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An improvement of derivative-free point-to-point iterative processes with central divided differences

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Abstract: In this article, we introduce a new Steffensen-type method with the advantage that its behavior is very similar to Newton's method; therefore, it is a very remarkable way of avoiding the drawback that Newton's method presents for nondifferentiable operators. In our study, we perform an exhaustive comparative study between the semilocal convergence of Newton's method and the derivative-free point-to-point iterative process considered; in the case of differentiable operators, we use the majoring sequences and the majorant principle. In the nondifferentiable case, we impose conditions on the starting point and on the nonlinear operator to obtain a semilocal convergence result for the iterative process considered. In both cases, we complete the theoretical convergence proofs with a dynamical study and a numerical test. In the case of differentiable operators, this study confirms that the accessibility and numerical behavior of both iterative processes, Newton's method and the derivative-free point-to-point iterative process considered, are very similar.

Keywords: derivative-free iterative processes; divided differences; iterative processes; semilocal convergence.

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1 Introduction

One of the most common problems in mathematics is the solution of a system of nonlinear equations.

$$F(x) = 0, \tag{1}$$

where $F\colon \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear operator, $F \equiv (F_1, F_2, \ldots, F_m)$ with $F_i\colon \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, $1 \le i \le m$, and Ω is a nonempty open convex domain.

Many applied problems can be reduced to solving systems of nonlinear equations, which is one of the most basic problems in mathematics. These problems arise in all scientific areas including both mathematics and physics, especially in a diverse range of engineering applications. This task has applications in many

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scientific fields [3–5, 8, 14, 23]. Therefore, great efforts have been made by a lot of researchers, and many constructive theories and algorithms are proposed to solve systems of nonlinear equations.

This problem is not always easy to solve since we cannot frequently obtain an exact solution to the previous system of nonlinear equation as in Eq. (1) so that we usually look for a numerical approximation to a solution. In this case, we use approximation methods, which are generally iterative. The best known iteration to solve nonlinear equations is undoubtedly the Newton's method.

$$\begin{cases} x_0 \quad \text{given in } \Omega, \\ x_{n+1} = x_n - \left[F'(x_n)\right]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{2}$$

The low operational cost and the quadratic convergence of the method guarantee a good computational efficiency. In addition, this method has good accessibility such that the domain of starting points of the method is large. However, the Newton's method has a serious shortcoming: the operator Fneeds to be differentiable for the method to be applied. This points out that this method is not applicable to nondifferentiable system of nonlinear equations.

Our main goal in this work is to consider an iterative process that has the characteristics of Newton's method in the case that the operator F is differentiable, but is also applicable in situations where the operator F is not. By maintaining the nondifferentiable situation, the important properties that the iterative process considered is verified in the differentiable case. To achieve this goal, the first step is to approximate the operator F' when the operator F is nondifferentiable. For this, it is common to approximate the derivatives by divided differences using a numerical derivation formula, and as a consequence, iterative processes that use divided differences instead of derivatives are obtained. Remember that the operator $[u, v; F]: \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, $u, v \in \Omega$, with $u \neq v$, is a first-order divided difference, ([2, 13]) if the following conditions are verified:

$$[u, v; F] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m) \quad \text{and} \quad [u, v; F](u - v) = F(u) - F(v), \tag{3}$$

where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is the set of bounded linear operators in $\mathbb{R}^m$. Note that in $\mathbb{R}^m$, there are several divided differences that can be considered (see [7, 13]).

If $d_1(x_n)$ and $d_2(x_n)$ are known data at the point x_n, then we can consider the approximation $F'(x_n) \sim [d_1(x_n), d_2(x_n);\ F]$ and define the following iterative process, [16]

$$\begin{cases} x_0 \quad \text{given in } \Omega, \\ x_{n+1} = x_n - [d_1(x_n), d_2(x_n); F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{4}$$

Obviously, this approximation will improve depending on the data considered. We can see examples of the choice of these data in the Steffensen method ($d_1(x_n) = x_n$ and $d_2(x_n) = x_n + F(x_n)$) [1], the Backward–Steffensen method ($d_1(x_n) = x_n - F(x_n)$ and $d_2(x_n) = x_n$) [16], and the Central–Steffensen method ($d_1(x_n) = x_n - F(x_n)$ and $d_2(x_n) = x_n + F(x_n)$) [16]. We can also consider data dependent on the previous approximation, that is, $d_1(x_{n-1}, x_n)$ and $d_2(x_{n-1}, x_n)$. This situation appears in iterative processes with memory, such as the Secant method ($d_1(x_{n-1}, x_n) = x_{n-1}$ and $d_2(x_{n-1}, x_n) = x_n$) [2], the Secant-type methods ($d_1(x_{n-1}, x_n) = \lambda x_{n-1} + (1 - \lambda)x_{n-1}$ and $d_2(x_{n-1}, x_n) = x_n$, with $\lambda \in [0, 1)$) [15], or the Kurchatov method ($d_1(x_{n-1}, x_n) = x_{n-1}$ and $d_2(x_{n-1}, x_n) = 2x_n - x_{n-1}$) [24].

