d(T(f\chi_A), T(g\chi_A))$$
$$= \sup_{A\in\Sigma} d(\widetilde{T}(i_0(f\chi_A)), \widetilde{T}(i_0(g\chi_A))) \le \sup_{A\in\Sigma} \overline{\rho}(i_0(f\chi_A), i_0(g\chi_A)) \le \overline{\rho}(i_0(f), i_0(g)).$$

Now we extend k so defined to the whole $J(\nu)$ using the density of $i_0(I(\mu))$ in $J(\nu)$. So extended, we have that k is a Lipschitz map and $Lip(k) \le 1$ by the inequalities above.

Let us now assume that $Lip(\widetilde{T}) = 1$ and let us see that $Lip(k) = 1$. Since $Lip(\widetilde{T}) = 1$ we have that given $\varepsilon > 0$ there are $f, g \in J(\nu)$ such that $d(\widetilde{T}(f), \widetilde{T}(g)) > (1-\varepsilon)\overline{\rho}(f, g)$. Using both the continuity of $\widetilde{T}$ and the continuity of k, we can find $\delta > 0$ such that if $f_0, g_0 \in I(\nu)$ satisfy $\overline{\rho}(f, i_0(f_0)) < \delta$ and $\overline{\rho}(g, i_0(g_0)) < \delta$ then

$$d\big(\widetilde{T}(i_0(f_0)), \widetilde{T}(f)\big) < \varepsilon, \quad d\big(\widetilde{T}(i_0(g_0)), \widetilde{T}(g)\big) < \varepsilon$$

and

$$\overline{d}_T\big(k(i_0(f_0)), k(f)\big) < \varepsilon, \quad \overline{d}_T\big(k(i_0(g_0)), k(g)\big) < \varepsilon.$$

Hence,

$$\begin{aligned}(1-\varepsilon)\overline{\rho}(f, g) &< d\big(\widetilde{T}(f), \widetilde{T}(g)\big) \le d\big(\widetilde{T}(f), \widetilde{T}(i_0(f_0))\big) \\ &\quad + d\big(\widetilde{T}(i_0(f_0)), \widetilde{T}(i_0(g_0))\big) + d\big(\widetilde{T}(i_0(g_0)), \widetilde{T}(g)\big) \\ &< \varepsilon + \sup_{A\in\Sigma} d\big(\widetilde{T}(i_0(f_0\chi_A)), \widetilde{T}(i_0(g_0\chi_A))\big) + \varepsilon \\ &= 2\varepsilon + \sup_{A\in\Sigma} d\big(T(f_0\chi_A), T(g_0\chi_A)\big) \\ &= 2\varepsilon + \overline{d}_T\big(i(f_0), i(g_0)\big) = 2\varepsilon + \overline{d}_T\big(k(i_0(f_0)), k(i_0(g_0))\big) \\ &\le 4\varepsilon + \overline{d}_T\big(k(f), k(g)\big).\end{aligned}$$

Since ε is arbitrary, we get that $Lip(k) \ge 1$ and thus $Lip(k) = 1$. $\square$

Note that the maximal metric space $I(\mu)_T$ is not necessarily formed by classes of measurable functions with respect to μ. However, if the operator T appearing in Lemma 2.1 is injective, we have that d_T is a metric, and so the functions in $I(\mu)$ (as equivalence classes of functions with respect to μ) are held when considered in $I(\mu)_T$. Any other measure ν and space $J(\nu)$ allowing a factorization as above must satisfy that is equivalent to μ, at least in the support of the functions of $I(\mu)$. However, these arguments are delicate and will be treated in the next section.

3. Measure-type aspects concerning the optimal factorization of Lipschitz maps acting in subsets of metric function spaces

In what follows, we will analyze the measure theoretic aspects of the concepts appeared in the previous results. Adapting the notion of μ-determined operator that is used in the linear case, we introduce the following definition.

Recall that a sequence (f_n) of measurable functions is said to be a Cauchy sequence in μ-measure if, given $\varepsilon > 0$, there is an N such that for all $m, n \ge N$ we have $\mu\{w \in \Omega : |f_n(w) - f_m(w)| \ge \varepsilon\} < \varepsilon$.

