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Homomorphisms of the lattice of slowly oscillating functions on the half-line

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Abstract

We study the space $H(\mathcal{SO})$ of all homomorphisms of the vector lattice of all slowly oscillating functions on the half-line $\mathbb{H} = [0, \infty)$. In contrast to the case of homomorphisms of uniformly continuous functions, it is shown that a homomorphism in $H(\mathcal{SO})$ which maps the unit to zero must be the zero-homomorphism. Consequently, we show that the space $H(\mathcal{SO})$ without the zero-homomorphism is homeomorphic to $\mathbb{H} \times (0, \infty)$. By describing a neighborhood base of the zero-homomorphism, we show that $H(\mathcal{SO})$ is homeomorphic to the space $\mathbb{H} \times (0, \infty)$ with one point added.

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1. Introduction

The aim of this note is to describe the real-valued homomorphisms of the vector lattice of all slowly oscillating functions on the half-line $\mathbb{H} = [0, \infty)$.

Slowly oscillating functions are used to define Higson compactifications [6] and are functions that appear frequently in coarse geometry. By analyzing slowly oscillating functions on $\mathbb{H}$, it follows that its Higson corona $\nu\mathbb{H}$ is a non-metrizable indecomposable continuum. Although this fact is topologically

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interesting in its own right, in the context of geometric group theory, it is applied to characterize the number of ends of finitely generated groups by whether the components of its Higson corona are decomposable or not [5].

Let $\mathcal{U}$ be the vector lattice of all uniformly continuous functions on $\mathbb{H}$ and $\mathcal{U}^*$ the sublattice of bounded functions. In [1], Félix Cabello Sánchez analyzed the space $H(\mathcal{U})$ of all homomorphisms of $\mathcal{U}$ and gave a fine description of it as follows: $H(\mathcal{U})$ is homeomorphic to a quotient space[1] obtained from $[1, 2] \times \beta\mathbb{N} \times (0, \infty)$ with one point added, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of natural numbers. Also, by considering $H(\mathcal{U}^*)$, he gave a description of the Samuel-Smirnov compactification of $\mathbb{H}$ (cf. [2], [9]).

Inspired by his work, we study the space $H(\mathcal{SO})$ of all homomorphisms of the vector lattice of slowly oscillating functions on $\mathbb{H}$. In contrast to the case of homomorphisms of uniformly continuous functions, it is shown that a homomorphism in $H(\mathcal{SO})$ which maps the unit to zero must be the zero-homomorphism (Proposition 3.9). Consequently, we show that the space $H(\mathcal{SO})$ without the zero-homomorphism is homeomorphic to $\mathbb{H} \times (0, \infty)$. By describing a neighborhood base of the zero-homomorphism, we show that $H(\mathcal{SO})$ is homeomorphic to the space $\mathbb{H} \times (0, \infty)$ with one point added (Theorem 3.10).

2. Preliminaries

Throughout this note, $\mathbb{H}$ denotes the half-line $[0, \infty)$ with the metric given by the absolute value $|x - y|,\ x, y \in \mathbb{H}$, and $\mathbb{N}$ denotes the space of natural numbers with the subspace metric. Also, $X = (X, d_X)$ is assumed to be a metric space.

Let $\mathcal{L} \subset C(X)$ be a unital vector lattice, that is, $\mathcal{L}$ contains the unit $\mathbf{1} : X \to \mathbb{R}$. The sublattice of all bounded functions of $\mathcal{L}$ is denoted by $\mathcal{L}^*$. A function $\phi : \mathcal{L} \to \mathbb{R}$ is called a *homomorphism* if it is a linear map preserving joins and meets, that is, ϕ satisfies

(i) $\phi(f \vee g) = \phi(f) \vee \phi(g),\ \ \phi(f \wedge g) = \phi(f) \wedge \phi(g)$, and
(ii) $\phi(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \phi(f) + \mu \cdot \phi(g)$

for all $f, g \in \mathcal{L},\ \lambda, \mu \in \mathbb{R}$. Note that (i) and (ii) implies

(iii) $\phi(|f|) = |\phi(f)|$ for all $f \in \mathcal{L}$.

Indeed, the formulation

$$|f| = f \vee \mathbf{0} - f \wedge \mathbf{0}$$

implies that

$$\begin{aligned}\phi(|f|) &= \phi(f) \vee \phi(\mathbf{0}) - \phi(f) \wedge \phi(\mathbf{0}) \\ &= \phi(f) \vee \mathbf{0} - \phi(f) \wedge \mathbf{0} \\ &= |\phi(f)|.\end{aligned}$$

Recall that join and meet induce a partial order $\leq$ on $H(\mathcal{L})$, that is,

$$f \leq g \Longleftrightarrow f = f \wedge g$$

[1]Detailed equivalence relations in the quotient space are not described here because they require preparation that is not needed in this note. See [1] for details.

or equivalently,

$$f \le g \Longleftrightarrow g = f \vee g.$$

Then (i) implies that

(iv) $\phi(f) \le \phi(g)$ whenever $f \le g$.

Besides, (iii) implies that a homomorphism ϕ is *positive*, that is,

(v) $\phi(f) \ge 0$ whenever $f \in \mathcal{L}$ satisfies $f \ge 0$.

The set of all homomorphisms $\phi : \mathcal{L} \to \mathbb{R}$ is denoted by $H(\mathcal{L})$. Note that $H(\mathcal{L})$ is a subset of $\mathbb{R}^{\mathcal{L}}$. We consider the topology on $H(\mathcal{L})$ inherited from $\mathbb{R}^{\mathcal{L}}$. Hence, a basic neighborhood of $\phi \in H(\mathcal{L})$ is given by

$$V(\phi; f_1, \dots, f_n; \varepsilon) = \{\varphi \in H(\mathcal{L}) : |\varphi(f_i) - \phi(f_i)| < \varepsilon,\ \forall i = 1, \dots, n\},$$

where $\varepsilon > 0$ and $f_i \in \mathcal{L}$, $i = 1, \dots, n$. Put

$$K(\mathcal{L}) = \{\phi \in H(\mathcal{L}) : \phi(\mathbf{1}) = 1\}.$$

Then it is easy to see that $K(\mathcal{L}) \subset H(\mathcal{L})$, and $H(\mathcal{L})$ and $K(\mathcal{L})$ are closed subspaces of $\mathbb{R}^{\mathcal{L}}$. In particular, $H(\mathcal{L}^*)$ and $K(\mathcal{L}^*)$ are compact spaces. Indeed, they are closed subspaces of the Cartesian product