Symmetric divided differences generally perform better. We can see this in the Central–Steffensen and Kurchatov methods, that is, both maintain the quadratic convergence of Newton's method by approximating the derivative through symmetric divided differences with respect to x_n. Following this idea, in this article, we consider the derivative-free point-to-point iterative process given by

$$\begin{cases} y_0 \quad \text{given in } \Omega, \\ y_{n+1} = y_n - [y_n - \|F(y_n)\|\mathbf{Tol}, y_n + \|F(y_n)\|\mathbf{Tol}; F]^{-1} F(y_n), \quad n \geq 0, \end{cases} \tag{5}$$

where $\mathbf{Tol} = (tol, tol, \ldots, tol) \in \mathbb{R}^m$ for a real number $tol > 0$. Thus, we are considering a symmetric divided difference to approximate the derivative in Newton's method. Furthermore, by varying the parameter tol, we can approach the value $F'(y_n)$.

As we have already indicated previously, our objective in this work focuses on verifying that this iterative process has a behavior like Newton's method in differentiable situations and maintains this behavior for nondifferentiable situations, where Newton's method is not applicable.

The article is structured as follows. After introducing the method in Section 1, we dedicate Section 2 to give lemmas and theorems needed for obtaining the semilocal convergence study in the differentiable case. Section 3 is devoted to theoretical convergence study in the nondifferentiable case. Moreover, dynamical studies and numerical tests are performed in each corresponding section. Finally, in Section 4, we give some conclusions for our work.

Along the article, we denote $\overline{B(x,\varrho)} = \{y \in \mathbb{R}^m; \|y - x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in \mathbb{R}^m; \|y - x\| < \varrho\}$, respectively, for the closed and open balls with center in x and of radius $\varrho > 0$.

2 A comparative study between Newton's method and iterative method (5)

In this section, we obtain a semilocal convergence result for the method defined by the derivative-free point-to-point iterative process (5), assuming that the nonlinear operator F is differentiable, and we compare it with the well-known Newton–Kantorowich result [10] that ensures the semilocal convergence of Newton's method. On obtaining the similarity between both results, we will test the quadratic convergence of method (5) and verify that the computational efficiency of this method (5) coincides with that of Newton's method. We will also see, through a dynamic study and a numerical test, that the accessibility and numerical behavior of both methods are similar.

In what follows, we consider this method (5) in the following form:

$$\begin{cases} y_0 \quad \text{given in } \Omega, \\ y_{n+1} = y_n - [y_n - \phi_n, y_n + \phi_n; F]^{-1} F(y_n), \quad n \geq 0, \end{cases} \tag{6}$$

where $\phi_n = \|F(y_n)\| \mathbf{Tol}$, such that $\mathbf{Tol} = (tol, tol, \ldots, tol) \in \mathbb{R}^m$ for a real number tol > 0.

2.1 Semilocal convergence

Under certain conditions, Newton's method (2), for a given x_0, gives a sequence $\{x_n\}$ which converges to a solution x^* of $F(x) = 0$. We know as a *majorant sequence* [10], a sequence of scalar numbers $\{t_n\}$*majorizes* a sequence $\{x_n\}$ defined in $\mathbb{R}^m$ if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n \geq 0. \tag{7}$$

The interest of the majorizing sequence is that the convergence of $\{x_n\}$ in $\mathbb{R}^m$ is deduced from that of the scalar sequence $\{t_n\}$. Indeed, if $\{t_n\}$ converges to t^*, then there exists $x^* \in X$, so that the sequence $\{x_n\}$ converges to x^* and

$$\|x^* - x_n\| \leq t^* - t_n, \quad n \geq 0.$$

Next, we focus our attention on the version of the Newton–Kantorovich theorem given by Ortega [20], which is also known as the Newton–Kantorovich theorem and established under the following conditions:

(K1) For $x_0 \in \mathbb{R}^m$, there exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is the set of bounded linear operators from $\mathbb{R}^m$ to $\mathbb{R}^m$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$.

(K2) There exists $L > 0$ such that $\|F'(x) - F'(y)\| \leq L\|x - y\|$, for $x, y \in \Omega$.

Theorem 1. (The Newton–Kantorovich theorem). *Let $F\colon \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable operator defined on a nonempty open convex domain Ω. Suppose the conditions* (K1) *and* (K2) *are satisfied and if $L\beta\eta \leq \frac{1}{2}$, then Newton's method, given by* (2), *converges to a solution x^* of equation $F(x) = 0$, starting* *at* x_0, *and* $x_n, x^* \in \overline{B(x_0, t^*)}$, *for all* $n \in \mathbb{N}$, *where* $t^* = \frac{1-\sqrt{1-2L\beta\eta}}{L\beta}$ *is the smallest positive zero of polynomial* $p(t) = \frac{L}{2}t^2 - \frac{t}{\beta} + d\frac{\eta}{\beta}$.

