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Presymmetric w-Distances on Metric Spaces

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Abstract: In an outstanding article published in 2008, Suzuki obtained a nice generalization of the Banach contraction principle, from which a characterization of metric completeness was derived. Although Suzuki's theorem has been successfully generalized and extended in several directions and contexts, we here show by means of a simple example that the problem of achieving, in an obvious way, its full extension to the framework of w-distances does not have an emphatic response. Motivated by this fact, we introduce the concept of presymmetric w-distance on metric spaces. We also give some properties and examples of this new structure and show that it provides a reasonable setting to obtain a real and hardly forced w-distance generalization of Suzuki's theorem. This is realized in our main result, which consists of a fixed point theorem that involves presymmetric w-distances and certain contractions of Suzuki-type. We also discuss the relationship between our main result and the well-known w-distance full generalization of the Banach contraction principle, due to Suzuki and Takahashi. Connected to this approach, we prove another fixed point result that compares with our main result through some examples. Finally, we state a characterization of metric completeness by using our fixed point results.

Keywords: metric space; presymmetric w-distance; basic p-contraction of Suzuki-type; p-contractive self map; fixed point

MSC: 54H25; 54E50; 47H10

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1. Introduction

It has been widely recognized for many years that the so-called w-distances, introduced and discussed by Kada et al. in [1], provide an impactful and very accurate kind of generalized metric. This structure allows us to refine, enhance, and extend classical and important results as Caristi's fixed point theorem and its "equivalent" Ekeland's Variational Principle, Nadler's fixed point theorem, and many others (see, e.g., [2–10]), as well as to characterize complete metric spaces and complete fuzzy metric spaces by means of fixed point results [11,12]. The recent monograph by Rakočević [13] and the references therein provide a valuable and updated source to the study of w-distances and their applications to the fixed point theory.

In an article that is already a classic [14], Suzuki presented an elegant generalization of Banach's contraction principle that he used to characterize complete metric spaces. Although Suzuki's theorem has been successfully generalized and extended in several directions and contexts (see, e.g., [15–27]), we here show by means of an easy example that its full extension to the framework of w-distances presents suggestive difficulties that we think deserve attention.

Motivated by this scenario, we introduce and analyze the concept of presymmetric w-distance on metric spaces. In particular, we give some properties and examples of this new structure and show that it provides a reasonable setting to obtain a real and hardly forced w-distance generalization of Suzuki's theorem. This is realized in our main result, which consists of a fixed point theorem that involves presymmetric w-distances and certain contractions of Suzuki-type. We also discuss the relationship between this theorem and the w-distance full generalization of the Banach contraction principle, obtained by Suzuki and Takahashi in [11] (Theorem 1) and which they used to characterize complete metric spaces (see [11] (Theorem 4)). In this context, we introduce and examine an alternative notion of basic contraction of Suzuki-type, by attaining a new fixed point theorem which we compare with our main result through some examples. The last part of the paper is devoted to obtaining necessary and sufficient conditions for a metric space to be complete, which is made by combining our fixed point results with both Suzuki's characterization and Suzuki–Takahashi's characterization.

At this point, we wish to remark the following: There are many publications, mainly about the fixed point theory, that deal (separately) with w-distances and contractions of Suzuki-type. For this reason, and in order not to saturate the amount of bibliographical references cited in this paper, we have stuck to those given in the first two paragraphs of this section because we think they provide a sufficient background to help the reader.

2. Preliminaries

As far as possible, our notation and terminology will be standard. In what follows, we design by $\mathbb{N}$ the set of natural numbers.

Next, we establish the following special case of Suzuki's theorem [14] (Theorem 2), which will be sufficient for our targets here.

Theorem 1 ([14]). *Let (X,d) be a complete metric space and T be a self map of X. If there exists a constant $r \in (0,1)$ such that, for any $x,y \in X$, the next contraction condition holds:*

$$d(x,Tx) \le 2d(x,y) \implies d(Tx,Ty) \le rd(x,y), \tag{1}$$

then, T has a unique fixed point in X.

In line with [27], a self map T of a metric space (X,d) satisfying the contraction condition (1) will be called a basic contraction of Suzuki-type (on (X,d)).

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X,d) that converges to an $x \in X$, we simply will write $x_n \to x$ or $d(x,x_n) \to x$.

Remind ([1]) that a w-distance on a metric space (X,d) is a function $p : X \times X \to [0,+\infty)$ that verifies the following conditions:

(w1) $p(x,y) \le p(x,z) + p(z,y)$, for all $x,y,z \in X$;
(w2) for each $x \in X$, the function $p(x,\cdot) : X \to [0,+\infty)$ is lower semicontinuous;
(w3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $p(x,y) \le \delta$ and $p(x,z) \le \delta$ imply $d(y,z) \le \varepsilon$.

Given a w-distance p on a metric space (X,d), we shall denote by p^s the function defined on $X \times X$ as $p^s(x,y) = \max\{p(x,y), p(y,x)\}$ for all $x,y \in X$. Notice that p^s clearly satisfies conditions (w1) and (w3) above, but not condition (w2) in general, as the following example shows.