$$\prod_{f \in \mathcal{L}^*} \big[\inf f, \sup f\big].$$

For each $x \in X$, let $\delta_x : \mathcal{L} \to \mathbb{R}$ be the evaluation homomorphism defined by $\delta_x(f) = f(x)$ for every $f \in \mathcal{L}$. We note that $\delta_x(\mathbf{1}) = 1$ for every $x \in X$, i.e., $\delta_x \in K(\mathcal{L})$. Then define

$$\delta : X \to K(\mathcal{L})$$

by $\delta(x) = \delta_x$ for each $x \in X$. When we treat $\mathcal{L}^*$, consider the map

$$e_{\mathcal{L}^*} : X \to \prod_{f \in \mathcal{L}^*} \big[\inf f, \sup f\big],$$

defined by $e_{\mathcal{L}^*}(x) = (f(x))_{f \in \mathcal{L}^*}$ for every $x \in X$. One should note that the two maps $\delta : X \to K(\mathcal{L}^*) \subset \mathbb{R}^{\mathcal{L}}$ and $e_{\mathcal{L}^*} : X \to \prod_{f \in \mathcal{L}^*}[\inf f, \sup f] \subset \mathbb{R}^{\mathcal{L}}$ are essentially the same correspondence.

A unital vector lattice $\mathcal{L} \subset C(X)$ is said to *separate points and closed sets* in X provided that, for each closed set $F \subset X$ and each point $p \in X \setminus F$, there exists $f \in \mathcal{L}$ such that $f(p) \notin \mathrm{cl}_{\mathbb{R}} f(F)$.

The following is a fundamental fact concerning $K(\mathcal{L})$ (see [4, pp. 129–130], [7, 1.7 (j)]).

Proposition 2.1. *If $\mathcal{L}$ separates points and closed sets in X, then $\delta : X \to K(\mathcal{L})$ is a dense topological embedding.*

Though $K(\mathcal{L})$ is not compact in general, it can be considered as a realcompactification of X by Proposition 2.1. See [4] for more information about realcompactifications.

Let $\mathcal{U}(X)$ denote the lattice of all real-valued uniformly continuous functions on X. We write $\mathcal{U}$ (resp. $\mathcal{U}^*$) instead of $\mathcal{U}(\mathbb{H})$ (resp. $\mathcal{U}(\mathbb{H})^*$) for notational simplicity. The family $\mathcal{U}^*(X)$ has a ring structure with respect to $\mathbb{R}$, but $\mathcal{U}(X)$

does not. Therefore, when considering unbounded vector lattices, we need to consider lattice homomorphisms instead of ring homomorphisms.

Let αX and γX be compactifications of X. We say $\alpha X \succeq \gamma X$ provided that there is a continuous map $f : \alpha X \to \gamma X$ such that $f|_X = \mathrm{id}_X$. If $\alpha X \preceq \gamma X$ and $\alpha X \succeq \gamma X$ then we say that αX and γX are *equivalent compactifications* of X. Of course, two equivalent compactifications of X are homeomorphic.

It is easy to check that $\mathcal{U}^*(X)$ contains all constant maps, separates points from closed sets, and is a closed subring of $C^*(X)$ with respect to the sup-metric, i.e., $\mathcal{U}^*(X)$ is a *complete ring on functions* (see [3, 3.12.22(e)]). Hence, $\mathcal{U}^*(X)$ uniquely determines a compactification uX of X (see [3, 3.12.22 (e)], [7, 4.5]), which is called the *Samuel-Smirnov compactification* of X (see [1], [9]). We note that uX is equivalent to $K(\mathcal{U}^*(X)) = \mathrm{cl}_{\mathbb{R}^{\mathcal{U}^*(X)}} \delta(X)$ because of the equivalence of two maps $\delta : X \to K(\mathcal{U}^*(X))$ and $e_{\mathcal{U}^*(X)} : X \to \prod_{f \in \mathcal{U}^*(X)} [\inf f, \sup f]$.

Let (X, d_X) be a metric space and let $B_{d_X}(x, r)$ be the closed ball of radius r centered at $x \in X$. A metric d_X on X is called *proper* if $B_{d_X}(x, r)$ is compact for every $x \in X$ and $r > 0$.

Let (X, d_X) and (Y, d_Y) be proper metric spaces. A map $f : X \to Y$ is said to be *slowly oscillating* provided that, given $R > 0$ and $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that

$$\mathrm{diam}_{d_Y} f(B_{d_X}(x, R)) < \varepsilon$$

for every $x \in X \setminus K$, where $\mathrm{diam}_d A = \sup\{d(x, y) : x, y \in A\}$. Let $\mathcal{SO}(X)$ denote the lattice of all real-valued slowly oscillating continuous functions on a proper metric space X. The sublattice of all bounded functions of $\mathcal{SO}(X)$ is denoted by $\mathcal{SO}(X)^*$. When $X = \mathbb{H}$ we just write $\mathcal{SO}$ (resp. $\mathcal{SO}^*$) instead of $\mathcal{SO}(\mathbb{H})$ (resp. $\mathcal{SO}(\mathbb{H})^*$) for notational simplicity. It is easy to check that $\mathcal{SO}^*(X)$ is a closed subring of $C^*(X)$ with respect to the sup-metric, namely, a complete ring on functions. Hence, $\mathcal{SO}^*(X)$ uniquely determines a compactification hX of X, which is called the *Higson compactification* of X. The remainder $\nu X = hX \setminus X$ is called the *Higson corona* of X (cf. [8], [6]). We note that νX is compact and that hX and $K(\mathcal{SO}^*(X))$ are equivalent compactifications of X.

