Applied General Topology

Appl. Gen. Topol. 25, no. 1 (2024), 183-198
doi:10.4995/agt.2024.19760

A study of new dimensions for ideal topological spaces

F. Sereti

University of Western Macedonia, Department of Mathematics, 52100 Kastoria, Greece (seretifot@gmail.com)

Communicated by S. Romaguera

Abstract

In this paper new notions of dimensions for ideal topological spaces are inserted, called ∗-quasi covering dimension and ideal quasi covering dimension. We study several of their properties and investigate their relations with types of covering dimensions like the ∗-covering dimension and the ideal covering dimension.

2020 MSC: *54F45; 54A05; 54A10.*

Keywords: *topological space; quasi covering dimension; ideal.*

1. Introduction

Various meanings of dimensions for topological spaces like the covering dimension, the quasi covering dimension, the small inductive dimension and the large inductive dimension have been studied (see for example [1,2,5–10,15–17]).

Recently, meanings of dimensions are studied for the so-called ideal topological spaces (see [14, 18]), developing a new branch of topological dimension theory. Especially, the ideal types of the covering dimension, small inductive dimension and large inductive dimension have been investigated.

In this paper, new notions of dimensions for ideal topological spaces are introduced and studied. They are based on the notion of the quasi cover of a topological space. Especially, in Section 2 the basic notation and terminology that is useful for the following sections are introduced. In Section 3, we insert

Received 23 May 2023 – Accepted 1 August 2023

and study the meaning of $*$-quasi covering dimension and in Section 4, we insert and study the meaning of ideal quasi covering dimension.

2. Preliminaries

In this section we remind the main notions and notations that will be used in the rest of this study. We state that the standard notation of Dimension Theory is referred to [1,4,5,17].

A non empty family $\mathcal{I}$ of subsets of a set X is called an *ideal* on X if it satisfies the following properties:
(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

A topological space (X, τ) with an ideal $\mathcal{I}$ is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$ [13]. In [12] the authors defined a new topology τ^* on X in terms of the Kuratowski closure operator cl^*. The family

$$\beta^* = \{U \setminus I : U \in \tau,\ I \in \mathcal{I}\}$$

is a basis for τ^* and the topology τ^* is finer than τ. Especially, if $\mathcal{I} = \{\varnothing\}$, then $\tau^* = \tau$ and if $\mathcal{I} = P(X)$ (that is, the power set of X), then τ^* is the discrete topology.

In what follows, by an *open set* (resp. *closed set*), we mean an open set (resp. closed set) in the topology τ. If a set U is open in the topology τ^*, then we say that U is a $*$-*open set*. Similarly, we define $*$-*closed sets*. In addition, by a *dense set* we mean a dense set in the topology τ and if a set is dense in the topology τ^*, then we refer to this set as $*$-*dense*.

The *order* of a family r of subsets of a topological space X is defined as follows:
(1) $\mathrm{ord}(r) = -1$, if r consists of the empty set only.
(2) $\mathrm{ord}(r) = n$, where $n \in \{0, 1, \ldots\}$, if the intersection of any $n + 2$ distinct elements of r is empty and there exist $n + 1$ distinct elements of r whose intersection is not empty.
(3) $\mathrm{ord}(r) = \infty$, if for every $n \in \{1, 2, \ldots\}$ there exist n distinct elements of r whose intersection is not empty.

Let (X, τ) be a topological space. By τ-*cover* we mean a family of open sets whose union is X. By τ-*quasi cover* we mean a family of open sets whose union is a dense subset of X. Moreover, we define a τ^c-*cover* as a family consisting of closed sets whose union is X and a τ^c-*quasi cover* as a family consisting of closed sets whose union is a dense subset of X.

Similarly, if $(X, \tau, \mathcal{I})$ is an ideal topological space, then by τ^*-*cover* we mean a family of $*$-open sets whose union is X. By τ^*-*quasi cover* we mean a family of $*$-open sets whose union is a $*$-dense subset of X. A $(\tau^*)^c$-*cover* is a family of $*$-closed sets whose union is X and a $(\tau^*)^c$-*quasi cover* is a family of $*$-closed sets whose union is a $*$-dense subset of X.

Two families c_1 and c_2 of a set X are said to be *similar*, writing $c_1 \sim c_2$, if their unions are the same subset of X. Also, a family r of subsets of a set X

is said to be a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

Definition 2.1. The *covering dimension* of a topological space (X, τ), denoted by $\dim X$, is defined as follows:
(i) $\dim X = -1$, if $X = \varnothing$.
(ii) $\dim X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ-cover c of X there exists a finite τ-cover r of X, which is a refinement of c and $\operatorname{ord}(r) \leqslant n$.
(iii) $\dim X = n$, where $n \in \{0, 1, \ldots\}$, if $\dim X \leqslant n$ and $\dim X \nleqslant n - 1$.
(iv) $\dim X = \infty$, if it is false that $\dim X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

Definition 2.2. The *quasi covering dimension* of a topological space (X, τ), denoted by $\dim_q X$, is defined as follows:
(i) $\dim_q X = -1$, if $X = \varnothing$.
(ii) $\dim_q X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ-quasi cover c of X there exists a finite τ-quasi cover r of X, which is similar to c, refinement of c and $\operatorname{ord}(r) \leqslant n$.
(iii) $\dim_q X = n$, where $n \in \{0, 1, \ldots\}$, if $\dim_q X \leqslant n$ and $\dim_q X \nleqslant n - 1$.
(iv) $\dim_q X = \infty$, if it is false that $\dim_q X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

In the paper [10] it was proved that for any topological space X,

$$\dim_q X = \sup\{\dim D : D \text{ is an open and dense subset of } X\}$$

(see [10, Proposition 2.1]), verifying that the quasi covering dimension is a topological dimension greater than or equal to the covering dimension. Also, it was proved that in the classes $\mathbf{T}(\mathbf{t.n.})$ of totally normal, $\mathbf{T}(\mathbf{p.n.})$ of perfectly normal and $\mathbf{T}(\mathbf{met.})$ of metrizable spaces the dimension $\dim_q$ coincides with the dimension dim (see [10, Remark 2.1 (4) and (5)]).

3. The $*$-quasi covering dimension

In this section, based on the notion of τ^*-quasi cover, we insert and study the $*$-quasi covering dimension for an ideal topological space $(X, \tau, \mathcal{I})$.

