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Iterated function system of generalized cyclic F-contractive mappings

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Abstract

The aim of this paper is to study the sufficient conditions for the existence of attractor of a generalized cyclic iterated function system composed of a complete metric space and a finite collection of generalized cyclic $\mathcal{F}$-contraction mappings. Some examples are presented to support our main results and concepts defined herein. The results proved in the paper extend and generalize various well known results in the existing literature.

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1. Introduction and preliminaries

Hutchinson [8] proposed and studied iterated function system. Using a finite family of contraction maps on Euclidean space $\mathbb{R}^n$, the Hutchinson operator was defined which has a closed and bounded subset of $\mathbb{R}^n$ and has a

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fixed point, known as attractor of iterated function system, see for example in [5, 6, 11, 18, 19, 20]. Banakh and Nowak [3] introduced the concept of topological iterated function system and attractor attractor, that generalizes the usual iterated function system in metric spaces. That is to say, every iterated function system is a topological iterated function system but not conversely. Particularly, in [13], the authors proved that a space, called "shark teeth" [4] is a topological iterated function system but is not homeomorphic to the usual iterated function system.
The concept of a new type of contraction mapping known as $\mathcal{F}$-contraction mapping was initiated by Wardoski [24]. He then proved a fixed point result as an interesting generalization of the Banach contraction principle. Nazir et al. [22] defined a Hutchinson operator with the help of a finite family of $\mathcal{F}$-contractions on a complete metric space which is itself generalized $\mathcal{F}$-contraction on the family of compact sets. The concept of cyclic contraction mapping was introduced by Rus [16]. Pecurar and Rus [14] proved some fixed point results for cyclic φ-contraction mappings on a metric space. Karapinar [9] obtained the unique fixed points of cyclic weak φ-contraction mappings and established well-posedness problem of these mappings. Other useful results of cyclic contraction mappings were obtained in [7, 10, 25].

In this paper, by using generalized cyclic $\mathcal{F}$-contraction mappings, we defined generalized Hutchinson operators on the class of compact subsets of a metric space. We obtained several results on the existence of attractors of these generalized Hutchinson operators. Some examples are presented to support results proved herein. Our results extend and unify various well known comparable results in the existing literature. Consistent with Searcoid [17], the following definitions and results will be needed in the sequel.

Let X be any nonempty set. Then for any $x \in X$ and $\varepsilon > 0$, the open ball in a metric space $(X, \boldsymbol{d})$ is defined as

$$B_\varepsilon(x) = \{y \in X : \boldsymbol{d}(x, y) < \varepsilon\}.$$

The topology τ on a metric space $(X, \boldsymbol{d})$ is as follows:

$$\tau = \{U \subseteq X : \ \forall\, u \in U, \ \exists\, \varepsilon > 0 \text{ suh that } B_\varepsilon(x) \subseteq U\}.$$

A subset Y in a metric space $(X, \boldsymbol{d})$ is said to be bounded if and only if the set $\{\boldsymbol{d}(x, y) : x, y \in Y\}$ is bounded above.

Let $\overline{C}$ be a closure of C with respect to a metric space $(X, \boldsymbol{d})$. Then

$$c \in \overline{C} \Longleftrightarrow B_\varepsilon(c) \cap C \neq \varnothing \quad \text{for all } \varepsilon > 0.$$

A set C in a metric space $(X, \boldsymbol{d})$ is closed if and only if $\overline{C} = C$.

Definition 1.1. Let $(X, \boldsymbol{d})$ be a metric space. A subset K of X is said to be compact if and only if every open cover of K (by open sets in M) has a finite subcover. If M itself has this property, then we say that M is a compact metric space.

Theorem 1.2. *Let $(X, \boldsymbol{d})$ be a dislocated metric space, and let K be a compact subset of X. Then K is a closed subset of X, and K is bounded.*

Definition 1.3. A metric space $(X, \boldsymbol{d})$ is sequentially compact if every sequence has a convergent subsequence.

Theorem 1.4. *A metric space $(X, \boldsymbol{d})$ is compact if and only if it is sequentially compact.*

Theorem 1.5. *Let f be a continuous selfmap on compact set X in a metric space $(X, \boldsymbol{d})$ into itself. Then the range $f(X)$ of f is also compact.*

Denote by $\mathcal{C}(X)$, the collection of all nonempty compact subsets of a metric space $(X, \boldsymbol{d})$. For $M_1, N_1 \in \mathcal{C}(X)$,

$$H(M_1, N_1) = \max\{\sup_{n_1 \in N_1} \boldsymbol{d}(n_1, M_1), \sup_{m_1 \in M_1} \boldsymbol{d}(m_1, N_1)\},$$

where

$$\boldsymbol{d}(n_1, M_1) = \inf\{\boldsymbol{d}(n_1, m_1) : m_1 \in M_1\}$$

is the distance of a point n_1 from the set M_1. The mapping $H : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}_+$ defined above, is known as Pompeiu-Hausdorff metric induced by $\boldsymbol{d}$. If $(X, \boldsymbol{d})$ is a complete metric space, then $(\mathcal{C}(X), H)$ is also a complete metric space.

If the range X of a mapping is replaced with the class of sets having some specific topological properties, then we have the concept of a point to set mapping. This initiates the study of metric fixed point theory for multivalued mappings. Nadler [12] obtained fixed point of multivalued mapping satisfying certain contractive condition on a complete metric space and hence multivalued version of Banach contraction principle was obtained (see also in [1], [2], [21]).

We need the following Lemma in the sequel.

Lemma 1.6 ([22])**.** *In a metric space $(X, \boldsymbol{d})$, for $R, S, U, V \in \mathcal{C}(X)$, the following hold:*

- **(i):** *If $S \subseteq U$, then* $\sup_{r \in R} \boldsymbol{d}(r, U) \leq \sup_{r \in R} \boldsymbol{d}(r, S)$.
- **(ii):** $\sup_{x \in R \cup S} \boldsymbol{d}(x, U) = \max\{\sup_{r \in R} \boldsymbol{d}(r, U), \sup_{s \in S} \boldsymbol{d}(s, U)\}$.
- **(iii):** $H(R \cup S, U \cup V) \leq \max\{H(R, U), H(S, V)\}$.

Lemma 1.7 ([15])**.** *In a complete metric space $(X, \boldsymbol{d})$, if B is a closed subset of X, then $\mathcal{C}(B)$ is also a closed subset of the complete metric space $(\mathcal{C}(X), H)$.*

Denote by $\digamma$, the collection of all continuous mappings $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F_1) $\mathcal{F}$ is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}_+$, $\alpha < \beta$ implies that $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$.

(F_2) For any sequence $\{\lambda_n\} \subseteq \mathbb{R}_+$,

$$\lim_{n \to \infty} \lambda_n = 0 \text{ and } \lim_{n \to \infty} \mathcal{F}(\lambda_n) = -\infty \text{ are equivalent.}$$

(F_3) For any $\eta \in (0,1)$, we have $\lim_{\eta \to 0^+} \lambda^\eta \mathcal{F}(\lambda) = 0$.

Let us recall the concept of $\mathcal{F}$-contraction mapping [23].

Definition 1.8. Let $(X, \boldsymbol{d})$ be a metric space. A mapping $f : X \to X$ is known as an $\mathcal{F}$-contraction if for $m_1, n_1 \in X$, we have $\mathcal{F} \in \digamma$ and $\tau > 0$ such that

$$\tau + \mathcal{F}(\boldsymbol{d}(fm_1, fn_1)) \leq \mathcal{F}(\boldsymbol{d}(m_1, n_1)),$$

whenever $\boldsymbol{d}(fm_1, fn_1) > 0$.