Proposition 2.2. *If (X, d) is a proper metric space, then $\mathcal{SO}(X) \subset \mathcal{U}(X)$.*

Proof. Let $f \in \mathcal{SO}(X)$. Given $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that $\mathrm{diam}\, f(B(x, 1)) < \varepsilon$ whenever $x \in X \setminus K$. Put $K' = \mathrm{cl}\, B(K, 1)$. Since X is a proper metric space, K' is compact. Consider a family

$$\mathscr{U} = \{f^{-1}(B(f(x), \varepsilon/2)) : x \in K'\}.$$

Since $K' \subset \bigcup \mathscr{U}$, we can take a Lebesgue number $\delta_0 > 0$ of $\mathscr{U}$, that is, every δ_0-neighborhood of $x \in K'$ is contained in some element of $\mathscr{U}$. Let $\delta = \min\{\delta_0, 1\}$. Then $d(x, y) < \delta$ implies that $x, y \in K'$ or $x, y \in X \setminus K$. Hence, $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. ∎

3. Homomorphisms of the lattice of slowly oscillating functions on the half-line

We note that $f : \mathbb{H} \to \mathbb{R}$ is a slowly oscillating function ($f \in \mathcal{SO}$) if and only if for every $R > 0$ and $\varepsilon > 0$ there exists $M > 0$ such that

$$\operatorname{diam} f([x, x+R]) < \varepsilon \text{ for every } x > M.$$

Let $\tau : \mathbb{H} \to \mathbb{R}$ be the map defined by

$$\tau(x) = x + 1$$

for every $x \in \mathbb{H}$. One should note that $\tau^\alpha \in \mathcal{SO}$ for every $0 < \alpha < 1$.

For each $f \in \mathcal{SO}$, we consider the map $f_* : H(\mathcal{SO}) \to \mathbb{R}$ defined by

$$f_*(\phi) = \phi(f)$$

for every $\phi \in H(\mathcal{SO})$. Recall that a basic neighborhood of $\phi \in H(\mathcal{SO})$ is of the form

$$V(\phi; f_1, \dots, f_n; \varepsilon) = \{\varphi \in H(\mathcal{SO}) : |\varphi(f_i) - \phi(f_i)| < \varepsilon,\ \forall i = 1, \dots, n\},$$

where $\varepsilon > 0$ and $f_i \in \mathcal{SO}$, $i = 1, \dots, n$. Now it is easy to see that f_* is continuous.

Proposition 3.1. $K(\mathcal{SO}) = \delta(\mathbb{H})$.

Proof. It is obvious that $\delta(\mathbb{H}) \subset K(\mathcal{SO})$. We shall show that $K(\mathcal{SO}) \subset \delta(\mathbb{H})$. Let $\phi \in K(\mathcal{SO})$. Note that $\delta(\mathbb{H})$ is dense in $K(\mathcal{SO})$ by Proposition 2.1. Thus we can take a net $(x_\alpha)_\alpha$ in $\mathbb{H}$ such that $(\delta_{x_\alpha})_\alpha$ converges to ϕ. For each $f \in \mathcal{SO}$, the net $(f_*(\delta_{x_\alpha}))_\alpha = (\delta_{x_\alpha}(f))_\alpha = (f(x_\alpha))_\alpha$ converges to $f_*(\phi) = \phi(f)$ because f_* is continuous, that is,

$$\phi(f) = \lim_\alpha f(x_\alpha).$$

Taking $f = \sqrt{\tau} \in \mathcal{SO}$, we have

$$\phi(\sqrt{\tau}) = \lim_\alpha \sqrt{x_\alpha + 1}.$$

Put $x_\phi = (\phi(\sqrt{\tau}))^2 - 1$. Then we have $x_\phi = \lim_\alpha x_\alpha$. Hence, we conclude that $\phi = \delta_{x_\phi} \in \delta(\mathbb{H})$, i.e., $K(\mathcal{SO}) \subset \delta(\mathbb{H})$. $\square$

Corollary 3.2. *For each $\phi \in H(\mathcal{SO})$, $\phi(\mathbf{1}) > 0$ if and only if there exist $x_\phi \in \mathbb{H}$ and $c > 0$ such that $\phi = c \cdot \delta_{x_\phi}$. In particular, if $\phi(\mathbf{1}) > 0$ then the point $x_\phi \in \mathbb{H}$ is uniquely determined.*

Proof. If $\phi(\mathbf{1}) > 0$ then $\phi(\mathbf{1})^{-1} \cdot \phi \in K(\mathcal{SO})$. By Proposition 3.1, there exists $x_\phi \in \mathbb{H}$ such that $\phi(\mathbf{1})^{-1} \cdot \phi = \delta_{x_\phi}$, i.e., $\phi = \phi(\mathbf{1}) \cdot \delta_{x_\phi}$. The reverse implication is trivial.

Suppose that $\phi(\mathbf{1}) > 0$ and $\phi = \phi(\mathbf{1}) \cdot \delta_s = \phi(\mathbf{1}) \cdot \delta_t$ for some $s, t \in \mathbb{H}$ then the equation $\phi(\tau) = \phi(\mathbf{1}) \cdot (s+1) = \phi(\mathbf{1}) \cdot (t+1)$ implies that $s = t$. $\square$

The following two lemmas are modifications of those stated in [1, p. 418].

Lemma 3.3. *Let $f \in \mathcal{SO}$ be a map such that $f \geq \mathbf{1}$. If there exits $\phi \in H(\mathcal{SO})$ such that $\phi(f) = 1$ and $\phi(\mathbf{1}) = 0$, then ϕ is contained in the closure of $\{f(n)^{-1} \cdot \delta_n : n \in \mathbb{N}\}$ in $H(\mathcal{SO})$.*

Proof. Suppose that there exists $\phi \in H(\mathcal{SO})$ such that $\phi(f) = 1$ and $\phi(\mathbf{1}) = 0$ but which is not contained in the closure of $\{f(n)^{-1} \cdot \delta_n : n \in \mathbb{N}\}$ in $H(\mathcal{SO})$. Then there exist $\varepsilon > 0$ and $g_1, \dots, g_k \in \mathcal{SO}$ such that $f(n)^{-1} \cdot \delta_n \notin V(\phi; g_1, \dots, g_k; \varepsilon)$ for every $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, there exists $i \in \{1, \dots, k\}$ such that

$$\left|\phi(g_i) - f(n)^{-1} \cdot g_i(n)\right| \geq \varepsilon.$$

Hence, we have

$$\bigvee_{i=1}^{k} |\phi(g_i) \cdot f(n) - g_i(n)| \geq \varepsilon \cdot f(n)$$

for every $n \in \mathbb{N}$. Let $c_i = \phi(g_i)$ for each $i = 1, \dots, k$. Put

$$h = 0 \wedge \left(\bigvee_{i=1}^{k} |c_i \cdot f - g_i| - \varepsilon \cdot f \right).$$