Definition 3.1. A Lipschitz map $T : I(\mu) \to E$ is μ-determined if and only if every d_T-Cauchy sequence (f_n) in $I(\mu)$ is Cauchy in μ-measure.

Remark 3.2.

1) We may assume that both metric spaces are pointed and $T(0) = 0$. Note that being T μ-determined implies that given $A \in \Sigma$, then $d(T(f\chi_A), 0) = 0$ for all $f \in I(\mu)$ if and only if $\mu(A) = 0$. Indeed, if we fix $A \in \Sigma$ such that $d(T(f\chi_A), 0) = 0$ for all $f \in I(\mu)$, in particular $d(T(\chi_B\chi_A), 0) = 0$ for all $B \in \Sigma$. Then,

$$d_T(\chi_A, 0) = \sup_{B\in\Sigma} d(T(\chi_A\chi_B), T(0\chi_B)) = \sup_{B\in\Sigma} d(T(\chi_A\chi_B), 0) = 0.$$

Therefore, the alternate sequence (f_n), where $f_n = \chi_A$ if n is even and $f_n = 0$ if n is odd is d_T-Cauchy. Hence, assuming that T is μ-determined, it follows that (f_n) is Cauchy in μ-measure. In particular, for any $\varepsilon > 0$,

$$\mu(\{w : |\chi_A(w)| \geq \varepsilon\}) < \varepsilon,$$

and so $\mu(A) = 0$. The converse is obvious.

2) In the particular case that T is linear, $(I(\mu), \rho)$ is a Banach function space $X(\mu)$ and $(E, \|.\|_E)$ is a Banach space, Definition 3.1 coincides with the usual definition of μ-determined operator (see for example [25, 4.2, p.187]). Indeed, changing the distances in the argument above by the corresponding norms, we get that if A is a measurable set and the semivariation of the vector measure m_T associated to the operator T in A,

$$\|m_T\|(A) := \sup_{B\in\Sigma} \|T(\chi_{A\cap B})\|$$

is 0, then $\mu(A) = 0$. This means that T is μ-determined in the standard sense.

In the case that a linear continuous operator from a Banach function space $X(\mu)$ to a Banach space E is μ-determined, we have that the map i in the maximal factorization through the space $L^1(m_T)$ of integrable functions that has been explained in the introduction, is injective. The next result provides the corresponding Lipschitz version. However, to get it we need another additional property for the Lipschitz operator T that plays the role of the order semi continuity of the operator in the linear case. This is usually obtained as a consequence of the order continuity of the Banach function space $X(\mu)$ (see for example [24, I]). Let us give the corresponding property for Lipschitz maps.

Definition 3.3. Let $I(\mu)$ a metric function space and E a metric space. We say that a Lipschitz map $T : I(\mu) \to E$ is order semi continuous if for each pair of d_T-convergent sequences $(f_n), (g_n)$ in $I(\mu)$, if we have that $|f_n - g_n| \downarrow_n 0$ μ-a.e., then $d(T(f_n), T(g_n)) \to_n 0$.

If we introduce the hypothesis that the Lipschitz map T is μ-determined and order semi continuous, we get a better factorization theorem, which improve the previous results in two different ways. The first one is that the factorization is properly an extension. The second one is that the maximal domain is also a metric function space.

Theorem 3.4. *Let $(I(\mu), \rho)$ be a metric function space and consider a Lipschitz map $T : I(\mu) \to E$ with values in the metric space (E, d). Suppose that T is order semi continuous and μ-determined. Then T can be extended to a Lipschitz map $\overline{T} : I(\mu)_T \to E$ with Lipschitz constant 1, that is,*

$$\begin{tikzcd} I(\mu) \arrow[rr, "T"] \arrow[dr, "i"'] & & E, \\ & I(\mu)_T \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

where $(I(\mu)_T, d_T)$ is a complete metric function space in which $I(\mu)$ is dense, d_T is defined by

$$d_T(f,g) := \sup_{A\in\Sigma} d(T(f\chi_A), T(g\chi_A)), \quad f,g \in I(\mu),$$

and i is an inclusion map with $Lip(i) = Lip(T)$.