Definition 3.1. The $*$-*quasi covering dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\dim_q^* X$, is defined as follows:
(i) $\dim_q^* X = -1$, if $X = \varnothing$.
(ii) $\dim_q^* X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ^*-quasi cover c of X there exists a finite τ^*-quasi cover r of X, which is similar to c, refinement of c and $\operatorname{ord}(r) \leqslant n$.
(iii) $\dim_q^* X = n$, where $n \in \{0, 1, \ldots\}$, if $\dim_q^* X \leqslant n$ and $\dim_q^* X \nleqslant n - 1$.
(iv) $\dim_q^* X = \infty$, if it is false that $\dim_q^* X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

We observe that the dimension $\dim_q^* X$ is the quasi covering dimension of the topological space (X, τ^*). Thus, we can obtain corresponding results for this dimension similar to those of [10]. For example, we state the closed subspace property.

Proposition 3.2. *If A is a closed subspace of X, then,* $\dim_q^* A \leqslant \dim_q^* X$.

Proof. It can be proved in a similar way as in [10, Proposition 2.4] taking into consideration that every closed set is also $*$-closed. □

In addition, we can observe that the dimensions $\dim_q^* X$ and $\dim_q X$ are different in general and the following examples prove this claim.

Example 3.3. (1) We consider the set $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Then $\dim_q X = 1$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, then the space (X, τ^*) is the discrete space and thus, $\dim_q^* X = 0$.

(2) We consider the set $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, X\}$. Then $\dim_q X = 0$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, then $\dim_q^* X = 1$.

(3) Generalizing the above example (2), if we consider an infinite countable set $X = \{x_1, x_2, \ldots\}$ with the topology $\tau = \{\varnothing, X\}$ and the ideal

$$\mathcal{I}_{i,j} = \{I \subseteq X : I \subseteq \{x_i, x_j\}\},$$

whenever $i \neq j$, then $\dim_q X = 0$ and $\dim_q^* X = 1$.

Remark 3.4. For any ideal topological space $(X, \tau, \mathcal{I})$ for which $\tau = \tau^*$ we have

$$\dim_q X = \dim_q^* X.$$

However, the converse of Remark 3.4 is not generally true and the following example proves this claim.

Example 3.5. We consider the set $X = \{1, 2, 3, \ldots\}$ of positive integers with the topology

$$\tau = \{\varnothing, X\} \cup \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots\}$$

and the ideal $\mathcal{I}_f$ of all finite subsets of X. Then clearly $\tau \neq \tau^*$ (as τ^* is the discrete topology) but $\dim_q X = \dim_q^* X = 0$.

In the following propositions we can prove further relations between the quasi covering dimensions $\dim_q$, $\dim_q^*$ and the covering dimensions dim and $\dim^*$, the last of which is reminded as follows.

The $*$-*covering dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\dim^* X$, is defined as follows:
(i) $\dim^* X = -1$, if $X = \varnothing$.
(ii) $\dim^* X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ^*-cover c of X there exists a finite τ^*-cover r of X, which is a refinement of c and $\mathrm{ord}(r) \leqslant n$.
(iii) $\dim^* X = n$, where $n \in \{0, 1, \ldots\}$, if $\dim^* X \leqslant n$ and $\dim^* X \nleqslant n - 1$.
(iv) $\dim^* X = \infty$, if it is false that $\dim^* X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

That is, the dimension $\dim^*$ is the covering dimension of the space (X, τ^*) and thus, we can succeed to have results which are similar to that of the covering dimension dim. Among these results, we state the following which will be useful for the study of the dimension $\dim_q^*$.

Proposition 3.6. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space such that* $\dim^* X \in \{0, 1, \ldots\}$ *and* $n \in \{0, 1, \ldots\}$*. Then, the following conditions are equivalent:*
(1) $\dim^* X \leqslant n$*.*
(2) For every finite τ^**-cover* $c = \{U_1, \ldots, U_k\}$ *of* X *there exists a finite* τ^**-cover* $r = \{V_1, \ldots, V_k\}$ *of* X *such that* $V_i \subseteq U_i$ *for* $i = 1, \ldots, k$*, and* $\operatorname{ord}(r) \leqslant n$*.*
(3) For every τ^**-cover* $c = \{U_1, \ldots, U_{n+2}\}$ *of* X *there exists a* τ^**-cover* $r = \{V_1, \ldots, V_{n+2}\}$ *of* X *such that* $V_i \subseteq U_i$*, for* $i = 1, \ldots, n+2$*, and* $\bigcap_{i=1}^{n+2} V_i = \varnothing$*.*

Proof. It is similar to [17, Proposition 3.1.2]. □

Proposition 3.7. *For every ideal topological space* $(X, \tau, \mathcal{I})$*, where* $\mathcal{I} \subseteq \tau^c$*, we have*

$$\dim_q X = \dim_q^* X.$$

Proof. Since $\mathcal{I} \subseteq \tau^c$, we have that $\beta^* \subseteq \tau$. Therefore, $\tau = \tau^*$ and by Remark 3.4 we have that $\dim_q X = \dim_q^* X$. □

Corollary 3.8. *For every ideal topological space* $(X, \tau, \mathcal{I})$*, where* $\mathcal{I} \subseteq \tau^c$ *and* $(X, \tau) \in \mathbb{P}$*, where* $\mathbb{P}$ *is any of the classes* **T(t.n.)**, **T(p.n.)**, **T(met.)**, *we have*

$$\dim X = \dim^* X = \dim_q X = \dim_q^* X.$$

Proposition 3.9. *For every ideal topological* T_1*-space* $(X, \tau, \mathcal{I})$*, where* $\mathcal{I} \subseteq \mathcal{I}_f$*, we have*

$$\dim_q X = \dim_q^* X.$$

Proof. Since the topological space (X, τ) is T_1, every $I \in \mathcal{I}$ is closed in (X, τ). Therefore, by Proposition 3.7 we have that $\dim_q X = \dim_q^* X$. □

Corollary 3.10. *For every ideal topological* T_1*-space* $(X, \tau, \mathcal{I})$*, where* $\mathcal{I} \subseteq \mathcal{I}_f$ *and* $(X, \tau) \in \mathbb{P}$*, where* $\mathbb{P}$ *is any of the classes* **T(t.n.)**, **T(p.n.)**, **T(met.)**, *we have*

$$\dim X = \dim^* X = \dim_q X = \dim_q^* X.$$

For an ideal topological space $(X, \tau, \mathcal{I})$ if $A \subseteq X$, then the family

$$\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$$

is an ideal on A. So, we can consider the ideal topological space $(A, \tau_A, \mathcal{I}_A)$, where τ_A is the subspace topology on A. We state that the topology $(\tau_A)^*$ is equal to the subspace topology $(\tau^*)_A$ on A [12].