Then $h \in \mathcal{SO} \subset \mathcal{U}$ and $h(n) = 0$ for every $n \in \mathbb{N}$. It follows from uniformity that h is a bounded function. So, there exists $c > 0$ such that $|h| \leq c \cdot \mathbf{1}$. Thus, we have $|\phi(h)| = \phi(|h|) \leq c \cdot \phi(\mathbf{1}) = 0$, i.e., $\phi(h) = 0$. We note that

$$\bigvee_{i=1}^{k} |c_i \cdot f - g_i| \geq h + \varepsilon \cdot f$$

and

$$\phi\left(\bigvee_{i=1}^{k} |c_i \cdot f - g_i| \right) = \bigvee_{i=1}^{k} |c_i \cdot \phi(f) - \phi(g_i)| = 0.$$

However, we have $\phi(h + \varepsilon \cdot f) = \phi(h) + \varepsilon \cdot \phi(f) = \varepsilon > 0$, a contradiction. □

Let $\mathscr{F}$ be an ultrafilter on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then we define the operation $\lim_{\mathscr{F}(n)}$ by

$$\lim_{\mathscr{F}(n)} f(n) = \bigcap_{F \in \mathscr{F}} \mathrm{cl}\, \{f(n) : n \in F\}$$

for every $f \in C(\mathbb{H})$ (cf. [1]). If $f \in C(\mathbb{H})$ is a map such that $\lim\limits_{\mathscr{F}(n)} f(n) \neq \varnothing$ then the set $\lim\limits_{\mathscr{F}(n)} f(n)$ consists of a single point since $\mathscr{F}$ is an ultrafilter.

Recall that the Stone-Čech compactification $\beta\mathbb{N}_0$ of $\mathbb{N}_0$ can be considered as the space of all ultrafilters on $\mathbb{N}_0$.

Lemma 3.4. *Let $f \in \mathcal{SO}$ be a map such that $f \geq \mathbf{1}$. Suppose that there exists a homomorphism $\phi \in H(\mathcal{SO})$ such that $\phi(f) = 1$ and $\phi(\mathbf{1}) = 0$. Then there exists a free ultrafilter $\mathscr{F}$ such that the function $\phi^f_{\mathscr{F}} : \mathcal{SO} \to \mathbb{R}$ defined by*

$$\phi^f_{\mathscr{F}}(g) = \lim_{\mathscr{F}(n)} \frac{g(n)}{f(n)}, \quad (g \in \mathcal{SO})$$

is a well-defined homomorphism that fulfils $\phi^f_{\mathscr{F}} = \phi$.

Proof. Suppose that there exists a homomorphism $\phi \in H(\mathcal{SO})$ such that $\phi(f) = 1$ and $\phi(\mathbf{1}) = 0$ for some $f \in \mathcal{SO}$ with $f \geq \mathbf{1}$. For each neighborhood V of ϕ in $H(\mathcal{SO})$, let $N_V = \left\{n \in \mathbb{N} : f(n)^{-1} \cdot \delta_n \in V\right\}$. Then $N_V \neq \varnothing$ by Lemma 3.3. Put

$$\mathscr{G} = \{N_V : V \text{ is a neighborhood of } \phi\}.$$

Then $\mathscr{G}$ becomes a filter on $\mathbb{N}$. Let $\mathscr{F}$ be an ultrafilter on $\mathbb{N}$ refining $\mathscr{G}$. We note that $\mathscr{F}$ must be a free ultrafilter since $\phi(\mathbf{1}) = 0$. Indeed, if $\mathscr{G}$ is a fixed ultrafilter, say $\lim \mathscr{G} = n_0$, then $\phi = f(n_0)^{-1}\delta_{n_0}$. Hence, we have $\phi(\mathbf{1}) = f(n_0)^{-1} \neq 0$, a contradiction.

Given $\varepsilon > 0$ and $g \in \mathcal{SO}$, we consider a neighborhood $V_\varepsilon = V(\phi; g; \varepsilon)$ of ϕ in $H(\mathcal{SO})$, that is,

$$V_\varepsilon = \{\varphi \in H(\mathcal{SO}) : |\varphi(g) - \phi(g)| < \varepsilon\}.$$

Since $\left\{n \in \mathbb{N} : f(n)^{-1} \cdot \delta_n \in V_\varepsilon\right\} \in \mathscr{G} \subset \mathscr{F}$, we have

$$\phi^f_{\mathscr{F}}(g) = \bigcap_{F \in \mathscr{F}} \mathrm{cl}\left\{\frac{g(n)}{f(n)} : n \in F\right\} \subset \bigcap_{G \in \mathscr{G}} \mathrm{cl}\left\{\frac{g(n)}{f(n)} : n \in G\right\} \subset B(\phi(g), \varepsilon).$$

Then $\phi^f_{\mathscr{F}}(g)$ is not empty by the compactness of $B(\phi(g), \varepsilon)$ and it is uniquely determined because $\mathscr{F}$ is an ultrafilter. Since ε is arbitrary, it follows that $\phi^f_{\mathscr{F}}(g) = \phi(g)$, that is, $\phi^f_{\mathscr{F}}$ is a well-defined homomorphism that fulfils $\phi^f_{\mathscr{F}} = \phi$. □

The following lemma is the key to this note, as it implies that a homomorphism in $H(\mathcal{SO})$ which maps the unit to zero must be the zero-homomorphism (Proposition 3.9). Using this fact, we will derive our main result (Theorem 3.10).

Lemma 3.5 (Vanishing Criterion)**.** *Let* $\phi \in H(\mathcal{SO})$. *If there are two maps* $f, g \in \mathcal{SO}$ *such that* $\mathbf{1} \leq f \leq g$ *and* $\lim\limits_{n \to \infty} f(n)^{-1} \cdot g(n) = \infty$, *then the condition* $\phi(\mathbf{1}) = 0$ *implies that* $\phi(f) = 0$.