We can see in [10] that we can interpolate conditions (K1)–(K2) of the Newton–Kantorovich theorem to define the polynomial p. Then, we construct the majorizing sequence used to prove the convergence of Newton's method in $\mathbb{R}^m$ from the applying method

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)}, \quad n \geq 0, \tag{8}$$

to the polynomial p appearing in the previous theorem. To obtain this polynomial, we consider $p(t) = a_0 + a_1 t + a_2 t^2$. In order to obtain $\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leq t_1 - t_0 = -\frac{p(t_0)}{p'(t_0)}$, we will demand that

$$\|\Gamma_0\| \leq \beta \leq -\frac{1}{p'(t_0)} \quad \text{and} \quad \|\Gamma_0 F(x_0)\| \leq \eta \leq -\frac{p(t_0)}{p'(t_0)},$$

taking $t_0 = 0$, we obtain that $a_0 = \frac{\eta}{\beta}$ and $a_1 = -\frac{1}{\beta}$. On the other hand, to set the value of a_2, we can consider, for example, that it is verified that $\|F(x_1)\| \leq p(t_1)$. However,

$$\begin{aligned} F(x_1) &= F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + \tau(x_1 - x_0)) - F'(x_0))\, \mathrm{d}\tau (x_1 - x_0) \\ &= \int_0^1 (F'(x_0 + \tau(x_1 - x_0)) - F'(x_0))\, \mathrm{d}\tau (x_1 - x_0), \end{aligned}$$

and then

$$\|F(x_1)\| \leq \frac{L}{2}\|x_1 - x_0\|^2.$$

Analogously,

$$p(t_1) = p(t_0) + p'(t_0)(t_1 - t_0) + \frac{p''(t_0)}{2!}(t_1 - t_0)^2 = \frac{a_2}{2!}(t_1 - t_0)^2, \tag{9}$$

then it must be verified that $a_2 = L$. Therefore, we obtain the polynomial $p(t)$ given in the Newton–Kantorowich theorem.

To analyze the semilocal convergence of the iterative process given in (6), we consider the characterization of divided difference of the first order of the function F at the points x, y of $\Omega \subseteq \mathbb{R}^m$ ($x \neq y$), introduced in [21], given by

$$[x, y; F] = \int_0^1 F'(\tau x + (1-\tau)y)\, \mathrm{d}\tau. \tag{10}$$

Notice that $[x, x;\ F] = F'(x)$, if F is differentiable.

From now, we establish the semilocal convergence of the sequence $\{y_n\}$, given by the iterative process as in (6), by using the majorant principle (see [10, 17]). For this, we are going to define a second-degree polynomial $q(t)$ that allows us to define a real majorizing sequence. Our idea is to consider

$$\xi_0 = 0, \quad \xi_{n+1} = \xi_n - \frac{q(\xi_n)}{q'(\xi_n)}, \quad n \geq 0. \tag{11}$$

Then, considering the same conditions that we have demanded for the semilocal convergence of Newton's method, that is (K1) and (K2), we have

$$\begin{aligned} \|I - \Gamma_0[y_0 - \phi_0, y_0 + \phi_0; F]\| &\leq \|\Gamma_0\| \|F'(y_0) - [y_0 - \phi_0, y_0 + \phi_0; F]\ \| \\ &\leq \|\Gamma_0\| \left\| F'(y_0) - \int_0^1 F'(\tau(y_0 - \phi_0) + (1-\tau)(y_0 + \phi_0)) \mathrm{d}\tau \right\| \\ &\leq \|\Gamma_0\| \left\| \int_0^1 \left[F'(y_0) - F'(y_0 - (1-2\tau)\phi_0)\right] \mathrm{d}\tau \right\| \\ &\leq \beta L \int_0^1 |1 - 2\tau| \|\phi_0\| \mathrm{d}\tau \leq \frac{tol \beta L \delta}{2}, \end{aligned} \tag{12}$$

where $\|F(y_0)\| \leq \delta$, with which we can consider $\eta = \beta\delta$. Then, if $tol\, L\beta\delta < 2$, there exists the operator $[y_0 - \phi_0, y_0 + \phi_0; F]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, for $y_0 - \phi_0, y_0 + \phi_0 \in \Omega$, and if

$$\|[y_0 - \phi_0, y_0 + \phi_0; F]^{-1}\| \leq \frac{2\beta}{2 - tol\, \beta L \delta},$$

we denote $\widetilde{\beta} = \frac{2\beta}{2 - tol \beta L \delta}$.