Definition 1.9 ([14]). Let X be any nonempty set and $f : X \to X$. A finite family $\{X_1, X_2, X_3, ..., X_r\}$ of nonempty subsets of X with $X = \cup_{i=1}^r X_i$ is called a cyclic representation of X with respect to f if

$$f(X_1) \subset X_2, ..., f(X_{r-1}) \subset X_r, \text{ and } f(X_r) \subset X_1.$$

2. Cyclic $\mathcal{F}$-contraction iterated function systems

In this section, we introduce the notion of a cyclic $\mathcal{F}$-contraction iterated function system in a metric space.

Definition 2.1 ([22]). In a complete metric space $(X, \boldsymbol{d})$, a set $\{X; f_n,\ n = 1, 2, ..., k\}$ is said to be a generalized iterated function system if each $f_n : X \to X$ is $\mathcal{F}$-contraction for $n \in \{1, 2, 3, ..., k\}$.

Definition 2.2. Let $(X, \boldsymbol{d})$ be a complete metric space. A set $\{X; f_n,\ n = 1, 2, ..., k\}$ is said to be a generalized cyclic iterated function system if each $f_n : X \to X$ is cyclic $\mathcal{F}$-contraction for $n \in \{1, 2, 3, ..., k\}$.

Definition 2.3. Let $\{B_i\}_{i=1}^r$ be a collection of nonempty closed subsets of a metric space $(X, \boldsymbol{d})$. A self-mapping $f : \cup_{i=1}^r B_i \to \cup_{i=1}^r B_i$ is known as cyclic $\mathcal{F}$-contraction on $\{B_i\}_{i=1}^r$ if there exists $\mathcal{F} \in \digamma$ and $\tau > 0$ such that

a) $f(B_i) \subseteq B_{i+1}$ for $i \in \mathbb{N}_r$, where $B_{r+1} = B_1$;
b) $\tau + \mathcal{F}(\boldsymbol{d}(fm_1, fn_1)) \leq \mathcal{F}(\boldsymbol{d}(m_1, n_1))$ for all $m_1 \in B_i$, $n_1 \in B_{i+1}$ for $i \in \mathbb{N}_r$ provided that $\boldsymbol{d}(fm_1, fn_1) > 0$.

If f satisfies condition (a), then f is called a cyclic function.

We now present an example of a cyclic $\mathcal{F}$-contraction mapping which is neither contraction nor $\mathcal{F}$-contraction.

Example 2.4. Let $X = [0, 2]$ be equipped with the usual metric $\boldsymbol{d}$ on X, $B_1 = [0, 1]$, and $B_2 = [0, 2]$.
Define $f : B_1 \cup B_2 \to B_1 \cup B_2$ by

$$f(m) = \begin{cases} \frac{m}{4} & \text{if } m \in [0, 1], \\ \frac{1}{4} & \text{if } m \in (1, \frac{3}{2}] \\ \frac{1}{6} & \text{if } m \in (\frac{3}{2}, 2]. \end{cases}$$

Note that, f is neither a contraction nor an $\mathcal{F}$-contraction. Indeed, if we take $m = \frac{3}{2}$, $n = \frac{26}{17}$, then

$$\boldsymbol{d}(f(\frac{3}{2}), f(\frac{26}{17})) = \frac{1}{12} \geq \frac{\lambda}{34} = \lambda \boldsymbol{d}(\frac{3}{2}, \frac{26}{17}) \text{ for any } \lambda \in [0,1).$$

Define $\mathcal{F} \in \digamma$ by $\mathcal{F}(\lambda) = \ln(\lambda) + \lambda$. We show that f is a cyclic $\mathcal{F}$-contraction on $X = B_1 \cup B_2$.
Clearly,

$$\begin{aligned} f(B_1) &= [0, \frac{1}{4}] \subseteq [0,2] = B_2, \text{ and} \\ f(B_2) &= [0, \frac{1}{2}] \subseteq [0,1] = B_1. \end{aligned}$$

Now, we consider the following cases:
<u>*Case 1*</u>:
Let $m \in B_1$, $n \in B_2$.
If $n \in [0,1]$, then

$$|f(m) - f(n)| = |\frac{m}{4} - \frac{n}{4}| = \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.
In case $n \in (1, \frac{3}{2}]$, then

$$|f(m) - f(n)| = |\frac{m}{4} - \frac{1}{4}| = \frac{1}{4}|m - 1| \leq \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.
When $n \in (\frac{3}{2}, 2]$, then

$$|f(m) - f(n)| = |\frac{m}{4} - \frac{1}{6}| = \frac{1}{4}|m - \frac{2}{3}| \leq \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.

<u>*Case 2*</u>:
For $m \in B_2$, $n \in B_1$.
Let $m \in [0,1]$. Then

$$|f(m) - f(n)| = |\frac{m}{4} - \frac{n}{4}| = \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.
If $m \in (1, \frac{3}{2}]$, then

$$|f(m) - f(n)| = |\frac{1}{4} - \frac{n}{4}| = \frac{1}{4}|1 - n| \leq \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.
In case $m \in (\frac{3}{2}, 2]$, then

$$|f(m) - f(n)| = |\frac{1}{6} - \frac{n}{6}| = \frac{1}{4}|\frac{2}{3} - n| \leq \frac{1}{4}|m - n| = e^{-\tau}\boldsymbol{d}(m,n),$$

where $\tau = \ln(4)$.

Here $e^{-\tau} = \dfrac{1}{4}$, that is, $\tau = \ln(4)$. Thus, f is a cyclic $\mathcal{F}$-contraction on $B_1 \cup B_2$.

Theorem 2.5. *Let $\{B_i\}_{i=1}^r$ be the collection of nonempty closed subsets of a metric space $(X, \boldsymbol{d})$ and $f : \cup_{i=1}^r B_i \to \cup_{i=1}^r B_i$ a continuous cyclic $\mathcal{F}$-contraction. Then the map defined by $f : \mathcal{C}(\cup_{i=1}^r B_i) \to \mathcal{C}(\cup_{i=1}^r B_i)$ is also a cyclic $\mathcal{F}$-contraction with respect to Hausdorff metric H with the same $\mathcal{F}$.*

Proof. We take $K \in B_i$ for some $i \in \mathbb{N}_r$. Applying the definition of cyclic map, we obtain $f(K) \subseteq B_{i+1}$. Also, the continuity of f implies that $f(K)$ is a compact set. Therefore, $f(K) \in \mathcal{C}(B_{i+1})$ which implies that $f(\mathcal{C}(B_i)) \subseteq \mathcal{C}(B_{i+1})$ for each $i \in \mathbb{N}_r$.
We take $M_1 \in \mathcal{C}(B_i)$ and $N_1 \in \mathcal{C}(B_{i+1})$ for some $i \in \mathbb{N}_r$. First we claim that

$$\tau + \mathcal{F}(\sup_{fm_1 \in f(M_1)} \boldsymbol{d}(fm_1, f(N_1))) \leq \mathcal{F}(\sup_{m_1 \in M_1} \boldsymbol{d}(m_1, N_1)).$$