Proof. Let $f, g \in \mathcal{SO}$ be such that $\mathbf{1} \leq f \leq g$ and $\lim_{n\to\infty} f(n)^{-1} \cdot g(n) = \infty$. Let $\phi \in H(\mathcal{SO})$ be such that $\phi(\mathbf{1}) = 0$. Suppose that $\phi(f) \neq 0$. Replacing ϕ by $\phi(f)^{-1} \cdot \phi$, we may assume that $\phi(f) = 1$. Then, by Lemma 3.4, there exists a free ultrafilter $\mathscr{F}$ such that $\phi = \phi^f_{\mathscr{F}}$. However, since $\mathscr{F}$ is a free ultrafilter, we have

$$\phi(g) = \phi^f_{\mathscr{F}}(g) = \lim_{\mathscr{F}(n)} \frac{g(n)}{f(n)} = \infty,$$

a contradiction. □

Definition 3.6. A sequence $\mathfrak{a} = (a_n) \subset \mathbb{N}$ is called a *strictly increasing sequence* provided that $a_n < a_{n+1}$ for every $n \in \mathbb{N}$. Note that if $\mathfrak{a} = (a_n)$ is a strictly increasing sequence then $\lim\limits_{n\to\infty} a_n = \infty$ since $\mathfrak{a} \subset \mathbb{N}$.

Let $\mathfrak{a}$ be a strictly increasing sequence. Let $\eta_{\mathfrak{a}}^0 = \tau : \mathbb{H} \to \mathbb{R}$. Suppose that $\eta_{\mathfrak{a}}^{n-1}$ has been defined for $n \geq 1$. Then we define $\eta_{\mathfrak{a}}^n : \mathbb{H} \to \mathbb{R}$ by

$$\eta_{\mathfrak{a}}^n(x) = \begin{cases} \eta_{\mathfrak{a}}^{n-1}(x), & 0 \leq x < a_n, \\ \eta_{\mathfrak{a}}^{n-1}(a_n) + (x - a_n)/n, & a_n \leq x, \end{cases}$$

for every $x \in \mathbb{H}$ (see Figure 1). Note that $\eta_{\mathfrak{a}}^{n-1} \geq \eta_{\mathfrak{a}}^n \geq 1$ for every $n \in \mathbb{N}$.

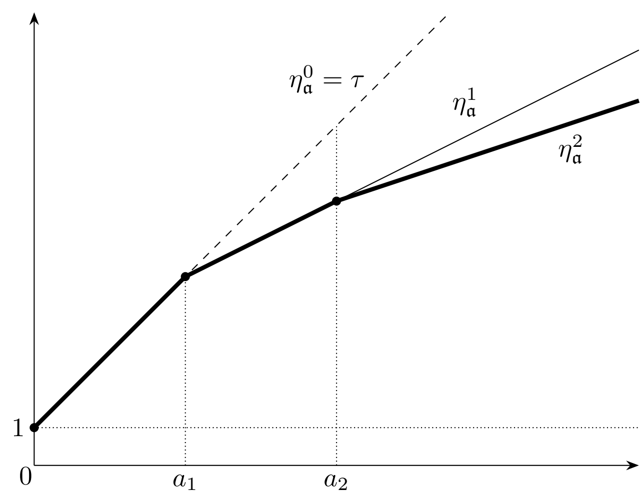

FIGURE 1. The graphs of $\eta_{\mathfrak{a}}^0$, $\eta_{\mathfrak{a}}^1$ and $\eta_{\mathfrak{a}}^2$.

We define $\eta_{\mathfrak{a}} : \mathbb{H} \to \mathbb{R}$ by

$$\eta_{\mathfrak{a}}(x) = \lim_{n\to\infty} \eta_{\mathfrak{a}}^n(x)$$

for every $x \in \mathbb{H}$. We note that if $x \leq a_n$ then

$$\eta_{\mathfrak{a}}(x) = \eta_{\mathfrak{a}}^n(x) = \eta_{\mathfrak{a}}^{n-1}(x).$$

It is easy to see that $\eta_{\mathfrak{a}} : \mathbb{H} \to \mathbb{R}$ is a well-defined slowly oscillating continuous function such that $\eta_{\mathfrak{a}} \geq 1$. We call $\eta_{\mathfrak{a}}$ the *slowly oscillating function with respect to* $\mathfrak{a}$.

Proposition 3.7. *For each $f \in \mathcal{SO}$, there exists a strictly increasing sequence $\mathfrak{a} \subset \mathbb{N}$ and $L > 0$ such that $|f| \leq L \cdot \eta_{\mathfrak{a}}$.*

Proof. Since $f \in \mathcal{SO}$, we can take a strictly increasing sequence $\mathfrak{a} = (a_n) \subset \mathbb{N}$ such that

(1) $\operatorname{diam} f(B(x,1)) < (n+1)^{-4}$ for every $x \geq a_n$.

Let $L = 1 + \sup\{|f(x)| : x \leq a_1\}$. Then we have

$$|f(x)| + 1 \leq L \leq L \cdot \tau(x) = L \cdot \eta_{\mathfrak{a}}^0(x)$$

for every $x \leq a_1$. Suppose that we have shown that

$(2)_n$ $|f(x)| + n^{-2} \le L \cdot \eta_{\mathfrak{a}}^{n-1}(x)$ for every $x \le a_n$.