As shown in the previous situation of Newton's method, we need that $\|y_1 - y_0\| = \|[y_0 - \phi_0, y_0 + \phi_0; F]^{-1} F(y_0)\| \leq \xi_1 - \xi_0 = -\frac{q(\xi_0)}{q'(\xi_0)}$, and we will demand that

$$\|[y_0 - \phi_0, y_0 + \phi_0; F]^{-1}\| \leq \widetilde{\beta} \leq -\frac{1}{q'(\xi_0)} \quad \text{and} \quad \|[y_0 - \phi_0, y_0 + \phi_0; F]^{-1} F(y_0)\| \leq \eta \leq -\frac{q(\xi_0)}{q'(\xi_0)},$$

taking $\xi_0 = 0$, and proceeding as in the case of Newton's method, we obtain the majorant polynomial

$$q(\xi) = K\xi^2 - \frac{\xi}{\widetilde{\beta}} + \delta$$

Now, we have to obtain K, the director coefficient of the polynomial. For this, from the algorithm of the iterativemethod (6), we have

$$F(y_1) = \int_0^1 (F'(y_0 + \tau(y_1 - y_0)) - F'(y_0)) \mathrm{d}\tau (y_1 - y_0) + \left(F'(y_0) - [y_0 - \phi_0, y_0 + \phi_0; F]\right)(y_1 - y_0), \tag{13}$$

and then, it follows

$$\begin{aligned} \|F(y_1)\| &\leq \frac{L}{2} \|y_1 - y_0\|^2 + \frac{tol\, L\delta}{2} \cdot \|y_1 - y_0\| \\ &\leq \frac{L}{2} (\xi_1 - \xi_0)^2 + \frac{tol\, L}{2} \cdot (-q'(\xi_0))(\xi_1 - \xi_0)^2 \\ &\leq \frac{L}{2} \left(1 + \frac{tol}{\widetilde{\beta}}\right) (\xi_1 - \xi_0)^2. \end{aligned}$$

Taking into account (9) for the polynomial $q(t)$, we obtain $K = L\left(1 + \frac{tol}{\widetilde{\beta}}\right)$.

Now, we define the polynomial

$$q(\xi) = \frac{K}{2}\xi^2 - \frac{\xi}{\widetilde{\beta}} + \delta. \tag{14}$$

Note that polynomial (14) has two positive roots, $\xi^* = \frac{1-\sqrt{1-2K\widetilde{\beta}\eta}}{\widetilde{\beta}K}$ and $\xi^{**} = \frac{1+\sqrt{1-2K\widetilde{\beta}\eta}}{\widetilde{\beta}}$, such that $\xi^* \leq \xi^{**}$ if $K\widetilde{\beta}\eta \leq \frac{1}{2}$. Therefore, we consider $\xi \in [0, \xi']$ with $\xi' > \xi^{**}$.

Now, by the usual reasoning for the majorizing polynomials, as shown in [10], the following result is easily proved.

Lemma 2. *Let $q(\xi)$ be the polynomial given by $q(\xi) = \frac{K}{2}\xi^2 - \frac{\xi}{\widetilde{\beta}} + \delta$ and consider the real iterative process*

$$\xi_0 = 0, \quad \xi_{n+1} = \xi_n - \frac{q(\xi_n)}{q'(\xi_n)}, \quad n \geq 0. \tag{15}$$

If $K\widetilde{\beta}\eta \leq \frac{1}{2}$, the real sequence $\{\xi_n\}$ increases and converges to ξ^*. □

Next, we will prove that the sequence $\{\xi_n\}$ majorizes the sequence $\{y_n\}$ given by method (6).

Lemma 3. *If $y_n, y_n - \phi_n, y_n + \phi_n \in \Omega$, for all $n \in \mathbb{N}$, it follows*
(i) $\|F(y_n)\| \leq q(\xi_n)$,
(ii) *There exists $[y_n - \phi_n, y_n + \phi_n; F]^{-1}$, with $\|[y_n - \phi_n, y_n + \phi_n; F]^{-1}\|\| \leq -\frac{1}{q'(\xi_n)}$, and*
(iii) $\|y_{n+1} - y_n\| \leq \xi_{n+1} - \xi_n$.

Proof. We prove these conditions by means of an inductive procedure. In the first place, (i) is trivially verified since it has been used to define K. To prove (ii), as in (12), we have

$$\begin{aligned}
\|I - \Gamma_0[y_1 - \phi_1, y_1 + \phi_1; F]\| &\leq \|\Gamma_0\| \left\| \in \xi_0^1 \left[F'(y_0) - F'(y_1 - (1-2\tau)\phi_1)\right] \mathrm{d}\tau \right\| \\
&\leq \beta L \left(\|y_1 - y_0\| + \int_0^1 |1 - 2\tau| \|\phi_1\| \mathrm{d}\tau \right) \leq \beta \left(L(\xi_1 - \xi_0) + \frac{tol\, L}{2} q(\xi_1) \right) \\
&\leq \beta L \left((\xi_1 - \xi_0) + \frac{tol}{2}(-q'(\xi_0)(\xi_1 - \xi_0)) \right) < \beta K(\xi_1 - \xi_0) = \beta \left(q'(\xi_1) + \frac{1}{\widetilde{\beta}} \right) \\
&= \frac{\beta}{\widetilde{\beta}} + \beta q'(\xi_1) < 1 + \beta q'(\xi_1) < 1.
\end{aligned} \tag{16}$$

Then, by the Banach Lemma for inverse operators [17], the operator $[y_1 - \phi_1, y_1 + \phi_1; F]^{-1}$ exists and is such that

$$\|[y_1 - \phi_1, y_1 + \phi_1; F]^{-1}\| \leq -\frac{1}{q'(\xi_1)}.$$

Therefore, from (i) and (ii) for $n = 1$, it is easy to follow (iii).

