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Kurchatov-type methods for non-differentiable Hammerstein-type integral equations

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Abstract

We consider a generic type of nonlinear Hammerstein-type integral equations with the particularity of having non-differentiable kernel of Nemystkii type. So, in order to solve it we consider a uniparametric family of iterative processes derivative free, with the main advantage that for a special value of the involved parameter the iterative method obtained coincides with Newton's method, that is due to the fact of evaluating the divided difference operator when the two values are the same. We perform a qualitative convergence study by choosing an auxiliary point, that allow us to obtain the existence and separation of solutions of the given equation, that is, local and semilocal convergence balls can be obtained.

Keywords Hammerstein-type integral equations · Divided differences · Kurchatov-type iterative processes

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1 Introduction

In this paper, we focus on the study of the existence of a solution for nonlinear Hammerstein-type integral equations, as well as the uniqueness of this solution and its approximation. So, we consider nonlinear Hammerstein-type integral equations of the form [5, 28, 31]

$$z(s) = h(s) + \theta \int_a^b \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \quad s \in [a,b], \quad \theta \in \mathbb{R}, \tag{1}$$

where $h \in \mathcal{C}[a,b]$, the kernel $K(s,t)$ is a known function in $[a,b] \times [a,b]$, $\mathcal{N}$ is the Nemytskii operator $\mathcal{N} : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ such that $\mathcal{N}(\phi)(x) = N(\phi(x))$, where $N : \mathbb{R} \to \mathbb{R}$, and z is the unknown function to be determined. In our case, we will require that the Nemystkii operator $\mathcal{N}$ be a simply continuous operator.

A commonly used procedure to prove the existence and uniqueness of a solution of the (1) consists in transforming said equation into an equivalent fixed point problem (see [30, 31]). Thus, considering the operator $\mathcal{T} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ with

$$\mathcal{T}(z)(s) = h(s) + \theta \int_a^b \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \quad s \in [a,b], \quad \theta \in \mathbb{R}, \tag{2}$$

by using a fixed point theorem, it is proved that the method of Successive Approximations converges to a fixed point of (2) and therefore a solution of (1). Other iterative methods are also used for this purpose. Thus, for example, the Picard method has been used (see [10]), testing its convergence to a solution of the equation $\mathcal{G}(z) = 0$, equivalent to (1), where we consider the operator $\mathcal{G} : \Omega \subseteq \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ with

$$[\mathcal{G}(z)](x) = z(s) - h(s) - \theta \int_a^b \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \quad s \in [a,b], \quad \theta \in \mathbb{R}, \tag{3}$$

Other methods such as Newton's method [12] and Newton-type methods [7, 25, 29] have been used to prove the existence and uniqueness of a solution for the (1). These studies are based on the qualitative results, of existence and uniqueness of solution, provided by the study of the convergence of iterative processes considered.

In our case, since the kernel of Nemystkii $\mathcal{N}$ is a continuous operator, there may be a possibility that it is not differentiable. In [16], the non-differentiable case has been studied. Also in this work, we will consider an iterative process that does not use derivatives in its algorithm, that is, a derivative-free iterative process. So, one of the aims of this paper is the qualitative study that we can obtain from the uniparametric family of iterative processes

$$\begin{cases} z_0, z_{-1} \text{ given in } \Omega, \lambda \in [0,1], \\ x_n = (1-\lambda)z_n + \lambda z_{n-1}, \\ y_n = (1+\lambda)z_n - \lambda z_{n-1}, \\ z_{n+1} = z_n - \left[x_n, y_n; \mathcal{G}\right]^{-1}\mathcal{G}(z_n), \quad n \geq 0. \end{cases} \tag{4}$$

Note that this uniparametric family of iterative processes can be considered as a combination of the Kurchatov method [4, 22]($\lambda = 1$) and, for differentiable case, Newton's method [8] ($\lambda = 0$). But, the main advantage for considering this scheme is that we have a family of derivative-free iterative methods that can be used in the non differential case. We use a first-order divided difference [1, 6, 14]. It is well known that, if we denote by $\mathscr{L}(X, Y)$ the space of bounded linear operators from X to Y, an operator $\left[x, y; D\right] \in \mathscr{L}(X, Y)$ is called a first-order divided difference for the operator $D : \Omega \subseteq X \to Y$ on the points x and y ($x \neq y$) if

$$\left[x, y; D\right](x - y) = D(x) - D(y). \tag{5}$$

To realize the qualitative study for the uniparametric family of iterative processes (4), we do an analysis of the convergence for (4), so that we can obtain the existence of a solution of (1) in a certain domain. Moreover, we obtain a result on the uniqueness of solution that allows separating solutions of (1). We analyze the convergence of the method by a technique based on recurrence relations that use an auxiliary function [9, 11]. So, we use the theoretical results obtained from the convergence of the method (4) to draw conclusions about the existence and separation of solutions of (1).

Other of the aims of this paper is to approximate a solution of (1). To obtain this objective in an appropriate way, we have considered the family of iterative processes given in (4), which is formed by iterative processes with quadratic convergence and low operational cost, therefore efficient iterative processes that also have good accessibility [19]

Our main result in the paper is to perform a complete convergence study of the scheme that we state and prove in Section 2.1 along with some necessary lemmas. In Section 2.2 we introduce the assumptions to make the semilocal convergence analysis and give some preliminary results before to set the main theorem. Then, in Section 2.3 by choosing an adequate auxiliary point we get the local convergence results.

In Section 3, we apply the theoretical results obtained in previous sections for obtaining domains of existence and uniqueness of solutions for the nonlinear Hammerstein-type integral equation, giving in the last subsection a numerical experiment.

Finally, in Section 4 we drawn some conclusions.

2 Convergence of Kurchatov-type methods

In this section, to make our study of convergence as general as possible, we consider a nonlinear equation

$$G(x) = 0, \tag{6}$$

where $G : \Omega \subseteq X \to Y$ is a continuous operator defined on a nonempty convex subset Ω of a Banach space X to a Banach space Y.

In previous works [2, 3, 8, 13], the convergence analysis for these kind of iterative methods, (4), have been analyzed from two different points of view. On the one hand, the semilocal convergence study, where we assume conditions on the initial guess z_0 and on the operator G, in order to obtain the existence ball, that is, this process assures the existence of solution z^* of the equation $G(z) = 0$ remaining all the iterates and the solution in the cited ball. On the other hand, the local convergence study, where we must assume the existence of a solution z^* of $G(z) = 0$ and then, with additional assumptions on the involved operators, we obtain the convergence domain, that is the ball centered at the z^* where we can take a possible starting guess for the iterative process. Moreover, we also have in the literature the technique based on auxiliary points [9, 10]. We use it in this paper. So, we assume some conditions on the operator G and on an auxiliary point $\tilde{z}$ in Ω for getting the existence of a solution z^* of $G(z) = 0$ and to prove the convergence of (4) to z^*. So, this technique is much general driving us to obtain results of semilocal and local convergence for iterative process (4) by choosing adequately two particular auxiliary point $\tilde{z}$.

2.1 Convergence from an auxiliary point

Throughout our study, we will assume that there is a first-order divided difference in the Banach space X. Now, we perform a convergence result for the iterative process defined in (4), for a fixed value $\lambda \in (0,1]$, by assuming that G is a continuous operator in Ω and such that the following conditions are satisfied.

- (C1) Let $z_0, \tilde{z} \in \Omega$, with $z_0 \in B(\tilde{z}, \mu)$, $z_0 \neq \tilde{z}$, and there exists $[\tilde{z}, z_0; G]^{-1}$ verifying $\|[\tilde{z}, z_0; G]^{-1} G(z_0)\| \leq \alpha$.
- (C2) Let $z_{-1} \in \Omega$, with $z_{-1} \in B(z_0, \alpha)$ and $z_{-1} \neq z_0$,
- (C3) For all $x, y, u, v \in \Omega$, with $x \neq y$ and $u \neq v$, holds: $\|[\tilde{z}, z_0; G]^{-1}([x, y; G] - [u, v; G])\| \leq \psi(\|x - u\|, \|y - v\|)$, where $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in its two arguments.

We notice that from $(C3)$, we deduce the following condition:

- (C3’) For all $x, y \in \Omega$, with $x \neq y$ holds: $\| \leq \psi_0(\|x - \tilde{z}\|, \|y - z_0\|) \|[\tilde{z}, z_0; G]^{-1}([x, y; G] - [\tilde{z}, z_0; G])$, with $\psi_0 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in two arguments.

Therefore, $(C3')$ is not an extra condition.

In addition, we assume that the following items are verified:

- (C4) The auxiliary real equation

$$(g_0(t) + 1 - g(t))\eta - (1 - g(t))t = 0, \tag{7}$$

where

$$\eta = \frac{\alpha}{1 - \psi_0(\mu + \lambda\alpha, \lambda\alpha)},$$

$$g_0(t) = \frac{\psi(\eta + \lambda\alpha, \lambda\alpha)}{1 - \psi_0(\mu + \lambda\eta + t, \lambda\eta + t)}$$

and

$$g(t) = \frac{\psi(\eta + \lambda\eta, \lambda\eta)}{1 - \psi_0(\mu + \lambda\eta + t, \lambda\eta + t)},$$

has at least one positive real root and we denote by r the smallest positive real root.