Theorem 3.11. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *we have*

$$\dim_q^* X = \sup\{\dim^* D : D \text{ is a } *\text{-open and } *\text{-dense subset of } X\}.$$

Proof. Since the dimension $\dim_q^*$ is the corresponding quasi covering dimension for the topological space (X, τ^*), the statement of the theorem can be proved in a similar way as in [10, Proposition 2.1]. □

Corollary 3.12. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *we have*

$$\dim^* X \leqslant \dim_q^* X.$$

However, we mention that the dimensions $\dim^*$ and $\dim_q^*$ are different and the following examples prove this claim.

Example 3.13. (1) We consider the set $X = \{a, b, c, d, e\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$. Then the topology τ^* has as a basis the family

$$\{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d, e\}\}$$

with $\dim^* X = 1$ and $\dim_q^* X = 2$.

(2) Generalizing the above example (1), if we consider a set $X = \{x_1, x_2, \dots, x_n\}$, where $n \geqslant 5$, with the topology which has as a basis the family

$$\{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \dots, \{x_1, x_2, x_{n-1}\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{x_2\}\}$, then the topology τ^* has as a basis the family

$$\{\{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \dots, \{x_1, x_{n-1}\}, \{x_1, x_3, \dots, x_{n-1}, x_n\}\}$$

with $\dim^* X = 1$ and $\dim_q^* X = n - 3$.

Proposition 3.14. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space such that* $\dim_q^* X \in \{0, 1, \dots\}$ *and* $n \in \{0, 1, \dots\}$. *Then, the following conditions are equivalent:*
(1) $\dim_q^* X \leqslant n$.
(2) For every finite τ^**-quasi cover* $c = \{U_1, \dots, U_k\}$ *of* X *there exists a finite* τ^**-quasi cover* $r = \{V_1, \dots, V_k\}$ *of* X *such that* $r \sim c$, $V_i \subseteq U_i$ *for* $i = 1, \dots, k$, *and* $\operatorname{ord}(r) \leqslant n$.
(3) For every τ^**-quasi cover* $c = \{U_1, \dots, U_{n+2}\}$ *of* X *there exists a* τ^**-quasi cover* $r = \{V_1, \dots, V_{n+2}\}$ *of* X *such that* $r \sim c$, $V_i \subseteq U_i$, *for* $i = 1, \dots, n+2$, *and* $\bigcap_{i=1}^{n+2} V_i = \varnothing$.

Proof. Since the dimension $\dim_q^*$ is the corresponding quasi covering dimension for the topological space (X, τ^*), the statement of the proposition can be proved in a similar way as in [10, Proposition 2.5], taking into consideration Proposition 3.6. □

Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If β^* is a topology on X (and hence $\tau^* = \beta^*$), then the ideal $\mathcal{I}$ is called τ-*simple* [11]. Also, the ideal $\mathcal{I}$ is called τ-*codense* if $\mathcal{I} \cap \tau = \{\varnothing\}$, that is each member of $\mathcal{I}$ has empty interior with respect to the topology τ [3].

Clearly, for an arbitrary ideal topological space $(X, \tau, \mathcal{I})$, every $*$-dense subset of X is dense. The converse is not generally true.

Lemma 3.15. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* $\mathcal{I}$ *is* τ*-codense, then every open and dense subset of* X *is* $*$*-open and* $*$*-dense.*

Proof. Let D be an open and dense subset of X. Clearly, D is $*$-open set. We shall prove that D is $*$-dense. In the opposite side we suppose that D is not $*$-dense. Then there exists a non empty $*$-open set U such that $D \cap U = \varnothing$.

Since β^* is a base for the topology τ^*, there exist an open set V and $I \in \mathcal{I}$ such that $V \setminus I \subseteq U$. Then

$$D \cap (V \setminus I) \subseteq D \cap U = \varnothing,$$

and thus,

$$D \cap (V \setminus I) = (D \cap V) \setminus I = \varnothing.$$

Hence, $D \cap V \subseteq I$. We also have that $D \cap V$ is a non empty open set. Hence, the element I of $\mathcal{I}$ has non empty interior with respect to τ. The last contradicts the assumption that the ideal $\mathcal{I}$ is τ-codense. □

Lemma 3.16. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that the ideal $\mathcal{I}$ is τ-simple and let also $A \subseteq X$. Then the ideal $\mathcal{I}_A$ is τ_A-simple.*

Proof. It suffices to prove that $(\tau_A)^* \subseteq (\beta_A)^*$. Let $V_A \in (\tau_A)^*$. Then $V_A \in (\tau^*)_A$. That is, there exists $V \in \tau^*$ such that $V_A = V \cap A$. Since the ideal $\mathcal{I}$ is τ-simple, we have $\tau^* = \beta^*$ and hence, there exist $U \in \tau$ and $I \in \mathcal{I}$ such that $V = U \setminus I$. Thus,

$$V_A = V \cap A = (U \setminus I) \cap A = (U \cap A) \setminus (I \cap A),$$

where $U \cap A \in \tau_A$ and $I \cap A \in \mathcal{I}_A$. Thus, $V_A \in (\beta_A)^*$. □

Lemma 3.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that the ideal $\mathcal{I}$ is τ-codense and let also A be an open subset of X. Then the ideal $\mathcal{I}_A$ is τ_A-codense.*

Proof. Let $V \cap A = I \cap A \in \mathcal{I}_A \cap \tau_A$, where $V \in \tau$ and $I \in \mathcal{I}$. Since $I \cap A \subseteq I$, we have that $I \cap A \in \mathcal{I}$. Also, since A is an open subset of X, we have that $V \cap A \in \tau$. Thus,

$$V \cap A = I \cap A \in \mathcal{I} \cap \tau.$$

Finally, since the ideal $\mathcal{I}$ is τ-codense, we have that $\mathcal{I} \cap \tau = \{\varnothing\}$. Therefore, $\mathcal{I}_A \cap \tau_A = \{\varnothing\}$. □

