APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 25, no. 1 (2024), 1-15
doi:10.4995/agt.2024.19855

Expansive homeomorphisms on quasi-metric spaces

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Communicated by J. Rodríguez-López

Abstract

The investigation of expansive homeomorphisms in metric spaces began with Utz in 1950. Thereafter, several authors have extensively studied this concept for different motivations. In this current article, we study expansive homeomorphism in the context of quasi-pseudometric spaces. This is motivated by the fact that any expansive homeomorphism on quasi-pseudometric space is again expansive homeomorphism on its induced pseudometric space but the converse is not true in general. Moreover, the study of orbit structures has been taken to consideration in this article. For instance, we investigate the denseness of orbits in the context of quasi-metric spaces.

2020 MSC: *54H20; 37B20.*

Keywords: *expansive homeomorphism; canonical coordinates; hyperbolic coordinates.*

1. Introduction

The phenomenon of expansivity occurs when the orbits of nearby points are separated by the dynamical system. A homeomorphism of a compact metric space is expansive if it does in the complement of finitely many orbits. The study of expansive homeomorphisms using generators has proved that every expansive homeomorphism of a compact metric space has a nonnecessarily unique

Received 8 June 2023 – Accepted 2 November 2023

measure of maximal entropy. The notion of expansive homeomorphisms in metric spaces was initiated by Utz [15] in 1950. Since then many other authors (for instance [2, 3, 4, 6, 11, 12, 13, 15]) got motivated and started to develop the concept of expansive homeomorphism. Expansive homeomorphisms have lots of applications in topology, ergodic theory etc. Utz defined a homeomorphism f of a metric space (X, d) to be unstable on X provided there is a number $\delta > 0$ (called an instability constant) such that for each pair of distinct points x, y of X then $d(f^n(x), f^n(y)) > \delta$ for $n \in \mathbb{Z}$.

An unstable homeomorphism followed by an unstable homeomorphism is not necessarily unstable. However, under some conditions a combination of homeomorphisms is always unstable. Many authors extended the notion of expansivity in the sense of Utz, for instance Bryant [6] studied or investigated expansive self-homeomorphisms of compact metric spaces. He discussed some of the special properties of the set of expansive self-homeomorphisms considered as a subset of group of all self-homeomorphisms of compact metric space. Bryant showed that an arc cannot carry an expansive self-homeomorphism, and he used this fact to show that the possession of an expansive self-homeomorphism is not a topological property. He also disclosed that there is a certain uniformity associated with an expansive homeomorphism.

In [13] William Reddy studied expansiveness by looking into expansive canonical coordinates which are hyperbolic. In particular, he proved that for an expansive homeomorphism of compact metric space with canonical coordinates there exists a metric compatible with the topology of X with respect to which the canonical coordinates are hyperbolic. William Reddy also generalized the notions of source, sink and saddle to any point in the phase space of an expansive homeomorphism which has canonical coordinates. Canonical coordinates were introduced by R. Bowen [2]. He used expansive homeomorphisms having canonical coordinates to study Axiom A diffeomorphisms. This notion was and still fruitful for ergodic theory, entropy and topological dynamics.

Furthermore, Morales and Sirvent [11] studied expansiveness of Borel measures in metric spaces. Morales and Sirvent proved the existence of expansive invariant measures for homeomorphisms of compact metric spaces. Indeed, every homeomorphism of a compact metric space carries invariant measures, but not necessarily expansive. More precisely, the authors in [11] showed every homeomorphism exhibiting expansive probability measures of compact metric space also exhibit expansive invariant probability measures. In [12], Morales and Sirvent continue to extend the notion of expansivity for measures on uniform spaces. They show that such measures can exist for measurable and bimeasurable maps on compact non-Hausdorff uniform space. Morales and Sirvent considered a notion of expansiveness, located between sensitivity and expansivity, in which Borel probability measures μ will play a fundamental role. Indeed μ is an expansive measure of a homeomorphism if the probability of two orbits remain closed to each other up to a prefixed radius which is zero. Analogously, for continuous maps, the authors define positively expansive measure by considering positive orbits instead.

It is noted that for a quasi-metric space (X, q), if $\psi : (X, q) \to (X, q)$ is expansive homeomorphism, then $\psi : (X, q^s) \to (X, q^s)$ is expansive homeomorphism, but the converse is not true in general (see Example 3.3 below). As might be expected these have led to possibilities of studying the notion of expansive homeomorphisms in quasi-metric space.

Since the notion of expansive homeomorphism of join-compact quasi-metric space has not yet been studied, this motivated us to study the concept of an expansive homeomorphism of a quasi-metric space (X, q). In this paper, we generalize some results from metric point of view to quasi-metric settings. For instance, we prove that if an expansive homeomorphism on a quasi-metric space that has canonical coordinates, then the canonical coordinates are hyperbolic (see Theorem 5.9). Moreover, we study expansive measures on the Borel structure generated by $\tau(q) \cup \tau(q^t)$ on quasi-metric space (X, q). In addition, we extend the notion of orbit structures obtained in [1]. It is noted that a self-homeomorphism of a quasi-metric space (X, q) is minimal if for all $x \in X$, the ψ-orbit set $\mathcal{O}_\psi(x)$ of x is doubly dense in X. Furthermore, we show that the homeomorphisms $\psi : (X, q) \longrightarrow (X, q)$ and $\psi : (X, q^s) \longrightarrow (X, q^s)$ have the same minimal set (see Proposition 6.6).