Example 1. *Let $X = \{0,2\} \cup \{1/n : n \in \mathbb{N}\}$, and d be restriction of the usual metric to X. It is routine to check that the function $p : X \times X \to [0,+\infty)$ defined as $p(0,2) = 2$ and $p(x,y) = 1$ otherwise, is a w-distance on (X,d). However, we obtain $p^s(2,0) = 2$ and $p^s(2,1/n) = 1$ for all $n \in \mathbb{N}$, so $p^s(2,\cdot) \to [0,+\infty)$ is not lower semicontinuous. Therefore, condition (w2) does not hold, and, thus, p^s is not a w-distance on (X,d).*

According to [28] (p. 3118), we say that a w-distance p on a metric space (X,d) is symmetric if $p(x,y) = p(y,x)$ for all $x,y \in X$. Obviously, $p = p^s$ whenever p is symmetric.

Several interesting examples of w-distances on metric spaces are given, among others, in [1,11,13]. In fact, every metric d on a set X is a symmetric w-distance on the metric space (X,d). Another very interesting example of a symmetric w-distance can be found in [1] (Example 7) and in [11] (Lemma 2).

We conclude this section by recalling a couple of representative instances of w-distances, which will be useful later on.

Example 2 (compare [1] (Example 4))**.** *Let X be a (non-empty) subset of $[0, +\infty)$. Then, the function $p : X \times X \to [0, +\infty)$ defined as $p(x, y) = y$ for all $x, y \in X$, is a w-distance for the metric space (X, d), where by d we denote the restriction of the usual metric to X.*

Example 3 (compare [1] (Example 3))**.** *Let X be a (non-empty) subset of $[0, +\infty)$ and let a, b, be constants such that $a, b > 0$. Then, the function $p : X \times X \to [0, +\infty)$ defined as $p(x, y) = ax + by$ for all $x, y \in X$, is a w-distance for the metric space (X, d), where by d we denote the restriction of the usual metric to X. Clearly, w is symmetric whenever $a = b$.*

3. Presymmetric w-Distances and a Generalization of Suzuki's Theorem

We begin by generalizing the notion of a basic contraction of Suzuki-type to the framework of w-distances.

Definition 1. *Let p be a w-distance on a metric space (X, d). We say that a self map T of X is a basic p-contraction of Suzuki-type (on (X, d)) if there exists a constant $r \in (0, 1)$ such that, for any $x, y \in X$, the next contraction condition holds:*

$$p(x, Tx) \leq 2p(x, y) \implies p(Tx, Ty) \leq rp(x, y). \tag{2}$$

In light of Theorem 1, the following question naturally arises:

Let p be a w-distance on a complete metric space (X, d) and T be a basic p-contraction of Suzuki-type (on (X, d)). Under these conditions, has T a fixed point in X?

The next example shows that, unfortunately, this question has a negative answer, in general.

Example 4. *Let $X = [0, 1]$ and let d be the restriction of the usual metric to X. Consider the self map T of X defined as $T0 = 1$ and $Tx = x/2$ for all $x \in (0, 1]$.*

Obviously, T has no fixed points. We shall show that, nevertheless, it is a basic p-contraction of Suzuki-type, where $r = 1/2$ and p is the w-distance on (X, d) given in Example 2, i.e., $p(x, y) = y$ for all $x, y \in X$.

Indeed,

- *If $x = y = 0$, we obtain $p(x, Tx) = p(0, 1) = 1 > 0 = 2p(x, y)$.*
- *If $x \in (0, 1]$ and $y = 0$, we obtain $p(x, Tx) = Tx = x/2 > 0 = 2p(x, y)$.*
- *If $y \in (0, 1]$, we obtain $p(Tx, Ty) = Ty = y/2 = p(x, y)/2$.*

With the aim of lessening the difficulties evidenced by Example 2, we propose the following notion.

Definition 2. *A w-distance p on a metric space (X, d) is called presymmetric provided that it fulfills the next property:*

Whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $x_n \to x$ and $p(x_n, x) \to 0$ for some $x \in X$, then there is a subsequence $(x_{k(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ satisfying $p(x, x_{k(n)+1}) \leq p(x_{k(n)}, x)$ for all $n \in \mathbb{N}$.