Proposition 3.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is τ-simple and τ-codense, then*

$$\dim_q X \leqslant \dim_q^* X.$$

Proof. Clearly, if $\dim_q^* X = -1$ or $\dim_q^* X = \infty$, then the inequality holds. We suppose that $\dim_q^* X = n$, where $n \in \{0, 1, \ldots\}$, and we shall prove that $\dim_q X \leqslant n$. By [10, Proposition 2.1] we have that

$$\dim_q X = \sup\{\dim D : D \text{ is an open and dense subset of } X\}.$$

Thus, it suffices to prove that $\dim D \leqslant n$, for each open and dense subset D of X. Let D be an open and dense subset of X. Since $\mathcal{I}$ is τ-codense, by Lemma

3.15 we have that D is a $*$-open and $*$-dense subset of X. Also, by Theorem 3.11 we have that

$$\dim_q^* X = \sup\{\dim^* D : \text{ is a } *\text{-open and } *\text{-dense subset of } X\}.$$

Since $\dim_q^* X = n$, we have $\dim^* D \leqslant n$. By Lemma 3.16 and Lemma 3.17, the ideal $\mathcal{I}_D$ is τ_D-simple and τ_D-codense, and thus, by [14, Proposition 3.8] we have that

$$\dim D \leqslant \dim^* D.$$

Therefore, $\dim D \leqslant n$. □

4. The ideal quasi covering dimension

In this section, the notion of the ideal quasi covering dimension, $\mathcal{I}\text{-}\dim_q$, of an ideal topological space $(X, \tau, \mathcal{I})$, is defined, combining the topologies τ and τ^*, and various of its properties are investigated.

For the meaning of the ideal quasi covering dimension it is necessary to define the notion of τ^*-family for an ideal topological space $(X, \tau, \mathcal{I})$. A family c of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is called *τ^*-family* if all elements of c are elements of τ^*.

Definition 4.1. The *ideal quasi covering dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\mathcal{I}\text{-}\dim_q X$, is defined as follows:
(i) $\mathcal{I}\text{-}\dim_q X = -1$, if $X = \varnothing$.
(ii) $\mathcal{I}\text{-}\dim_q X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ-quasi cover c of X there exists a finite τ^*-family r of subsets of X such that $r \sim c$, r is a refinement of c and $\text{ord}(r) \leqslant n$.
(iii) $\mathcal{I}\text{-}\dim_q X = n$, where $n \in \{0, 1, \ldots\}$, if $\mathcal{I}\text{-}\dim_q X \leqslant n$ and $\mathcal{I}\text{-}\dim_q X \nleqslant n - 1$.
(iv) $\mathcal{I}\text{-}\dim_q X = \infty$, if it is false that $\mathcal{I}\text{-}\dim_q X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

Proposition 4.2. *For any ideal topological space $(X, \tau, \mathcal{I})$ we have that*

$$\mathcal{I}\text{-}\dim_q X \leqslant \dim_q X.$$

Proof. Clearly, if $\dim_q X \in \{-1, \infty\}$, the inequality holds. We suppose that $\dim_q X = n$, where $n \in \{0, 1, \ldots\}$, and we shall prove that $\mathcal{I}\text{-}\dim_q X \leqslant n$. Let c be a finite τ-quasi cover of X. Since $\dim_q X = n$, there exists a finite τ-quasi cover r of X such that $r \sim c$, r is a refinement of c and $\text{ord}(r) \leqslant n$. The family r is also a τ^*-family of subsets of X such that $r \sim c$, r is refinement of c and $\text{ord}(r) \leqslant n$. Thus, $\mathcal{I}\text{-}\dim_q X \leqslant n$. □

Proposition 4.3. *For any ideal topological space $(X, \tau, \mathcal{I})$ for which the ideal $\mathcal{I}$ is τ-codense we have that*

$$\mathcal{I}\text{-}\dim_q X \leqslant \dim_q^* X.$$

Proof. Clearly, if $\dim_q^* X \in \{-1, \infty\}$, the inequality holds. We suppose that $\dim_q^* X = n$, where $n \in \{0, 1, \ldots\}$, and we shall prove that $\mathcal{I}\text{-}\dim_q X \leqslant n$. Let c be a finite τ-quasi cover of X. Then by Lemma 3.15 c is a τ^*-quasi cover of X. Since $\dim_q^* X = n$, there exists a finite τ^*-quasi cover r of X such that $r \sim c$, r is a refinement of c and $\mathrm{ord}(r) \leqslant n$. Thus, $\mathcal{I}\text{-}\dim_q X \leqslant n$. ☐

Corollary 4.4. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *for which the ideal* $\mathcal{I}$ *is* τ*-codense we have that*

$$\mathcal{I}\text{-}\dim_q X \leqslant \min\{\dim_q X, \dim_q^* X\}.$$

It is observed that the ideal quasi covering dimension $\mathcal{I}\text{-}\dim_q$ is different from the dimensions $\dim_q$ and $\dim_q^*$ and the following examples prove this assertion.

Example 4.5. (1) We consider the set $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $\mathcal{I}\text{-}\dim_q X = 0$ and $\dim_q X = 1$.

(2) We consider the indiscrete space $(\mathbb{R}, \tau)$ and the ideal $\mathcal{I} = \{I \subseteq \mathbb{R} : 0 \notin I\}$. Then $\tau^* = \{\varnothing\} \cup \{U \subseteq \mathbb{R} : 0 \in U\}$. Since $\dim^*\mathbb{R}$ can be arbitrary large [2, Example 1.1.11], by Theorem 3.11 we have that $\dim_q^* \mathbb{R}$ can also be arbitrary large. In addition, $\mathcal{I}\text{-}\dim_q \mathbb{R} = 0$.

(3) We consider the indiscrete space $(\mathbb{R}, \tau)$ and the ideal $\mathcal{I}_f$ of finite subsets of $\mathbb{R}$. Then τ^* is the cofinite topology on $\mathbb{R}$ for which $\dim^* \mathbb{R} = \infty$ (see [2, Example 1.1.12]) and thus, by Theorem 3.11 we have that $\dim_q^* \mathbb{R} = \infty$. Also, $\mathcal{I}\text{-}\dim_q \mathbb{R} = 0$.

Example 4.6. We consider the following ideal topological spaces:
(1) $(X, \tau_X, \mathcal{I}_X)$, where $X = \{a, b, c, d\}$,

$\tau_X = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\mathcal{I}_X = \{I \subseteq X : I \subseteq \{a, b\}\}$.