Moreover, $\overline{T}$ is maximal, in the sense that $I(\mu)_T$ is the biggest metric function space in which $I(\mu)$ is dense and to which the operator T can be extended as a Lipschitz operator.

Proof. As a consequence of Lemma 2.1, we have the factorization through the space $I(\mu)_T$ with $Lip(\overline{T}) = 1$ and $Lip(i) = Lip(T)$. We only have to show that the fact that T is μ-determined implies that $(I(\mu)_T, \overline{d}_T)$ is a metric function space and that $i : I(\mu) \to I(\mu)_T$ is an inclusion map. We have for $f,g \in I(\mu)$ that $d_T(f,g) \le Lip(T)\rho(f,g)$. Consider a Cauchy sequence (f_n) in the quotient space $I(\mu)/d_T$ that converges to $x \in I(\mu)_T$ with respect the quotient metric $\overline{d}_T$. Since T is μ-determined, we have that (f_n) is Cauchy in μ-measure, and so converges to a function $h_0 \in L^0(\mu)$. We identify the function h_0 with the element of the completion x as follows. Note that the function h_0 does not depend on the particular sequence that we choose: if (g_n) is another sequence with limit x, the sequence defined as $h_n = f_{n/2}$ if n is even, and $h_n = g_{(n+1)/2}$ if n is odd, d_T converges also to x and so is d_T-Cauchy. Therefore, converges to h_0 in the Hausdorff topology given by the convergence in μ-measure, and so each subsequence (f_n) and (g_n) must have the same limit.

This allows to construct a map $k : I(\mu)_T \to L^0(\mu)$ by $x \mapsto h_0$. To see that k is in fact injective we need to use the order semi continuity of T. Indeed, if there are two different elements $x, y \in I_T(\mu)$ such that $k(x) = k(y)$, there are two different sequences (f_n) and (g_n) that converge in μ-measure to the same function $h_0 = k(x) = k(y) \in L^0(\mu)$. Therefore, there are subsequences (f_{n_k}) and (g_{n_k}) of (f_n) and (g_n), respectively, such that

- $\lim_k \overline{d}_T(f_{n_k}, x) = 0$,
- $\lim_k \overline{d}_T(g_{n_k}, y) = 0$, and
- $\lim_k f_{n_k} = \lim_k g_{n_k} = h_0$ μ-a.e.

Consider the inequalities

$$\overline{d}_T(x,y) \le \overline{d}_T(f_{n_k}, x) + \overline{d}_T(g_{n_k}, y) + d_T(f_{n_k}, g_{n_k}).$$

By the definition of d_T, there are $A_k \in \Sigma$ such that

$$d_T(f_{n_k}, g_{n_k}) \le d(T(f_{n_k}\chi_{A_k}), T(g_{n_k}\chi_{A_k})) + 1/k.$$

Note that

$$|f_{n_k}\chi_{A_k} - g_{n_k}\chi_{A_k}| = |f_{n_k} - g_{n_k}|\chi_{A_k} \le |f_{n_k} - g_{n_k}| \downarrow_k 0$$

μ-a.e., and then using the fact that T is order semi continuous, we get that $\lim_k d(T(f_{n_k}\chi_{A_k}), T(g_{n_k}\chi_{A_k})) = 0$. Thus

$$\begin{aligned}&\overline{d}_T(x,y)\\ &\leq \lim_k \overline{d}_T(f_{n_k},x)+\lim_k \overline{d}_T(g_{n_k},y)+\lim_k d(T(f_{n_k}\chi_{A_k}),T(g_{n_k}\chi_{A_k}))+\lim_k 1/k=0,\end{aligned}$$

what gives a contradiction with the fact that $\overline{d}_T(x,y)>0$.