2. Preliminaries

We start by recalling some useful concepts that we are going to use in the sequel.

Definition 2.1. Let X be a nonempty set and $q : X \times X \to [0, \infty)$ be a map. Then q is a quasi-pseudometric on X if
(a) $q(x, x) = 0$ whenever $x \in X$, and
(b) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

If q is a quasi-pseudometric on a set X, then the pair (X, q) is called a quasi-pseudometric space. Moreover, we say that q is a T_0-quasi-metric (quasi-metric) provided that it satisfies the additional condition that for any $x, y \in X$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$. The set together with a T_0-quasi-metric on X is a quasi-metric space.

Furthermore, if q is a quasi-pseudometric on X, then the function $q^t : X \times X \to [0, \infty)$ defined by $q^t(x, y) = q(y, x)$, for all $x, y \in X$ is also an ultra-quasi-pseudometric on X and it is called the *conjugate quasi-pseudometric* of q. Note that for any q quasi-pseudometric on X, the function q^s defined by $q^s(x, y) := \max\{q(x, y), q^t(x, y)\}$ is a pseudometric on X.

Let (X, q) be a quasi-pseudometric space and $x \in X$ and $r \in [0, \infty)$. Then the *q-closed ball* $C_q(x, r)$ with the centre $x \in X$ and radius $r \geq 0$ defined by $C_q(x, r) =: \{y \in X : q(x, y) \leq r\}$ is $\tau(q^t)$-closed ball but not q-closed in general. Furthermore, the open ball $B_q(x, r)$ with the centre $x \in X$ and radius $r \geq 0$ which is represented by $B_q(x, r) = \{y \in X : q(x, y) < r\}$, is q-open. The open ball and the closed ball with respect to q^t are defined by duality.

A quasi-pseudometric space (X, q) is called *join-compact* if the $\tau(q^s)$ is compact.

Let us recall the following important definition that one can find in [10].

Definition 2.2 (compare [10, Definition 7])**.** Let (X, q) be a quasi-pseudometric space and $G \subseteq X$. The set $\mathrm{cl}_{\tau(q)} G \cap \mathrm{cl}_{\tau(q^t)} G$ is called the *double closure* of G. We say that G is *doubly closed* if $G = \mathrm{cl}_{\tau(q)} G \cap \mathrm{cl}_{\tau(q^t)} G$.
Moreover, we say that G is *doubly dense* in X if $\mathrm{cl}_{\tau(q)} G \cap \mathrm{cl}_{\tau(q^t)} G = X$.

It is noted that for any G subset of a quasi-metric space (X, q), we have

$$\mathrm{cl}_{\tau(q^s)} G \subseteq \mathrm{cl}_{\tau(q)} G \tag{2.1}$$

and

$$\mathrm{cl}_{\tau(q^s)} G \subseteq \mathrm{cl}_{\tau(q^t)} G. \tag{2.2}$$

A map $\psi : (X, q) \longrightarrow (Y, p)$ is called *uniformly continuous* if for any $\eta > 0$, there exists $\mu > 0$ such that if $q(x, x') < \mu$ with $x, x' \in X$, then $p(\psi(x), \psi(x')) < \eta$.

The following observation can be obtained in [8, p.3].

Lemma 2.3. *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of reflexive relations on X such that for each $n \in \mathbb{N}$,*

$$U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n.$$

Then there is a quasi-pseudometric q for X such that

$$U_{n+1} \subset \{(x, y) \in X \times X : q(x, y) < 2^{-n}\} \subset U_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

3. Expansive homeomorphisms

Definition 3.1. Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a homeomorphism. We say that ψ is *expansive* if there is $\delta > 0$ such that for any $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{Z}$ such that $q(\psi^n(x), \psi^n(y)) > \delta$. The constant δ is called an *expansive constant* for ψ.

The following observations are crucial.

Remark 3.2. For any quasi-pseudometric space (X, q), it is easy to see that:

(1) a map $\psi : (X, q) \to (X, q)$ is an expansive homeomorphism if and only if $\psi : (X, q^t) \to (X, q^t)$ is an expansive homeomorphism.

(2) if a map $\psi : (X, q) \to (X, q)$ is an expansive homeomorphism, then $\psi : (X, q^s) \to (X, q^s)$ is an expansive homeomorphism too but the converse is not true in general (see Example 3.3).

Example 3.3. Let us equip $X = \mathbb{R}$ with its standard quasi-metric $u(x, y) = \max\{0, x - y\}$ whenever $x, y \in X$. Then it turned out that the map $\psi : (X, u^s) \longrightarrow (X, u^s)$ defined by $\psi(x) = 2x$ for all $x \in X$ is expansive with expansive constant any number in X (see [6, Example 2]). But we observe that the map $\psi : (X, u) \longrightarrow (X, u)$ is not expansive because for any $x, y \in X$ with $x \neq y$, let us say $x < y$ and for any $n \in \mathbb{N}$ we have

$$u(\psi^n(x), \psi^n(y)) = u(2^n x, 2^n y) = 2^n u(x, y) = 0.$$

Therefore, we cannot find a $\delta > 0$ such that $u(\psi^n(x), \psi^n(y)) > \delta$.