(2) $(Y, \tau_Y, \mathcal{I}_Y)$, where $Y = \{x, y, z, w\}$ such that $X \cap Y = \varnothing$,

$$\tau_Y = \{\varnothing, Y\} \text{ and } \mathcal{I}_Y = \{I \subseteq Y : I \subseteq \{y, z, w\}\}.$$

Then

$$\dim_q X = 1,\ \dim_q^* X = 0,\ \dim_q Y = 0 \text{ and } \dim_q^* Y = 2.$$

We consider the ideal topological space $(Z, \tau_Z, \mathcal{I}_Z)$, where $Z = X \cup Y$,

$$\tau_Z = \{U \subseteq Z : U \cap X \in \tau_X,\ U \cap Y \in \tau_Y\}$$

and

$$\mathcal{I}_Z = \{I \subseteq Z : I \subseteq \{a, b, y, z, w\}\}.$$

Then $\mathcal{I}_Z\text{-}\dim_q Z = 0$, $\dim_q Z = 1$ and $\dim_q^* Z = 2$.

Remark 4.7. We can generalize Example 4.6 in order to succeed a construction of an ideal topological space with different dimensions $\mathcal{I}\text{-}\dim_q$, $\dim_q$ and $\dim_q^*$. For that, we consider two ideal topological spaces $(X, \tau_X, \mathcal{I}_X)$ and $(Y, \tau_Y, \mathcal{I}_Y)$ with the following properties:

(1) $X \cap Y = \varnothing$,
(2) $\mathcal{I}_X = \{I \subseteq X : I \subseteq A\}$ for some $A \subseteq X$,
(3) $\mathcal{I}_Y = \{I \subseteq Y : I \subseteq B\}$ for some $B \subseteq Y$,

(4) $\dim_q^* X < \min\{\dim_q X, \dim_q^* Y\}$ and
(5) $\dim_q Y < \min\{\dim_q X, \dim_q^* Y\}$.
Then we construct the ideal topological space $(Z, \tau_Z, \mathcal{I}_Z)$ as $Z = X \cup Y$,

$$\tau_Z = \{U \subseteq Z : U \cap X \in \tau_X,\ U \cap Y \in \tau_Y\}$$

and

$$\mathcal{I}_Z = \{I \subseteq Z : I \subseteq A \cup B\}.$$

We have that

$$\dim_q Z = \max\{\dim_q X,\ \dim_q Y\} = \dim_q X$$

and

$$\dim_q^* Z = \max\{\dim_q^* X,\ \dim_q^* Y\} = \dim_q^* Y.$$

Also,

$$\mathcal{I}_Z\text{-}\dim_q Z \leqslant \max\{\dim_q^* X,\ \dim_q Y\},$$

proving that

$$\mathcal{I}_Z\text{-}\dim_q Z < \dim_q X = \dim_q Z$$

and

$$\mathcal{I}_Z\text{-}\dim_q Z < \dim_q^* Y = \dim_q^* Z.$$

Proposition 4.8. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a closed subset of X, then*

$$\mathcal{I}_A\text{-}\dim_q A \leqslant \mathcal{I}\text{-}\dim_q X.$$

Proof. Obviously, if $\mathcal{I}\text{-}\dim_q X = -1$ or $\mathcal{I}\text{-}\dim_q X = \infty$, then the inequality holds. We suppose that $\mathcal{I}\text{-}\dim_q X = n$, where $n \in \{0, 1, \ldots\}$, and we shall prove that $\mathcal{I}_A\text{-}\dim_q A \leqslant n$.

Let $c_A = \{U_1^A, \ldots, U_k^A\}$ be a finite τ_A-quasi cover of A, that is

$$\bigcup_{i=1}^{k} U_i^A = D^A,$$

where D^A is an open and dense subset of A. For every $i \in \{1, \ldots, k\}$ there exist $U_i \in \tau$ such that $U_i^A = A \cap U_i$ and an open subset D of X such that $D^A = A \cap D$. We consider the finite τ-quasi cover

$$c = \{U_i \cap D : i = 1, \ldots, k\} \cup \{X \setminus A\}$$

of the space X. We observe that the family c consists of open subsets of X whose union is the dense set $D \cup (X \setminus A)$.

Since $\mathcal{I}\text{-}\dim_q X = n$, there exists a finite τ^*-family r of subsets of X such that $r \sim c$, r is a refinement of c and $\mathrm{ord}(r) \leqslant n$.

We consider the family

$$r_A = \{A \cap V : V \in r\}.$$

Then r_A is a finite $(\tau_A)^*$-family of subsets of A such that $r_A \sim c_A$, r_A is a refinement of c_A, and $\mathrm{ord}(r_A) \leqslant n$. Thus, $\mathcal{I}_A\text{-}\dim_q A \leqslant n$. □

In order to present a different approach of the ideal quasi covering dimension it is considered to be necessary to remind the meaning of the ideal covering dimension [14].

Especially, the *ideal covering dimension*, denoted by $\mathcal{I}\text{-}\dim$, is defined as follows:
(i) $\mathcal{I}\text{-}\dim X = -1$, if $X = \varnothing$.
(ii) $\mathcal{I}\text{-}\dim X \leqslant n$, where $n \in \{0, 1, \ldots\}$, if for every finite τ-cover c of X there exists a finite τ^*-cover r of X, which is a refinement of c with $\operatorname{ord}(r) \leqslant n$.
(iii) $\mathcal{I}\text{-}\dim X = n$, where $n \in \{0, 1, \ldots\}$, if $\mathcal{I}\text{-}\dim X \leqslant n$ and $\mathcal{I}\text{-}\dim X \nleqslant n-1$.
(iv) $\mathcal{I}\text{-}\dim X = \infty$, if it is false that $\mathcal{I}\text{-}\dim X \leqslant n$ for every $n \in \{-1, 0, 1, \ldots\}$.

Theorem 4.9. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *we have*

$$\mathcal{I}\text{-}\dim_q X = \sup\{\mathcal{I}_D\text{-}\dim D : D \textit{ is an open and dense subset of } X\}.$$

Proof. Firstly, we shall prove the inequality

$$\sup\{\mathcal{I}_D\text{-}\dim D : D \text{ is an open and dense subset of } X\} \leqslant \mathcal{I}\text{-}\dim_q X.$$

Clearly, if $\mathcal{I}\text{-}\dim_q X \in \{-1, \infty\}$, then this inequality is true. We suppose that $\mathcal{I}\text{-}\dim_q X = n$, where $n \in \{0, 1, \ldots\}$. We shall prove that

$$\sup\{\mathcal{I}_D\text{-}\dim D : D \text{ is an open and dense subset of } X\} \leqslant n.$$

Let D be an open and dense subset of X. It suffices to prove that $\mathcal{I}_D\text{-}\dim D \leqslant n$. Let c be a finite τ_D-cover of D. Since every element of c is open set in D and D is open set in X, we have that c consists of open subsets of X. That is, c is a finite τ-quasi cover of X. Therefore, there exists a finite τ^*-family r of subsets of X such that $r \sim c$, r is a refinement of c and $\operatorname{ord}(r) \leqslant n$. Since $r \sim c$, r is a finite $(\tau_D)^*$-cover of D (as $(\tau_D)^* = (\tau^*)_D$). Hence, $\mathcal{I}_D\text{-}\dim D \leqslant n$.