(C5) $B(\tilde{z}, \mu + r + \lambda\eta) \subseteq \Omega$, $\psi_0(\mu + r + \lambda\eta, r + \lambda\eta) < 1$ and $\max\{g_0(r), g(r)\} < 1$.

Firstly, notice that parameter η is well defined since that $\psi_0(\mu + \lambda\alpha, \lambda\alpha) < \psi_0(\mu + r + \lambda\eta, r + \lambda\eta) < 1$.

Secondly, we will consider that $z_{k+1} \neq z_k$ for all $k \geqslant 0$, because, in other case, $z_{k+1} = z_k$ for some $k \geqslant 0$, and then the sequence $\{z_n\}$ converges to z^* with $z^* = z_n = z_{k+1} = z_k$ for all $n \geqslant k+2$. Moreover, if $z_{k+1} \neq z_k$ we obtain that $x_{k+1} \neq y_{k+1}$. Therefore, the operators $[x_{k+1}, y_{k+1}; G]$ are always well defined.

Finally, with respect to the first-order divided differences, [1], we include the boundedness process by means of ω-functions, that is used in the non-differentiable case, (see [17]), and it is a generalization of the case in which $[x,y;G]$ is Lipschitz-continuous or Hölder-continuous condition [20]. In the above cases, the Fréchet derivative of G exists in Ω and satisfies $[x, x; G] = G'(x)$, see[1]. Moreover, it is already well known, see [18], that if $\psi(0,0) = 0$ then G is differentiable, so in non-differentiable situations we have that $\psi(0,0) > 0$.

We will begin our convergence study by considering $n = 0$. Firstly, notice that

$$\begin{aligned} \|x_0 - \tilde{z}\| &\leq \|x_0 - z_0\| + \|z_0 - \tilde{z}\| \\ &\leq \|(1-\lambda)z_0 + \lambda z_{-1} - z_0\| + \|z_0 - \tilde{z}\| \\ &\leq \|z_0 - \tilde{z}\| + \lambda\|z_0 - z_{-1}\| \\ &< \mu + \lambda\alpha, \end{aligned}$$

and

$$\|y_0 - z_0\| \leq \|(1+\lambda)z_0 - \lambda z_{-1} - z_0\| \leq \lambda\|z_0 - z_{-1}\| < \lambda\alpha.$$

So, it follows that $x_0, y_0 \in B(\tilde{z}, \mu + r + \lambda\eta)$, and as $x_0 \neq y_0$ then $[x_n, y_n; G]$ is well defined.

Secondly, by using $(C3')$, we have that

$$\begin{aligned} \|[\tilde{z}, z_0; G]^{-1}[x_0, y_0; G] - I\| &\leq \|[\tilde{z}, z_0; G]^{-1}([x_0, y_0; G] - [\tilde{z}, z_0; G])\| \\ &\leq \psi_0\big(\|x_0 - \tilde{z}\|, \|y_0 - z_0\|\big) \\ &\leq \psi_0(\mu + \lambda\alpha, \lambda\alpha) \\ &\leq \psi_0(\mu + r + \lambda\eta, r + \lambda\eta) < 1. \end{aligned}$$

Therefore, by applying Banach Lemma, one gets the existence of $[x_0,y_0;G]^{-1}$ and

$$\|z_1 - z_0\| \leq \|[x_0, y_0;G]^{-1}[\tilde{z}, z_0;G]\|\|[\tilde{z}, z_0;G]^{-1}G(z_0)\| \leq \frac{\alpha}{1-\psi_0(\mu+\lambda\alpha,\lambda\alpha)} = \eta.$$

Next, we consider two technical lemmas that we use later.

Lemma 1 *If* $z_n,z_{n+1} \in \Omega$, *then*

$$G(z_{n+1}) = ([z_{n+1}, z_n;G] - [x_n, y_n;G])(z_{n+1} - z_n) \tag{8}$$

Proof As $z_{n+1} \neq z_n$ and $x_n \neq y_n$, then $[z_{n+1},z_n;G]$ and $[x_n,y_n;G]$ are well defined. Then, by taking into account the algorithm (4), we have

$$G(z_{n+1}) = G(z_{n+1}) - G(z_n) - [x_n, y_n;G](z_{n+1} - z_n).$$

Next, as $[z_{n+1},z_n;G](z_{n+1} - z_n) = G(z_{n+1}) - G(z_n)$, the result is proved.

Lemma 2 *Let G be a continuous operator in* Ω *such that the conditions* $(C1)$–$(C5)$ *are satisfied,* $z_n,z_{n-1} \in B(z_0,r)$ *and* $\|z_n - z_{n-1}\| \leq \|z_1 - z_0\|$, *for* $n \geqslant 1$, *then* $x_n, y_n \in B(\tilde{z}, \mu + r + \lambda\eta)$, *and there exists* $[x_n,y_n;G]^{-1}$ *with*

$$\|[x_n, y_n;G]^{-1}[\tilde{z}, z_0;G]\| \leq \frac{1}{1 - \psi_0(\mu + r + \lambda\eta, r + \lambda\eta)}.$$

Proof Firstly, notice that

$$\begin{aligned} \|x_n - \tilde{z}\| &\leq \|x_n - z_0\| + \|z_0 - \tilde{z}\| \\ &\leq \|(1 - \lambda)z_n + \lambda z_{n-1} - z_0\| + \|z_0 - \tilde{z}\| \\ &\leq \|z_0 - \tilde{z}\| + \|z_n - z_0\| + \lambda\|z_n - z_{n-1}\| \\ &< \mu + r + \lambda\eta, \end{aligned} \tag{9}$$

and

$$\|y_n - z_0\| \leq \|(1 + \lambda)z_n - \lambda z_{n-1} - z_0\| \leq \|z_n - z_0\| + \lambda\|z_n - z_{n-1}\| < r + \lambda\eta. \tag{10}$$

It follows that $x_n, y_n \in B(\tilde{z}, \mu + r + \lambda\eta)$, and as $x_n \neq y_n$ then $[x_n,y_n;G]$ is well defined.

Secondly, by using $(C3')$ and (10), we have that

$$\begin{aligned} \|[\tilde{z}, z_0;G]^{-1}([x_n, y_n;G] - [\tilde{z}, z_0;G])\| &\leq \psi_0\big(\|x_n - \tilde{z}\|, \|y_n - z_0\|\big) \\ &\leq \psi_0(\mu + r + \lambda\eta, r + \lambda\eta) < 1. \end{aligned}$$

Therefore, by applying Banach Lemma, one gets the existence of $[x_n,y_n;G]^{-1}$ and the result is obtained.

To continue, we set $n = 1$. From (9) and (10), we obtain that $x_1, y_1 \in B(\tilde{z}, \mu + r + \lambda\eta) \subset \Omega$. Next, as $z_0 \neq z_1$ then $x_1 \neq y_1$, and it follows that $[x_1,y_1;G]$ is well defined.

In addition, from (7), it follows that $\|z_1 - z_0\| \leq \eta < r$, then $z_1 \in B(z_0,r)$. So, by applying Lemma 2, there exists $[x_1,y_1;G]^{-1}$.

On the other hand, we have

$$\|z_1 - x_0\| = \|z_1 - (1-\lambda)z_0 - \lambda z_{-1}\| \leq \|z_1 - z_0\| + \lambda\|z_0 - z_{-1}\| < \eta + \lambda\alpha$$

and

$$\|z_0 - y_0\| = \|z_0 - (1+\lambda)z_0 + \lambda z_{-1}\| \leq \lambda\|z_0 - z_{-1}\| < \lambda\alpha.$$

Then, from Lemma 1, (9) and (10),

$$\begin{aligned}
\|z_2 - z_1\| &= \|[x_1,y_1;G]^{-1}G(z_1)\| \\
&\leq \|[x_1,y_1;G]^{-1}([z_1,z_0;G] - [x_0,y_0;G])(z_1 - z_0)\| \\
&\leq \|[x_1,y_1;G]^{-1}[\tilde{z},z_0;G]\|\,\|[\tilde{z},z_0;G]^{-1}([z_1,z_0;G] - [x_0,y_0;G])\|\,\|z_1 - z_0\| \\
&\leq \frac{1}{1-\psi_0(\|x_1-\tilde{z}\|,\|y_1-z_0\|)}\psi(\|z_1 - x_0\|,\|z_0 - y_0\|)\|z_1 - z_0\| \\
&\leq \frac{\psi(\eta+\lambda\alpha,\lambda\alpha)}{1-\psi_0(\mu+r+\lambda\eta,r+\lambda\eta)}\|z_1 - z_0\|
\end{aligned} \tag{11}$$

$$= g_0(r)\|z_1 - z_0\| \tag{12}$$

$$< \|z_1 - z_0\|. \tag{13}$$

Thus, as $g_0(r) < 1$ by $(C5)$, from (7) and (11), we get

$$\|z_2 - z_0\| \leq \|z_2 - z_1\| + \|z_1 - z_0\| \leq (g_0(r)+1)\|z_1 - z_0\| < \left(\frac{g_0(r)}{1-g(r)} + 1\right)\eta = r.$$

Therefore, iterate $z_2 \in B(z_0,r)$ and $\|z_2 - z_1\| < \|z_1 - z_0\| \leq \eta$.