Let us prove now that $k(I(\mu)_T)$ is a function ideal set; that is, for every $h_0 \in k(I(\mu)_T)$ and $A \in \Sigma$, $h_0\chi_A \in k(I(\mu)_T)$. Let $x \in I(\mu)_T$ be so that $k(x)=h_0$ and take a sequence (f_n) in $I(\mu)/d_T$ converging to x. Then $d_T(f_n\chi_A, f_m\chi_A) \leq d_T(f_n,f_m)$ and so $(f_n\chi_A)$ is d_T-Cauchy. Since T is μ-determined, we have that this sequence converges also in μ-measure to a function $h_A \in L^0(\mu)$. On the other hand, (f_n) converges in μ-measure to h_0 and so $(f_n\chi_A)$ converges in μ-measure to $h_0\chi_A$ too. Therefore, we have the μ-a.e equality $h_0\chi_A=h_A \in k(I(\mu)_T)$ as we wanted to show.

We have already identified via k the space $I(\mu)_T$ with a subset of $L^0(\mu)$ that satisfies that $k(I(\mu)_T)\chi_A \subset k(I(\mu)_T)$. Thus, we can use this identification to consider from now on $I(\mu)_T$ as a subset of $L^0(\mu)$ endowed with the metric function $\overline{d}_T$ to define a metric function space.

Moreover, let us see that if $f,g \in I(\mu)$ are such that $d_T(f,g)=0$ then $f=g$ μ-a.e.; that is, the map $i: I(\mu) \to I(\mu)_T$ is injective. Indeed, if we define $f_n=f$ when n is odd and $f_n=g$ when n is even then the sequence (f_n) is clearly d_T-Cauchy. As T is μ-determined it follows that (f_n) is Cauchy in μ-measure and so $f=g$ μ-a.e.

Identifying $I(\mu)_T$ with $k(I(\mu)_T)$, let us consider the distance

$$\overline{d}_{T,k}: k(I(\mu)_T)\times k(I(\mu)_T) \to \mathbb{R}$$

given by $\overline{d}_{T,k}(k(x),k(y))=\overline{d}_T(x,y)$. We prove that $(k(I(\mu)_T),\overline{d}_{T,k})$ is a metric function space, i.e.

$$\overline{d}_{T,k}(k(x)\chi_A,k(y)\chi_A) \leq \overline{d}_{T,k}(k(x),k(y))$$

for any $x,y \in I(\mu)_T$. Take $h_n,f_n \in I(\mu)$ $\overline{d}_T$-converging to x,y respectively. As above we have that h_n,f_n converge μ-a.e. to $h_0=k(x)$ and $f_0=k(y)$ respectively. The sequences $\chi_A h_n$ and $\chi_A f_n$ converge μ-a.e. to $\chi_A h_0$ and $\chi_A f_0$ respectively and since both sequences are d_T-Cauchy and T is μ-determined then $\chi_A h_n$ and $\chi_A f_n$ converge μ-a.e. to some $h_A, f_A \in L^0(\mu)$ respectively. Since $I(\mu)_T$ is complete, the sequences $\chi_A h_n$ and $\chi_A f_n$ also converge to some x_A and y_A in $I(\mu)_T$ respectively. Then,

$$h_A=h_0\chi_A=k(x_A) \qquad \text{and} \qquad f_A=f_0\chi_A=k(y_A).$$

Hence,

$$\begin{aligned}\overline{d}_{T,k}(h_0\chi_A,f_0\chi_A)&=\overline{d}_T(x_A,y_A)=\lim_n \overline{d}_T(\chi_A h_n,\chi_A f_n)\\ &=\lim_n d_T(\chi_A h_n,\chi_A f_n)\leq \lim_n d_T(h_n,f_n)\\ &=\lim_n \overline{d}_T(h_n,f_n)=\overline{d}_T(x,y)=\overline{d}_{T,k}(h_0,f_0).\end{aligned}$$

Clearly $\overline{T}|_{I(\mu)}=T$.