Now, suppose (i)–(iii) hold for $n = 1, \ldots, k-1$, then let's see whether it holds for $n = k$. Following (13), we have

$$\begin{aligned}
\|F(y_k)\| &\leq \frac{L}{2}\|y_k - y_{k-1}\|^2 + \left\| \int_0^1 \left[F'(y_{k-1}) - F'(y_{k-1} - (1-2\tau)\phi_{k-1})\right] \mathrm{d}\tau \right\| \|y_k - y_{k-1}\| \\
&\leq \frac{L}{2}\|y_k - y_{k-1}\|^2 + L\int_0^1 |1 - 2\tau| \|\phi_{k-1}\| \mathrm{d}\tau \|y_k - y_{k-1}\| \\
&\leq \frac{L}{2}(\xi_k - \xi_{k-1})^2 + \frac{tol\, L}{2} q(\xi_{k-1})(\xi_k - \xi_{k-1})
\end{aligned}$$

$$\leq \frac{L}{2}(\xi_k - \xi_{k-1})^2 + \frac{tol\,L}{2} - q'(\xi_{k-1})(\xi_k - \xi_{k-1})^2$$
$$\leq \frac{L}{2}(\xi_k - \xi_{k-1})^2 + \frac{tol\,L}{2} \cdot (-q'(\xi_0))(\xi_k - \xi_{k-1})^2$$
$$\leq \frac{L}{2}\left(1 + \frac{tol}{\widetilde{\beta}}\right)(\xi_k - \xi_{k-1})^2 = q(\xi_k),$$

which proves (i) for $n = k$.

To prove (ii) for $n = k$, as in (16), we have

$$\begin{aligned}\|I - \Gamma_0[y_k - \phi_k, y_k + \phi_k; F]\| &\leq \|\Gamma_0\| \left\| \int_0^1 \left[F'(y_0) - F'(y_k - (1-2\tau)\phi_k)\right] \mathrm{d}\tau \right\| \\ &\leq \beta L \left(\|y_k - y_0\| + \int_0^1 |1-2\tau| \|\phi_k\| \mathrm{d}\tau \right) \leq \beta \left(L(\xi_k - \xi_0) + \frac{tol\,L}{2} q(\xi_k) \right) \\ &\leq \beta L \left((\xi_k - \xi_0) + \frac{tol}{2}(-q'(\xi_0)(\xi_1 - \xi_0)) \right) < \beta K(\xi_k - \xi_0) = \beta \left(q'(\xi_k) + \frac{1}{\widetilde{\beta}} \right) \\ &= \frac{\beta}{\widetilde{\beta}} + \beta q'(\xi_k) < 1 + \beta q'(\xi_k) < 1.\end{aligned}$$

Then, by the Banach Lemma for inverse operators, the operator $[y_k - \phi_k, y_k + \phi_k; F]^{-1}$ exists and is such that

$$\|[y_k - \phi_k, y_k + \phi_k; F]^{-1}\| \leq -\frac{1}{q'(\xi_k)},$$

which proves (ii) for $n = k$.

As (iii) is easily followed from (i) and (ii), the result is proved. □

Now, from the previous results, we prove the semilocal convergence of iterative process given in (6).

Theorem 4. *Let $F\colon \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable operator defined on a nonempty open convex domain Ω of $\mathbb{R}^m$. Suppose, conditions* (K1) *and* (K2), *with $tol < \frac{2}{L\beta\delta}$, are satisfied and if $K\widetilde{\beta}\eta \leq \frac{1}{2}$ and $B(y_0, \xi^* + tol\,\delta) \subset \Omega$, then method* (6) *converges to a solution x^* of the equation $F(x) = 0$, starting at y_0, and $y_n, y_n - \phi_n, y_n + \phi_n, x^* \in \overline{B(y_0, \xi^* + tol\,\delta)}$, for all $n \geq 0$.*

Proof. To prove the semilocal convergence of the method given by (6), we use an inductive process.

In the first place, it is obvious that $\|y_0 - \phi_0 - y_0\| = \|y_0 + \phi_0 - y_0\| < \xi^* + tol\ \delta$. Moreover,

$$\|y_1 - y_0\| \leq \widetilde{\beta}\delta = \xi_1 - \xi_0 < \xi^* + tol\,\delta.$$

Therefore, it follows that $y_0 - \phi_0, y_0 + \phi_0, y_1 \in B(y_0, \xi^* + tol\,\delta) \subset \Omega$ and then, we can define y_2. Next, from Lemma 3, $\|F(y_1)\| \leq q(\xi_1) \leq q(\xi_0) = \delta$. Then, we get

$$\begin{aligned}\|y_1 - \phi_1 - y_0\| &\leq \|y_1 - y_0\| + tol\,\|F(y_1)\| < \xi^* + tol\,q(\xi_0) = \xi^* + tol\,\delta,\\ \|y_1 + \phi_1 - y_0\| &\leq \|y_1 - y_0\| + tol\,\|F(y_1)\| < \xi^* + tol\,q(\xi_0) = \xi^* + tol\,\delta.\end{aligned}$$