Next, we establish the recurrence relations that verify the elements of the sequence $\{x_n\}$ generated by the method (4) for a fixed $\lambda \in (0,1]$.

Lemma 3 *Let G be a continuous operator in Ω, $z_0 \in B(\tilde{z},\mu)$ and $z_{-1} \in B(z_0,\alpha)$ such that the conditions $(C1)$–$(C5)$ are satisfied, then the following items hold, for $j \geq 3$, by the sequence $\{z_n\}$:*

($\mathrm{i_j}$) $x_{j-1},y_{j-1} \in B(z_0,r+\lambda\eta)$ and $x_{j-1},y_{j-1} \in B(\tilde{z},\mu+r+\lambda\eta)$.
($\mathrm{ii_j}$) $\|z_{j-1} - x_{j-2}\| \leq \eta + \lambda\eta$ and $\|z_{j-2} - y_{j-2}\| \leq \lambda\eta$.
($\mathrm{iii_j}$) $\|z_j - z_{j-1}\| \leq g(r)\|z_{j-1} - z_{j-2}\| \leq g(r)^{j-2}g_0(r)\|z_1 - z_0\| \leq g(r)^{j-2}g_0(r)\eta < \eta$.
($\mathrm{iv_j}$) $\|z_j - z_0\| \leq \left(\frac{g_0(r)}{1-g(r)} + 1\right)\eta = r$, then $z_j \in B(z_0,r)$.

Proof Note that previously we have proved (i_j)–(iii_j) for $j = 1,2$.

We consider $j = 3$. From (9) and (10), we obtain that $x_2,y_2 \in B(z_0,r+\lambda\eta)$ and $x_2,y_2 \in B(\tilde{z},\mu+r+\lambda\eta)$, which proves item (i_3).

On the one hand, we have

$$\|z_2 - x_1\| = \|z_2 - (1-\lambda)z_1 - \lambda z_0\| \leq \|z_2 - z_1\| + \lambda\|z_1 - z_0\| < \eta + \lambda\eta$$

and

$$\|z_1 - y_1\| = \|z_1 - (1+\lambda)z_1 + \lambda z_0\| \le \lambda\|z_1 - z_0\| < \lambda\eta.$$

Then (ii_3) is satisfied.

On the other hand,

$$\begin{aligned}
\|z_3 - z_2\| &\le \|[x_2, y_2;G]^{-1}G(z_2)\| \\
&\le \|[x_2, y_2;G]^{-1}([z_2, z_1;G] - [x_1, y_1;G])(z_2 - z_1)\| \\
&\le \|[x_2, y_2;G]^{-1}[\tilde{z}, z_0;G]\|\|[\tilde{z}, z_0;G]^{-1}([z_2, z_1;G] - [x_1, y_1;G])\|\|z_2 - z_1\| \\
&\le \frac{1}{1-\psi_0(\|x_2-\tilde{z}\|,\|y_2-z_0\|)}\psi(\|z_2 - x_1\|, \|z_1 - y_1\|)\|z_2 - z_1\| \\
&\le \frac{\psi(\eta+\lambda\eta,\lambda\eta)}{1-\psi_0(\mu+r+\lambda\eta,r+\lambda\eta)}\|z_2 - z_1\| = g(r)\|z_2 - z_1\| \\
&< \|z_2 - z_1\|.
\end{aligned}$$

So, as $g(r) < 1$, we obtain that $\|z_3 - z_2\| < \|z_2 - z_1\| < \eta$. Therefore, (iii_3) is proved.

To continue, we prove (iv_3), this is that iterate z_3 remains in the ball $B(z_0,r)$. For this, noting that the fact of being $g(r) < 1$ allow us to sum a geometric progression of reason $g(r)$, we have

$$\begin{aligned}
\|z_3 - z_0\| &\le \|z_3 - z_2\| + \|z_2 - z_1\| + \|z_1 - z_0\| \le (g(r) + 1)g_0(r) + 1)\|z_1 - z_0\| \\
&\le (g(r) + 1)g_0(r) + 1)\eta \ < (\frac{g_0(r)}{1-g(r)} + 1)\eta = r.
\end{aligned}$$

In order to complete the proof we apply an inductive procedure. So, we suppose that the items are satisfied, for $k \ge j \ge 3$ and similarly to the case $j = 3$, we prove that these items hold for $j = k + 1$.

Theorem 4 *Let G be a continuous operator in Ω, for each $z_0 \in B(\tilde{z}, \mu)$ and $z_{-1} \in B(z_0,\alpha)$ such that the conditions $(C1)$–$(C5)$ are satisfied, then the sequence $\{z_n\}$, given by (4), converges to z^* a solution of equation $G(z) = 0$. Moreover, $z_n, z^* \in \overline{B(z_0, r)}$ for all $n \geqslant 1$.*

Proof From the recurrence relations given in Lemma 3 we only have to prove the convergence of the sequence $\{z_n\}$, given by (4). As X is a Banach space, we will see that $\{z_n\}$ is a Cauchy sequence. For this, we consider

$$\begin{aligned}
\|z_{n+k} - z_n\| &\le \|z_{n+k} - z_{n+k-1}\| + \|z_{n+k-1} - z_{n+k-2}\| + \cdots + \|z_{n+2} - z_{n+1}\| + \|z_{n+1} - z_n\| \\
&\le \sum_{i=1}^{k} \|z_{n+i} - z_{n+i-1}\| \le \sum_{i=1}^{k} g(r)^{n+i-2}g_0(r)\|z_1 - z_0\| \\
&\le \left(\frac{1-g(r)^k}{1-g(r)}\right)g(r)^{n-1}g_0(r)\|z_1 - z_0\|.
\end{aligned}$$

By taking limits when $n \to \infty$, then $\|z_{n+k} - z_n\| \to 0$. Hence, $\{z_n\}$ is a Cauchy sequence which converges to $z^* \in \overline{B(z_0, r)}$.

Moreover, from the following

$$\begin{aligned}\|[\tilde{z}, z_0;G]^{-1}G(z_{k+1})\| &= \|[\tilde{z}, z_0;G]^{-1}([z_{k+1}, z_k;G] - [x_k, y_k;G])(z_{k+1} - z_k)\| \\ &\leq \psi(\|z_{k+1} - x_k\|, \|z_k - y_k\|)\|z_{k+1} - z_k\| \\ &\leq \psi(\|z_{k+1} - z_k + \lambda z_k - \lambda z_{k-1}\|, \|z_k - z_k - \lambda z_k + \lambda z_{k-1}\|)\|z_{k+1} - z_k\| \\ &\leq \psi(\eta + \lambda\eta, \lambda\eta)\|z_{k+1} - z_k\| \leq \psi(\eta + \lambda\eta, \lambda\eta)g(r)^{k-1}g_0(r)\eta.\end{aligned}$$

by the continuity of the operator G and as $g(r)^{k-1}$ tends to zero when k tends to infinity, we have that $G(z^*) = 0$.

Notice that $\overline{B(z_0, r)}$ is a domain of existence of solution for $G(z) = 0$.
To prove the uniqueness we give the following result,

Theorem 5 *Let G be a continuous operator in* Ω, $z_0 \in B(\tilde{z}, \mu)$ *and* $z_{-1} \in B(z_0, \alpha)$ *such that the conditions* $(C1)$–$(C5)$ *are satisfied. We assume that there exists* $r_1 \geq r$ *such that*

$$\psi_0(\mu + r + \lambda\eta, r_1 + \lambda\eta) < 1,$$

then, the limit point z^* *is the only solution of equation* $G(z) = 0$ *in* $\overline{B(z_0, r_1 + \lambda\eta)} \cap \Omega$.

Proof Let $y^* \in \overline{B(z_0, r_1 + \lambda\eta)} \cap \Omega$ be such that $G(y^*) = 0$. By defining $Q = [z^*, y^*; G]$ we get

$$\begin{aligned}\|[\tilde{z}, z_0;G]^{-1}([z^*, y^*;G] - [\tilde{z}, z_0;G])\| &\leq \psi_0(\|z^* - \tilde{z}\|, \|y^* - z_0\|) \\ &\leq \psi_0(\|z^* - z_0\| + \|z_0 - \tilde{z}\|, \|y^* - z_0\|) \\ &\leq \psi_0(\mu + r + \lambda\eta, r_1 + \lambda\eta) < 1.\end{aligned}$$

Hence, by Banach lemma the operator Q^{-1} exists and as

$$[z^*, y^*;G](z^* - y^*) = G(z^*) - G(y^*) = 0,$$

then $z^* = y^*$.

Note that it may happen that the hypotheses necessary to ensure convergence are not verified for all values of $\lambda \in (0,1]$.