We shall prove the inequality

$$\mathcal{I}\text{-}\dim_q X \leqslant \sup\{\mathcal{I}_D\text{-}\dim D : D \text{ is an open and dense subset of } X\}.$$

If

$$\sup\{\mathcal{I}_D\text{-}\dim D : D \text{ is an open and dense subset of } X\} \in \{-1, \infty\},$$

then this inequality holds. We suppose that

$$\sup\{\mathcal{I}_D\text{-}\dim D : D \text{ is an open and dense subset of } X\} = n,$$

where $n \in \{0, 1, \ldots\}$, and we shall prove that $\mathcal{I}\text{-}\dim_q X \leqslant n$. Let c be a finite τ-quasi cover of X. Then, the union of c is an open and dense subset D of X. Our assumption tends us to have $\mathcal{I}_D\text{-}\dim D \leqslant n$. Since c is a finite τ_D-cover of D, there exists a finite $(\tau_D)^*$-cover r of D, refinement of c, such that $\operatorname{ord}(r) \leqslant n$. Since every element of r is $*$-open set in D and D is an open subset of X (and thus, $*$-open in X), we have that every element of r is $*$-open set in X. Then r is a finite τ^*-family of subsets of X such that $r \sim c$ and $\operatorname{ord}(r) \leqslant n$. Therefore, $\mathcal{I}\text{-}\dim_q X \leqslant n$. ☐

Corollary 4.10. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *we have*

$$\mathcal{I}\text{-}\dim X \leqslant \mathcal{I}\text{-}\dim_q X.$$

The following examples show that these types of ideal dimensions are different.

Example 4.11. (1) We consider the set $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$. Then $\mathcal{I}\text{-}\dim X = 0$ and $\mathcal{I}\text{-}\dim_q X = 1$.

(2) Let $n \in \mathbb{N} \setminus \{0\}$, (Y, τ_Y) be a metrizable space with $\dim Y = n$ and the ideal $\mathcal{I}_Y = \{\varnothing\}$ on Y. According to Corollary 3.8 we have that

$$\dim Y = \dim^* Y = \dim_q Y = \dim_q^* Y = n.$$

Also, $\mathcal{I}_Y\text{-}\dim_q Y = \dim_q Y = n$. Then we consider a point p that does not belong to Y, the set $X = Y \cup \{p\}$ with the topology

$$\tau_X = \{U \subseteq X : U \in \tau_Y \text{ or } U = X\}$$

and the ideal $\mathcal{I}_X = \{\varnothing, \{p\}\}$ on X (with respect to the meaning of the ideal on subsets of X we clearly have that $(\mathcal{I}_X)_Y = \mathcal{I}_Y$) Then we observe that the τ_X^*-cover $\{X\}$ of X is a refinement of every finite τ_X-cover of X. Thus, $\mathcal{I}_X\text{-}\dim X = 0$. Also, by the definition of the space X a proper subset of X is open and dense in X if and only if it is open and dense in Y. Hence, by Theorem 4.9, we have that $\mathcal{I}_X\text{-}\dim_q X = \mathcal{I}_Y\text{-}\dim_q Y = n$.

Proposition 4.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\mathcal{I}\text{-}\dim_q X \in \{0, 1, \ldots\}$ and $n \in \{0, 1, \ldots\}$. Then, the following statements are equivalent:*
(1) $\mathcal{I}\text{-}\dim_q X \leqslant n$.
(2) For every finite τ-quasi cover $c = \{U_1, \ldots, U_k\}$ of X there exists a finite τ^*-family $r = \{V_1, \ldots, V_k\}$ of subsets of X such that $r \sim c$, $V_i \subseteq U_i$ for $i = 1, \ldots, k$, and $\operatorname{ord}(r) \leqslant n$.
(3) For every τ-quasi cover $c = \{U_1, \ldots, U_{n+2}\}$ of X there exists a τ^*-family $r = \{V_1, \ldots, V_{n+2}\}$ of subsets of X such that $r \sim c$, $V_i \subseteq U_i$ for $i = 1, \ldots, n+2$ and $\bigcap_{i=1}^{n+2} V_i = \varnothing$.

Proof. Firstly, we shall prove the implication (1) $\Rightarrow$ (2). Let $c = \{U_1, \ldots, U_k\}$ be a finite τ-quasi cover of X. Then $\bigcup_{i=1}^{k} U_i = D$, where D is dense in X. By Theorem 4.9 we have that $\mathcal{I}_D\text{-}\dim D \leqslant n$. Thus, by [14, Proposition 4.4] there exists a finite $(\tau_D)^*$-cover $r = \{V_1, \ldots, V_k\}$ of D such that $V_i \subseteq U_i$ for $i = 1, \ldots, k$, and $\operatorname{ord}(r) \leqslant n$. Since every element of r is $*$-open set in D and D is an open subset of X (and thus, $*$-open in X), we have that every element of r is $*$-open set in X. Also, $r \sim c$ and hence r is the desired τ^*-family.

The implication (2) $\Rightarrow$ (3) follows directly by the meaning of the order.