Proposition 1. *Every symmetric w-distance is presymmetric.*

Proof. Let p be a symmetric w-distance on a metric space (X, d). If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $x_n \to x$ and $p(x_n, x) \to 0$ for some $x \in X$, there is a subsequence $(x_{k(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $p(x_{k(n)+1}, x) \leq p(x_{k(n)}, x)$ for all $n \in \mathbb{N}$. By the symmetry of p, we conclude that $p(x, x_{k(n)+1}) \leq p(x_{k(n)}, x)$ for all $n \in \mathbb{N}$. □

Example 5. *Let p be a w-distance on a metric space (X,d) for which there is a constant $c>0$ such that $p(x,y)\geq c$ for all $x,y\in X$. Then, it is clear that p is a presymmetric w-distance on (X,d).*

Example 6. *Let X be a subset of $[0,+\infty)$ be such that $[0,1]\subseteq X$, let d be the restriction of the usual metric to X, and let p be the w-distance on (X,d) given by $p(x,y)=ax+by$ for all $x,y\in X$, where a and b are real constants such that $a\geq 0$ and $b>0$ (compare Examples 2 and 3).*

- *If $a<b$, the w-distance p is not presymmetric: Indeed, take the sequence $(1/n)_{n\in\mathbb{N}}$. Then, $1/n\to 0$ and $p(1/n,0)=a/n$ for all $n\in\mathbb{N}$, so $p(1/n,0)\to 0$. Since $b/a>1$, there exists an $n_0\in\mathbb{N}$ such that $b/a>(n+1)/n$ for all $n\geq n_0$. Therefore, $p(0,1/(n+1))=b/(n+1)>a/n=p(1/n,0)$ for all $n\geq n_0$.*
- *If $a=b$, the w-distance p is symmetric, and, hence, it is presymmetric by Proposition 1.*
- *If $a>b$, the w-distance p is presymmetric: Indeed, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X fulfilling $x_n\to x$ and $p(x_n,x)\to 0$ for some $x\in X$. Then, $x=0$ and, hence, $p(x,x_n)=bx_n\leq ax_n=p(x_n,x)$ for all $n\in\mathbb{N}$.*
  Note that, in this case, p is not symmetric.

Notice that in both Proposition 1 and Examples 5 and 6, the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x is not essential. We now give an example, which may be compared with Examples 3 and 6, where such convergence plays a decisive role.

Example 7. *Let $c>0$, Y be a non-empty subset of $[c,+\infty)$, $X=\{0\}\cup Y$ and d be the discrete metric on X. Define $p:X\times X\to[0,+\infty)$ as $p(x,y)=ax+by$ for all $x,y\in X$, with a,b constants such that $0\leq a<b$. Then, p is a non-symmetric presymmetric w-distance on (X,d).*

We first show that p satisfies conditions (w1), (w2), and (w3):

(w1) *It is trivially satisfied.*

(w2) *Let $x,y\in X$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $y_n\to y$. Since d is the discrete metric on X, $y_n=y$, eventually. Therefore, $p(x,y)=p(x,y_n)$, eventually.*

(w3) *Given $\varepsilon>0$, put $\delta=bc/2$. Let $p(x,y)\leq\delta$ and $p(x,z)\leq\delta$. Then, $ax+by\leq bc/2$ and $ax+bz\leq bc/2$; therefore, in particular, $y\leq c/2<c$ and $z\leq c/2<c$, which implies that $y=z=0$.*

We have shown that p is a w-distance on (X,d). Clearly, it is not symmetric.

Now let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n\to x$ and $p(x_n,x)\to 0$ for some $x\in X$. Since d is the discrete metric on X we infer that $x_n=x$, eventually, which obviously implies that p is presymmetric.

Next we state and show the main result of this paper. In the first part of its proof, we will adapt the methods customary applied when both contractions of Suzuki-type and w-distances are involved.

Theorem 2. *Let p be a presymmetric w-distance on a complete metric space (X,d) and let T be a basic p-contraction of Suzuki-type. Then, T has a unique fixed point $\xi\in X$. Furthermore, $p(\xi,\xi)=0$.*

Proof. By assumption, there exists a constant $r\in(0,1)$ for which the contraction condition (2) holds.

Fix $x_0\in X$. Put $x_n=T^n x_0$ for all $n\in\mathbb{N}\cup\{0\}$. By standard arguments, we first show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d).

As $p(x_{n-1},x_n)\leq 2p(x_{n-1},x_n)$, it follows from (2) that $p(x_n,x_{n+1})\leq rp(x_{n-1},x_n)$, so, $p(x_n,x_{n+1})\leq r^n p(x_0,x_1)$ for all $n\in\mathbb{N}$. Therefore, by the triangle inequality (w1),

$$p(x_n,x_{n+k})\leq\frac{r^n}{1-r}p(x_0,x_1),\tag{3}$$

for all $n\in\mathbb{N}$ and $k>n$.

Given $\varepsilon > 0$ let $\delta := \delta(\varepsilon) > 0$, for which condition (w3) holds. Take an $n_\delta \in \mathbb{N}$ such that

$$\frac{r^{n_\delta}}{1-r} p(x_0, x_1) < \delta. \tag{4}$$

Then, it follows from (3) that $p(x_{n_\delta}, x_n) < \delta$ and $p(x_{n_\delta}, x_m) < \delta$ whenever $n, m > n_\delta$, which implies, by (w3), that $d(x_n, x_m) \leq \varepsilon$ whenever $n, m > n_\delta$. Since ε is arbitrary, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, d).

Let $\xi \in X$ be such that $x_n \to \xi$.