In the sequel we assume that (X, q) is a join-compact quasi-metric space and $\psi : (X, q) \longrightarrow (X, q)$ is an expansive homeomorphism.

Definition 3.4 (compare [13, p.205])**.** Let $\delta > 0$ and $x \in X$. A δ-*stable* set of x with respect to ψ and q noted $S_q(x, \delta, \psi)$ is defined by

$$S_q(x, \delta, \psi) := \{y \in X : q(\psi^n(x), \psi^n(y)) \leq \delta, \text{ for } n \in \mathbb{N}\}.$$

Similarly, one can define δ-stable set with respect to ψ and q^t and q^s.

Remark 3.5. It is easy to see that

$$S_{q^s}(x, \delta, \psi) \subseteq S_q(x, \delta, \psi) \tag{3.1}$$

and

$$S_{q^s}(x, \delta, \psi) \subseteq S_{q^t}(x, \delta, \psi). \tag{3.2}$$

Moreover,

$$S_{q^s}(x, \delta, \psi) = S_q(x, \delta, \psi) \cap S_{q^t}(x, \delta, \psi). \tag{3.3}$$

Definition 3.6 (compare [13, p.205])**.** Let $\delta > 0$ and $x \in X$. A δ-*unstable* set of x with respect to ψ and q noted $U_q(x, \delta, \psi)$ is defined by

$$U_q(x, \delta, \psi) := \{y \in X : q(\psi^{-n}(x), \psi^{-n}(y)) \leq \delta, \text{ for } n \in \mathbb{N}\}.$$

Similarly, one can define δ-unstable set with respect to ψ and q^t and q^s. Note that inclusions (3.1), (3.2) and (3.3) in Remark 3.5 for δ-stable sets are also satisfied for δ-unstable sets.

Example 3.7. Let $X = \mathbb{R}$. If we equip X with the T_0-quasi-metric (Sorgenfrey line)

$$v(x, y) = \begin{cases} x - y & x \geq y \\ 1 & x < y, \end{cases}$$

and $\psi(x) = 2x$. Then for $x \geq y$ and $n \in \mathbb{N}$, we have

$$v(\psi^n(x), \psi^n(y)) = 2^n v(x, y)$$

and for $x < y$, we have $v(\psi^n(x), \psi^n(y)) = 2^n$. Thus ψ is expansive and any real number smaller than 2^n is an expansive constant for ψ. Moreover, for $\delta > 0$, we have

$$S_v(x, \delta, \psi) = \{y \in \mathbb{R} : v(\psi^n(x), \psi^n(y)) = 2^n v(x, y) \leq \delta\} = \{x\}.$$

The following observation is a consequence of Remark 3.2 and [2, Remark 1.2].

Remark 3.8. If (X, q) is a quasi-metric space and $\psi : (X, q) \to (X, q)$ is a homeomorphism, then ψ is expansive if and only if $S_{q^s}(x, \delta, \psi) = \{x\}$ whenever $x \in X$ and some $\delta > 0$.

Example 3.9. Let us equip $\mathbb{R}$ with the T_0-quasi-metric $u(x, y) = \max\{x - y, 0\}$. Then the homeomorphism identity map $\mathrm{Id}_X : (X, u) \longrightarrow (X, u)$ is not expansive. Indeed, for $\delta > 0$ and $x \in X$ we have $S_u(x, \delta, \mathrm{Id}_X) = [x - \delta, \infty)$ and $S_{u^t}(x, \delta, \mathrm{Id}_X) = (-\infty, x + \delta]$, thus $S_{u^s}(x, \delta, \mathrm{Id}_X) = [x - \delta, x + \delta] \neq \{x\}$.

The following result holds in the case of metric spaces "see [6, Theorem 1]".

Theorem 3.10. *Let (X, q) and (Y, p) be join compact quasi-metric spaces and $\psi : (X, q) \longrightarrow (X, q)$ be an expansive homeomorphism with δ the expansive constant. If $\varphi : (X, q) \longrightarrow (Y, p)$ is a homeomorphism such that $\varphi^{-1} : (Y, p) \longrightarrow (X, q)$ is uniformly continuous, then $\varphi \circ \psi \circ \varphi^{-1} : (Y, p) \longrightarrow (Y, p)$ is an expansive homeomorphism with δ its expansive constant.*

Proof. Suppose that $\psi : (X, q) \longrightarrow (X, q)$ is an expansive homeomorphism with $\delta > 0$ the expansive constant. Let $x, x' \in X$. Since $\varphi^{-1} : (Y, p) \longrightarrow (X, q)$ is uniformly continuous, then there exists $\epsilon > 0$ such that if $p(\varphi(x), \varphi(x')) \leq \epsilon$, then

$$q(x, x') = q(\varphi(\varphi^{-1}(x)), \varphi(\varphi^{-1}(x'))) < \delta.$$

Thus

$$q(x, x') > \delta \text{ implies that } p(\varphi(x), \varphi(x')) > \epsilon. \tag{3.4}$$

Moreover, for any $y, y' \in Y$ with $y \neq y'$, then $\varphi^{-1}(y), \varphi^{-1}(y') \in X$, so by expansiveness of ψ it follows that there exists $n \in \mathbb{N}$ such that

$$q(\psi^n(\varphi^{-1}(y)), \psi^n(\varphi^{-1}(y'))) > \delta. \tag{3.5}$$

By combining (3.4) and (3.5) we have that

$$p(\varphi(\psi^n(\varphi^{-1}(y))), \varphi(x\psi^n(\varphi^{-1}(y')))) > \epsilon.$$