2.2 Semilocal convergence

To establish another result for semilocal convergence, we consider $\tilde{z} = z_{-1}$ under the following conditions:

(S1) Let $z_{-1} \in B(z_0, \mu)$, with $\mu > 0$, $\overline{B(z_0, \mu)} \subset \Omega$ and there exists $[z_{-1}, z_0; G]^{-1}$ with $\|[z_{-1}, z_0; G]^{-1} G(z_0)\| \leq \alpha$.
(S2) $\|[z_{-1}, z_0; G]^{-1}([x, y; G] - [u, v; G])\| \leq \psi(\|x - u\|, \|y - v\|)$ holds for all $x, y, u, v \in \Omega$ with $x \neq y$ and $u \neq v$, where $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non decreasing function in its both arguments.
(S3) The scalar equation

$$t(1-g(t))-\eta=0, \tag{14}$$

where we have $\eta>0$, for

$$\frac{\alpha}{1-\psi_0(\mu+\lambda\mu,\lambda\mu)}=\eta,$$

$$g(t)=\frac{\tilde{g}}{1-\psi_0(\mu+t+\lambda\eta,t+\lambda\eta)},$$

and

$$\tilde{g}=\max\{\psi(\eta+\lambda\mu,\lambda\mu),\psi(\eta+\lambda\eta,\lambda\eta)\},$$

has at least one positive real root and we denote by r the smallest positive root.

(S4) $B(z_0,r+\lambda\eta)\subseteq\Omega$ and $0<g(r)<1$.

As the previous study, we notice that from ($S2$), we deduce the following condition:

(S2') $\|[z_{-1},z_0;G]^{-1}([x,y;G]-[z_{-1},z_0;G])\|\leq\psi_0(\|x-z_{-1}\|,\|y-z_0\|)$ holds for all $x,y\in\Omega$ with $x\neq y$, where $\psi_0:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a continuous non decreasing function in both arguments.

We assume that $z_n\neq z_{n-1}$, for all $n\geq 1$, otherwise the sequence $\{z_n\}$ is convergent. If $z_n\neq z_{n-1}$, then we obtain $x_n\neq y_n$. By using the definition of the method (4), we get

$$\|x_0-z_0\|\leq\lambda\|z_0-z_{-1}\|=\lambda\mu, \tag{15}$$

and

$$\|y_0-z_0\|\leq\lambda\|z_0-z_{-1}\|=\lambda\mu. \tag{16}$$

It shows that $x_0,y_0\in\overline{B(z_0,\mu)}\subset\Omega$ and as $x_0\neq y_0$, then $[x_0,y_0;G]$ is well defined. So, by using ($S2'$) as $\eta>0$, we obtain

$$\|I-[z_{-1},z_0;G]^{-1}[x_0,y_0;G]\|\leq\psi_0(\|x_0-z_{-1}\|,\|y_0-z_0\|)\leq\psi_0(\mu+\lambda\mu,\lambda\mu)<1.$$

Therefore, by the Banach Lemma on invertible operators, $[x_0,y_0;G]^{-1}$ exists and

$$\|[x_0,y_0;G]^{-1}[z_{-1},z_0;G]\|\leq\frac{1}{1-\psi_0(\mu+\lambda\mu,\lambda\mu)}. \tag{17}$$

Moreover,

$$\|z_1-z_0\|\leq\|[x_0,y_0;G]^{-1}[z_{-1},z_0;G]\|\|[z_{-1},z_0;G]^{-1}G(z_0)\|\leq\frac{\alpha}{1-\psi_0(\mu+\lambda\mu,\lambda\mu)}=\eta.$$

Lemma 6 *Assume that the conditions* ($S1$)–($S4$) *hold. If* $z_n,z_{n-1}\in B(z_0,r)$ *and* $\|z_n-z_{n-1}\|<\|z_1-z_0\|$ *for* $n\geq 1$, *then* $x_n,y_n\in B(z_0,r+\lambda\eta)$ *and there exists* $[x_n,y_n;G]^{-1}$ *such that*

$$\|[x_n, y_n; G]^{-1}[z_{-1}, z_0; G]\| \leq \frac{1}{1-\psi_0(\mu+r+\lambda\eta, r+\lambda\eta)}.$$

Proof Consider

$$\|x_n - z_0\| \leq \|z_n - z_0\| + \lambda\|z_n - z_{n-1}\| < r + \lambda\eta, \tag{18}$$

and

$$\|y_n - z_0\| \leq \|z_n - z_0\| + \lambda\|z_n - z_{n-1}\| < r + \lambda\eta. \tag{19}$$

Thus, $x_n, y_n \in B(z_0, r+\lambda\eta) \subset \Omega$ and as $x_n \neq y_n$, then $[x_n, y_n; G]$ is well defined. Using $(S1)$, $(S2^{'})$, (18) and (19), and $g(t) > 0$, we get

$$\|I - [z_{-1}, z_0; G]^{-1}[x_n, y_n; G]\| \leq \psi_0(\mu + r + \lambda\eta, r + \lambda\eta) < 1.$$

Therefore, by Banach Lemma, $[x_n, y_n; G]^{-1}$ exists and

$$\|[x_n, y_n; G]^{-1}[z_{-1}, z_0; G]\| \leq \frac{1}{1-\psi_0(\mu+r+\lambda\eta, r+\lambda\eta)}.$$

Using (18) and (19), we obtain $x_1, y_1 \in B(z_0, r+\lambda\eta) \subset \Omega$. As, $z_0 \neq z_1$ then $x_1 \neq y_1$, so $[x_1, y_1; G]$ is well defined. Also from (14), we obtain $\|z_1 - z_0\| \leq \eta < r$, then $z_1 \in B(z_0, r)$. So, by using Lemma 6, there exists $[x_1, y_1; G]^{-1}$ and we have

$$\begin{aligned}
\|z_2 - z_1\| &= \|[x_1, y_1; G]^{-1}G(z_1)\| \\
&\leq \|[x_1, y_1; G]^{-1}[z_{-1}, z_0; G]\| \left\|[z_{-1}, z_0; G]^{-1}([z_1, z_0; G] - [x_0, y_0; G])\right\| \|z_1 - z_0\| \\
&\leq \frac{\psi(\|z_1 - x_0\|, \|z_0 - y_0\|)}{1-\psi_0(\mu+r+\lambda\eta, r+\lambda\eta)}\|z_1 - z_0\| \\
&\leq \frac{\psi(\eta+\lambda\mu, \lambda\mu)}{1-\psi_0(\mu+r+\lambda\eta, r+\lambda\eta)}\|z_1 - z_0\| \\
&\leq g(r)\|z_1 - z_0\| < \eta,
\end{aligned}$$

since $g(r) < 1$. Further,

$$\|z_2 - z_0\| \leq \|z_2 - z_1\| + \|z_1 - z_0\| \leq (g(r) + 1)\|z_1 - z_0\| < \frac{\eta}{1-g(r)} = r.$$

So, $z_2 \in B(z_0, r) \subseteq B(z_0, r + \lambda\eta) \subset \Omega$ and $\|z_2 - z_1\| < \|z_1 - z_0\| \leq \eta$. In the next Lemma, we establish the recurrence relations to prove the convergence of the sequence $\{z_n\}$.

Lemma 7 *Assume that the conditions* $(S1)$–$(S4)$ *hold. Then for* $j \geq 3$, *the following items are satisfied by the sequence* $\{z_n\}$:

(i) $x_{j-1}, y_{j-1} \in B(z_0, r + \lambda\eta)$.
(ii) $\|z_j - z_{j-1}\| \leq g(r)\|z_{j-1} - z_{j-2}\| \leq g(r)^{j-1}\|z_1 - z_0\| \leq g(r)^{j-1}\eta < \eta$.
(iii) $\|z_j - z_0\| < \frac{\eta}{1-g(r)} = r$, then $z_j \in B(z_0, r)$.

Proof We have just shown that (i)–(iii) holds true for $j = 1,2$. From (18) and (19), we obtain $x_2, y_2 \in B(z_0, r + \lambda\eta)$ which proves (i) for $j = 3$. Then,

$$\begin{aligned}\|z_3 - z_2\| &\leq \|[x_2, y_2;G]^{-1}[z_{-1}, z_0;G]\|\|[z_{-1}, z_0;G]^{-1}\big([z_2, z_1;G] - [x_1, y_1;G]\big)\|\|z_2 - z_1\| \\ &\leq \frac{\psi(\|z_2-x_1\|,\|z_1-y_1\|)}{1-\psi_0(\mu+r+\lambda\eta,r+\lambda\eta)}\|z_2 - z_1\| \\ &\leq \frac{\psi(\eta+\lambda\eta,\lambda\eta)}{1-\psi_0(\mu+r+\lambda\eta,r+\lambda\eta)}\|z_2 - z_1\| \leq g(r)\|z_2 - z_1\|.\end{aligned}$$

As $g(r) < 1$, therefore, $\|z_3 - z_2\| < \|z_2 - z_1\| < \eta$. Hence, (ii) is proved for $j = 3$. Further,

$$\begin{aligned}\|z_3 - z_0\| &\leq \|z_3 - z_2\| + \|z_2 - z_1\| + \|z_1 - z_0\| \\ &\leq [(g(r) + 1)g(r) + 1]\|z_1 - z_0\| < \frac{\eta}{1-g(r)} = r.\end{aligned}$$

Therefore, $z_3 \in B(z_0,r)$ and hence (iii) is proved for $j = 3$. We suppose that the items are satisfied, for $k \geq j \geq 3$ and similarly to the case $j = 3$, we prove that these items hold for $j = k + 1$.