Moreover, from Lemma 3, there exists the operator $[y_1, z_1; F]^{-1}$, and we can to define y_2. As a consequence,

$$\|y_2 - y_1\| \leq \|[y_1, z_1; F]^{-1}\| \|F(y_1)\| \leq -\frac{q(\xi_1)}{q'(\xi_1)} \leq \xi_2 - \xi_1,$$
$$\|y_2 - y_0\| \leq \|y_2 - y_1\| + \|y_1 - y_0\| \leq \xi_2 - \xi_0 < \xi^* - \xi_0 < \xi^* + tol\,\delta,$$

$y_2 \in B(y_0, \xi^* + tol\,\delta) \subset \Omega$.

Now, we suppose that $y_n - \phi_n, y_k + \phi_n, y_{n+1} \in B(y_0, \xi^* + tol\,\delta)$ and $\|y_{n+1} - y_n\| < \xi_{n+1} - \xi_n$, for $n = 1, \ldots, k-1$.

Next, from Lemma 3 and proceeding as in the first step, we obtain that $y_k - \phi_k, y_k + \phi_k \in B(y_0, \xi^* + tol\,\delta)$ and

$$\|y_{k+1} - y_k\| \leq \|[y_k - \phi_k, y_k + \phi_k; F]^{-1}\| \|F(y_k)\| \leq -\frac{q(\xi_k)}{q'(\xi_k)} = \xi_{k+1} - \xi_k, \tag{17}$$

$$\|y_{k+1} - y_0\| \leq \|y_{kn+1} - y_k\| + \|y_k - y_0\| \leq \xi_{k+1} - \xi_0 < \xi^* - \xi_0 < \xi^* + tol\,\delta,$$

proves the induction.

After that, as $\{\xi_n\}$ converges to ξ^*, from (17), it follows that the sequence $\{y_n\}$ is convergent. Let $\lim_n y_n = x^* \in \overline{B(y_0, \xi^* + m\delta)}$, to see that x^* is a solution of $F(x) = 0$, it is enough to note that $\|F(y_n)\| \leq q(\xi_n)$, and by the continuities of F and q, it follows that $F(x^*) = 0$. □

Remark 5. Note that, if we consider tol small enough, the values of K and L are close, as is the case with $\widetilde{\beta}$ and β. Therefore, the convergence condition required for method (6) in Theorem 4 is close to the convergence condition required for Newton's method in Theorem 1. Therefore, the behavior of both methods turns out to be similar with regard to semilocal convergence.

Next, we get a unique result for method (6).

Theorem 6. *In the conditions of the previous theorem, the solution x^* is unique in $B(y_0, r) \cap \Omega$, where $r = \frac{2}{L\beta} - (\xi^* + tol\,\delta)$, provided that $L\beta(\xi^* + tol\,\delta) < 2$.*

Proof. To prove the uniqueness of the solution y^*, we suppose that we have a solution $z^* \in B(y_0, r) \cap \Omega$ of $F(x) = 0$ such that $z^* \neq y^*$. Consider

$$F(z^*) - F(y^*) = \int_{y^*}^{z^*} F'(x)\mathrm{d}x = \int_0^1 F'(y^* + \tau(z^* - y^*))\mathrm{d}\tau(z^* - y^*) = 0$$

and the operator $J = \int_0^1 F'(y^* + \tau(z^* - y^*))\mathrm{d}\tau$. If

$$\|I - \Gamma_0 J\| \leq \|\Gamma_0\| \int_0^1 \|F'(y^* + \tau(z^* - y^*)) - F'(y_0)\|\mathrm{d}\tau < \frac{L\beta}{2}(\xi^* + tol\delta + r) = 1,$$

then the operator J is invertible, provided $L\beta(\xi^* + tol\,\delta) < 2$. Therefore $z^* = y^*$. □

2.2 A numerical comparative study

We consider a special case of nonlinear Fredholm integral equation [12]

$$x(s) = f(s) + \lambda \int_a^b G(s,t)H(x(t))\mathrm{d}t, \quad s \in [a,b], \tag{18}$$

where $\lambda \in \mathbb{R}$ and $-\infty < a < b < +\infty$; the function $f(s)$ is continuous on $[a,b]$ and, given, the kernel of (18) is a Green's function defined as follows:

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t, \end{cases}$$

which is a continuous function in $[a,b] \times [a,b]$, H is a known continuous function in $\mathbb{R}$, and x is a solution to be determined in $\mathcal{C}[a,b]$, the set of continuous functions in $[a,b]$.