Therefore, the homeomorphism $\varphi \circ \psi \circ \varphi^{-1}$ is an expansive homeomorphism with ϵ the expansive constant. □

4. Expansive measures

We start this section with the following observation. If (X, q) is a quasi-metric space, the two topologies $\tau(q)$ and $\tau(q^t)$ are associated to X. It is appropriate to use the Borel structure $\mathcal{B}_{q^s}(X)$ generated by $\tau(q) \cup \tau(q^t)$ so that any countable union, intersection or difference of any $\tau(q)$-open or $\tau(q^t)$-open sets is measurable. Furthermore, the Borel structure generated by $\tau(q) \cup \tau(q^t)$

is equivalent to the Borel structure generated by $\tau(q^s)$ (see [14]). In the sequel, we are going to use $\mathcal{B}_{q^s}(X)$ as Borel structure on (X, q).

Definition 4.1. Let (X, q) be a quasi-metric space. A finite Borel measure on X is a map $\mu : \mathcal{B}_{q^s}(X) \to [0, \infty)$ such that
(a) $\mu(\varnothing) = 0$,
(b) if $A_1, A_2, A_3, ..., A_n \in \mathcal{B}_{q^s}(X)$ are mutually disjoint, then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i),$$

(c) $\mu(X) > 0$.
Furthermore, a finite Borel measure μ on a quasi-metric space (X, q) is called *probability* if $\mu(X) = 1$ and *non-atomic* if $\mu(\{x\}) = 0$ whenever $x \in X$.

The following definition is motivated by Remark 3.8.

Definition 4.2 (compare [2, definition 1.3])**.** Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a homeomorphism. Then we say that the finite Borel measure μ on $\mathcal{B}_{q^s}(X)$ is an *expansive measure* of ψ if there exists $\delta > 0$ such that $\mu(S_{q^s}(x, \delta, \psi)) = 0$ whenever $x \in X$. In this case the constant δ is called *expansivity constant* of μ.

Example 4.3. Every expansive measure is non-atomic. Therefore, every quasi-metric space carrying homeomorphisms with expansive (probability) measures also carries a non-atomic Borel (probability) measure.

Definition 4.4. If (X, q) is a quasi-metric space and $\psi : (X, q) \to (X, q)$ is a homeomorphism, the map ψ is a called *countably-expansive* if there exists $\delta > 0$ such that $S_{q^s}(x, \delta, \psi)$ is countable whenever $x \in X$.

De Brecht [5] introduced the concept of quasi-Polish spaces as follows. A topological space is quasi-Polish if and only if it is countably based and completely quasi-metrisable. He proved that a metrisable space is quasi-Polish if and only if it is Polish (see [5]).

Proposition 4.5 ([2, Proposition 1,7])**.** *Let (X, q) be a quasi-metric space which is quasi-Polish and $\psi : (X, q) \to (X, q)$ be a homeomorphism. Then the following are equivalent.*

(a) ψ is countably-expansive.

(b) All non-atomic Borel probability measures on X (if they exist) are expansive with common expansivity constant.

Proof. (a) $\Rightarrow$ (b) Let ψ be countably-expansive. Suppose that there exists a non-atomic Borel probability measure μ on X. For any $x \in X$. There exists $\delta > 0$ such that

$$S_{q^s}(x, \delta, \psi) = \{a_1, a_2, \cdots, a_i, \cdots\}.$$

Then

$$\mu(S_{q^s}(x, \delta, \psi)) = \mu(\amalg_{i \in \mathbb{N}}\{a_i\}) = \sum_{i \in \mathbb{N}} \mu(\{a_i\}) = 0.$$

So μ is non-atomic with the expansivity constant δ.

(b) $\Rightarrow$ (a) If there are non-atomic Borel probability measure on X which are expansive with common expansivity constant. Suppose that ψ is not countably-expansive. It follows that $\psi : (X, q^s) \to (X, q^s)$ is a homeomorphism by Remark 3.2. From the fact that (X, q) is Polish metric space. Then, we obtained the result from [12, Proposition 1.7]. □

Definition 4.6. Let (X, q) and (Y, p) be quasi-metric spaces. If μ is a Borel measure on (X, q) and $\phi : (X, q) \to (Y, p)$ is a homeomorphism, then we define the *pullback* $\phi_\star(\mu)$ of μ by

$$\phi_*(\mu)(A) = \mu(\phi^{-1}(A)) \text{ whenever } A \in \mathcal{B}_{q^s}(X).$$

We recall that a quasi-metric space (X, q) is join-compact if the topological space $(X, \tau(q^s))$ is compact.