Finally, the maximality of $\overline{T}$ is easy to see. Indeed, take another extension $T_0: J \to E$ such that $i_0: I(\mu) \to J$ is a continuous inclusion and its range is dense in $(J,\overline{\rho})$, which is a metric function space (and so included in $L^0(\mu)$). A direct application of Lemma 2.3 gives the result. □

In the case of Banach spaces, this result provides a "genuine Lipschitz" extension theorem not necessarily preserving linearity but acting on linear structures. We need an additional requirement for preserving the linear structure of the factorization space that is related with the d_T-continuity of the operations of the linear space.

Definition 3.5. Let E be a complete metric space and $X(\mu)$ be a Banach function space. We say that the Lipschitz operator $T : X(\mu) \to E$ is linear-space-preserving if for every pair of d_T-Cauchy sequences (f_n) and (g_n) in $X(\mu)$ and $a, b \in \mathbb{R}$, we have that the sequence $(af_n + bg_n)$ is also d_T-Cauchy.

Corollary 3.6. *Let $X(\mu)$ be a Banach function space, E a complete metric space, and let $T : X(\mu) \to E$ be a semi order continuous μ-determined Lipschitz map. Then T can be extended to a maximal Lipschitz map $\overline{T} : X(\mu)_T \to E$ —in the sense of Theorem 3.4—, with Lipschitz constant 1, where $(X(\mu)_T, d_T)$ is a complete metric function space in which the linear space $X(\mu)$ is dense.*

Moreover, if T is linear-space-preserving, then $X(\mu)_T$ is a linear metric function space.

Proof. It is a direct consequence of Theorem 3.4. Only the "moreover" part needs a proof. Take $f_0, g_0 \in X(\mu)_T$, $a, b \in \mathbb{R}$ and consider sequences (f_n) and (g_n) in $X(\mu)$ that d_T-converge to f_0 and g_0, respectively. Then by hypothesis we have that $(af_n + bg_n) \subset X(\mu)$ is d_T-Cauchy. Then it converges to an element h_0 of $X(\mu)_T$, that we know that can be represented as a (class of) measurable function(s).

On the other hand, we have that (af_n) converges in μ-measure to af_0, and (bg_n) to bg_0. Therefore, $(af_n + bg_n)$ converges in μ-measure to $af_0 + bg_0$. Since T is μ-determined, we have that $(af_n + bg_n)$ converges in μ-measure to h_0. Consequently, we have that $af_0 + bg_0 = h_0 \in X(\mu)_T$, and the result is proved. □

Remark 3.7.

1) Corollary 3.6 provides a useful application. Under the assumptions of Theorem 3.4, if there is another Banach function space $Z(\mu)$ containing $X(\mu)$ such that the second one is dense in the first one and T can be extended to $Z(\mu)$, then we know that $Z(\mu)$ is included in $X(\mu)_T$. Moreover, although there is no reason to assume that the topology in $X(\mu)_T$ is normable, in the case that T is linear-space-preserving we have that there is a (linear) inclusion of $Z(\mu)$ into $X(\mu)_T$. This opens the door to find new maximality results on Lipschitz maps acting in Banach function spaces, in cases in which at least the linear structure of the space is preserved.
2) If the range space E is a Banach space Y, we have an explicit formula for the metric function d_T as

$$d_T(f, g) = \sup_{A \in \Sigma} \|T(f\chi_A) - T(g\chi_A)\|_Y, \quad f, g \in X(\mu)$$

that suggests the relation with the linear case and the equivalent norm for the maximal space

$$\|f\|_T = \sup_{A \in \Sigma} \|T(f\chi_A)\|_Y$$

for any $f \in I(\mu)$. Recall that in this case, the maximal space is the space of integrable functions $L^1(m_T)$. Note that $\|.\|_T$ determines a norm in the quotient space $X(\mu)/\|.\|_T$ and $X(\mu)_T$ is its completion endowed with the extended norm. Therefore, $(X(\mu)_T, \|.\|_T)$ becomes a Banach space in this case (see [25, Ch.4]).