Theorem 8 *Suppose that the conditions* $(S1)$–$(S4)$ *hold, then the sequence* $\{z_n\}$ *given by* (4) *converges to* z^* *a solution of equation* $G(z) = 0$ *for each* $z_0 \in \Omega$ *and* $z_{-1} \in B(z_0,\mu)$. *Furthermore,* $z_n \in B(z_0,r)$ *for all* $n \geq 1$. *Moreover, if we suppose that there exists* $r_1 \geq r$ *such that* $\psi_0(\mu + r + \lambda\eta, r_1 + \lambda\eta) < 1$, *then* z^* *is the unique solution of equation* $G(z) = 0$ *in* $\overline{B(z_0, r_1 + \lambda\eta)} \cap \Omega$.

Proof To prove the convergence of the sequence, it is sufficient to prove that the sequence $\{z_n\}$ is a Cauchy sequence. For this, we consider

$$\begin{aligned}\|z_{n+k} - z_n\| &\leq \|z_{n+k} - z_{n+k-1}\| + \|z_{n+k-1} - z_{n+k-2}\| + \cdots + \|z_{n+2} - z_{n+1}\| + \|z_{n+1} - z_n\| \\ &\leq \sum_{i=1}^{k} \|z_{n+i} - z_{n+i-1}\| \leq \sum_{i=1}^{k} g(r)^{n+i-1}\|z_1 - z_0\| \\ &\leq \frac{1-g(r)^k}{1-g(r)} g(r)^n \|z_1 - z_0\|.\end{aligned}$$

As $n \to \infty$, then $\|z_{n+k} - z_n\| \to 0$, hence, $\{z_n\}$ is a Cauchy sequence which converges to $z^* \in B(z_0, r)$. Now,

$$\begin{aligned}\|[z_{-1}, z_0;G]^{-1}G(z_{n+1})\| &= \|[z_{-1}, z_0;G]^{-1}\big([z_{n+1}, z_n;G] - [x_n, y_n;G]\big)(z_{n+1} - z_n)\| \\ &\leq \psi(\|z_{n+1} - x_n\|, \|z_n - y_n\|)\|z_{n+1} - z_n\| \\ &\leq \psi(\eta + \lambda\eta, \lambda\eta)\|z_{n+1} - z_n\|.\end{aligned}$$

As $n \to \infty$, we obtain $G(z^*) = 0$ by using the continuity of G.

To prove the uniqueness, let y^* be another solution of $G(z) = 0$ in $\overline{B(z_0, r_1 + \lambda\eta)} \cap \Omega$. Define $Q = [z^*,y^*;G]$, then

$$\begin{aligned}\|[z_{-1}, z_0;G]^{-1}([z_{-1}, z_0;G] - [z^*, y^*;G])\| &\leq \psi_0(\|z_{-1} - z^*\|, \|z_0 - y^*\|) \\ &\leq \psi_0(\mu + r + \lambda\eta, r_1 + \lambda\eta) < 1.\end{aligned}$$

Thus, by the Banach Lemma, Q^{-1} exists and hence, $z^* = y^*$.

2.3 Local convergence

In this section, a local convergence result is obtained by considering $\tilde{z} = z^*$ under the following conditions:

(LC1) Let $z^* \in \Omega$ be a solution of $G(z) = 0$ and $z_0 \in B(z^*, \mu)$, with $\mu > 0$, such that $[z^*, z_0; G]^{-1}$ exists.
(LC2) Let $z_{-1} \in \Omega$, with $z_{-1} \in B(z_0, \alpha)$, $\alpha \leq 2\mu$ and $z_{-1} \neq z_0$.
(LC3) $\|[z^*, z_0; G]^{-1}([x,y;G] - [u,v;G])\| \leq \psi(\|x-u\|, \|y-v\|)$ holds for all $x,y,u,v \in \Omega$ with $x \neq y$ and $u \neq v$, where $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non decreasing function in its both arguments.
(LC4) $B(z^*, (1+2\lambda)\mu)) \subseteq \Omega$ with $\psi(2\lambda\mu, (1+2\lambda)\mu) + \psi_0((1+2\lambda)\mu, 2(1+\lambda)\mu) < 1$.

As previously, we notice that from $(LC3)$, we deduce the following condition:

(LC3') $\|[z^*, z_0; G]^{-1}([x,y;G] - [z^*, z_0; G])\| \leq \psi_0(\|x-z^*\|, \|y-z_0\|)$ holds for all $x,y \in \Omega$ with $x \neq y$, where $\psi_0 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non decreasing function in both arguments.

Firstly, we present a result on the inverse of the divided difference of the operator G.

Lemma 9 *Under conditions $(LC1)$ and $(LC3')$, then there exists $[x,y;G]^{-1}$ and*

$$\|[x,y;G]^{-1}[z^*, z_0; G]\| \leq \frac{1}{1-\psi_0((1+2\lambda)\mu, 2(1+\lambda)\mu)}, \tag{20}$$

for each pair of distinct points $(x,y) \in B(z^*, (1+2\lambda)\mu) \times B(z^*, (1+2\lambda)\mu)$.

Proof Using $(LC3')$, we get

$$\|[z^*, z_0; G]^{-1}([z^*, z_0; G] - [x,y;G])\| \leq \psi_0(\|x-z^*\|, \|y-z_0\|) \leq \psi_0((1+2\lambda)\mu, 2(1+\lambda)\mu) < 1.$$

Thus, by Banach Lemma, $[x,y;G]^{-1}$ exists and the result is obtained.

Now, by the definition of the method (4), it follows

$$\begin{aligned} \|x_0 - z^*\| &\leq \|x_0 - z_0\| + \|z_0 - z^*\| \\ &\leq \|(1-\lambda)z_0 + \lambda z_{-1} - z_0\| + \|z_0 - z^*\| \\ &\leq \|z_0 - z^*\| + \lambda\|z_0 - z_{-1}\| \\ &< \mu + \lambda\alpha \leq (1+2\lambda)\mu, \end{aligned}$$

and

$$\begin{aligned} \|y_0 - z^*\| \leq \|y_0 - z_0\| + \|z_0 - z^*\| &\leq \|(1+\lambda)z_0 - \lambda z_{-1} - z_0\| \\ + \|z_0 - z^*\| &\leq \lambda\|z_0 - z_{-1}\| \\ + \|z_0 - z^*\| &< (1+2\lambda)\mu. \end{aligned}$$

It follows that $x_0,y_0 \in B(z^*,(1+2\lambda)\mu)$, and as $x_0 \neq y_0$ then $[x_0,y_0;G]$ is well defined. Therefore, by using Lemma 9, $[x_0,y_0;G]^{-1}$ exists. Again, by using $(LC3)$ and $(LC3')$, we get

$$\begin{aligned} \|z_1 - z^*\| &= \|(z_0 - z^*) - [x_0, y_0;G]^{-1}(G(z_0) - G(z^*))\| \\ &\leq \|[x_0, y_0;G]^{-1}[z^*, z_0;G]\| \|[z^*, z_0;G]^{-1}([x_0, y_0;G] - [z_0, z^*;G])\| \|z_0 - z^*\| \\ &\leq \frac{\psi(\|x_0-z_0\|,\|y_0-z^*\|)}{1-\psi_0((1+2\lambda)\mu,2(1+\lambda)\mu)} \|z_0 - z^*\| \\ &\leq \frac{\psi(2\lambda\mu,(1+2\lambda)\mu)}{1-\psi_0((1+2\lambda)\mu,2(1+\lambda)\mu)} \|z_0 - z^*\| = g(\mu)\|z_0 - z^*\|, \end{aligned} \tag{21}$$

where $g(\mu) = \frac{\psi(2\lambda\mu,(1+2\lambda)\mu)}{1-\psi_0((1+2\lambda)\mu,2(1+\lambda)\mu)}$. As $g(\mu) < 1$, therefore

$$\|z_1 - z^*\| < \|z_0 - z^*\| < \mu.$$

In addition, we consider $z_1 \neq z_0$, then $x_1 \neq y_1$ and $[x_1,y_1;G]$ is well defined. Thus,

$$\begin{aligned} \|x_1 - z^*\| &\leq \|x_1 - z_1\| + \|z_1 - z^*\| \\ &\leq \|(1-\lambda)z_1 + \lambda z_0 - z_1\| + \|z_1 - z^*\| \\ &\leq \|z_1 - z^*\| + \lambda\|z_1 - z_0\| \\ &< \mu + \lambda\mu \leq (1+2\lambda)\mu \end{aligned}$$

and

$$\begin{aligned} \|y_1 - z^*\| \leq \|y_1 - z_1\| + \|z_1 - z^*\| \leq \|(1+\lambda)z_1 - \lambda z_0 - z_1\| \\ +\|z_1 - z^*\| \leq \lambda\|z_1 - z_0\| + \|z_1 - z^*\| < (1+2\lambda)\mu. \end{aligned}$$

Hence, iterates $x_1,y_1 \in B(z^*,(1+2\lambda)\mu)$ and, by using Lemma 9, there exists $[x_1,y_1;G]^{-1}$. Further, it follows that

$$\|z_2 - z^*\| < g(\mu)\|z_1 - z^*\| < g(\mu)^2\|z_0 - z^*\| < \|z_0 - z^*\| < \mu.$$

To continue this and by applying the mathematical induction, we get the following recurrence relations for the sequence $\{z_n\}$ given by (4).