Lemma 4.7. *Let (X, q) and (Y, p) be join-compact quasi-metric space and μ be an expansive measure of a homeomorphism $\psi : (X, q) \to (X, q)$. If $\phi : (X, q) \to (Y, p)$ is a homeomorphism, then $\phi_\star(\mu)$ is an expansive measure of $\phi \circ \psi \circ \phi^{-1}$.*

Proof. Suppose that $\phi : (X, q) \to (Y, p)$ is a homeomorphism of a join-compact quasi-metric spaces. It follows that ϕ is a uniformly continuous map. Let $\delta > 0$. Then for any $x, z \in X$, there exists $\varepsilon > 0$ such that $p(\phi(x), \phi(z)) < \delta$ whenever $q(x, z) < \varepsilon$.
We first prove that

$$S_{q^s}(y, \varepsilon, \phi \circ \psi \circ \phi^{-1}) \subseteq \phi(S_{q^s}((\phi^{-1}(y), \delta, \psi))) \text{ whenever } y \in X. \tag{4.1}$$

If $t \in S_{q^s}(y, \varepsilon, \phi \circ \psi \circ \phi^{-1})$, then $i = 1 \in \mathbb{Z}$ we have

$$p^s(\phi \circ \psi \circ \phi^{-1}(y), \phi \circ \psi \circ \phi^{-1}(t)) < \varepsilon$$

whenever

$$q^s(\psi \circ \phi^{-1}(y), \psi \circ \phi^{-1}(t)) < \delta.$$

It follows that $\phi^{-1}(t) \in S_{q^s}(\phi^{-1}(y), \delta, \psi)$. Hence $t \in \phi(S_{q^s}((\phi^{-1}(y), \delta, \psi)))$.
Moreover, from 4.1 we have

$$\begin{aligned} \phi_\star(\mu)(S_{q^s}(y, \varepsilon, \phi \circ \psi \circ \phi^{-1})) &\leq \phi_\star(\mu)[\phi[(S_{q^s}(\phi^{-1}(y), \delta, \psi))]] \\ &= \mu[\phi_\star[\phi(S_{q^s}(\phi^{-1}(y), \delta, \psi))]] \\ &= \mu[S_{q^s}(\phi^{-1}(y), \delta, \psi)]. \end{aligned}$$

Since μ is an expansive measure of a homeomorphism ψ, we have that

$$\mu[S_{q^s}(\phi^{-1}(y), \delta, \psi)] = 0,$$

Thus μ has δ as expansivity constant.
Hence,

$$\phi_\star(S_{q^s}(y, \varepsilon, \phi \circ \psi \circ \phi^{-1})) = 0.$$

Therefore, $\phi_\star(\mu)$ is an expansive measure of $\phi \circ \psi \circ \phi^{-1}$ and $\phi_\star(\mu)$ has ε as expansivity constant. □

Definition 4.8. Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a continuous mapping. Then we say that a Borel measure μ of X is invariant if $\psi_\star \circ \mu = \mu$.

Lemma 4.9 (compare [2, Lemma 1.15])**.** *Let $\psi : (X, q) \to (X, q)$ be a homeomorphism of a quasi-metric space (X, q). If μ is an expansive measure with expansivity constant δ of ψ, then so does $\psi_\star\mu$.*

Proof. Assume that ψ is a bijective map, in other words

$$\psi(S_q(x, \delta, \psi)) = S_q(\psi(x), \delta, \psi)$$

then,

$$\begin{aligned} \psi_\star\mu(S_q(x, \delta, \psi) &= \mu(\psi^{-1}(S_q(x, \delta, \psi))) \\ &= \mu(S_q(\psi^{-1}(x), \delta, \psi)) \\ &= 0 \end{aligned}$$

whenever $x \in X$. □

Theorem 4.10. *Let (X, q) be a join-compact quasi-pseudometric space and let $\psi : (X, q) \longrightarrow (X, q)$ be a homeomorphism. Then ψ has an expansive probability measure if and only if ψ has an expansive invariant probability measure.*

Proof. Let $\psi : (X, q) \longrightarrow (X, q)$ be a homeomorphism. From Lemma 3.2 we know that $\psi : (X, q^s) \longrightarrow (X, q^s)$ is a homeomorphism. Furthermore, from [11, Theorem 1.18] and by the join-compactness of (X, q) we have that $\psi : (X, q^s) \longrightarrow (X, q^s)$ has an expansive probability measure if and only if $\psi : (X, q^s) \longrightarrow (X, q^s)$ has an expansive invariant probability measure. □

5. Canonical coordinates

Definition 5.1. Let (X, q) be a quasi-pseudometric space and $\psi : (X, q) \longrightarrow (X, q)$ be an expansive homeomorphism. We say that ψ has *canonical coordinates* if for any $\delta > 0$, there exists $\epsilon > 0$ such that $q(x, y) \leq \delta$, then

$$S_q(x, \delta, \psi) \cap U_q(y, \delta, \psi) \neq \varnothing.$$

Definition 5.2 (compare [13, p. 206])**.** We say that the canonical coordinates of $\psi : (X, q) \longrightarrow (X, q)$ are *hyperbolic* if there exists $\mu > 0$, $0 < \lambda < 1$ and $c \geq 1$ such that the following two conditions hold:

(a) If $y \in S_q(x, \delta, \psi)$, then $q(\psi^n(x), \psi^n(y)) \leq c\lambda^n q(x, y)$ for $n \geq 0$.
(b) If $y \in U_q(x, \delta, \psi)$, then $q(\psi^n(x), \psi^n(y)) \leq c\lambda^{-n} q(x, y)$ for $n \leq 0$.