As f is cyclic $\mathcal{F}$-contraction of f, we obtain

$$\tau + \mathcal{F}(\boldsymbol{d}(fm_1, fn_1)) \leq \mathcal{F}(\boldsymbol{d}(m_1, n_1)) \text{ for all } m_1 \in B_i,\ n_1 \in B_{i+1} \text{ for } i \in \mathbb{N}_r.$$

Thus

$$\begin{aligned}\tau + \mathcal{F}(\sup_{fm_1 \in f(M_1)} \boldsymbol{d}(fm_1, f(N_1))) &= \tau + \mathcal{F}(\sup_{fm_1 \in f(M_1)} \inf_{fn_1 \in f(N_1)} \boldsymbol{d}(fm_1, fn_1)) \\ &\leq \mathcal{F}(\sup_{m_1 \in M_1} \inf_{n_1 \in N_1} \boldsymbol{d}(m_1, n_1)) \\ &\leq \mathcal{F}(\sup_{m_1 \in M_1} \boldsymbol{d}(m_1, N_1)).\end{aligned}$$

Similarly, we have

$$\tau + \mathcal{F}(\sup_{fn_1 \in f(N_1)} \boldsymbol{d}(fn_1, f(M_1)) \leq \mathcal{F}(\sup_{n_1 \in N_1} (n_1, M_1)).$$

So

$$\begin{aligned}& \tau + \mathcal{F}(H(f(M_1), f(N_1))) \\ = \;& \tau + \mathcal{F}(\max\{\sup_{fm_1 \in f(M_1)} \boldsymbol{d}(fm_1, f(N_1)), \sup_{fn_1 \in f(N_1)} \boldsymbol{d}(fn_1, f(M_1))\}) \\ \leq \;& \mathcal{F}(\max\{\sup_{m_1 \in M_1} \boldsymbol{d}(m_1, N_1), \sup_{n_1 \in N_1} \boldsymbol{d}(n_1, M_1)\}) \\ = \;& \mathcal{F}(H(M_1, N_1)).\end{aligned}$$

Hence, f is cyclic $\mathcal{F}$-contraction on $\{B_i\}_{i=1}^r$. □

Theorem 2.6. *Let $\{B_i\}_{i=1}^r$ be the collection of nonempty closed subsets of a metric space $(X, \boldsymbol{d})$, and $\mathbf{N}$ a fixed natural number. If $f_n : \cup_{i=1}^r B_i \to \cup_{i=1}^r B_i$ for all $n \in \mathbb{N}_{\mathbf{N}}$ are cyclic $\mathcal{F}$-contractions, then the map $T : \mathcal{C}(\cup_{i=1}^r B_i) \to \mathcal{C}(\cup_{i=1}^r B_i)$ defined by $T(M) = \cup_{n=1}^N f_n(M)$ for $M \in \mathcal{C}(\cup_{i=1}^r B_i)$ is also a cyclic $\mathcal{F}$-contraction.*

Proof. Let $K \in \mathcal{C}(B_i)$ for some $i \in \mathbb{N}_r$. By Theorem 2.9, for each $n \in \mathbb{N}_{\mathbf{N}}$, f_n is a cyclic $\mathcal{F}$-contraction. Therefore $f_n(K) \in \mathcal{C}(B_{i+1})$ for all $n \in \mathbb{N}_{\mathbf{N}}$ which implies

that $T(K) = \cup_{n=1}^{\mathbf{N}} f_n(K) \in \mathcal{C}(B_{i+1})$, and consequently, $T(\mathcal{C}(B_i)) \subseteq \mathcal{C}(B_{i+1})$ for $i \in \mathbb{N}_r$.
Since f_n is cyclic $\mathcal{F}$-contraction for each $n \in \mathbb{N}_{\mathbf{N}}$, we have

$$\begin{aligned}\tau + \mathcal{F}(H(f_n(M_1), f_n(N_1))) &\leq \mathcal{F}(H(M_1, N_1)) \\ \text{for all } M_1 &\in \mathcal{C}(B_i),\ N_1 \in \mathcal{C}(B_{i+1}) \text{ for each } i \in \mathbb{N}_r.\end{aligned}$$

If $M_1 \in \mathcal{C}(B_i)$, and $N_1 \in \mathcal{C}(B_{i+1})$ for some $i \in \mathbb{N}_r$, then we have

$$\begin{aligned}H(T(M_1), T(N_1)) &= H(\cup_{n=1}^{N} f_n(M_1), \cup_{n=1}^{N} f_n(N_1)) \\ &\leq \max\{H(f_1(M_1), f_1(N_1)), ..., H(f_{\mathbf{N}}(M_1), f_{\mathbf{N}}(N_1))\}.\end{aligned}$$

By applying $\mathcal{F}$ on bothe sides of the above inequality, we obtain that

$$\begin{aligned}&\tau + \mathcal{F}(H(T(M_1), T(N_1))) \\ =\ &\tau + \mathcal{F}(H(\cup_{n=1}^{N} f_n(M_1), \cup_{n=1}^{N} f_n(N_1))) \\ \leq\ &\tau + \mathcal{F}(\max\{H(f_1(M_1), f_1(N_1)), ..., H(f_{\mathbf{N}}(M_1), f_{\mathbf{N}}(N_1))\}) \\ \leq\ &\mathcal{F}(H(M_1, N_1)).\end{aligned}$$

□

Definition 2.7. Let $\{B_i\}_{i=1}^{r}$ be a collection of nonempty closed subsets of X. A mapping $T : \cup_{i=1}^{r}\mathcal{C}(B_i) \to \cup_{i=1}^{r}\mathcal{C}(B_i)$ is known as

(1) a generalized cyclic $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \digamma$ and $\tau > 0$ such that for $M_1 \in \mathcal{C}(B_i)$, $N_1 \in \mathcal{C}(B_{i+1})$ with $H(T(M_1), T(N_1)) > 0$, we have

$$\tau + \mathcal{F}(H(T(M_1), T(N_1)) \leq \mathcal{F}(\mathcal{M}_T(M_1, N_1)),$$

where

$$\begin{aligned}\mathcal{M}_T(M_1, N_1) &= \max\{H(M_1, N_1), H(M_1, T(M_1)), H(N_1, T(N_1)), \\ &\frac{H(M_1, T(N_1)) + H(N_1, T(M_1))}{2}, H(T^2(M_1), T(M_1)), \\ &H(T^2(M_1), N_1), H(T^2(M_1), T(N_1))\}.\end{aligned}$$

(2) a generalized rational cyclic $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \digamma$ and $\tau > 0$ such that for $M_1 \in \mathcal{C}(B_i)$, $N_1 \in \mathcal{C}(B_{i+1})$ with $H(T(M_1), T(N_1)) > 0$, we have

$$\tau + \mathcal{F}(H(T(M_1), T(N_1)) \leq \mathcal{F}(\mathcal{N}_T(M_1, N_1)),$$

where

$$\begin{aligned}\mathcal{N}_T(M_1, N_1) &= \max\{\frac{H(N_1, T(N_1))\,[1 + H(M_1, T(M_1))]}{1 + H(M_1, N_1)}, \\ &\frac{H(N_1, T(M_1))\,[1 + H(M_1, T(M_1))]}{1 + H(M_1, N_1)}, \\ &\frac{H(M_1, T(M_1))\,[1 + H(M_1, T(M_1))]}{1 + H(M_1, N_1)}\}.\end{aligned}$$

An operator T defined above in (1) is called a generalized cyclic $\mathcal{F}$-Hutchinson operator, whearas in (2) T is called a generalized rational cyclic $\mathcal{F}$-Hutchinson operator.