For the given ε, δ, and n_δ above, choose any $j \in \mathbb{N}$ with $j > n_\delta$. By (w2), we can find $m > j$ such that $p(x_j, \xi) < p(x_j, x_m) + \varepsilon$.

Combining the relations (3) and (4), we obtain

$$p(x_j, x_m) \leq \frac{r^j}{1-r} p(x_0, x_1) < \frac{r^{n_\delta}}{1-r} p(x_0, x_1) < \delta,$$

so, $p(x_j, \xi) < \delta + \varepsilon$ for all $j > n_\delta$, which implies that $p(x_n, \xi) \to 0$.

Next we prove that ξ is a fixed point of T.

To this end, suppose that there is an $n_0 \in \mathbb{N}$ verifying that $p(x_n, x_{n+1}) > 2p(x_n, \xi)$ for all $n > n_0$. Then, we obtain

$$2p(x_n, \xi) < p(x_n, \xi) + p(\xi, x_{n+1}),$$

for all $n > n_0$. Thus, $p(x_n, \xi) < p(\xi, x_{n+1})$ for all $n > n_0$. As $p(x_n, \xi) \to 0$, we have reached a contradiction because, by assumption, p is presymmetric.

Hence, there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $p(x_{k(n)}, x_{k(n)+1}) \leq 2p(x_{k(n)}, \xi)$ for all $n \in \mathbb{N}$. By condition (2), we obtain

$$p(x_{k(n)+1}, T\xi) \leq rp(x_{k(n)}, \xi),$$

for all $n \in \mathbb{N}$. Since $p(x_{k(n)}, \xi) \to 0$ we deduce that $p(x_{k(n)+1}, T\xi) \to 0$.

Again, given $\varepsilon > 0$ let $\delta := \delta(\varepsilon) > 0$, for which condition (w3) holds. Since $p(x_n, \xi) \to 0$, there is $n_0 \in \mathbb{N}$ such that $p(x_n, \xi) \leq \delta$ for all $n > n_0$.

Choose $n \in \mathbb{N}$ be such that $k(n) + 1 > n_0$ and $p(x_{k(n)+1}, T\xi) \leq \delta$. Therefore, $p(x_{k(n)+1}, x) \leq \delta$, and, by condition (w3), we obtain $d(\xi, T\xi) \leq \varepsilon$. Since ε is arbitrary, we conclude that $\xi = T\xi$.

Now we check that $p(\xi, \xi) = 0$. Indeed, we have $p(\xi, T\xi) = p(\xi, \xi) \leq 2p(\xi, \xi)$, so, by condition (2), $p(\xi, \xi) = p(T\xi, T\xi) \leq rp(\xi, \xi)$, which implies that $p(\xi, \xi) = 0$.

Finally, we see that ξ is the unique fixed point of T. Suppose that $\vartheta \in X$ satisfies $\vartheta = T\vartheta$. As $p(\xi, T\xi) = 0$, we obtain $p(\xi, T\xi) \leq 2p(\xi, \vartheta)$. From condition (2) it follows that $p(\xi, \vartheta) \leq rp(\xi, \vartheta)$. Hence, $p(\xi, \vartheta) = 0$. Thus, $\xi = \vartheta$ by condition (w3). □

Corollary 1. *Let p be a symmetric w-distance on a complete metric space (X, d) and let T be a basic p-contraction of Suzuki-type. Then, T has a unique fixed point $\xi \in X$. Furthermore, $p(\xi, \xi) = 0$.*

The following example shows that Corollary 1, and, hence, Theorem 2, is a real generalization of Theorem 1.

Example 8. *Let $X = [0, 1]$ and let d be the restriction of the usual metric to X. Consider the self map T of X defined as $T1 = 1/2$ and $Tx = 0$ for all $x \in [0, 1)$.*

We first show that we cannot apply Theorem 1 to this case. Indeed, for $x = 1/2$ and $y = 1$, we obtain $d(x, Tx) = 1/2 = d(x, y) < 2d(x, y)$, but $d(Tx, Ty) = 1/2 = d(x, y)$.

Next we show that we can apply Corollary 1 and, thus, Theorem 2 for the symmetric w-distance on (X, d) given by $p(x, y) = x + y$ for all $x, y \in X$ (see Example 6). Indeed,

- *If $x = 1$, we obtain $p(Tx, Ty) \leq p(x, y)/2$.*
- *If $x, y \in [0, 1)$, we obtain $p(Tx, Ty) = p(0, 0) = 0$.*

- *If $x \in [0,1)$ and $y = 1$, we obtain $p(Tx, T1) = p(0, 1/2) = 1/2 \leq (x+1)/2 = p(x,y)/2$.*

4. On the Relationship between *p*-Contractive Self Maps, Basic *p*-Contractions of Suzuki-Type, and the Corresponding Fixed Point Theorems

In accordance with Suzuki and Takahashi [11], given a w-distance p on a metric space (X, d), a self map T of X is called p-contractive (or weakly contractive) if there is a constant $r \in (0,1)$ such that $p(Tx, Ty) \leq rp(x,y)$ for all $x, y \in X$. Then, they proved the following w-distance generalization of Banach's contraction principle.