Suppose that $\psi : (X, q) \to (X, q)$ is an expansive homeomorphism with the expansive constant $c > 0$ and ψ has canonical coordinates. We define the sequence $(W_n)_{n\geq 0}$ of $\tau(q^t)$-closed neighborhoods of the diagonal $\Delta_X = \{(x, x) : x \in X\}$ as follows: $W_0 = X \times X$ and for any $n \geq 1$, W_n is defined by

$$W_n := \{(x, y) \in X \times X : q(\psi^i(x), \psi^i(y)) \leq c \quad \text{for} \quad |i| < n\}.$$

The following lemma can be compared with [13, Lemma 1].

Lemma 5.3. *The sequence of* $(W_n)_{n\geq 0}$ *is a nested sequence of* $\tau(q^t)$*-closed neighborhoods of the diagonal* Δ_X *such that* $\bigcap_{n=0}^{\infty} W_n = \Delta_X$.

Proof. It is easy to see that the sequence $(W_n)_{n\geq 0}$ is a nested sequence of $\tau(q^t)$-closed neighborhoods of the diagonal Δ_X. Then $\Delta_X \subseteq \bigcap_{n=0}^{\infty} W_n$.

Now we show that $\Delta_X \supseteq \bigcap_{n=0}^{\infty} W_n$. Suppose that $(x, y) \notin \Delta_X$. Then $x \neq y$, it follows that $q(\psi^n(x), \psi^n(y)) > c$ for some $n \in \mathbb{Z}$ since ψ is expansive. Thus $(x, y) \notin W_n$ for some $n \in \mathbb{Z}$. □

Note that from Lemma 5.3 there exists $\epsilon > 0$ and $N > 1$ such that

$$W_{1+N} \subset N^q_{\epsilon/3} \subset N^q_{\epsilon} \subset W_1,$$

where

$$N^q_{\epsilon} := \{(x, y) \in X \times X : q(x, y) < \epsilon\}$$

and for $W \subset X \times X$ and $x \in X$, the set $W[x]$ is defined by

$$W[x] = \{y : (x, y) \in W\}.$$

Furthermore, we define a sequence $(V_k)_{k\geq 0}$ of $\tau(q^t)$-closed neighborhoods of the diagonal Δ_X by:

(a) $V_0 = W_0$,
(b) $V_k = W_{1+(k-1)N}$ for $k > 0$.

Lemma 5.4. *The sequence* $(V_k)_{k\geq 0}$ *is a nested sequence of* $\tau(q^t)$*-closed neighborhoods of the diagonal* Δ_X *such that*

$$V_{n+1} \circ V_{n+1} \circ V_{n+1} \subset V_n \quad \textit{for each} \quad n.$$

Remark 5.5. From Lemma 5.4 and Lemma 2.3, we observe that there exists a quasi-pseudometric p on X such that

$$N^p_{2^{-(n+1)}} \subset V_n \subset N^p_{2^{-n}} \quad \text{for} \quad n \geq 1.$$

Proposition 5.6. *Let* (X, q) *be a join-compact quasi-metric space and* $\psi : (X, q) \longrightarrow (X, q)$ *be an expansive homeomorphism with expansive constant* $c > 0$. *Then*

$$\psi(S_q(x, c, \psi) \cap W_n[x]) = S_q(\psi(x), c, \psi) \cap W_{n+1}[\psi(x)].$$

Proof. Let $t \in \psi(S_q(x, c, \psi) \cap W_n[x])$ if and only if there exists $y \in S_q(x, c, \psi) \cap W_n[x]$ such that $\psi(y) = t$.

Equivalently for $n \geq 0$, $q(\psi^n(x), \psi^n(y)) \leq c$ and $(x, y) \in W_n$ such that $\psi(y) = t$.

Moreover, for $n \geq 0$, $q(\psi^n(x), \psi^n(y)) \leq c$ and $q(\psi^i(x), \psi^i(y)) \leq c$ with $|i| < n$ such that $\psi(y) = t$.

Thus for $m = n-1 \geq 0$, $q(\psi^m(\psi(x)), \psi^m(\psi(t))) \leq c$ and $q(\psi^k(\psi(x)), \psi^k(\psi(t))) \leq c$ with $|k| < m$.

Thus $t \in S_q(\psi(x), c, \psi) \cap W_{n+1}[\psi(x)]$. □

Proposition 5.7. *Let (X, q) be a join-compact quasi-metric space and $\psi : (X, q) \longrightarrow (X, q)$ be an expansive homeomorphism with expansive constant $c > 0$. Then*

$$\psi^{-1}(U_q(x, c, \psi) \cap W_n[x]) = S_q(\psi^{-1}(x), c, \psi) \cap W_{n+1}[\psi^{-1}(x)].$$

Corollary 5.8. *Let (X, q) be a join-compact quasi-metric space and $\psi : (X, q) \longrightarrow (X, q)$ be an expansive homeomorphism with expansive constant $c > 0$. Then for any $m, n \in \mathbb{N}$, we have:*

(a) $\psi^m(S_q(x, c, \psi) \cap V_n[x]) = S_q(\psi^m(x), c, \psi) \cap V_{n+1}[\psi^m(x)]$.

(b) $\psi^{-m}(U_q(x, c, \psi) \cap V_n[x]) = S_q(\psi^{-m}(x), c, \psi) \cap V_{n+1}[\psi^{-m}(x)]$.