Theorem 2.8 ([22]). *In a complete metric space $(X, \boldsymbol{d})$, suppose that $\{X : f_n, : n = 1, 2, ..., k\}$ is a generalized iterated function system. If $T : \mathcal{C}(X) \to \mathcal{C}(X)$ is defined by $T(M_1) = \cup_{n=1}^{N} f_n(M_1)$ for $M_1 \in \mathcal{C}(X)$ and there exists $\mathcal{F} \in \digamma$ and $\tau > 0$ such that for $M_1, N_1 \in \mathcal{C}(X)$ with $H(T(M_1), T(N_1)) > 0$, we have*

$$\tau + \mathcal{F}(\mathrm{H}(\mathrm{T}(\mathrm{M}_1),\mathrm{T}(\mathrm{N}_1)) \leq \mathcal{F}(\mathcal{M}_T(\mathrm{M}_1,\mathrm{N}_1)),$$

where

$$\begin{aligned}\mathcal{M}_T(\mathrm{M}_1,\mathrm{N}_1) = \quad & \max\{H(\mathrm{M}_1,\mathrm{N}_1), H(\mathrm{M}_1,\mathrm{T}(\mathrm{M}_1)), H(\mathrm{N}_1,\mathrm{T}(\mathrm{N}_1)), \\ & \frac{H(\mathrm{M}_1,\mathrm{T}(\mathrm{N}_1)) + H(\mathrm{N}_1,\mathrm{T}(\mathrm{M}_1))}{2}, H(\mathrm{T}^2(\mathrm{M}_1),\mathrm{T}(\mathrm{M}_1)), \\ & H(\mathrm{T}^2(\mathrm{M}_1),\mathrm{N}_1), H(\mathrm{T}^2(\mathrm{M}_1),\mathrm{T}(\mathrm{N}_1))\}.\end{aligned}$$

Then T has a unique fixed point $U \in \mathcal{C}(X)$, that is,

$$U = T(U) = \cup_{n=1}^{k} f_n(U).$$

Moreover, for any initial set $M_0 \in \mathcal{C}(X)$, the sequence of compact sets

$$\{\mathrm{M}_0,\mathrm{T}(\mathrm{M}_0),\mathrm{T}^2(\mathrm{M}_0),...\}$$

converges to the fixed point of T.

Definition 2.9. Let $\mathcal{A}$ be a nonempty compact set of $(X, \mathbf{d})$. Then $\mathcal{A}$ is called an attractor of the iterated function system if

(i) $T(\mathcal{A}) = \mathcal{A}$ and
(ii) there exists an open set $V_1 \subseteq X$ such that $\mathcal{A} \subseteq V_1$ and $\lim_{k\to\infty} T^k(\mathcal{B}) = \mathcal{A}$ for any compact set $\mathcal{B} \subseteq V_1$, where the limit is taken with respect to the Pompeiu-Hausdorff metric.

3. Main Results

In the following, we obtain the existence of a unique attractor of generalized cyclic $\mathcal{F}-$contraction operator.

Theorem 3.1. *In a complete metric space $(X, \boldsymbol{d})$, suppose that $\{B_i\}_{i=1}^{r}$ is the collection of nonempty closed subsets of X and $\{X; f_n, n = 1, 2, ..., k\}$ is a generalized cyclic iterated function system. If $\mathrm{T} : \mathcal{C}(\cup_{i=1}^{r} B_i) \to \mathcal{C}(\cup_{i=1}^{r} B_i)$ defined by*

$$T(\mathcal{L}) = \cup_{n=1}^{N} f_n(\mathcal{L}) \text{ for each } \mathcal{L} \in \mathcal{C}(\cup_{i=1}^{r} B_i)$$

is a generalized cyclic $\mathcal{F}$-Hutchinson operator, then T has unique attractor $U \in \mathcal{C}(B_i)$, that is,

$$U = T(U) = \cup_{i=1}^{k} f_n(U).$$

Moreover, for an initial set $\mathcal{L}_0 \in \mathcal{C}(\cup_{i=1}^{r} B_i)$, the sequence of compact sets

$$\{\mathcal{L}_0, T(\mathcal{L}_0), T^2(\mathcal{L}_0), ...\}$$

converges to an attractor of T.

Proof. Suppose that $\mathcal{L}_0$ is an arbitrary element in $\mathcal{C}(\cup_{i=1}^r B_i)$. Then, there exists some i_0 such that $\mathcal{L}_0 \in \mathcal{C}(B_{i_0})$. Also $T(\mathcal{C}(B_{i_0})) \subseteq \mathcal{C}(B_{i_{0+1}})$ implies that $T(\mathcal{L}_0) \in \mathcal{C}\left(B_{i_{0+1}}\right)$. Thus there exists $\mathcal{L}_1 \in \mathcal{C}(B_{i_{0+1}})$ such that $T(\mathcal{L}_0) = \mathcal{L}_1$. Also, $T(\mathcal{C}(B_{i_0+1})) \subseteq \mathcal{C}(B_{i_{0+2}})$ implies that $\mathcal{L}_2 = T(\mathcal{L}_1) \in \mathcal{C}\left(B_{i_{0+2}}\right)$. Continuing this way, we define a sequence of sets $\{\mathcal{L}_m\}$ by

$$\mathcal{L}_1 = T(\mathcal{L}_0),\ \mathcal{L}_2 = T(\mathcal{L}_1), ..., \mathcal{L}_{m+1} = T(\mathcal{L}_m)$$

for $m \in \mathbb{N} \cup \{0\}$.

Now we assume that $\mathcal{L}_m \neq \mathcal{L}_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. If not, then $\mathcal{L}_k = \mathcal{L}_{k+1}$ for some k implies that $\mathcal{L}_k = T(\mathcal{L}_k)$ which completes the Proof.

Thus, $\mathcal{L}_m \neq \mathcal{L}_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. By using (1) of Definition 2.7, for $\mathcal{L}_m \in \mathcal{C}(B_{i_{m+1}})$ and $\mathcal{L}_{m+1} = T(\mathcal{L}_m) \in \mathcal{C}(B_{i_m+2})$, we obtain that

$$\begin{aligned}\tau + \mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) &= \tau + \mathcal{F}(H(T(\mathcal{L}_m), T(\mathcal{L}_{m+1})))\\ &\leq \mathcal{F}(\mathcal{M}_T(\mathcal{L}_m,\mathcal{L}_{m+1})),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_T(\mathcal{L}_m,\mathcal{L}_{m+1}) &= \max\{H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_m, T(\mathcal{L}_m)), H(\mathcal{L}_{m+1}, T(\mathcal{L}_{m+1})),\\ &\quad \frac{H(\mathcal{L}_m, T(\mathcal{L}_{m+1})) + H(\mathcal{L}_{m+1}, T(\mathcal{L}_m))}{2},\\ &\quad H(T^2(\mathcal{L}_m), T(\mathcal{L}_m)), H\left(T^2(\mathcal{L}_m), \mathcal{L}_{m+1}\right),\\ &\quad H\left(T^2(\mathcal{L}_m), T(\mathcal{L}_{m+1})\right)\}\end{aligned}$$