These equations are related to boundary value problems for differential equations, since they can be reformulated as two-point boundary value problems or elliptic partial differential equations with nonlinear boundary conditions [9]. Moreover, these equations appear in several applications to the real world: the theory of elasticity, engineering, mathematical physics, potential theory, electrostatics, and radiative heat-transfer problems [6].

Now, our aim is to apply the theoretical result obtained in previous section in order to solve a nonlinear problem. We consider the nonlinear integral Eq. (18) defined in $C[0,1]$ with the max norm, as shown in [3], where $f(x) = -1$, $\lambda \in \mathbb{R}$, $H(x(t)) = x(t)^3$ and $s, t \in [0,1]$.

First of all, we have to discretize the problem for transforming it into a finite dimensional system in $\mathbb{R}^m$. For this purpose, we approximate the integral by means of Gauss–Legendre quadrature:

$$\int_a^b \Phi(t)\mathrm{d}t = \sum_{j=1}^{m} w_j \Phi(t_j),$$

where w_j and t_j are the corresponding weights and nodes, respectively. Therefore, by denoting the approximation $x(s_i)$ by x_i for $i = 1, \ldots, m$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, we can define the following nonlinear operator $F\colon \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ describing our problem:

$$F_i(\mathbf{x}) = x_i - 1 - \lambda \sum_{j=1}^{m} w_{ij} x_j^3, \quad i = 1, 2, \ldots, m$$

where $w_{ij} = w_j G(t_i, t_j), i, j = 1, \ldots, m$, that is,

$$F(\mathbf{x}) = \begin{pmatrix} x_1 - 1 - \lambda \sum_{j=1}^{m} w_{1j} x_j^3 \\ x_2 - 1 - \lambda \sum_{j=1}^{m} w_{2j} x_j^3 \\ \vdots \\ x_m - 1 - \lambda \sum_{j=1}^{m} w_{mj} x_j^3 \end{pmatrix}.$$

The Jacobian matrix of F can be expressed as

$$F'(\mathbf{x}) = I - 3\lambda W \ \operatorname{diag}\left(x_1^2, x_2^2, \ldots, x_m^2\right)$$

where $W = (w_{ij}), i, j = 1, \ldots, m$. Therefore, by choosing $\lambda \in \mathbb{R}$ and the starting guess $\mathbf{x}_0 \in \mathbb{R}^m$ such that

$$3\|W\|\,|\lambda|\,\|\operatorname{diag}\left(x_1^2, x_2^2, \ldots, x_m^2\right)\| \le 1,$$

by applying Banach lemma, we found that Γ_0 exists and

$$\|\Gamma_0\| \le \frac{1}{1 - 3\|W\|\,|\lambda|\,\|\operatorname{diag}\left(x_1^2, x_2^2, \ldots, x_m^2\right)\|}.$$

On the other hand, we have that

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le 3\,|\lambda|\,\|W\|\|\operatorname{diag}(x_1 + y_1, x_2 + y_1, \ldots, x_m + y_m)\|\|x - y\|$$

Then, by taking $m = 8$, $\mathbf{y}_0 = (0.001, \ldots, 0.001)$, $\lambda = 0.1$ and working in the domain $\Omega = B(0, 3/2)$, with the infinite norm, following the semilocal convergence study previously developed, we have

$$\|W\| \le 0.1173, \quad \|\Gamma_0\| \le \beta = 1.000000035, \quad \|\Gamma_0 F(\mathbf{y}_0)\| \le \eta = 1.001000035, \quad L \le 0.10557$$

Then, with these values, we can apply Theorem 1 and obtain the semilocal convergence radius for Newton's method. We compare this value with the radius obtained by using Theorems 4 and 6 established for the derivative-free point-to-point iterative process introduced in this article, given by (6). The results are shown in Table 1. It is observed that, for the iterative process (6), the convergence radius is quite similar to the Newton's method radius that is always the smaller value, showing a good behavior of these methods, even in the case of the iterative process (6), where we do not need the condition of differentiability for operator F. Moreover, as it was expected, when the value of the parameter tol decreases for the iterative process (6), both the radius of convergence and that of uniqueness improve.

Finally, we approximate the solution of this nonlinear system by using the iterative methods mentioned. We run the algorithms in Matlab 2019 by using variable precision arithmetic with 100 digits and stopping criteria 10^{-50}. In Tables 2–6, we can observe that k is the number of iterations needed, the distance between the last two iterates, $\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$, and the value of the operator F at the approximated solution $\|F(\mathbf{y}_{n+1})\|$. We have considered different starting points and different values for parameter tol. The results show that the behavior of the iterative process (6) analyzed in this article gets practically the same results as the Newton's method. The solution of the problem with six decimal digits is $\mathbf{y}_{n+1} = (1.000426, 1.003878, 1.011652, 1.024517, 1.033415, 1.025248, 1.042486, 1.031837)$.

2.3 Efficiency and accessibility analysis

Next, we compare the computational efficiency of iterative processes (2) and (6). As shown in [25], it is well-known that the computational efficiency index of an iterative process is $\mathrm{CE} = q_1^{1/q_2}$, where q_1 is the order

Table 1: Semilocal convergence radius: Newton versus iterative process (6).