If $x \le a_n$ then $(2)_{n+1}$ follows from $(2)_n$ since $|f(x)| + (n+1)^{-2} \le |f(x)| + n^{-2}$ and $\eta_{\mathfrak{a}}^n(x) = \eta_{\mathfrak{a}}^{n-1}(x)$. Now suppose that $a_n \le x \le a_{n+1}$. Then we have

$$\begin{aligned}
|f(x)| + \frac{1}{(n+1)^2} &\le |f(a_n)| + \frac{x-a_n}{(n+1)^4} + \frac{1}{(n+1)^4} + \frac{1}{(n+1)^2} && \text{(by (1))}\\
&< |f(a_n)| + \frac{x-a_n}{n+1} + \frac{1}{n^2}\\
&\le L \cdot \eta_{\mathfrak{a}}^n(a_n) + \frac{x-a_n}{n+1} && \text{(by } (2)_n)\\
&\le L \cdot \eta_{\mathfrak{a}}^{n-1}(a_n) + \frac{x-a_n}{n} && (\because\ \eta_{\mathfrak{a}}^n(a_n) = \eta_{\mathfrak{a}}^{n-1}(a_n))\\
&\le L \cdot \left(\eta_{\mathfrak{a}}^{n-1}(a_n) + \frac{x-a_n}{n}\right) && (\because\ L \ge 1)\\
&= L \cdot \eta_{\mathfrak{a}}^n(x).
\end{aligned}$$

Thus $(2)_{n+1}$ holds. Consequently, we have $|f| \le L \cdot \eta_{\mathfrak{a}}$ since $\lim_{n\to\infty} \eta_{\mathfrak{a}}^n = \eta_{\mathfrak{a}}$ and $\lim_{n\to\infty} a_n = \infty$. $\square$

Proposition 3.8. *For each strictly increasing sequence $\mathfrak{a} \subset \mathbb{N}$, there exists a strictly increasing sequence $\mathfrak{b} \subset \mathbb{N}$ such that $\eta_{\mathfrak{a}} \le \eta_{\mathfrak{b}}$ and $\lim_{n\to\infty} \eta_{\mathfrak{a}}(n)^{-1} \cdot \eta_{\mathfrak{b}}(n) = \infty$.*

Proof. Let $\mathfrak{a} = (a_n) \subset \mathbb{N}$ be a strictly increasing sequence. Let $\mathfrak{b} = (b_n)$ be a strictly increasing sequence such that

(1) $b_0 = a_1$ and
(2) $b_n \ge n^2 \cdot \eta_{\mathfrak{a}}(b_{n-1}) + b_{n-1} + a_{(n+1)^3}$ for each $n \in \mathbb{N}$.

We shall show that $\eta_{\mathfrak{b}}(x) \ge n \cdot \eta_{\mathfrak{a}}(x)$ for every $x \in [b_n, b_{n+1}]$.

Since $b_i > a_i$ for $i = 1, 2$, we have $\eta_{\mathfrak{b}}(x) \ge 1 \cdot \eta_{\mathfrak{a}}(x)$ for every $x \in [b_1, b_2]$. Suppose that we have shown that

(3) $\eta_{\mathfrak{b}}(x) \ge (n-1) \cdot \eta_{\mathfrak{a}}(x)$ whenever $x \in [b_{n-1}, b_n]$ for $n \ge 2$.

Let $x \in [b_n, b_{n+1}]$. We write $x = b_{n-1} + t$, $t > 0$ for some technical reason. Then we have

$$\frac{t}{n^2} > \eta_{\mathfrak{a}}(b_{n-1}) \tag{4}$$

Indeed, since $x = b_{n-1} + t \ge b_n$, we have

$$\begin{aligned}
t &\ge b_n - b_{n-1}\\
&> n^2 \cdot \eta_{\mathfrak{a}}(b_{n-1}) + a_{(n+1)^3} && \text{(by (2))}\\
&> n^2 \cdot \eta_{\mathfrak{a}}(b_{n-1}).
\end{aligned}$$

Then we have

$$\begin{aligned}
\eta_{\mathfrak{b}}(x) &= \eta_{\mathfrak{b}}^{n}(x) = \eta_{\mathfrak{b}}^{n-1}(b_n) + \frac{1}{n}(x - b_n) \\
&= \eta_{\mathfrak{b}}^{n-2}(b_{n-1}) + \frac{1}{n-1}(b_n - b_{n-1}) + \frac{1}{n}(x - b_n) \\
&\geq \eta_{\mathfrak{b}}^{n-2}(b_{n-1}) + \frac{1}{n}(b_n - b_{n-1} + x - b_n) \\
&= \eta_{\mathfrak{b}}^{n-2}(b_{n-1}) + \frac{1}{n}(x - b_{n-1}) \\
&= \eta_{\mathfrak{b}}^{n-2}(b_{n-1}) + \frac{t}{n} \\
&= \eta_{\mathfrak{b}}(b_{n-1}) + \frac{t}{n} \\
&\geq (n-1) \cdot \eta_{\mathfrak{a}}(b_{n-1}) + \frac{t}{n}. \qquad (5)
\end{aligned}$$

The last inequality follows from (3). Since $b_{n-1} > a_{n^3}$, there exists $k \geq n^3$ such that $a_k \leq b_{n-1} < a_{k+1}$. Then we have

$$\begin{aligned}
\eta_{\mathfrak{a}}(x) &\leq \eta_{\mathfrak{a}}^{k}(x) = \eta_{\mathfrak{a}}^{k-1}(a_k) + \frac{1}{k}(x - a_k) \\
&= \eta_{\mathfrak{a}}^{k-1}(a_k) + \frac{1}{k}(b_{n-1} - a_k) + \frac{1}{k}(x - b_{n-1}) \\
&= \eta_{\mathfrak{a}}^{k}(b_{n-1}) + \frac{1}{k}(x - b_{n-1}) \\
&= \eta_{\mathfrak{a}}^{k}(b_{n-1}) + \frac{t}{k} \\
&\leq \eta_{\mathfrak{a}}(b_{n-1}) + \frac{t}{n^3}. \qquad (6)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\eta_{\mathfrak{b}}(x) - n \cdot \eta_{\mathfrak{a}}(x) &\geq (n-1) \cdot \eta_{\mathfrak{a}}(b_{n-1}) + \frac{t}{n} - n \cdot \eta_{\mathfrak{a}}(x) \qquad \text{(by (5))} \\
&\geq (n-1) \cdot \eta_{\mathfrak{a}}(b_{n-1}) + \frac{t}{n} - n \cdot \left(\eta_{\mathfrak{a}}(b_{n-1}) + \frac{t}{n^3}\right) \qquad \text{(by (6))} \\
&= (n-1) \cdot \frac{t}{n^2} - \eta_{\mathfrak{a}}(b_{n-1}) \\
&> (n-1) \cdot \eta_{\mathfrak{a}}(b_{n-1}) - \eta_{\mathfrak{a}}(b_{n-1}) \qquad \text{(by (4))} \\
&= (n-2) \cdot \eta_{\mathfrak{a}}(b_{n-1}) \\
&\geq 0.
\end{aligned}$$