$$\begin{aligned}&= \max\{H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_{m+1},\mathcal{L}_{m+2}),\\ &\quad \frac{H(\mathcal{L}_m,\mathcal{L}_{m+2}) + H(\mathcal{L}_{m+1},\mathcal{L}_{m+1})}{2}, H(\mathcal{L}_{m+2},\mathcal{L}_{m+1}),\\ &\quad H(\mathcal{L}_{m+2},\mathcal{L}_{m+1}), H(\mathcal{L}_{m+2},\mathcal{L}_{m+2})\}\\ &\leq \max\{H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_{m+1},\mathcal{L}_{m+2}),\\ &\quad \frac{H(\mathcal{L}_m,\mathcal{L}_{m+1}) + H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})}{2}\}\\ &= \max\{H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\tau + \mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) &\leq \mathcal{F}(\max\{H(\mathcal{L}_m,\mathcal{L}_{m+1}), H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})\})\\ &= \mathcal{F}(H(\mathcal{L}_m,\mathcal{L}_{m+1})),\end{aligned}$$

that is,

$$\mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) \leq \mathcal{F}(H(\mathcal{L}_m,\mathcal{L}_{m+1})) - \tau$$

for all $m \in \mathbb{N} \cup \{0\}$. Therefore, we have

$$\begin{aligned}\mathcal{F}(H(\mathcal{L}_n,\mathcal{L}_{n+1})) &\leq \mathcal{F}(H(\mathcal{L}_{n-1},\mathcal{L}_n)) - \tau\\ &\leq \mathcal{F}(H(\mathcal{L}_{n-2},\mathcal{L}_{n-1})) - 2\tau\\ &\leq ... \leq \mathcal{F}(H(\mathcal{L}_0,\mathcal{L}_1)) - n\tau\end{aligned}$$

and we get $\lim_{n\to\infty} \mathcal{F}\left(H(\mathcal{L}_n,\mathcal{L}_{n+1})\right) = -\infty$ which by $(\mathcal{F}_2)$ implies that

$$\lim_{n\to\infty} H(\mathcal{L}_n,\mathcal{L}_{n+1}) = 0.$$

Now by $(\mathcal{F}_3)$, there exists $h \in (0,1)$ such that

$$\lim_{n\to\infty} [H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h \mathcal{F}\left(H(\mathcal{L}_n,\mathcal{L}_{n+1})\right) = 0.$$

Thus we have

$$\begin{array}{rl} & [H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h \mathcal{F}\left(H(\mathcal{L}_n,\mathcal{L}_{n+1})\right) - [H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h \mathcal{F}\left(H(\mathcal{L}_0,\mathcal{L}_{n+1})\right) \\ \leq & -n\tau[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h \leq 0. \end{array}$$

On taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} n[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h = 0.$$

As $\lim_{n\to\infty} n^{\frac{1}{h}} H(\mathcal{L}_n,\mathcal{L}_{n+1}) = 0$, there exists $n_1 \in \mathbb{N} \cup \{0\}$ such that

$$n^{\frac{1}{h}} H(\mathcal{L}_n,\mathcal{L}_{n+1}) \leq 1$$

for all $n \geq n_1$. So we have

$$H(\mathcal{L}_n,\mathcal{L}_{n+1}) \leq \frac{1}{n^{1/h}}$$

for all $n \geq n_1$. For $m, n \in \mathbb{N} \cup \{0\}$ with $m > n \geq n_1$, we have

$$\begin{array}{rcl} H(\mathcal{L}_n,\mathcal{L}_m) & \leq & H(\mathcal{L}_n,\mathcal{L}_{n+1}) + H(\mathcal{L}_{n+1},\mathcal{L}_{n+2}) + \ldots + H(\mathcal{L}_{m-1},\mathcal{L}_m) \\ & \leq & \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{array}$$

As, $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}} < \infty$, we get $H(\mathcal{L}_n,\mathcal{L}_m) \to 0$ as $n, m \to \infty$. Therefore $\{\mathcal{L}_n\}$ is a Cauchy sequence in $\mathcal{C}(\cup_{i=1}^r B_i)$. As $(\mathcal{C}(\cup_{i=1}^r B_i), H)$ is complete, $\mathcal{L}_n \to U$ as $n \to \infty$ for some $U \in \mathcal{C}(\cup_{i=1}^r B_i)$.
Note that the iterative sequence $\{\mathcal{L}_n\}$ has infinite number of terms in $\mathcal{C}(B_i)$ for each $i = 1, 2, ..., r$. Hence, in each $\mathcal{C}(B_i)$ for $i = 1, 2, ..., r$, we can construct a subsequence of $\{\mathcal{L}_n\}$ that converges to U. Since each element in $\mathcal{C}(B_i)$ for $i = 1, 2, ..., r$, is closed, we conclude $U \in \cap_{i=1}^m \mathcal{C}(B_i) \neq \varnothing$.

Let $V = \cap_{i=1}^r \mathcal{C}(B_i)$ and denote by $\mathcal{C}(V)$, the collection of all nonempty compact sets of V. Then $T|_{\mathcal{C}(V)} : \mathcal{C}(V) \to \mathcal{C}(V)$ is a self mapping with domain of compact sets. It follows from Theorem 2.8 that $T|_{\mathcal{C}(V)}$ has a unique attractor in $\mathcal{C}(V)$. □

Remark 3.2. In Theorem 3.1, if $\mathcal{S}(\cup_{i=1}^r X_i)$ is the union of the collection of all singleton subsets of X, where $X = \cup_{i=1}^r X_i$. Then, clearly $\mathcal{S}(\cup_{i=1}^r X_i) \subseteq \mathcal{C}(\cup_{i=1}^r X_i)$. Moreover, if we take $f_n = f$ for each n where $f = f_1$, then the mapping T becomes

$$T(m) = f(m).$$

By using the above Remark, we obtain the following results for the existence of fixed point.

Corollary 3.3. *In a complete metric space* $(X, \boldsymbol{d})$*, suppose that* $\{X; f_n, n = 1, 2, ..., k\}$ *is a generalized cyclic iterated function system. If* $f : X \to X$ *is defined as in the Remark 3.2 and there exists some* $\mathcal{F} \in \digamma$ *and* $\tau > 0$ *such that for* $\mathrm{m}_1 \in \mathcal{C}(X_i)$ *and* $\mathrm{n}_1 \in \mathcal{C}(X_{i+1})$ *with* $\boldsymbol{d}(f(m_1), f(n_1)) > 0$*, the following condition holds*

$$\tau + \mathcal{F}(\boldsymbol{d}(fm_1, fn_1)) \leq \mathcal{F}(\mathcal{M}_f(m_1, n_1)),$$

where

$$\begin{aligned}\mathcal{M}_f(m_1, n_1) &= \max\{\boldsymbol{d}(m_1, n_1), \boldsymbol{d}(m_1, fm_1), \boldsymbol{d}(n_1, fn_1), \\ &\quad \frac{\boldsymbol{d}(m_1, fn_1) + \boldsymbol{d}(n_1, fm_1)}{2}, \boldsymbol{d}(f^2m_1, n_1), \\ &\quad \boldsymbol{d}(f^2m_1, fm_1), \boldsymbol{d}(f^2m_1, fn_1)\}.\end{aligned}$$

Then f *has a unique fixed point* $\mathrm{x} \in X$*. Moreover, for any initial point* $\mathrm{x}_0 \in X$*, the sequence* $\{x_0, fx_0, f^2x_0, ...\}$ *converges to the fixed point of* f*.*