Lemma 10 *Assume that the conditions* $(LC1)$–$(LC4)$ *hold, then, for* $n \geq 1$, *it follows*:

(i) $x_n, y_n \in B(z^*, (1+2\lambda)\mu) \subseteq \Omega$ with $x_n \neq y_n$,
(ii) $\|z_n - z^*\| < g(\mu)\|z_{n-1} - z^*\| < g(\mu)^n\|z_0 - z^*\| < \|z_0 - z^*\| < \mu$.

Next, by using Lemma 10, we obtain a local convergence result for the sequence $\{z_n\}$ given by (4).

Theorem 11 *Assume that the conditions* $(LC1)$–$(LC4)$ *hold. For each* $z_0 \in B(z^*,\mu)$, *with* $z_{-1} \in B(z_0,\alpha)$, *the sequence* $\{z_n\}$ *given by* (4) *remains in* $B(z^*,\mu)$ *and converges to* z^*, *a solution of equation* $G(z) = 0$. *Furthermore, if we assume that there exists* $r_1 \geq \mu$ *such that* $\psi_0(0,\mu + r_1) < 1$. *Then,* z^* *is the unique solution of the equation* $G(z) = 0$ *in* $B(z^*,r_1) \cap \Omega$.

Proof From the previous Lemma, the sequence $\{z_n\}$ is well defined remains in $B(z^*,\mu)$ and converges to z^*. To prove the uniqueness of solution, suppose that $y^* \in B(z^*,r_1) \cap \Omega$ be such that $G(y^*) = 0$. Then by using $(LC2')$, we get

$$\|[z^*, z_0;G]^{-1}([z^*, z_0;G] - [z^*, y^*;G])\| \le \psi_0(\|z^* - z^*\|, \|z_0 - y^*\|) \le \psi_0(0, \mu + r_1) < 1.$$

Therefore, $[z^*,y^*;G]^{-1}$ exists and hence $z^* = y^*$.

3 Non-differentiable Hammerstein-type integral equations

One of the most important mathematical tool to describe applied problems is related with nonlinear integral equations. We can mention among others science applied problems, fracture mechanics problems, aerodynamics, the theory of porous filtering, antenna problems in electromagnetic theory and others. These complex and practical situations can be formulated as integral equations of the first, second and third kind. It is well known that, the obtainment of a solution of these equations are used to be very difficult and sometimes impossible, so numerical procedures to approximate the solutions are the angular stone in Numerical Methods. In the literature, we find different Fredholm-type integral equations [27, 31], Volterra-Fredholm integral equations [15, 26], nonlinear Fredholm integro-differential equations [23], systems of Fredholm-Volterra integral equations [24], etc.

Next, we deal with the following case of nonlinear Hammerstein-type integral equation [7, 25, 27]:

$$z(s) = h(s) + \theta \int_a^b \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \quad s \in [a,b], \tag{22}$$

where $\theta \in \mathbb{R}$, $-\infty < a < b < +\infty$, the function $h(s)$ is a given continuous function on $[a,b]$, the kernel $\mathcal{K}(s,t)$ is a known continuous function in $[a,b] \times [a,b]$, the Nemytskii operator $\mathcal{N} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, where Ω is a nonempty open convex domain in $\mathscr{C}([a,b])$, given by $\mathcal{N}(z)(t) = N(z(t))$, where N is a known continuous but non-differentiable function in $\mathbb{R}$ and z is a solution to be determined in $\mathscr{C}([a,b])$, where $\mathscr{C}([a,b])$ denotes the space of continuous real functions in $[a,b]$.

In this section, our first objective is to carry out a qualitative study of (22), obtaining domains of existence and uniqueness of solutions. The second objective is to use an iterative process of (4) for a fixed value $\lambda \in (0,1]$, and by direct application to approximate a solution of (22). For this, we observe that the (22) can be defined as $\mathcal{G}(z) = 0$ for $\mathcal{G} : \Omega \subseteq \mathscr{C}([a,b]) \to \mathscr{C}([a,b])$, where Ω is a nonempty open convex domain in $\mathscr{C}([a,b])$ and

$$[\mathcal{G}(z)](s) = z(s) - h(s) - \theta \int_a^b \mathcal{K}(s,t)\mathcal{N}(z)(t)dt, \quad s \in [a,b]. \tag{23}$$

Obviously, a solution of $\mathcal{G}(z) = 0$ is a solution of (22).

It is clear that, in order to apply an iterative process of (4) for a fixed value $\lambda \in (0,1]$, we will first need to define a first-order divided difference for the $\mathcal{G}$ operator. Taking into account that $\mathcal{N}$ is a non-derivable real function, we can define the first-order divided difference in $[a, b] \subset \mathbb{R}$ given by

$$[u, v; N] = \begin{cases} \frac{N(u)-N(v)}{u-v} & \text{if} u, v \in [a, b] \text{such that} u \neq v, \\ 0 & \text{if} u, v \in [a, b] \text{such that} u = v. \end{cases}$$

Then, from this definition, we can define $[x, y; \mathcal{G}] : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b]$ with

$$[x, y; \mathcal{G}](u)(s) = u(s) - \theta \int_a^b \mathcal{K}(s, t)[x, y; \mathcal{N}](u)(t)dt, \tag{24}$$

where we consider

$$[x, y; \mathcal{N}](u)(t) = \begin{cases} \frac{N(x(t))-N(y(t))}{x(t)-y(t)} u(t) & \text{if} t \in [a, b] \text{such that} x(t) \neq y(t), \\ 0 & \text{if} t \in [a, b] \text{such that} x(t) = y(t). \end{cases} \tag{25}$$

Therefore, for the continuous real functions x and $y (x \neq y)$, obviously $[x, y; \mathcal{G}] \in \mathscr{L}(\Omega, \mathscr{C}([a, b]))$ and

$$\left[x, y; \mathcal{G}\right](x - y) = \mathcal{G}(x) - \mathcal{G}(y).$$

Then, $\left[x, y; \mathcal{G}\right]$ is a first-order divided difference for the operator $\mathcal{G} : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b])$. In our study, the max-norm has been considered in $\mathscr{C}([a, b])$.

3.1 Domain of existence of solution

As a consequence of Theorem 4 we saw that for each $z_0 \in B(\tilde{z}, \mu)$ and $z_{-1} \in B(z_0, \alpha)$ such that the conditions $(C1)$-$(C5)$ are satisfied, from the iterative processes given in (4), we obtain a domain of existence of solution, $\overline{B(z_0, r)}$, for the equation $G(z) = 0$. Therefore, we study what conditions must be verified on the operator $\mathcal{G}$, given in (23) from the integral (22), so that the corresponding conditions $(C1)$-$(C5)$ are satisfied and then, we apply the Theorem 4.

Firstly, we suppose that $z_0, \tilde{z} \in \Omega$, with $z_0 \neq \tilde{z}$, such that $[\tilde{z}, z_0; \mathcal{G}]^{-1}$ exists.

Notice that as $z_0 \neq \tilde{z}$ then, $[\tilde{z}, z_0; \mathcal{N}]$ is well defined and if $\mathcal{I}$ is the identity on $\mathscr{C}([a, a])$, we have

$$(\mathcal{I} - [\tilde{z}, z_0; \mathcal{G}])(u)(s) = \theta \int_a^b \mathcal{K}(s, t)[\tilde{z}, z_0; \mathcal{N}](t)u(t)dt.$$

Then, $\|\mathcal{I} - [\tilde{z}, z_0;\mathcal{G}]\| < |\theta|M\|[\tilde{z}, z_0;\mathcal{N}]\|$, where $M = \left\|\int_a^b \mathcal{K}(s,t)dt\right\|$.

Thus, if $\|[\tilde{z}, z_0;\mathcal{N}]\| < 1/(|\theta|M)$, by the Banach Lemma for inverse operators [21], we obtain that there exists $[\tilde{z}, z_0;\mathcal{G}]^{-1}$ with

$$\|[\tilde{z}, z_0;\mathcal{G}]^{-1}\| \leq \frac{1}{1 - |\theta|M\|[\tilde{z}, z_0;\mathcal{N}]\|},$$

so, we denote $\beta = \frac{1}{1-|\theta|M\|[\tilde{z},z_0;\mathcal{N}]\|}$.

Secondly, in order to analyze the domain of existence of iterative processes that do not use derivatives in their algorithms, the conditions are usually required on the operator divided difference. For this, we have that for each pair of distinct points x,y $\in\Omega$, there exists a first-order divided difference of $\mathcal{G}$ at these points given in (24). Then, if we suppose that the following condition hold:

$$\|[x, y;\mathcal{N}] - [u, v;\mathcal{N}]\| \leq \omega(\|x - u\|, \|y - v\|), x, y, u, v \in \Omega, \tag{26}$$

where $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in its two arguments, then we deduce that

$$[\tilde{z}, z_0;\mathcal{G}]^{-1}\Big([x, y;\mathcal{G}] - [u, v;\mathcal{G}]\Big)(w)(s) = [\tilde{z}, z_0;\mathcal{G}]^{-1}\Big(\theta\int_a^b \mathcal{K}(s,t)\Big([u, v;\mathcal{N}](t) - [x, y;\mathcal{N}](t)\Big)w(t)dt\Big),$$

and therefore

$$\|[\tilde{z}, z_0;\mathcal{G}]^{-1}\Big([x, y;\mathcal{G}] - [u, v;\mathcal{G}]\Big)\| \leq |\theta|M\beta\omega(\|x - u\|, \|y - v\|). \tag{27}$$

Notice that, from (26), we deduce the following condition: For all x,y $\in\Omega$, with $x\neq y$ holds:

$$\|[x, y;\mathcal{N}] - [\tilde{z}, z_0, \mathcal{N}])\| \leq \omega_0\big(\|x - \tilde{z}\|, \|y - z_0\|\big), \tag{28}$$

where $\omega_0 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in its two arguments.