Theorem 5.9. *Let (X, q) be a join-compact quasi-metric space and $\psi : (X, q) \longrightarrow (X, q)$ be an expansive homeomorphism with expansive constant $c > 0$. Moreover, if ψ has canonical coordinates, then the canonical coordinates are hyperbolic with respect to the quasi-pseudometric p from Remark 5.5.*

Proof. Suppose that ψ has canonical coordinates with respect to p. Then there exists $\eta > 0$ such that $p(x, y) \leq \eta$. Then whenever $x, y \in X$ we have $q(x, y) \leq c$.

Moreover, we have $S_p(x, \eta, \psi) \subseteq S_q(x, c, \psi)$. Let $\mu = \min\{\eta, 1/4\}$. Let $y \in S_p(x, \mu, \psi)$. Then there exists $n \geq 0$ such that $(x, y) \in V_n \setminus V_{n+1}$. It follows that

$$(x, y) \notin V_{n+1} \supset N^p_{2^{-(n+1)}},$$

then

$$1/2^{n+1} < p(x, y) \leq 1/4.$$

Since $y \in S_q(x, c, \psi) \cap V_n[x]$, then we have

$$\psi^{3N}(y) \in S_q(\psi^{3N}(x), c, \psi) \cap V_{n+3}(\psi^{3N}) \subset N^p_{2^{-(n+1)}}[\psi^{3N}].$$

Thus

$$p(\psi^{3N}(x), \psi^{3N}(y)) < 1/2^{n+3} = (1/2)(1/2^{n+2}) < (1/2)p(x, y).$$

If we let $M = 3N$, then we have

$$p(\psi^M(x), \psi^M(y)) \leq (1/2)p(x, y).$$

By induction for $k \geq 0$, we have

$$p(\psi^{kM}(x), \psi^{kM}(y)) \leq (1/2)^k p(x, y). \tag{5.1}$$

Let α be chosen such that $\alpha^M = 1/2$ it follows that $0 \leq \alpha < 1$. For $y \in S_p(x, \mu, \psi) \cap U_p(x, \mu, \psi)$, we claim that:

$$p(\psi^m(x), \psi^m(y)) \leq 8\alpha^m p(x, y) \quad \text{for} \quad m \geq 0 \tag{5.2}$$

and

$$p(\psi^m(x), \psi^m(y)) \leq 8\alpha^{-m} p(x, y) \quad \text{for} \quad m \leq 0 \tag{5.3}$$

We first prove the claim (5.2). Let $m = kM + i$ with $0 \leq i \leq M-1$ and suppose $y \in S_p(x, \mu, \psi)$.

Since $(\psi^{kM}(x), \psi^{kM}(y)) \in V_n \setminus V_{n+1}$ for some $n \geq 1$, it follows that

$$\psi^{kM}(y) \in S_p(\psi^{kM}(x), c, \psi) \cap W_{1+(n-1)N}(\psi^{kM}(x)).$$

Thus

$$\psi^{kM+i}(y) \in S_p(\psi^{kM+i}(x), c, \psi) \cap W_{1+(n-1)N+i}(\psi^{kM+i}(x)).$$

Moreover, we have

$$(\psi^{kM+i}(x), \psi^{kM+i}(y)) \in W_{1+(n-1)N+i} \subset W_{1+(n-1)N} = V_n \subset N^p_{2^{-n}} \tag{5.4}$$

and since

$$(\psi^{kM+i}(x), \psi^{kM+i}(y)) \notin V_{n+1} \supset N^p_{2^{-(n+2)}}$$

it follows that

$$p(\psi^{kM}(x), \psi^{kM}(y)) \geq 1/2^{n+2}. \tag{5.5}$$

Furthermore, from (5.4) and (5.5) we have

$$p(\psi^{kM+i}(x), \psi^{kM+i}(y)) \leq 1/2^n \leq 4p(\psi^{kM}(x), \psi^{kM}(y)). \tag{5.6}$$

Hence, from inequalities (5.1) and (5.6) we have

$$\begin{aligned} p(\psi^{kM+i}(x), \psi^{kM+i}(y)) &\leq 4p(\psi^{kM}(x), \psi^{kM}(y)) \leq 4(1/2)^k p(x,y) \\ &= 4\alpha^{kM} p(x,y) \\ &< 8\alpha^{kM+i} p(x,y). \end{aligned}$$

Therefore, since $m = kM + i$ we have

$$p(\psi^m(x), \psi^m(y)) \leq 8\alpha^m p(x,y).$$

Now let $y \in U_p(x, \mu, \psi)$. Then claim (5.3) follows by similar arguments as in the proof of claim (5.2) by replacing ψ by ψ^{-1}. □

6. Orbit

In this section, we attempt to study dynamical phenomena of a self-homeomorphism of a quasi-metric space.

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a homeomorphism. For any $x_0 \in X$, the ψ-orbit of x_0 denoted $\mathcal{O}_\psi(x_0)$ is defined by $\mathcal{O}_\psi(x_0) := \{\psi^p(x_0) : p \in \mathbb{Z}\}$.

We introduce the following definition.