Corollary 3.4. *In a complete metric space* $(X, \boldsymbol{d})$*, suppose that* $\{B_i\}_{i=1}^r$ *is the nonempty collection of closed subsets of* X*. Suppose that* $(X; f_n, n = 1, 2, ..., k)$ *is a cyclic iterated function system, where each* f_i *for* $i = 1, 2, ..., k$ *is a cyclic contraction. Then the map* $T : \mathcal{C}(\cup_{i=1}^r B_i) \to \mathcal{C}(\cup_{i=1}^r B_i)$ *defined in Theorem 3.1 has a unique attractor. Moreover, for any set* $\mathcal{L}_0 \in \mathcal{C}(B_i)$*, the sequence of compact sets* $\{\mathcal{L}_0, T(\mathcal{L}_0), T^2(\mathcal{L}_0), ...\}$ *converges to an attractor of* T*.*

Proof. It follows from Theorem 2.5 that each f_i for $i = 1, 2, ..., k$ is a cyclic contraction on X. Moreover, the mapping $T : \mathcal{C}(\cup_{i=1}^r B_i) \to \mathcal{C}(\cup_{i=1}^r B_i)$ defined by

$$T(\mathcal{L}) = \cup_{n=1}^k f_n(\mathcal{L}), \text{ for all } \mathcal{L} \in \mathcal{C}(B_i)$$

is also cyclic contraction on $\mathcal{C}(B_i)$ with respect to Hausdorff metric H. The result then follows from Theorem 3.1. □

Theorem 3.5. *In a complete metric space* $(X, \boldsymbol{d})$*, suppose* $\{B_i\}_{i=1}^r$ *is the collection of nonempty closed subsets* X *and* $\{X; f_n, n = 1, 2, ..., k\}$ *is a generalized cyclic iterated function system. If* $\mathrm{T} : \mathcal{C}(\cup_{i=1}^r B_i) \to \mathcal{C}(B_i)$ *defined by*

$$T(\mathcal{L}) = \cup_{n=1}^N f_n(\mathcal{L}) \text{ for each } \mathcal{L} \in \mathcal{C}(\cup_{i=1}^r B_i)$$

is a generalized rational cyclic $\mathcal{F}$*-Hutchinson operator, then* T *has a unique attractor* $U \in \mathcal{C}(\cup_{i=1}^r B_i)$*, that is,*

$$U = T(U) = \cup_{i=1}^k f_n(U).$$

Moreover, for an initial set $\mathcal{L}_0 \in \mathcal{C}(\cup_{i=1}^r B_i)$*, the sequence of compact sets*

$$\{\mathcal{L}_0, T(\mathcal{L}_0), T^2(\mathcal{L}_0), ...\}$$

converges to an attractor of T*.*

Proof. Let $\mathcal{L}_0$ be an arbitrary element in $\mathcal{C}(\cup_{i=1}^r B_i)$. Then, there exists some i_0 such that $\mathcal{L}_0 \in \mathcal{C}(B_{i_0})$. Also $T(\mathcal{C}(B_{i_0})) \subseteq \mathcal{C}(B_{i_{0+1}})$ implies that $T(\mathcal{L}_0) \in B_{i_{0+1}}$. Thus there exists $\mathcal{L}_1 \in \mathcal{C}(B_{i_{0+1}})$ such that $T(\mathcal{L}_0) = \mathcal{L}_1$. Also, $T(\mathcal{C}(B_{i_0+1})) \subseteq$

$\mathcal{C}(B_{i_{0+2}})$ implies that $\mathcal{L}_2 = T(\mathcal{L}_1) \in B_{i_{0+2}}$. Continuing this way, we define a sequence of sets $\{\mathcal{L}_m\}$ by

$$\mathcal{L}_1 = T(\mathcal{L}_0),\ \mathcal{L}_2 = T(\mathcal{L}_1), ..., \mathcal{L}_{m+1} = T(\mathcal{L}_m)$$

for $m \in \mathbb{N} \cup \{0\}$.

Now assume that $\mathcal{L}_m \neq \mathcal{L}_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. If not, then $\mathcal{L}_k = \mathcal{L}_{k+1}$ for some k, which implies that $\mathcal{L}_k = T(\mathcal{L}_k)$ and hence the proof. Thus, $\mathcal{L}_m \neq \mathcal{L}_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. Using (2) of Definition 2.7, for $\mathcal{L}_m \in \mathcal{C}(B_{i_{m+1}})$ and $\mathcal{L}_{m+1} = T(\mathcal{L}_m) \in \mathcal{C}(B_{i_m+2})$, we obtain that

$$\begin{aligned} \tau + \mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) &= \tau + \mathcal{F}(H(T(\mathcal{L}_m), T(\mathcal{L}_{m+1}))) \\ &\leq \mathcal{F}(\mathcal{N}_T(\mathcal{L}_m,\mathcal{L}_{m+1})), \end{aligned}$$

where

$$\begin{aligned} \mathcal{N}_T(\mathcal{L}_m,\mathcal{L}_{m+1}) &= \max\{\frac{H(\mathcal{L}_{m+1}, T(\mathcal{L}_{m+1}))\,[1 + H(\mathcal{L}_m, T(\mathcal{L}_m))]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}, \\ &\quad \frac{H(\mathcal{L}_{m+1}, T(\mathcal{L}_m))\,[1 + H(\mathcal{L}_m, T(\mathcal{L}_m))]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}, \\ &\quad \frac{H(\mathcal{L}_m, T(\mathcal{L}_m))\,[1 + H(\mathcal{L}_m, T(\mathcal{L}_m))]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}\} \\ &= \max\{\frac{H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})\,[1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}, \\ &\quad \frac{H(\mathcal{L}_{m+1},\mathcal{L}_{m+1})\,[1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}, \\ &\quad \frac{H(\mathcal{L}_m,\mathcal{L}_{m+1})\,[1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})]}{1 + H(\mathcal{L}_m,\mathcal{L}_{m+1})}\} \\ &= \max\{H(\mathcal{L}_{m+1},\mathcal{L}_{m+2}), H(\mathcal{L}_m,\mathcal{L}_{m+1})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \tau + \mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) &\leq \mathcal{F}(\max\{H(\mathcal{L}_{m+1},\mathcal{L}_{m+2}), H(\mathcal{L}_m,\mathcal{L}_{m+1})\}) \\ &= \mathcal{F}(H(\mathcal{L}_m,\mathcal{L}_{m+1})), \end{aligned}$$

that is,

$$\mathcal{F}(H(\mathcal{L}_{m+1},\mathcal{L}_{m+2})) \leq \mathcal{F}(H(\mathcal{L}_m,\mathcal{L}_{m+1})) - \tau$$

for all $m \in \mathbb{N} \cup \{0\}$. Therefore

$$\begin{aligned} \mathcal{F}(H(\mathcal{L}_n,\mathcal{L}_{n+1})) &\leq \mathcal{F}(H(\mathcal{L}_{n-1},\mathcal{L}_n)) - \tau \\ &\leq \mathcal{F}(H(\mathcal{L}_{n-2},\mathcal{L}_{n-1})) - 2\tau \\ &\leq ... \leq \mathcal{F}(H(\mathcal{L}_0,\mathcal{L}_1)) - n\tau \end{aligned}$$

gives that $\lim\limits_{n\to\infty} \mathcal{F}(H(\mathcal{L}_n,\mathcal{L}_{n+1})) = -\infty$ which by $(\mathcal{F}_2)$ becomes

$$\lim_{n\to\infty} H(\mathcal{L}_n,\mathcal{L}_{n+1}) = 0.$$

From $(\mathcal{F}_3)$, there exists $h \in (0,1)$ such that

$$\lim_{n\to\infty} [H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h \mathcal{F}(H(\mathcal{L}_n,\mathcal{L}_{n+1})) = 0.$$