Theorem 3 ([11]). *Let p be a w-distance on a complete metric space (X, d). Then, each p-contractive self map T of X has a unique fixed point $\xi \in X$. Furthermore, $p(\xi, \xi) = 0$.*

It is clear that the Banach contraction principle is a direct consequence of both Theorems 1 and 3. In contrast to this fact, we have that Theorem 3 is not a direct consequence of Theorem 2 as the following easy example shows.

Example 9. *Let $X = [0,1]$, d be the restriction of the usual metric to X, p be the w-distance on (X, d) given in Example 2, and T be the self map of X defined as $Tx = x/2$ for all $x \in X$. Since for each $x, y \in X$, $p(Tx, Ty) = y/2 = p(x,y)/2$, we deduce that T is p-contractive. Hence, all conditions of Theorem 3 are satisfied. However, we cannot apply Theorem 2 because p is not presymmetric.*

Remark 1. *Evidently, every p-contractive self map is a basic p-contraction of Suzuki-type. Example 4 furnishes an instance of a basic p-contraction of Suzuki-type T (on a complete metric space (X, d)) without fixed point, so, by Theorem 3, it is not p-contractive. In fact, we have $p(Tx, T0) = 1 > 0 = p(x, 0)$ for all $x \in X$. In addition, and as an immediate consequence of Theorem 2, we conclude that the involved w-distance p is not presymmetric.*

Remark 2. *In [14] (Example 1), Suzuki presented an instance of a complete metric space (X, d) and a self map T of X that is a basic p-contraction of Suzuki-type but not p-contractive, for $p = d$.*

Despite Example 9, in Remark 3 below we shall see that it is still possible to deduce Theorem 3 from Theorem 2 by applying the following consequence of a result from Shioji et al. [28] (Theorem 1).

Theorem 4 ([28]). *Let p be a w-distance on a metric space (X, d) and let T be a p-contractive self map of X. Then, there exists a symmetric w-distance q on (X, d) such that T is q-contractive.*

Remark 3. *Let p be a w-distance on a complete metric space (X, d) and T be a p-contractive self map of X (with contraction constant $r \in (0,1)$). By Theorem 4, there exists a symmetric w-distance q on (X, d) such that T is q-contractive. Therefore, T is a basic q-contraction of Suzuki-type, so, by Corollary 1, it has a unique fixed point $\xi \in X$ and $q(\xi, \xi) = 0$. We also have $p(\xi, \xi) = 0$ because $p(\xi, \xi) = p(T\xi, T\xi) \leq rp(\xi, \xi)$.*

Shioji et al. also explored in [28] a class of contractions which we here define as follows: Given a w-distance p on a metric space (X, d), a self map T of X is p^--contractive provided that there is a constant $r \in (0,1)$ such that $p(Tx, Ty) \leq rp(y, x)$ for all $x, y \in X$. Then, they proved the following.

Theorem 5 ([28]). *Let p be a w-distance on a metric space (X, d) and let T be a p-contractive self map of X. Then, there exists a w-distance q on (X, d) such that T is q^--contractive.*

From Theorem 5, it follows that the relevant Theorem 3 is a consequence of the next result. We emphasize that, to our surprise, we have not found that result in the literature. In any case, we formulate and prove it for the sake of completeness.

Theorem 6. *Let p be a w-distance on a complete metric space (X,d). Then, each p^--contractive self map T of X has a unique fixed point $\xi \in X$. Furthermore, $p(\xi,\xi) = 0$.*

Proof. Let T be a p^--contractive self map of X. Then, there exists a constant $r \in (0,1)$ such that

$$p(Tx,Ty) \leq rp(x,y), \tag{5}$$

for all $x,y \in X$.

Now, fix $x_0 \in X$ and put $x_n = T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

By the contraction condition (5), we obtain

$$p(x_1,x_2) \leq rp(x_1,x_0) \text{ and } p(x_2,x_1) \leq rp(x_0,x_1).$$

Similarly,

$$p(x_2,x_3) \leq rp(x_2,x_1) \text{ and } p(x_3,x_2) \leq rp(x_1,x_2),$$

so, $p(x_2,x_3) \leq r^2 p(x_0,x_1)$.

Continuing this process, we obtain

$$p(x_n,x_{n+1}) \leq r^n p(x_1,x_0) \text{ if } n \text{ is odd, and } p(x_n,x_{n+1}) \leq r^n p(x_0,x_1) \text{ if } n \text{ is even.}$$

Hence,

$$p(x_n,x_{n+1}) \leq r^n p^s(x_0,x_1),$$

for all $n \in \mathbb{N}$. Hence, exactly as in the proof of Theorem 2, we deduce the existence of a point $\xi \in X$ such that $x_n \to \xi$ and $p(x_n,\xi) \to 0$.

We shall prove that ξ is the unique fixed point of T, and also that $p(\xi,\xi) = 0$.