4. The optimal metric for Lipschitz operators acting in subsets of metric function spaces

In Section 2 we have shown how to factor Lipschitz maps T that are defined on subsets S of metric function spaces $I(\mu)$. To get such a factorization, the operator had to be defined on the whole metric function space $I(\mu)$. In this section we will see that we can get such a factorization just considering Lipschitz maps defined on S instead of in the whole $I(\mu)$. In order to get this, we need to ask for a lattice structure on the range space. The order relation in the range space will play a relevant role.

Let ν be a measure. We say that a metric space (E, d), where E is a vector lattice of (classes of ν-a.e. equal) measurable real functions, is a *metric vector lattice* if for each $x, y \in E$, $d(x, y) \le \|y - x\|_{L^\infty(\nu)}$.

Definition 4.1. Let $(I(\mu), \rho)$ be a metric function space, and let $\mathcal{A} \subset I(\mu)$ be a subset. We define the metric function space generated by $\mathcal{A}$ as the space

$$I(\mathcal{A}, \mu) := \Big\{ f\chi_A : \, f \in \mathcal{A}, \; A \in \Sigma \Big\}$$

endowed with the metric ρ.

In a vector lattice E, the supremum $u \in E$ of an order bounded set $\{x_r\}_{r\in R}$ set is the unique element satisfying $x_r \leq u$ for every $r \in R$. Recall that a vector lattice E is order complete if every order bounded set has a supremum.

Theorem 4.2. *Let $(I(\mu), \rho)$ be a metric function space and consider a subset $\mathcal{A} \subseteq I(\mu)$. Let (E, d) be an order complete metric vector lattice of real valued functions that contains all constant functions. Consider a Lipschitz map $T : \mathcal{A} \to E$ such that $T(\mathcal{A})$ is order bounded. Then there is a factorization of T as*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "T"] \arrow[dr, "i"'] & & E, \\ & \mathcal{A}_T \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

such that $(\mathcal{A}_T, \rho_T)$ is a complete metric space containing $i(\mathcal{A})$ as a dense subspace, $\overline{T}$ is a Lipschitz map with constant 1, and i is a Lipschitz map with $Lip(i) \leq Lip(T)$.

Proof. Consider the metric function space $I(\mathcal{A}, \mu) \subseteq I(\mu)$ generated by $\mathcal{A}$. The set $\{T(g) : g \in \mathcal{A}\}$ is by hypothesis order bounded, and so by the order completeness of E there is a lower upper bound u for it.

Now we *claim* that the formula

$$\hat{T}(f\chi_A) := \sup_{g\in\mathcal{A}} \{T(g) - Lip(T)\rho(f\chi_A, g)\}$$

gives an extension of T to $I(\mathcal{A}, \mu)$ that is Lipschitz with constant $Lip(\hat{T}) = Lip(T)$. Recall that we are assuming that the range is a metric space having a linear lattice structure, and it is also order bounded by an element $u \in E$. We have that for each $g \in \mathcal{A}$,

$$T(g) - Lip(T)\rho(f\chi_A, g) \leq T(g) \leq u$$

and so

$$\{T(g) - Lip(T)\rho(f\chi_A, g) : \, g \in \mathcal{A}\}$$

is also order bounded. By the order completeness of E there is a supremum for this set, what gives that $\hat{T}$ is well-defined.

Let us show now that 1) $\hat{T}$ is an extension of T, and 2) that $\hat{T}$ is Lipschitz with $Lip(\hat{T}) = Lip(T)$.

1) First suppose that $f \in \mathcal{A}$. Then $\hat{T}(f) = \sup_{g\in I}\{T(g) - Lip(T)\rho(f, g)\} \geq T(f)$, and for every $g \in \mathcal{A}$,

$$T(g) - T(f) \leq Lip(T)\rho(f, g)$$

and so

$$T(g) - Lip(T)\rho(f,g) \le T(f),$$

what gives $\hat{T}(f) = T(f)$.