Thus we conclude that

$$\lim_{n\to\infty} \frac{\eta_{\mathfrak{b}}(n)}{\eta_{\mathfrak{a}}(n)} \geq \lim_{n\to\infty} n = \infty.$$

□

Proposition 3.9. *If $\phi \in H(\mathcal{SO})$ satisfies $\phi(\mathbf{1}) = 0$ then $\phi = \mathbf{0}$.*

Proof. Let $f \in \mathcal{SO}$. Suppose that $\phi(\mathbf{1}) = 0$. By Proposition 3.7, there exists a strictly increasing sequence $\mathfrak{a} \in \mathbb{N}$ and $L > 0$ such that $|f| \leq L \cdot \eta_{\mathfrak{a}}$. Recall that $\eta_{\mathfrak{a}} \geq \mathbf{1}$. By Proposition 3.8, there exists a strictly increasing sequence $\mathfrak{b} \in \mathbb{N}$ so that $\mathbf{1} \leq \eta_{\mathfrak{a}} \leq \eta_{\mathfrak{b}}$ and

$$\lim_{n \to \infty} \frac{\eta_{\mathfrak{b}}(n)}{\eta_{\mathfrak{a}}(n)} = \infty.$$

Thus, the condition $\phi(\mathbf{1}) = 0$ implies that $\phi(\eta_{\mathfrak{a}}) = 0$ by the Vanishing Criterion (Lemma 3.5). Hence, we have

$$|\phi(f)| = \phi(|f|) \leq L \cdot \phi(\eta_{\mathfrak{a}}) = 0,$$

i.e., $\phi(f) = 0$. Since f can be taken arbitrary, ϕ must be the zero-homomorphism. □

By Proposition 3.9, it follows that the structure of $H(\mathcal{SO})$ is very simple in contrast to the case of uniformly continuous functions [1] (see also [2]).

Theorem 3.10. *The space $H(\mathcal{SO})$ of all homomorphisms of the vector lattice of all slowly oscillating functions on the half-line $\mathbb{H}$ is homeomorphic to the space $(\mathbb{H} \times (0, \infty)) \cup \{\mathbf{0}\}$ where a neighborhood base of the point $\mathbf{0}$ consists of sets of the form:*

$$\{(x, y) \in \mathbb{H} \times (0, \infty) : y \leq \varepsilon \cdot \eta_{\mathfrak{a}}(x)^{-1}\} \cup \{\mathbf{0}\}$$

for some $\varepsilon > 0$ and the slowly oscillating function $\eta_{\mathfrak{a}}$ with respect to some strictly increasing sequence $\mathfrak{a}$.

Proof. By Proposition 3.9, every non-zero homomorphism $\phi \in H(\mathcal{SO})$ satisfies $\phi(\mathbf{1}) > 0$ since ϕ is positive. Hence, every $\phi \in H(\mathcal{SO}) \setminus \{\mathbf{0}\}$ is uniquely expressed as $\phi = \phi(\mathbf{1}) \cdot \delta_{x_\phi}$ for some $x_\phi \in \mathbb{H}$ by Corollary 3.2. Therefore, the function $\Phi : H(\mathcal{SO}) \setminus \{\mathbf{0}\} \to \mathbb{H} \times (0, \infty)$ defined by

$$\Phi(\phi) = (x_\phi, \phi(\mathbf{1}))$$

is a well-defined bijection.

The function Φ is continuous. To see this, fix $\phi \in H(\mathcal{SO})$ and let $\varepsilon > 0$. We consider following two neighborhoods of ϕ:

$$V(\phi; \mathbf{1}; \varepsilon_1) = \{\varphi \in H(\mathcal{SO}) : |\varphi(\mathbf{1}) - \phi(\mathbf{1})| < \varepsilon_1\},$$
$$V(\phi; \tau; \varepsilon_2) = \{\varphi \in H(\mathcal{SO}) : |\varphi(\tau) - \phi(\tau)| < \varepsilon_2\},$$

where

(1) $\varepsilon_1 < \min\left\{\varepsilon/2, \ \frac{\phi(\mathbf{1}) \cdot \varepsilon}{2(x_\phi + 1)}\right\}$ and
(2) $\varepsilon_2 < \frac{\phi(\mathbf{1}) \cdot \varepsilon}{2} - \varepsilon_1(x_\phi + 1)$.

Then for each $\varphi \in V(\phi; \mathbf{1}; \varepsilon_1) \cap V(\phi; \tau; \varepsilon_2)$, we have

$$\begin{aligned}
\phi(\mathbf{1})|x_\phi - x_\varphi| &= |\phi(\mathbf{1})(x_\phi + 1) - \phi(\mathbf{1})(x_\varphi + 1)| \\
&\leq |\phi(\mathbf{1})(x_\phi + 1) - \varphi(\mathbf{1})(x_\phi + 1)| \\
&\quad + |\varphi(\mathbf{1})(x_\phi + 1) - \phi(\mathbf{1})(x_\varphi + 1)| \\
&= |\phi(\mathbf{1}) - \varphi(\mathbf{1})|(x_\varphi + 1) + |\varphi(\tau) - \phi(\tau)| \\
&< \varepsilon_1(x_\varphi + 1) + \varepsilon_2.
\end{aligned}$$

By (2), we have

$$|x_\phi - x_\varphi| < \frac{\varepsilon_1(x_\phi + 1) + \varepsilon_2}{\phi(\mathbf{1})} < \varepsilon/2.$$

Thus we have

$$\begin{aligned}
d(\Phi(\phi), \Phi(\varphi)) &\leq d((x_\phi, \phi(\mathbf{1})), (x_\phi, \varphi(\mathbf{1}))) + d((x_\phi, \varphi(\mathbf{1})), (x_\varphi, \varphi(\mathbf{1}))) \\
&= |\varphi(\mathbf{1}) - \phi(\mathbf{1})| + |x_\varphi - x_\phi| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Next we shall show that $\Phi^{-1} : \mathbb{H} \times (0, \infty) \ni (x, s) \mapsto s \cdot \delta_x \in H(\mathcal{SO}) \setminus \{\mathbf{0}\}$ is continuous. Given $(x, s) \in \mathbb{H} \times (0, \infty)$ and $\varepsilon > 0$, let $f \in \mathcal{SO}$ and consider a basic neighborhood

$$V(s \cdot \delta_x; f; \varepsilon) = \{\varphi \in H(\mathcal{SO}) : |\varphi(f) - s \cdot f(x)| < \varepsilon\}$$

of $\Phi^{-1}(x, s) = s \cdot \delta_x$. We take $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_0 > 0$ so that

(3) $\lambda_1 \cdot |f(x)| < \varepsilon/2$,
(4) $(s + \lambda_1) \cdot \lambda_2 < \varepsilon/2$,
(5) $\lambda_0 < \min\{\lambda_1, \lambda_2\}$ and
(6) $|f(x) - f(y)| < \lambda_2$ whenever $|x - y| < \lambda_0$.