From condition (5) we deduce that $p(T\xi,x_{n+1}) \to 0$, and, thus, $p(T\xi,\xi) = 0$ by the triangle inequality (w1).

Since $p(T\xi,T^2\xi) \leq rp(T\xi,\xi)$, we obtain $p(T\xi,T^2\xi) = 0$. So, by condition (w3), $d(\xi,T^2\xi) \leq \varepsilon$ for all $\varepsilon > 0$, i.e., $\xi = T^2\xi$.

Consequently,

$$p(\xi,T\xi) = p(T^2\xi,T\xi) \leq rp(\xi,T\xi),$$

which implies that $p(\xi,T\xi) = 0$. Therefore, by the triangle inequality (w1), $p(\xi,\xi) = 0$ and $p(T\xi,T\xi) = 0$, so, $\xi = T\xi$, by condition (w3).

Finally, suppose that $\vartheta \in X$ satisfies $\vartheta = T\vartheta$. Then,

$$p((\xi,\vartheta) = p(T\xi,T\vartheta) \leq rp(\vartheta,\xi) \leq r^2 p(\xi,\vartheta).$$

Therefore, $p(\xi,\vartheta) = 0$. Since $p(\xi,\xi) = 0$, we obtain $\xi = \vartheta$ by condition (w3). □

In [28] (Proposition 1), it was presented an example of a w-distance p on a complete metric space (X,d) and a p^--contractive self map of X that has a unique fixed point, but that is not a q-contractive self map for any w-distance q on (X,d). Hence, this example shows that Theorem 6 is a real generalization of Theorem 3.

Since Theorem 2 can be interpreted as an extension of Theorem 3, it seems natural to ask whether it is possible to obtain an analogous extension of Theorem 6. We shall prove that this question has an affirmative answer.

Definition 3. *Let p be a w-distance on a metric space (X,d). We say that a self map T of X is a basic p^--contraction of Suzuki-type (on (X,d)) if there exists a constant $r \in (0,1)$ such that, for any $x,y \in X$, the next contraction condition holds:*

$$p(x,Tx) \leq 2p(x,y) \implies p(Tx,Ty) \leq rp(y,x). \tag{6}$$

Theorem 7. *Let p be a presymmetric w-distance on a complete metric space (X, d) and let T be a basic p^--contraction of Suzuki-type. Then, T has a unique fixed point $\xi \in X$. Furthermore, $p(\xi, \xi) = 0$.*

Proof. By assumption, there exists a constant $r \in (0, 1)$ for which the contraction condition (6) holds.

Fix $x_0 \in X$ and put $x_n = T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

Since $p(x_0, x_1) \leq 2p(x_0, x_1)$, we have $p(x_1, x_2) \leq rp(x_1, x_0)$. Hence, $p(x_1, x_2) \leq 2p(x_1, x_0)$, so, by condition (6), $p(x_2, x_1) \leq rp(x_0, x_1)$.

Again, condition (6) implies that $p(x_2, x_3) \leq rp(x_2, x_1)$. Therefore, $p(x_2, x_3) \leq r^2 p(x_0, x_1)$.

Following this process, we deduce that $p(x_n, x_{n+1}) \leq r^n p^s(x_0, x_1)$ for all $n \in \mathbb{N}$.

Then, and as in the proof of Theorem 2, there exists $\xi \in X$ such that $x_n \to \xi$ and $p(x_n, \xi) \to 0$.

By the presymmetry of p, there exists a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $p(\xi, x_{k(n)+1}) \leq p(x_{k(n)}, \xi)$ for all $n \in \mathbb{N}$. This implies that $p(\xi, x_{k(n)+1}) \to 0$.

Since $p(x_{k(n)+1}, \xi) \to 0$ we can repeat the argument given in the proof of Theorem 2 to deduce the existence of a subsequence $(x_{k(j(n))})_{n\in\mathbb{N}}$ of $(x_{k(n)+1})_{n\in\mathbb{N}}$ satisfying $p(x_{k(j(n))}, x_{k(j(n))+1}) \leq 2p(x_{k(j(n))}, \xi)$ for all $n \in \mathbb{N}$. Hence, condition (6) implies that

$$p(x_{k(j(n))+1}, T\xi) \leq rp(\xi, x_{k(j(n))}),$$

for all $n \in \mathbb{N}$. Since $p(\xi, x_{k(j(n))}) \to 0$, we deduce that $p(x_{k(j(n))+1}, T\xi) \to 0$. Taking into account that $p(x_{k(j(n))+1}, \xi) \to 0$, we obtain $\xi = T\xi$ by condition (w3). Moreover, $p(\xi, \xi) = 0$ by condition (6).

Finally, suppose that $\vartheta \in X$ satisfies $\vartheta = T\vartheta$. Once again, condition (6) implies that

$$p(\xi, \vartheta) \leq rp(\vartheta, \xi) \leq r^2 p(\xi, \vartheta),$$

so, $p(\xi, \vartheta) = 0$. By condition (w3), we conclude that $\xi = \vartheta$. □

We finish this section by exemplifying that Theorems 2 and 7 are of a different value from each other.