2) If $A, B \in \Sigma$ and $f, h \in \mathcal{A}$,

$$\begin{aligned}\hat{T}(f\chi_A) - \hat{T}(h\chi_B) &\le \sup_{g\in\mathcal{A}}\{T(g) - Lip(T)\rho(f\chi_A, g)\} - \sup_{v\in\mathcal{A}}\{T(v) - Lip(T)\rho(h\chi_B, v)\} \\ &\le \sup_{w\in\mathcal{A}}\{T(w) - T(w) - Lip(T)\rho(f\chi_A, w) + Lip(T)\rho(h\chi_B, w)\} \\ &\le \sup_{w\in\mathcal{A}}\{Lip(T)\rho(h\chi_B, f\chi_A)\} = Lip(T)\rho(h\chi_B, f\chi_A).\end{aligned}$$

Thus, taking into account that f and h can be interchanged in the computations, we get

$$|\hat{T}(f\chi_A) - \hat{T}(h\chi_B)| \le Lip(T)\rho(h\chi_B, f\chi_A),$$

and so $Lip(\hat{T}) \le Lip(T)$. Since the converse inequality is direct, we get $Lip(\hat{T}) = Lip(T)$.

Then, by Lemma 2.1 applied to $I(\mathcal{A},\mu)$ and $\hat{T}$, the map T can be factored through a map i with Lipschitz constant $Lip(i) \le Lip(T)$ and a Lipschitz operator $\overline{T} : \mathcal{A}_T \to E$ with Lipschitz constant equal to 1, where $(\mathcal{A}_T, \overline{d}_T)$ is a complete metric space in which $i(\mathcal{A})$ is dense in $\mathcal{A}_T$, and $\overline{d}_T$ is the quotient metric associated to the pseudo-metric

$$d_T(f,g) := \sup_{A\in\Sigma} d(\hat{T}(f\chi_A), \hat{T}(g\chi_A)), \quad f, g \in \mathcal{A}. \quad \square$$

Basic examples of the result above are given when T is real valued, or the range of T is order bounded as a subset of an order complete Banach lattice, for example, a Banach function space.

Let us finish the paper by giving an extension theorem for Lipschitz operators on $L^0(\mu)$—endowed with its natural metric d_0—that can be obtained from our results. As we said in the Introduction, to provide new tools for the analysis of Lipschitz operators on this space was one of the main motivations of the present paper. Recall that

$$d_0(f,g) = \int_\Omega \frac{|f-g|}{|f-g|+1}\, d\mu, \quad f, g \in L^0(\mu),$$

and

$$d_0(f,g) \ge \int_A \frac{|f-g|}{|f-g|+1}\, d\mu = \int_\Omega \frac{|f\chi_A - g\chi_A|}{|f\chi_A - g\chi_A|+1}\, d\mu = d_0(f\chi_A, g\chi_A),$$

for all $A \in \Sigma$. Thus, $(L^0(\mu), d_0)$ is a (complete) metric function space.

Corollary 4.3. *Let $(I(\mu), \rho)$ a metric function space. Consider an injective Lipschitz map $T : I(\mu) \to L^0(\mu)$ such that $T(I(\mu))$ is order bounded. Then there exists a factorization for T as*

$$\begin{tikzcd} I(\mu) \arrow[rr, "T"] \arrow[dr, "i"'] & & L^0(\mu), \\ & I(\mu)_T \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

where $(I(\mu)_T, \rho_T)$ is a complete metric space, i is an injective Lipschitz map with $Lip(i) = Lip(T)$, $I(\mu)_T$ contains $i(I(\mu))$ as a dense subspace, and $\overline{T}$ is a Lipschitz map with constant 1.

Note that Theorem 3.4 asserts that $I(\mu)_T$ is a metric function space under the assumption of being T order semi continuous and μ-determined. In this case, the metric ρ_T is given by

$$\rho_T(f,g) := \sup_{A\in\Sigma} \int_\Omega \frac{|\overline{T}(f\chi_A) - \overline{T}(g\chi_A)|}{|\overline{T}(f\chi_A) - \overline{T}(g\chi_A)| + 1}\, d\mu, \quad f, g \in I(\mu)_T,$$

where

$$\overline{T}(f\chi_A) := \sup_{g\in I(\mu)} \{T(g) - \rho(f\chi_A, g)\}.$$

Declaration of competing interest

None.

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