Definition 6.1. Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a homeomorphism. We say that the ψ-orbit of $x_0 \in X$ is *doubly dense* in X if

$$\mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0)) = X.$$

Proposition 6.2. *Let (X,q) be a join-compact quasi-metric space and $\psi : (X,q) \to (X,q)$ be a homeomorphism. If $x_0 \in X$, then we have:*

(1) *$\psi : (X,q) \to (X,q)$ has the ψ-orbit of x_0 doubly dense in X if and only if $\psi : (X,q^t) \to (X,q^t)$ has the ψ-orbit of x_0 doubly dense in X.*
(2) *If $\psi : (X,q^s) \to (X,q^s)$ has the ψ-orbit of x_0 $\tau(q^s)$-dense in X, then $\psi : (X,q) \to (X,q)$ has the ψ-orbit of x_0 doubly dense in X and $\psi : (X,q^t) \to (X,q^t)$ has the ψ-orbit of x_0 doubly dense in X.*

Proof. (1) Follows from Definition 6.1.

(2) Suppose that $\psi : (X,q^s) \to (X,q^s)$ has the ψ-orbit of x_0 $\tau(q^s)$-dense in X. Then

$$X = \mathrm{cl}_{\tau(q^s)}(\mathcal{O}_\psi(x_0)) \subseteq \mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0)) \subseteq X.$$

Therefore, we have

$$\mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0)) = X.$$

Thus $\psi : (X,q) \to (X,q)$ has the ψ-orbit of x_0 doubly dense in X.

By similar arguments and by definition of double closure, we have that $\psi : (X,q^t) \to (X,q^t)$ has the ψ-orbit of x_0 doubly dense in X. □

Definition 6.3 (compare [1, Definition 1.1])**.** Let (X,q) be a quasi-metric space and $\psi : (X,q) \longrightarrow (X,q)$ be a homeomorphism. We say that ψ is *minimal* if for all $x \in X$, the set $\mathcal{O}_\psi(x)$ is doubly dense in X.

The following lemma is our version of [1, Remark 1.2].

Lemma 6.4. *Let (X,q) be a quasi-metric space and $\psi : (X,q) \longrightarrow (X,q)$ be a homeomorphism. Then ψ is minimal if and only if $\psi(G) = G$ and G is doubly closed, then $G = \varnothing$ or $G = X$.*

Proof. $\Longrightarrow$ Suppose that ψ is minimal and $G = \mathrm{cl}_{\tau(q)}G \cap \mathrm{cl}_{\tau(q^t)}G$. Moreover, we suppose that $\psi(G) = G \neq \varnothing$. We show that $G = X$.
Let $x \in G$. Since $\psi(G) = G$ we have $\mathcal{O}_\psi(x) \subset G$. It follows that

$$\begin{aligned} X &= \mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0)) \\ &\subseteq G = \mathrm{cl}_{\tau(q)}G \cap \mathrm{cl}_{\tau(q^t)}G \subseteq X. \end{aligned}$$

Thus $G = X$.
$\Longleftarrow$ We show that $X = \mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0))$. It is sufficient to show that $X \subset \mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0))$.
Let $x \in X$. Since $\mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0))$ is doubly closed and nonempty and ψ-invariant, hence

$$\mathrm{cl}_{\tau(q)}(\mathcal{O}_\psi(x_0)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_\psi(x_0)) = X.$$

Therefore, ψ is minimal. □

Definition 6.5. Let (X,q) be a quasi-metric space and $\psi : (X,q) \longrightarrow (X,q)$ be a homeomorphism. A subset A of X which is ψ-*invariant* (that is $\psi(A) = A$) is called a *minimal set* with respect to ψ if $\psi|_A$ is minimal.

The following proposition elaborates on the fact that the self-homeomorphism of a quasi-metric space and its induced metric self-homeomorphism have the same minimal set.

Proposition 6.6. *Let (X,q) be a quasi-metric space and $\psi : (X,q) \longrightarrow (X,q)$ be a homeomorphism. Then both $\psi : (X,q) \longrightarrow (X,q)$ and $\psi : (X,q^s) \longrightarrow (X,q^s)$ have the same minimal set.*

Proof. Since the map $\psi : (X,q^s) \longrightarrow (X,q^s)$ is a self-homeomorphism on a metric space (X,q^s), it follows that $\psi : (X,q^s) \longrightarrow (X,q^s)$ has a minimal set G by [1, Remark 1.4]. We now show that G is a minimal set of $\psi : (X,q^s) \longrightarrow (X,q^s)$.

Since G is a minimal set of $\psi : (X,q) \longrightarrow (X,q)$, then G is $\tau(q^s)$-closed subset of X and ψ-invariant. Furthermore, for any $x \in X$ we have $\mathrm{cl}_{\tau(q^s)}(\mathcal{O}_{\psi|_G}(x)) = X$.

Moreover, from inclusions (2.1) and (2.2) we have

$$\begin{aligned} X &= \mathrm{cl}_{\tau(q^s)}(\mathcal{O}_{\psi|_G}(x)) \\ &\subseteq \mathrm{cl}_{\tau(q)}(\mathcal{O}_{\psi|_G}(x))) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_{\psi|_G}(x)) \subseteq X. \end{aligned}$$

Thus

$$\mathrm{cl}_{\tau(q)}(\mathcal{O}_{\psi|_G}(x)) \cap \mathrm{cl}_{\tau(q^t)}(\mathcal{O}_{\psi|_G}(x)) = X.$$

Therefore, the set G is a minimal set of $\psi : (X,q) \longrightarrow (X,q)$. □

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