Example 10. *Let $X = \{0\} \cup (1, +\infty)$, d be the discrete metric on X, and $p : X \times X \to [0, +\infty)$ be defined as $p(x, y) = x + 2y$ for all $x, y \in X$.*

Then, p is a presymmetric w-distance on (X, d) as Example 7 shows.

Now let T be the self map of X defined as $Tx = 0$ if $x \in \{0\} \cup (1, 2)$, and $Tx = (x + 1)/2$ if $x \in [2, +\infty)$.

We prove that T is p-contractive with contraction constant $r = 3/4$, and, hence, it is a basic p-contraction of Suzuki-type.

- *If $x, y \in \{0\} \cup (1, 2)$, we obtain $p(Tx, Ty) = 0$.*
- *If $x \in \{0\} \cup (1, 2)$ and $y \geq 2$, we obtain $p(Tx, Ty) = y + 1 \leq 3y/2 \leq 3p(x, y)/4$.*
- *If $x \geq 2$ and $y \in \{0\} \cup (1, 2)$, we obtain $p(Tx, Ty) = (x + 1)/2 \leq 3x/4 \leq 3p(x, y)/4$.*
- *If $x, y \geq 2$, we obtain $p(Tx, Ty) = (x + 1)/2 + y + 1 \leq 3x/4 + 3y/2 = 3p(x, y)/4$.*

Therefore, all conditions of Theorem 2 (and of Theorem 3) are satisfied.

However, we have $p(0, T0) = 0 < 2p(0, 2)$, but $p(T0, T2) = p(0, 3/2) = 3 > 2 = p(2, 0)$. Thus, T is not a basic p^--contraction of Suzuki-type, so, we cannot apply Theorem 7 to these p and T.

Example 11. *Let $X = [0, 1] \cup \{2\}$, let d be the discrete metric on X, $c > 0$ be a constant such that $c(c + 2) < 1$, and $p : X \times X \to [0, +\infty)$ be defined as $p(0, 0) = 0$, $p(0, x) = 1$ and $p(x, 0) = 1 + c$ for all $x \in (0, 1]$, $p(x, y) = 1$ for all $x, y \in (0, 1]$, $p(x, 2) = 1 + c$ for all $x \in X$, $p(2, 0) = c(c + 2)$, and $p(2, x) = 1 + c(c + 2)$ for all $x \in (0, 1]$.*

It is routine to check that p is a presymmetric w-distance on the complete metric space (X, d) (for instance, (w2) follows immediately from the fact that d is the discrete metric, and to verify (w3), one must choose $\delta = c(c+2)/2)$ for any $\varepsilon > 0)$.

Now let T be self map of X defined as $Tx = 0$ for all $x \in [0,1]$ and $T2 = 1$.

We first show that T is a basic p-contraction of Suzuki-type (with contraction constant $r = 1/(1+c)$) and, unlike Example 10, it is not p-contractive.

Indeed,

- *If $x, y \in [0,1]$, we obtain $p(Tx, Ty) = p(0,0) = 0$.*
- *If $x \in [0,1]$ and $y = 2$, we obtain $p(Tx, Ty) = p(0,1) = 1 = rp(x,y)$.*
- *If $x = y = 2$, we obtain $p(Tx, Ty) = p(1,1) = 1 = rp(x,y)$.*
- *If $x = 2$ and $y \in (0,1]$, we obtain $p(Tx, Ty) = p(1,0) = 1 + c = r(1+c)^2 = rp(x,y)$.*
- *If $x = 2$ and $y = 0$, we obtain $p(x, Tx) = p(2,1) = 1 + 2c + c^2 > 2c(2+c) = 2p(x,y)$ (recall that $c(2+c) < 1$).*

We conclude that T is a basic p-contraction of Suzuki-type and, hence, all conditions of Theorem 2 are satisfied.

However, T is not p-contractive because $p(T2, T0) = p(1,0) = 1 + c > c(2+c) = p(2,0)$.

Finally, note that $p(0, T0) = 0 < 2p(0,2)$, but $p(T0, T2) = p(0,1) = 1 > c(2+c) = p(2,0)$, which implies that T is not a basic p^--contraction of Suzuki-type, so we cannot apply Theorem 6 to these p and T.

Example 12. *Denote by $\mathbb{N}^F$ the set of all finite sequences whose terms are natural numbers. Thus, if $x \in \mathbb{N}^F$, we put $x := x_1...x_k$, where $x_n \in \mathbb{N}$, for all $n \in \{1, ..., k\}$. The number $k \geq 1$ is called the length of x as is denoted by $\ell(x)$.*

Let $X = \{\omega\} \cup \mathbb{N}^F$, where by ω we denote the infinite sequence $(n)_{n \in \mathbb{N}}$, that we represent as follows: $\omega := 123...$

Set $Y = \{\omega\} \cup \{x \in \mathbb{N}^F : \ell(x) \text{ is even}\}$.