Suppose that $(y, t) \in \mathbb{H} \times (0, \infty)$ satisfies $d((x, s), (y, t)) < \lambda_0$. Then $|x - y| < \lambda_0$ and $|s - t| < \lambda_0 \leq \lambda_1$, in particular, $t \leq s + \lambda_1$. Hence, we have

$$\begin{aligned}
|\Phi^{-1}(x, s)(f) - \Phi^{-1}(y, t)(f)| &= |s \cdot f(x) - t \cdot f(y)| \\
&\leq |s \cdot f(x) - t \cdot f(x)| + |t \cdot f(x) - t \cdot f(y)| \\
&= |s - t| \cdot |f(x)| + t \cdot |f(x) - f(y)| \\
&\leq \lambda_1 \cdot |f(x)| + (s + \lambda_1) \cdot \lambda_2 \\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $\Phi^{-1}(y, t) \in V(s \cdot \delta_x; f; \varepsilon)$. Consequently, Φ is a homeomorphism.

Finally, we shall consider neighborhoods of $\mathbf{0} \in H(\mathcal{SO})$. We can take a subbase of neighborhoods of $\mathbf{0}$ in $H(\mathcal{SO})$ as a family which consists of sets of the form:

$$\begin{aligned}
V(\mathbf{0}; f; \varepsilon) &= \{\varphi \in H(\mathcal{SO}) : |\varphi(f) - \mathbf{0}(f)| < \varepsilon\} \\
&= \{\varphi \in H(\mathcal{SO}) : |\varphi(\mathbf{1}) \cdot \delta_{x_\varphi}(f)| < \varepsilon\} \\
&= \{\varphi \in H(\mathcal{SO}) : |\varphi(\mathbf{1}) \cdot f(x_\varphi)| < \varepsilon\}
\end{aligned}$$

for some $f \in \mathcal{SO}$ and $\varepsilon > 0$.

Let $f \in \mathcal{SO}$ and $\varepsilon > 0$. By Proposition 3.7, there exists $L > 0$ and a strictly increasing sequence $\mathfrak{a}$ such that $|f| \leq L \cdot \eta_{\mathfrak{a}}$. So, if $\varphi \in V(\mathbf{0}; L \cdot \eta_{\mathfrak{a}}; \varepsilon)$ then $|\varphi(\mathbf{1}) \cdot f(x_\varphi)| < |\varphi(\mathbf{1}) \cdot L \cdot \eta_{\mathfrak{a}}(x_\varphi)| < \varepsilon$, that is,

$$V(\mathbf{0}; L \cdot \eta_{\mathfrak{a}}; \varepsilon) \subset V(\mathbf{0}; f; \varepsilon).$$

Since $V(\mathbf{0}; L \cdot \eta_{\mathfrak{a}}; \varepsilon) = V(\mathbf{0}; \eta_{\mathfrak{a}}; \varepsilon \cdot L^{-1})$, we can take a subbase of neighborhoods of $\mathbf{0}$ in $H(\mathcal{SO})$ as a family which consists of sets of the form:

$$V(\mathbf{0}; \eta_{\mathfrak{a}}; \varepsilon) = \{\varphi \in H(\mathcal{SO}) : \varphi(\mathbf{1}) \cdot \eta_{\mathfrak{a}}(x_\varphi) < \varepsilon\}$$

for some $\varepsilon > 0$ and a strictly increasing sequence $\mathfrak{a} \subset \mathbb{N}$. Note that for any two strictly increasing sequences $\mathfrak{a} = (a_n)$ and $\mathfrak{b} = (b_n)$, if we take a strictly increasing sequence $\mathfrak{c} = (c_n)$ such that $c_n \geq \max\{a_n, b_n\}$ then we have $\eta_{\mathfrak{c}} \geq \eta_{\mathfrak{a}} \vee \eta_{\mathfrak{b}}$, i.e.,

$$V(\mathbf{0}; \eta_{\mathfrak{c}}; \varepsilon) \subset V(\mathbf{0}; \eta_{\mathfrak{a}}, \eta_{\mathfrak{b}}; \varepsilon).$$

Thus, we can take a base of neighborhoods of $\mathbf{0}$ in $H(\mathcal{SO})$ as a family which consists of sets of the form $V(\mathbf{0}; \eta_{\mathfrak{a}}; \varepsilon)$ for some $\varepsilon > 0$ and a strictly increasing sequence $\mathfrak{a} \subset \mathbb{N}$. Consequently, we can take a base of neighborhoods of $\mathbf{0}$ in $\mathbb{H} \times (0, \infty) \cup \{\mathbf{0}\}$ as a family which consists of sets of the form

$$\begin{aligned}\Phi(V(\mathbf{0}; \eta_{\mathfrak{a}}; \varepsilon)) \cup \{\mathbf{0}\} &= \{(x_\varphi, \varphi(\mathbf{1})) \in \mathbb{H} \times (0, \infty) : \varphi(\mathbf{1}) \cdot \eta_{\mathfrak{a}}(x_\varphi) < \varepsilon\} \cup \{\mathbf{0}\} \\ &= \{(x, y) \in \mathbb{H} \times (0, \infty) : y \leq \varepsilon \cdot \eta_{\mathfrak{a}}(x)^{-1}\} \cup \{\mathbf{0}\}\end{aligned}$$

for some $\varepsilon > 0$ and an increasing sequence $\mathfrak{a} \subset \mathbb{N}$. □

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