Next, we establish the following convergence result for the sequence $\{z_n\}$, given in (4). This sequence converges to a solution of equation $\mathcal{G}(z) = 0$ with $\mathcal{G}$ given in (23).

Theorem 12 *Let $\mathcal{G}$ be the continuous operator in $\Omega \subseteq \mathscr{C}([a,b])$ given in* (23). *Fixed λ* $\in$ (0,1], *we suppose that the following conditions are satisfied*:

(I) Let $z_0, \tilde{z} \in \Omega$, with $z_0 \in B(\tilde{z}, \mu)$ and $z_0 \neq \tilde{z}$, such that $[\tilde{z}, z_0;\mathcal{G}]^{-1}$ exists and $\|[\tilde{z}, z_0;\mathcal{G}]\|^{-1} < \beta$.
(II) Let z_{-1} $\in\Omega$, with $z_{-1} \in B(z_0,\beta\delta)$ and $z_{-1}\neq z_0$, where $\|\mathcal{G}(z_0)\| \leq \delta$.
(III) $\|[x, y;\mathcal{N}] - [u, v;\mathcal{N}]\| \leq \omega(\|x - u\|, \|y - v\|), x, y, u, v \in \Omega$, where $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in its two arguments.
(IV) The auxiliary real equation

$$(f_0(t) + 1 - f(t))\xi - (1 - f(t))t = 0, \tag{29}$$

where

$$\xi = \frac{\beta\delta}{1 - |\theta| M\beta\omega_0(\mu + \lambda\beta\delta, \lambda\beta\delta)},$$

$$f_0(t) = \frac{|\theta| M\beta\omega(\xi + \lambda\beta\delta, \lambda\beta\delta)}{1 - |\theta| M\beta\omega_0(\mu + \lambda\xi + t, \lambda\xi + t)}$$

and

$$f(t) = \frac{|\theta| M\beta\omega(\xi + \lambda\xi, \lambda\xi)}{1 - |\theta| M\beta\omega_0(\mu + \lambda\xi + t, \lambda\xi + t)},$$

has at least one positive real root and we denote by R the smallest positive real root.

(V) $B(\tilde{z}, \mu + R + \lambda\xi) \subseteq \Omega$, $|\theta| M\beta\omega_0(\mu + R + \lambda\xi, R + \lambda\xi) < 1$ and $\max\{f_0(R), f(R)\} < 1$.

Then, for each $z_0 \in B(\tilde{z}, \mu)$ *and* $z_{-1} \in B(z_0, \beta\delta)$ *satisfying the previous conditions, the sequence* $\{z_n\}$, *given by* (4), *converges to* z^* *a solution of equation* $\mathcal{G}(z) = 0$. *Moreover,* $z_n, z^* \in \overline{B(z_0, R)}$ *for all* $n \geqslant 1$.

Proof The idea of the proof is to apply Theorem 5 to the operator $\mathcal{G}$ given in (23).

From condition **(I)**, we have that there exists $[\tilde{z}, z_0; \mathcal{G}]^{-1}$ with $\|[\tilde{z}, z_0; \mathcal{G}]^{-1}| \leq \beta$. Then, if $\|\mathcal{G}(z_0)\| \leq \delta$, so that $(C1)$ is satisfied with $\alpha = \beta\delta$. Obviously, from **(II)**, with this notation $(C2)$ is verified.

Moreover, from condition **(III)**, taking into account (27), if we consider $\psi(-,-) = |\theta| M\beta\omega(-,-)$, the condition $(C3)$ is satisfied. Moreover, from (28), taking $\psi_0(-,-) = |\theta| M\beta\omega_0(-,-)$, the condition $(C3')$ is satisfied.

To finish, note that, for $\alpha = \beta\delta$, the real functions $\psi(-,-)$ and $\psi_0(-,-)$ indicated previously and $r = R$, from conditions **(IV)** and **(V)**, it follows that the conditions $(C4)$ and $(C5)$ are also satisfied. Therefore we can apply the Theorem 5 and the result is proved.

Note that, the ball $\overline{B(z_0, R)}$ is the domain of existence of solution for the equation $\mathcal{G}(z) = 0$.

3.2 Domain of uniqueness of solution

To obtain the domain of uniqueness of solution for the (23), it is enough to apply the Theorem 5 taking into account the Theorem 12 that we have just proved.

Theorem 13 *Let $\mathcal{G}$ be the continuous operator in Ω given in (23), $z_0 \in B(\tilde{z}, \mu)$ and $z_{-1} \in B(z_0,\beta\delta)$ such that the conditions* **(I)-(V)** *are satisfied. We assume that there exists $R_1 \geq R$ such that*

$$|\theta| M\beta\omega_0(\mu + R + \lambda\xi, R_1 + \lambda\xi) < 1,$$

then, the limit point z^* is the only solution of equation $\mathcal{G}(z) = 0$ in $\overline{B(z_0, R_1 + \lambda\xi)} \cap \Omega$

.

Therefore, $\overline{B(z_0, R_1 + \lambda\xi)} \cap \Omega$ is the domain of uniqueness of solution for the equation $\mathcal{G}(z) = 0$.

3.3 Numerical experiment

Next, we present a numerical experiment where we illustrate all the above results. We consider the following nonlinear and non-differentiable integral equation of Fredholm-type of the form given in (22):

$$z(s) = h(s) + \theta \int_0^1 st\left(z(t)^2 - \frac{|z(t)|}{5}\right)dt, \quad s \in [0, 1], \tag{30}$$

this is, for a fixed value $\theta \in \mathbb{R}$, we consider $\mathcal{K}(s, t) = st$ and $\mathcal{N}(z)(t) = z(t)^2 - \frac{|z(t)|}{5}$. The function $h(s)$ is chosen in order to $z^*(s) = s - \frac{1}{2}$ be a solution. In this case

$$h(s) = (1 - \frac{\theta}{60})s - \frac{1}{2} \tag{31}$$

Then, to solve this (30), we apply the iterative scheme (4) to the operator (23), with

$$[\mathcal{G}(z)](s) = z(s) - (1 - \frac{\theta}{60})s + \frac{1}{2} - \theta \int_0^1 st\left(z(t)^2 - \frac{|z(t)|}{5}\right)dt. \quad s \in [0, 1], \tag{32}$$

being $\mathcal{G} : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b])$, where Ω is a nonempty open convex domain in $\mathscr{C}([a, b])$

On the one hand, note that, if we consider $z_0 \neq \tilde{z}$ with $z_0, \tilde{z} \in \Omega$ and $\mathcal{I}$ is the identity on $\mathscr{C}([\alpha, \beta])$, it follows that

$$(\mathcal{I} - [\tilde{z}, z_0;\mathcal{G}])(u)(s) = \theta \int_0^1 st[\tilde{z}, z_0;\mathcal{N}](t)u(t)dt,$$

then, we have

$$\|\mathcal{I} - [\tilde{z}, z_0;\mathcal{G}]\| \leq \frac{|\theta|}{2} \|[\tilde{z}, z_0;\mathcal{N}]\| \leq \frac{|\theta|}{2}(\frac{1}{5} + \|\tilde{z}\| + \|z_0\|),$$

So, if $\frac{|\theta|}{2}\left(\frac{1}{5} + \|\tilde{z}\| + \|z_0\|\right) < 1$, by the Banach Lemma for inverse operators [21], we obtain that there exists $[\tilde{z}, z_0;\mathcal{G}]^{-1}$ with

Table 1 Radii of the balls of existence and uniqueness, $\mu = 1$, $\tilde{z} = 0, z_0 = 1/6, z_{-1} = 1/9, \beta = 1.0187, \delta = 0.6667$.

λ	R	R_1
0.2	0.8129	18.1235
0.4	0.8439	17.7865
0.6	0.8780	17.4368
0.8	0.9158	17.0733
1	0.9584	16.6947

$$\|[\tilde{z}, z_0;\mathcal{G}]^{-1}\| \leq \frac{1}{1 - \frac{|\theta|}{2}\left(\frac{1}{5} + \|\tilde{z}\| + \|z_0\|\right)}.$$

Then, we have $\beta = \frac{1}{1-\frac{|\theta|}{2}\left(\frac{1}{5}+\|\tilde{z}\|+\|z_0\|\right)}$.

On the other hand, in this case, $N(\tau) = \tau^2 - \frac{|\tau|}{5}$ and from (25), it is easy to check that

$$\|[x, y;\mathcal{N}] - [u, v;\mathcal{N}]\| \leq \frac{2}{5} + \|x - u\| + \|y - v\|, \tag{33}$$

therefore, for condition **(III)** we have $\omega(\tau, \zeta) = \frac{2}{5} + \tau + \zeta$, a non decreasing real function in its two variables.

To analyze the existence and uniqueness of solution for (23), we consider $\Omega = \mathscr{C}([0, 1])$, $\theta = 1/10$, and also $\omega_0 = \omega$.