We shall prove the implication (3) $\Rightarrow$ (1). Since $\mathcal{I}\text{-}\dim_q X \in \{0, 1, \ldots\}$, by Theorem 4.9 there exists an open and dense subset D_0 of X such that

$$\mathcal{I}\text{-}\dim_q X = \mathcal{I}_{D_0}\text{-}\dim D_0.$$

Let $c = \{U_1, \ldots, U_{n+2}\}$ be a τ_{D_0}-cover of D_0. Since every element of c is open set in D_0 and D_0 is open set in X, we have that c consists of open subsets of X. That is, c is a τ-quasi cover of X consisting of $n+2$ elements. By assumption, there exists a τ^*-family $r = \{V_1, \ldots, V_{n+2}\}$ of subsets of X such that $r \sim c$, $V_i \subseteq U_i$ for $i = 1, \ldots, n+2$, and $\bigcap_{i=1}^{n+2} V_i = \varnothing$. Since $r \sim c$, r is a $(\tau_{D_0})^*$-cover of D_0. Thus, by [14, Proposition 4.4] we have $\mathcal{I}_{D_0}\text{-}\dim D_0 \leqslant n$ and hence $\mathcal{I}\text{-}\dim_q X \leqslant n$. $\square$

In the following result we present a characterization of the ideal quasi covering dimension for hereditarily $*$-normal spaces. In particular, an ideal topological space $(X, \tau, \mathcal{I})$ is said to be *$*$-normal* if the space (X, τ^*) is normal (in the usual meaning of normal topological spaces).

Also, a property of a topological space X is said to be *hereditary* if each subspace of X has also this property. Especially, an ideal topological space $(X, \tau, \mathcal{I})$ is called *hereditarily $*$-normal* if each subspace of X is $*$-normal. Similarly, an ideal topological space $(X, \tau, \mathcal{I})$ is called *hereditarily normal* if each subspace of X is normal.

For the next result it is also useful the meaning of relatively closed family. We recall that for a topological space (X, τ) a family c of subsets of X is said to be *relatively closed* if all elements of c are closed in the subspace $\bigcup\{U : U \in c\}$ of X.

Similarly, for an ideal topological space $(X, \tau, \mathcal{I})$ a family c of subsets of X is said to be *relatively $*$-closed* if all elements of c are $*$-closed in the subspace $\bigcup\{U : U \in c\}$ of X.

Proposition 4.13. *Let $(X, \tau, \mathcal{I})$ be a hereditarily $*$-normal space such that $\mathcal{I}\text{-}\dim_q X \in \{0, 1, \ldots\}$ and $n \in \{0, 1, \ldots\}$. Then, the following conditions are equivalent:*
(1) $\mathcal{I}\text{-}\dim_q X \leqslant n$.
(2) For every finite τ-quasi cover $c = \{U_1, \ldots, U_k\}$ of X there exists a finite relatively $*$-closed family $r = \{F_1, \ldots, F_k\}$ of subsets of X such that $r \sim c$, $F_i \subseteq U_i$ for $i = 1, \ldots, k$, and $\operatorname{ord}(r) \leqslant n$.
(3) For every τ-quasi cover $c = \{U_1, \ldots, U_{n+2}\}$ of X there exists a relatively $*$-closed family $r = \{F_1, \ldots, F_{n+2}\}$ of subsets of X such that $r \sim c$, $F_i \subseteq U_i$ for $i = 1, \ldots, n+2$, and $\bigcap_{i=1}^{n+2} F_i = \varnothing$.

Proof. Firstly, we shall prove the implication (1) $\Rightarrow$ (2). Let $c = \{U_1, \ldots, U_k\}$ be a finite τ-quasi cover of X. Then $\bigcup_{i=1}^{k} U_i = D$, where D is dense in X. By Theorem 4.9 we have that $\mathcal{I}_D\text{-}\dim D \leqslant n$ and by [14, Proposition 4.21] there exists a finite $((\tau_D)^*)^c$-cover $r = \{F_1, \ldots, F_k\}$ of D such that $F_i \subseteq U_i$ for $i = 1, \ldots, k$, and $\operatorname{ord}(r) \leqslant n$. Since each element of r is $*$-closed in the subspace $F_1 \cup \ldots \cup F_k = D$, r is the desired relatively $*$-closed family.

The implication (2) ⇒ (3) follows directly by the meaning of the order.

We shall prove the implication (3) ⇒ (1). Since $\mathcal{I}\text{-}\dim_q X \in \{0, 1, \ldots\}$, by Theorem 4.9 there exists an open and dense subset D_0 of X such that

$$\mathcal{I}\text{-}\dim_q X = \mathcal{I}_{D_0}\text{-}\dim D_0.$$

Let $c = \{U_1, \ldots, U_{n+2}\}$ be a τ_{D_0}-cover of D_0. Since every element of c is open set in D_0 and D_0 is open set in X, we have that c consists of open sets in X. That is, c is a τ-quasi cover of X consisting of $n+2$ elements. By assumption, there exists a relatively $*$-closed family $r = \{F_1, \ldots, F_{n+2}\}$ of subsets of X such that $r \sim c$, $F_i \subseteq U_i$ for $i = 1, \ldots, n+2$, and $\bigcap_{i=1}^{n+2} F_i = \varnothing$. Since $r \sim c$ and r is relatively $*$-closed, r is a $((\tau_{D_0})^*)^c$-cover of D_0. Thus, by [14, Proposition 4.21] we have $\mathcal{I}_{D_0}\text{-}\dim D_0 \leqslant n$ and hence $\mathcal{I}\text{-}\dim_q X \leqslant n$. □

We also study the ideal quasi covering dimension in hereditarily $*$-normal spaces under the view of partitions. We recall that a subset L of a topological space (X, τ) is called a *partition* between two disjoint subsets A and B of X if there exist open subsets U and V of X satisfying the conditions $A \subseteq U$, $B \subseteq V$, $U \cap V = \varnothing$ and $X \setminus L = U \cup V$.

Proposition 4.14. *Let $(X, \tau, \mathcal{I})$ be a hereditarily $*$-normal space such that $\mathcal{I}\text{-}\dim_q X \leqslant n$, where $n \in \{0, 1, \ldots\}$. Then for every open and dense subset D of X and every family*

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

of $n+1$ pairs of disjoint closed subsets of D, there exist $*$-closed sets $L_1, \ldots, L_{n+1}$ in D such that L_i is a partition between A_i and B_i in the space $(D, (\tau_D)^*)$ for $i = 1, \ldots, n+1$ and $\bigcap_{i=1}^{n+1} L_i = \varnothing$.