Now, let d be the discrete metric on X and $p : X \times X \to [0, +\infty)$ be the function defined as $p(x, \omega) = 0$ if $x \in Y$, $p(x,y) = 1$ if $x \in Y$ and $y \in X \backslash \{\omega\}$, and $p(x,y) = 2$ otherwise.

It is straightforward to check that p is a presymmetric w-distance on the complete metric space (X, d).

Define a self map T of X as $Tx = \omega$ if $x \in Y$, and $Tx = z_x$ if $x \in X \backslash Y$, where z_x is the unique element of $\mathbb{N}^F$ obtained by repeating once the terms of x, i.e., if $x = x_1...x_k$, then $z_x = x_1...x_k x_1...x_k$. Thus, $\ell(z_x) = 2\ell(x)$.

It is routine to check that $p(Tx, Ty) \leq p(y,x)/2$ for all $x, y \in X$ (in particular, if $x \in Y$ and $y \in X \backslash Y$ we obtain $p(Tx, Ty) = p(0, z_y) = 1 = p(y,x)/2)$).

Hence, T is p^--contractive, so it is a basic p^--contraction of Suzuki-type. Thus, all conditions of Theorem 7 (and of Theorem 6) are fulfilled.

However, for $x \in Y \backslash \{\omega\}$ and $y \in X \backslash Y$, we obtain $p(x, Tx) = p(x, \omega) = 0 < p(x,y)$, but $p(Tx, Ty) = p(\omega, z_y) = 1 = p(x,y)$. This shows that T is not a basic p-contraction of Suzuki-type, so, we cannot apply Theorem 2 to these p and T.

5. Characterizing Complete Metric Spaces

Suzuki and Takahashi showed in [11] (Theorem 4) that Theorem 3 characterizes complete metric spaces. More precisely, they proved the following result.

Theorem 8 ([11])**.** *For a metric space (X, d) the following statements are equivalent:*

(1) *(X, d) is complete.*
(2) *For any w-distance p on (X, d), every p-contractive self map of X has a (unique) fixed point.*

Observe that by combining Theorems 5, 6, and 8, we obtain the following variant of Theorem 8.

Theorem 9. *For a metric space (X, d) the following statements are equivalent:*

(1) *(X, d) is complete.*
(2) *For any w-distance p on (X, d), every p^--contractive self map of X has a (unique) fixed point.*

On the other hand, Theorem 1 also characterizes metric completeness (compare [14] (Corollary 1)). More precisely, we have the following result.

Theorem 10 ([14])**.** *For a metric space (X, d) the following statements are equivalent:*

(1) *(X, d) is complete.*
(2) *Every basic contraction of Suzuki-type has a (unique) fixed point.*

By using Theorem 2, Corollary 1 and the preceding characterization, we deduce the next result.

Theorem 11. *For a metric space (X, d) the following statements are equivalent:*

(1) *(X, d) is complete.*
(2) *For any presymmetric w-distance p on (X, d), every basic p-contraction of Suzuki-type has a (unique) fixed point.*
(3) *For any presymmetric w-distance p on (X, d), every basic p^--contraction of Suzuki-type has a (unique) fixed point.*
(4) *For any symmetric w-distance p on (X, d), every basic p-contraction of Suzuki-type has a (unique) fixed point.*

Proof. $(1) \Rightarrow (2)$ Apply Theorem 2.

$(1) \Rightarrow (3)$ Apply Theorem 7.

$(2) \Rightarrow (4)$ and $(3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (1)$ Let T be a basic contraction of Suzuki-type on (X, d). Then, it is a basic p-contraction of Suzuki-type for $p = d$. Hence, by assumption, T has a unique fixed point. Therefore, (X, d) is complete by Theorem 10. □

Remark 4. $(4) \Rightarrow (1)$*, in Theorem 11, can be also deduced from Theorem 8. Indeed, suppose that (X, d) is not complete. Thus, there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) that does not converge. Then, following the proof of [11] (Theorem 4), we can construct a symmetric w-distance p on (X, d) and a p-contractive self map T of X that has no fixed point. Since T is a basic p-contraction of Suzuki-type, we have reached a contradiction.*

6. Conclusions

We have discussed the question of extending the classical Suzuki fixed point theorem to the framework of w-distances on metric spaces. We have presented a simple example showing that a possible natural extension does not hold. Then, we introduced the notion of a presymmetric w-distance. We gave some examples and proved a real generalization of Suzuki's theorem for this class of of w-distances. We have examined the relationship between the so-called p-contractive self maps of Suzuki and Takahashi, our basic p-contractions of Suzuki-type, and the corresponding fixed point results derived from them. We continued this study by introducing the notion of a basic p^--contraction of Suzuki-type, which allows us to obtain a new fixed point theorem which we compared with the obtained via basic p-contractions of Suzuki-type through some examples. Finally, a characterization of the metric completeness in terms of our fixed point results was given.

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