Thus we have

$$\begin{aligned}&[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h\mathcal{F}\left(H(\mathcal{L}_n,\mathcal{L}_{n+1})\right)-[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h\mathcal{F}\left(H(\mathcal{L}_0,\mathcal{L}_{n+1})\right)\\ \leq\;& -n\tau[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h\leq 0.\end{aligned}$$

On taking the limit as $n\to\infty$, we obtain that

$$\lim_{n\to\infty} n[H(\mathcal{L}_n,\mathcal{L}_{n+1})]^h=0.$$

As $\lim_{n\to\infty} n^{\frac{1}{h}}H(\mathcal{L}_n,\mathcal{L}_{n+1})=0$, there exists $n_1\in\mathbb{N}\cup\{0\}$ such that

$$n^{\frac{1}{h}}H(\mathcal{L}_n,\mathcal{L}_{n+1})\leq 1$$

for all $n\geq n_1$. So,

$$H(\mathcal{L}_n,\mathcal{L}_{n+1})\leq\frac{1}{n^{1/h}}$$

for all $n\geq n_1$. For $m,n\in\mathbb{N}\cup\{0\}$ with $m>n\geq n_1$, we have

$$\begin{aligned}H(\mathcal{L}_n,\mathcal{L}_m)&\leq H(\mathcal{L}_n,\mathcal{L}_{n+1})+H(\mathcal{L}_{n+1},\mathcal{L}_{n+2})+...+H(\mathcal{L}_{m-1},\mathcal{L}_m)\\&\leq\sum_{i=n}^{\infty}\frac{1}{i^{1/h}}.\end{aligned}$$

As $\sum_{i=1}^{\infty}\frac{1}{i^{1/h}}$ is convergent, $H(\mathcal{L}_n,\mathcal{L}_m)\to 0$ as $n,m\to\infty$. Therefore $\{\mathcal{L}_n\}$ is a Cauchy sequence in X. As $(\mathcal{C}(\cup_{i=1}^r B_i),H)$ is complete, $\mathcal{L}_n\to U^*$ as $n\to\infty$ for some $U^*\in\mathcal{C}(\cup_{i=1}^r B_i)$.
Note that a sequence $\{\mathcal{L}_n\}$ has infinite number of terms in $\mathcal{C}\left(\cup_{i=1}^r B_i\right)$, for each $i=1,2,...,r$. Hence, in each $\mathcal{C}\left(B_i\right)$ for $i=1,2,...,r$, we can construct a subsequence of $\{\mathcal{L}_n\}$ that converges to U^*. As each B_i, $i=1,2,...,r$, is closed, we conclude that $U^*\in\cap_{i=1}^r\mathcal{C}(B_i)\neq\varnothing$.

Let $V=\cap_{i=1}^r\mathcal{C}\left(B_i\right)$ and $\mathcal{C}(V)$ denotes the collection of all nonempty compact sets of V. Then $T|_{\mathcal{C}(V)}:\mathcal{C}(V)\to\mathcal{C}(V)$ is a self mapping on the family of compact sets of V.

Now by the Definition 2.7 (2) and following the similar arguments to those given in the proof of the Theorem 2.8, we obtain that the map $T|_{\mathcal{C}(V)}$ has a unique attractor in $\mathcal{C}(V)$. □

Corollary 3.6. *In a complete metric space $(X,\boldsymbol{d})$, suppose that $\{X;f_n,n=1,2,...,k\}$ is a generalized cyclic iterated function system. Let $f:X\to X$ be defined as in Remark 3.2. If there exists some $\mathcal{F}\in\digamma$ and $\tau>0$ such that for $\mathrm{m}_1\in\mathcal{C}\left(X_i\right)$ and $\mathrm{n}_1\in\mathcal{C}\left(X_{i+1}\right)$ with $\boldsymbol{d}(f(m_1),f(n_1))>0$, the following condition holds*

$$\tau+\mathcal{F}\left(\boldsymbol{d}\left(fm_1,fn_1\right)\right)\leq\mathcal{F}(\mathcal{N}_f(m_1,n_1)),$$

where

$$\begin{aligned}\mathcal{N}_f(m_1,n_1) &= \max\{\frac{\boldsymbol{d}(n_1,f\,(n_1))\,[1+\boldsymbol{d}(m_1,f(m_1))]}{1+\boldsymbol{d}(m_1,n_1)},\\ &\frac{\boldsymbol{d}(n_1,f\,(m_1))\,[1+\boldsymbol{d}(m_1,f(m_1))]}{1+\boldsymbol{d}(m_1,n_1)},\\ &\frac{\boldsymbol{d}(m_1,f\,(m_1))\,[1+\boldsymbol{d}(m_1,f(m_1))]}{1+\boldsymbol{d}(m_1,n_1)}\}.\end{aligned}$$

Then f has a unique fixed point $\mathrm{x} \in X$. Moreover, for any initial point $\mathrm{x}_0 \in X$, the sequence $\{x_0, fx_0, f^2x_0, ...\}$ converges to the fixed point of f.

Example 3.7. Let $X = [0,2] \times [0,2]$ be equipped with a Euclidian metric $\boldsymbol{d}$ on X, $A_1 = [0,1] \times [0,1]$ and $A_2 = [0,2] \times [0,2]$. Define $f_1, f_2 : X \to X$ by

$$\begin{aligned} f_1(x,y) &= \begin{cases} \left(\frac{x}{4},\frac{y}{4}\right) & x,y\in[0,1], \\ \left(\frac{1}{6},\frac{1}{6}\right) & x,y\in(1,2], \end{cases} \quad \text{and} \\ f_2(x,y) &= \begin{cases} \left(\frac{x}{3},\frac{y}{3}\right) & x,y\in[0,1], \\ \left(\frac{1}{5},\frac{1}{5}\right) & x,y\in(1,2]. \end{cases}\end{aligned}$$

Clearly, $f_i\,(A_1) \subseteq A_2$ and $f_i\,(A_2) \subseteq A_1$ for each $i = 1, 2$.
Now, for $\mathbf{x} = (x_1, y_1) \in [0,1]\times[0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) \in [0,2]\times[0,2] = A_2$, we have the following cases:
Case I: If $\mathbf{x} = (x_1, y_1) \in [0,1]\times[0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) \in [0,1]\times[0,1] \subseteq A_2$, then

$$\begin{aligned}\boldsymbol{d}(f_1\,(\mathbf{x})\,, f_1\,(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{4},\frac{y_1}{4}),(\frac{x_2}{4},\frac{y_2}{4})\right)\\ &= \sqrt{\left(\frac{x_1}{4}-\frac{x_2}{4}\right)^2+\left(\frac{y_1}{4}-\frac{y_2}{4}\right)^2}\\ &= \sqrt{\frac{1}{16}\,(x_1-x_2)^2+\frac{1}{16}\,(y_1-y_2)^2}\\ &< \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ &= \boldsymbol{d}(\mathbf{x},\mathbf{y}).\end{aligned}$$