Proof. Let D be an open and dense subset of X and

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

be a family of $n+1$ pairs of disjoint closed subsets of D. By Theorem 4.9, since $\mathcal{I}\text{-}\dim_q X \leqslant n$, we have $\mathcal{I}_D\text{-}\dim D \leqslant n$, Thus, by [14, Theorem 4.25], since the corresponding ideal topological space $(D, \tau_D, \mathcal{I}_D)$ is $*$-normal, there exist $*$-closed sets $L_1, \ldots, L_{n+1}$ in D such that L_i is a partition between A_i and B_i in the space $(D, (\tau_D)^*)$ for $i = 1, \ldots, n+1$ and $\bigcap_{i=1}^{n+1} L_i = \varnothing$. □

Proposition 4.15. *Let $(X, \tau, \mathcal{I})$ be a hereditarily normal ideal topological space and assume that for every open and dense set D of X and every family*

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

of $n+1$ pairs of disjoint closed subsets of D there exist $*$-closed sets $L_1, \ldots, L_{n+1}$ in D such that L_i is a partition between A_i and B_i in the space $(D, (\tau_D)^*)$ for $i = 1, \ldots, n+1$ and $\bigcap_{i=1}^{n+1} L_i = \varnothing$. Then $\mathcal{I}\text{-}\dim_q X \leqslant n$.

Proof. Let D be an open and dense subset of X and

$$\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$$

be a family of $n+1$ pairs of disjoint closed subsets of D. By assumption there exist $*$-closed sets $L_1, \ldots, L_{n+1}$ in D such that L_i is a partition between A_i and B_i in the space $(D, (\tau_D)^*)$ for $i = 1, \ldots, n+1$ and $\bigcap_{i=1}^{n+1} L_i = \varnothing$. Then by [14, Theorem 4.26], since the corresponding ideal topological space $(D, \tau_D, \mathcal{I}_D)$ is normal, we have $\mathcal{I}_D\text{-}\dim D \leqslant n$ and therefore by Theorem 4.9 we have $\mathcal{I}\text{-}\dim_q X \leqslant n$. □

In what follows, properties of the ideal quasi covering dimension, using different ideals on the underlying sets, are studied.

Proposition 4.16. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two ideals on X. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then*

$$\mathcal{I}_2\text{-}\dim_q X \leqslant \mathcal{I}_1\text{-}\dim_q X.$$

Proof. Clearly, if $\mathcal{I}_1\text{-}\dim_q X \in \{-1, \infty\}$, the inequality holds. We suppose that $\mathcal{I}_1\text{-}\dim_q X = n$, where $n \in \{0, 1, \ldots\}$, and we shall prove that $\mathcal{I}_2\text{-}\dim_q X \leqslant n$.

Let c be a finite τ-quasi cover of X. Since $\mathcal{I}_1\text{-}\dim_q X = n$, there exists a finite $(\tau_1)^*$-family r of subsets of X, which is similar to c, refinement of c and $\mathrm{ord}(r) \leqslant n$. (We state that the topology $(\tau_1)^*$ is referred to the ideal $\mathcal{I}_1$.) Also, since $\mathcal{I}_1 \subseteq \mathcal{I}_2$, r is a $(\tau_2)^*$-family of subsets of X (similarly, $(\tau_2)^*$ is referred to the ideal $\mathcal{I}_2$). Thus, $\mathcal{I}_2\text{-}\dim_q X \leqslant n$. □

Corollary 4.17. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two ideals on X. Then*

$$\max\{\mathcal{I}_1\text{-}\dim_q X,\ \mathcal{I}_2\text{-}\dim_q X\} \leqslant \mathcal{I}_1 \cap \mathcal{I}_2\text{-}\dim_q X.$$

Corollary 4.18. *Let (X, τ) be a topological space, A, B subsets of X and $\mathcal{I}_1 = P(A)$, $\mathcal{I}_2 = P(B)$ and $\mathcal{I}_3 = P(A \cup B)$ three ideals on X, where $P(A)$, $P(B)$ and $P(A \cup B)$ are the power sets of A, B and $A \cup B$, respectively. Then*

$$\mathcal{I}_3\text{-}\dim_q X \leqslant \min\{\mathcal{I}_1\text{-}\dim_q X,\ \mathcal{I}_2\text{-}\dim_q X\}.$$

ACKNOWLEDGEMENTS. *The author would like to thank the reviewer for the careful reading of the paper and the useful comments.*

References

[1] M. G. Charalambous, Dimension Theory, A Selection of Theorems and Counterexamples, Springer Nature Switzerland AG, Cham, Switzerland, 2019.

[2] M. Coornaert, Topological dimension and dynamical systems, Translated and revised from the 2005 French original, Universitext, Springer, Cham, 2015.

[3] J. Dontchev, M. Ganster, D. Rose, Ideal resolvability, Topology and its Applications 93, no. 1 (1999), 1–16.

[4] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.

[5] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, Berlin, 1995.

[6] D. N. Georgiou, A. C. Megaritis, Covering dimension and finite spaces, Applied Mathematics and Computation 218 (2014), 3122–3130.

[7] D. N. Georgiou, A. C. Megaritis, S. Moshokoa, Small inductive dimension and Alexandroff topological spaces, Topology and its Applications 168 (2014), 103–119.

[8] D. N. Georgiou, A. C. Megaritis, F. Sereti, A study of the quasi covering dimension for finite spaces through matrix theory, Hacettepe Journal of Mathematics and Statistics 46, no. 1 (2017), 111–125.

[9] D. N. Georgiou, A. C. Megaritis, F. Sereti, A study of the quasi covering dimension of Alexandroff countable spaces using matrices, Filomat 32, no. 18 (2018), 6327–6337.

[10] D. N. Georgiou, A. C. Megaritis, F. Sereti, A topological dimension greater than or equal to the classical covering dimension, Houston Journal of Mathematics 43, no. 1 (2017), 283–298.

[11] T. R. Hamlett, D. Rose, D. Janković, Paracompactness with respect to an ideal, Internat. J. Math. Math. Sci. 20, no. 3 (1997), 433–442.

[12] D. Janković, T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97, no. 4 (1990), 295–310.

[13] K. Kuratowski, Topologie I, Monografie Matematyczne 3, Warszawa-Lwów, 1933.

[14] A. C. Megaritis, Covering dimension and ideal topological spaces, Quaestiones Mathematicae 45, no. 2 (2022), 197–212.

[15] K. Nagami, Dimension theory, Pure and Applied Mathematics, vol. 37 Academic Press, New York-London, 1970.

[16] J. Nagata, Modern dimension theory, Revised edition, Sigma Series in Pure Mathematics, 2, Heldermann Verlag, Berlin, 1983.

[17] A. R. Pears, Dimension theory of general spaces, Cambridge University Press, Cambridge, 1975.

[18] F. Sereti, Small and large inductive dimension for ideal topological spaces, Applied General Topology 22, no. 2 (2021), 417–434.