Firstly, fixed $\mu = 1$, and taking different values for z_{-1} and z_0, by applying the theoretical results obtained in Theorems 6 and 7 we get the results given in Tables 1 and 2 . In this case, we observe that taking smaller values for the output points, better domains of existence and uniqueness are obtained. Besides, as one can check, when λ decreases the values slightly improve. The ball of existence (see R) is smaller, that is, we locate the solution better. While the ball of uniqueness (see R_1) grows, so we better separate the solutions.

Secondly, for $\mu = 1$, $\mu = 1/3$ and $\mu = 1/5$ with $\tilde{z} = 0, z_0 = 1/2\mu$ and $z_{-1} = 1/3\mu$, by applying the theoretical results obtained in Theorems 6 and 7 we get the results given in Tables 2, 3 and 4. This study allows us to affirm that, reducing the value of μ better domains of existence and uniqueness are obtained. Furthermore, in each case, we can observe that when λ decreases the radii of existence and uniqueness balls slightly improve.

Table 2 Radii of the balls of existence and uniqueness, $\mu = 1$, $\tilde{z} = 0, z_0 = 1/2\mu, z_{-1} = 1/3\mu, \beta = 1.0363, \delta = 1$.

λ	R	R_1
0.2	1.31875	17.1237
0.4	1.4135	16.5493
0.6	1.5305	15.9286
0.8	1.6834	15.2464
1	1.9039	14.4689

Table 3 Radii of the balls of existence and uniqueness, $\mu = 1/3$, $\tilde{z} = 0, z_0 = 1/2\mu, z_{-1} = 1/3\mu$, $\beta = 1.0187$, $\delta = 0.6667$.

λ	R	R_1
0.2	0.7792	18.6873
0.4	0.8072	18.3832
0.6	0.8378	18.0694
0.8	0.8714	17.7453
1	0.9087	17.4103

In general, since this situation is not differentiable, we observe that by approaching Newton's method ($\lambda = 0$) with the iterative processes given in (4), we obtain better qualitative results.

Next, our interest is focused on approximating a solution of the (22), for this we will apply the iterative processes (4) to the operator $\mathcal{G}$, given in (23), and thus obtain a solution of $\mathcal{G}(z) = 0$. In first place, we will need to calculate $[x_n, y_n;\mathcal{G}]^{-1}$. For this, taking into account (24), for $u, v, x, y \in \Omega = \mathscr{C}([0, 1])$, with $x \neq y$, we consider

$$[x, y;\mathcal{G}](u)(s) = u(s) - \theta \int_0^1 st[x, y;\mathcal{N}](u)(t)dt = v(s),$$

then we have $u(s) = v(s) + \theta \mathbb{I} s$, where $\mathbb{I} = \int_0^1 t[x, y;\mathcal{N}](u)(t)dt$, with $\mathbb{I} \in \mathbb{R}$. Therefore,

$$\int_0^1 s[x, y;\mathcal{N}](u)(s)ds = \int_0^1 s[x, y;\mathcal{N}](v)(s)ds + \theta \mathbb{I} \int_0^1 [x, y;\mathcal{N}](w)(s)ds,$$

where $w(s) = s^2$. So, we obtain

$$\mathbb{I} = \frac{\int_0^1 s[x, y;\mathcal{N}](v)(s)ds}{1 - \theta \int_0^1 [x, y;\mathcal{N}](w)(s)ds},$$

and then

$$u(s) = [x, y;\mathcal{G}]^{-1}(v)(s) = v(s) + \theta \frac{\int_0^1 s[x, y;\mathcal{N}](v)(s)ds}{1 - \theta \int_0^1 [x, y;\mathcal{N}](w)(s)ds},$$

as long as $\theta \int_0^1 [x, y;\mathcal{N}](w)(s)ds \neq 1$.

Table 4 Radii of the balls of existence and uniqueness, $\mu = 1/5$, $\tilde{z} = 0, z_0 = 1/2\mu, z_{-1} = 1/3\mu$, $\beta = 1.0152$, $\delta = 0.6$.

λ	R	R_1
0.2	0.6867	18.9087
0.4	0.7801	18.6435
0.6	0.7308	18.3711
0.8	0.7556	18.0915
1	0.7825	17.8017

Now, taking into account (25), as $\mathcal{N}(z)(t) = z(t)^2 - \frac{|z(t)|}{5}$ is easy to check that

$$[x, y;\mathcal{N}](v)(s) = \left(x(s) + y(s) - \frac{1}{5}\left[\frac{|x(s)| - |y(s)|}{x(s) - y(s)}\right]\right)v(s)$$

and

$$[x, y;\mathcal{N}](w)(s) = \left(x(s) + y(s) - \frac{1}{5}\left[\frac{|x(s)| - |y(s)|}{x(s) - y(s)}\right]\right)s^2,$$

where $\frac{|x(s)|-|y(s)|}{x(s)-y(s)}$ is zero if there exists $s \in [0,1]$ such that $x(s) = y(s)$.

To approximate a solution of the (22) by means the iterative process (4), with $\lambda \in (0,1]$, we take $z_{-1}, z_0 \in \Omega$ and as $x_n(t) + y_n(t) = 2z_n(t)$, fixed $\theta \in \mathbb{R}$ we apply the following algorithm for $n \geqslant 0$:

First step: Calculate:

$$[\mathcal{G}(z_n)](s) = z_n(s) - (1 - \frac{\theta}{60})s + \frac{1}{2} - \theta s\int_0^1 t\left(z_n(t)^2 - \frac{|z_n(t)|}{5}\right)dt.$$

Second step: Calculate:

$$A_n = \int_0^1 2tz_n(t)[\mathcal{G}(z_n)](t)dt, B_n = \int_0^1 t\left[\frac{|x_n(t)| - |y_n(t)|}{x_n(t) - y_n(t)}\right][\mathcal{G}(z_n)](t))dt.$$

$$C_n = \int_0^1 \left(2z_n(t) - \frac{1}{5}\left[\frac{|x_n(t)| - |y_n(t)|}{x_n(t) - y_n(t)}\right]\right)t^2dt.$$

Third step: Calculate:

$$W_n = \frac{A_n - \frac{1}{5}B_n}{1 - \theta C_n},$$

and finally set new iterate:

$$z_{n+1}(s) = z_n(s) - [\mathcal{G}(z_n)](s) - \theta W_n s.$$

We implement this algorithm working with Matlab R2019a setting variable precision arithmetic with 60 digits, by choosing $\theta = 1/10$, $z_0 = 1/2$ and $z_{-1} = 1/3$. So, by imposing as a stopping criteria $\|z_{n+1}(s) - z_n(s)\| \leq 10^{-30}$ we obtain the exact solution z^*. With the results of Table 5, it can be checked that for smaller values of λ the results show a slight improvement in accuracy although in all cases computational order of convergence $p = 2$ is reached with the same number of iterations, *iter*. Also, we can compare the distance between the last 2 iterates, $\|z_{n+1}(s) - z_n(s)\|$, the value of the function at the approximation solution,

Table 5 Numerical results for different value of λ with 60 digits

Method (4)	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
iter	6	6	6	6	6
p	2	2.0022	2.0548	2.0151	1.9246
$\Vert z_{n+1}(s) - z_n(s)\Vert$	1.2238e-58	1.3607e-59	3.7993e-60	3.0797e-59	1.6014e-56
$\Vert G(z_{n+1}(s))\Vert$	6.0931e-59	6.7723e-60	3.7993e-60	1.5344e-59	7.9730e-57
$\Vert z_{n+1}(s) - z^*(s)\Vert$	0	0	0	0	0

$\Vert G(z_{n+1})(s)\Vert$ and the distance between the approximated solution an the exact solution $\Vert z_{n+1}(s) - z^*(s)\Vert$.

Now, by following the theoretical study performed in section 2.3 we obtain the convergence ball centered at the exact solution $z^*(s) = s - 1/2$. We take the values $z_0(s) = 1/2$ and $z_{-1}(s) = 1/3$ so, $\alpha = \Vert z_{-1}(s) - z_0(s)\Vert = 1/6$ and then by imposing condition $(LC4)$ we solve the equation:

$$\psi(2\lambda t, (1 + 2\lambda)t) + \psi_0((1 + 2\lambda)t, 2(1 + \lambda)t) - 1 = 0,$$

which smallest positive root gives us the value of $\mu = 1.7548$, then we verify that $2\alpha \leq \mu$, that is we obtain the local convergence ball $B(z^*, 1.7)$ in case $\lambda = 0.8$. In Table 6 we can check the radius for other values of parameter λ. We notice that the domain of convergence is better for smaller values ofλ.

4 Conclusions

In this work, we consider a uniparametric family of iterative processes that are derivative free, so we can apply them to solve non-differentiable problems, as is the case of nonlinear Hammerstein-type integral equations with the Nemystkii operator continuous but maybe non-differentiable. We perform a qualitative convergence study with the particularity of using the technique based on auxiliary points, that allow us to obtain local and semilocal convergence balls. Finally, we apply the theoretical results for proving the existence of solution of an apply problem, obtaining the domain of existence and uniqueness. Moreover, the accessibility region is also provided.

Table 6 Radii of the convergence ball for different values of λ

(4)	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
μ	3.7	3.2	2.5	2	1.7

Author contribution These authors contributed equally.

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