Case II: In case $\mathbf{x} = (x_1, y_1) \in [0,1] \times [0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) \in (1,2] \times (1,2] \subseteq A_2$, we have

$$\begin{aligned}
\boldsymbol{d}(f_1(\mathbf{x}), f_1(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{4}, \frac{y_1}{4}), (\frac{1}{6}, \frac{1}{6})\right) \\
&= \sqrt{\left(\frac{x_1}{4} - \frac{1}{6}\right)^2 + \left(\frac{y_1}{4} - \frac{1}{6}\right)^2} \\
&= \sqrt{\frac{1}{16}\left(x_1 - \frac{2}{3}\right)^2 + \frac{1}{16}\left(y_1 - \frac{2}{3}\right)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= d(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Also, for $\mathbf{x} = (x_1, y_1) \in [0,2] \times [0,2] = A_2$ and $\mathbf{y} = (x_2, y_2) \in [0,1] \times [0,1] = A_1$, we consider the following two cases:
Case I: If $\mathbf{x} = (x_1, y_1) \in [0,1] \times [0,1] \subseteq A_2$ and $\mathbf{y} = (x_2, y_2) = [0,1] \times [0,1] = A_1$, then

$$\begin{aligned}
\boldsymbol{d}(f_1(\mathbf{x}), f_1(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{4}, \frac{y_1}{4}), (\frac{x_2}{4}, \frac{y_2}{4})\right) \\
&= \sqrt{\left(\frac{x_1}{4} - \frac{x_2}{4}\right)^2 + \left(\frac{y_1}{4} - \frac{y_2}{4}\right)^2} \\
&= \sqrt{\frac{1}{16}(x_1 - x_2)^2 + \frac{1}{16}(y_1 - y_2)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Case II: Finally for $\mathbf{x} = (x_1, y_1) \in (1, 2 \times (1,2] \subseteq A_2$ and $\mathbf{y} = (x_2, y_2) \in [0,1] \times [0,1] = A_1$, we have

$$\begin{aligned}
\boldsymbol{d}(f_1(\mathbf{x}), f_1(\mathbf{y})) &= \boldsymbol{d}\left((\frac{1}{6}, \frac{1}{6}), (\frac{x_2}{4}, \frac{y_2}{4})\right) \\
&= \sqrt{\left(\frac{1}{6} - \frac{x_2}{4}\right)^2 + \left(\frac{1}{6} - \frac{y_2}{4}\right)^2} \\
&= \sqrt{\frac{1}{16}\left(\frac{2}{3} - x_2\right)^2 + \frac{1}{16}\left(\frac{2}{3} - y_2\right)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Now, again for $\mathbf{x} = (x_1, y_1) \in [0,1] \times [0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) = [0,2] \times [0,2] = A_2$, we consider the following cases:

Case I: If $\mathbf{x} = (x_1, y_1) \in [0,1]\times[0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) \in [0,1]\times[0,1] \subseteq A_2$, then

$$\begin{aligned}
\boldsymbol{d}(f_2(\mathbf{x}), f_2(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{3}, \frac{y_1}{3}), (\frac{x_2}{3}, \frac{y_2}{3})\right) \\
&= \sqrt{\left(\frac{x_1}{3} - \frac{x_2}{3}\right)^2 + \left(\frac{y_1}{3} - \frac{y_2}{3}\right)^2} \\
&= \sqrt{\frac{1}{9}(x_1 - x_2)^2 + \frac{1}{9}(y_1 - y_2)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Case II: For $\mathbf{x} = (x_1, y_1) \in [0,1]\times[0,1] = A_1$ and $\mathbf{y} = (x_2, y_2) = (1,2]\times(1,2] \subseteq A_2$, we have

$$\begin{aligned}
\boldsymbol{d}(f_2(\mathbf{x}), f_2(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{3}, \frac{y_1}{3}), (\frac{1}{5}, \frac{1}{5})\right) \\
&= \sqrt{\left(\frac{x_1}{3} - \frac{1}{5}\right)^2 + \left(\frac{y_1}{3} - \frac{1}{5}\right)^2} \\
&= \sqrt{\frac{1}{9}\left(x_1 - \frac{3}{5}\right)^2 + \frac{1}{9}(y_1 - \frac{3}{5}^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Also, for $\mathbf{x} = (x_1, y_1) \in [0,2]\times[0,2] = A_2$ and $\mathbf{y} = (x_2, y_2) \in [0,1]\times[0,1] = A_1$, we consider the following two cases:
Case I: In case $\mathbf{x} = (x_1, y_1) \in [0,1]\times[0,1] \subseteq A_2$ and $\mathbf{y} = (x_2, y_2) = [0,1]\times[0,1] = A_1$, we obtain that

$$\begin{aligned}
\boldsymbol{d}(f_2(\mathbf{x}), f_2(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{3}, \frac{y_1}{3}), (\frac{x_2}{3}, \frac{y_2}{3})\right) \\
&= \sqrt{\left(\frac{x_1}{3} - \frac{x_2}{3}\right)^2 + \left(\frac{y_1}{3} - \frac{y_2}{3}\right)^2} \\
&= \sqrt{\frac{1}{9}(x_1 - x_2)^2 + \frac{1}{9}(y_1 - y_2)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Case II: Finally, if $\mathbf{x} = (x_1, y_1) \in (1,2] \times (1,2] \subseteq A_2$ and $\mathbf{y} = (x_2, y_2) \in [0,1] \times [0,1] = A_1$, then

$$\begin{aligned}
\boldsymbol{d}(f_2(\mathbf{x}), f_2(\mathbf{y})) &= \boldsymbol{d}\left((\frac{x_1}{3}, \frac{y_1}{3}), (\frac{1}{5}, \frac{1}{5}) \right) \\
&= \sqrt{\left(\frac{1}{5} - \frac{x_2}{3}\right)^2 + \left(\frac{1}{5} - \frac{y_2}{3}\right)^2} \\
&= \sqrt{\frac{1}{9}\left(\frac{3}{5} - x_2\right)^2 + \frac{1}{9}\left(\frac{3}{5} - y_2\right)^2} \\
&< \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Now there exists $\tau > 0$ such that

$$\begin{aligned}
\boldsymbol{d}(f_1(\mathbf{x}), f_1(\mathbf{y}))(1 + \tau\sqrt{\boldsymbol{d}(\mathbf{x}, \mathbf{y})})^2 &\leq \boldsymbol{d}(\mathbf{x}, \mathbf{y}) \text{ and} \\
\boldsymbol{d}(f_2(\mathbf{x}), f_2(\mathbf{y}))(1 + \tau\sqrt{\boldsymbol{d}(\mathbf{x}, \mathbf{y})})^2 &\leq \boldsymbol{d}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

If we consider the cyclic iterated function system $\{X; f_1, f_2\}$ with mapping $T : \mathcal{C}(\cup_{i=1}^r [0,2]^2) \to \mathcal{C}(\cup_{i=1}^r [0,2]^2)$ given by

$$T(A) = f_1(A) \cup f_2(A) \text{ for all } A \in \mathcal{C}(\cup_{i=1}^r [0,2]^2).$$

Then, for $A_1 \in \mathcal{C}_i([0,2]^2)$, $A_2 \in \mathcal{C}_{i+1}([0,2]^2)$ with $H(T(A_1), T(A_2)) > 0$, we have

$$H(T(A_1), T(A_2))(1 + \tau\sqrt{H(A_1, A_2)})^2 \leq H(A_1, A_2).$$

Thus, T is a generalized cyclic $\mathcal{F}$-Hutchinson operator. And, for any set $W_0 \in \mathcal{C}(\cup_{i=1}^r [0,2]^2)$, the sequence of sets $\{W_0, T(W_0), T^2(W_0), ...\}$ converges to the unique attractor of an operator T.

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