Method	(2) Newton	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
$\widetilde{\beta}$		1.0216	1.0161	1.0107	1.0053	1.0005
ξ^*		1.0902	1.0824	1.0748	1.0675	1.0611
t^*	1.0603					
Radius	1.0603	1.4906	1.38268	1.2750	1.1676	1.0710
Uniqueness radius	17.8844	17.4542	17.5621	17.669	17.7772	17.8737

Table 2: Numerical results with starting guess $\mathbf{y}_0 = (0.001, 0.001, \dots, 0.001)^T$.

Method	(2) Newton	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
k	7	6	7	7	7	7
$\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$	8.9634×10^{-83}	2.0316×10^{-51}	2.3494×10^{-98}	1.2074×10^{-87}	7.8274×10^{-84}	8.7558×10^{-83}
$\|F(\mathbf{y}_{n+1})\|$	8.2521×10^{-83}	1.8747×10^{-51}	2.1631×10^{-98}	1.1116×10^{-87}	7.2063×10^{-84}	8.0610×10^{-83}

Table 3: Numerical results with starting guess $\mathbf{y}_0 = (0.1, 0.1, \dots, 0.1)^T$.

Method	(2) Newton	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
k	7	6	7	6	6	6
$\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$	3.9050×10^{-83}	5.9943×10^{-58}	6.8271×10^{-93}	1.2677×10^{-86}	6.3478×10^{-84}	3.8368×10^{-83}
$\|F(\mathbf{y}_{n+1})\|$	3.5952×10^{-83}	5.5243×10^{-58}	6.2855×10^{-93}	1.1671×10^{-86}	5.8440×10^{-84}	3.5323×10^{-83}

Table 4: Numerical results with starting guess $\mathbf{y}_0 = (1, 1, \dots, 1)^T$.

Method	(2) Newton	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
k	6	6	6	6	6	6
$\Vert\mathbf{y}_{n+1} - \mathbf{y}_n\Vert$	8.9642×10^{-83}	7.9135×10^{-83}	8.3581×10^{-83}	8.6899×10^{-83}	8.8948×10^{-83}	8.9635×10^{-83}
$\Vert F(\mathbf{y}_{n+1})\Vert$	8.2528×10^{-83}	7.2855×10^{-83}	7.6948×10^{-83}	8.0003×10^{-83}	8.1890×10^{-83}	8.2521×10^{-83}

Table 5: Numerical results with bigger Tol and $\mathbf{y}_0 = (1, 1, \dots, 1)^T$.

Method	(2) Newton	(6) tol = 3	(6) tol = 2	(6) tol = 1	(6) tol = 0.75	(6) tol = 0.5
k	6	6	6	6	6	6
$\Vert\mathbf{y}_{n+1} - \mathbf{y}_n\Vert$	8.9642×10^{-83}	2.1116×10^{-83}	4.7903×10^{-83}	7.6817×10^{-83}	8.2200×10^{-83}	8.6259×10^{-83}
$\Vert F(\mathbf{y}_{n+1})\Vert$	8.2521×10^{-83}	1.9441×10^{-83}	4.4102×10^{-83}	7.0721×10^{-83}	7.5677×10^{-83}	7.9414×10^{-83}

Table 6: Numerical results with starting guess $\mathbf{y}_0 = (2, 2, \dots, 2)^T$.

Method	(2) Newton	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
k	7	7	7	7	7	7
$\Vert\mathbf{y}_{n+1} - \mathbf{y}_n\Vert$	2.0551×10^{-58}	2.2262×10^{-56}	3.4756×10^{-57}	7.6853×10^{-58}	2.8851×10^{-58}	2.0622×10^{-58}
$\Vert F(\mathbf{y}_{n+1})\Vert$	1.8922×10^{-58}	2.0498×10^{-56}	3.2001×10^{-57}	7.0762×10^{-58}	2.6564×10^{-58}	1.8987×10^{-58}

of convergence and q_2 is the number of operations (products and divisions) needed to apply it, which is defined as the computational cost of doing an iteration of the algorithm. This computational efficiency index represents a good measure of the efficiency of the iterative process.

Regarding the value of q_1, it is known [12] that the Newton's method has R-order of convergence [11] of at least two if $t^* \neq t^{**}$ or at least one if $t^* = t^{**}$. Therefore, Newton's method has at least quadratic convergence if $t^* \neq t^{**}$[10]. Now, we prove that the iterative process (6) verifies the same conditions as Newton's method and, therefore, it has the same value for q_1 as Newton's method.

Theorem 7. *Method* (6) *has R-order of convergence of at least two if $\xi^* \neq \xi^{**}$ or at least one if $\xi^* = \xi^{**}$.*

Proof. It is clear that $q(\xi) = \frac{K}{2}(\xi - \xi^*)(\xi - \xi^{**})$. Then, it is known by the Ostrowski's technique for *a priori* error estimates (see [12, 22]) that

– If $\xi^* < \xi^{**}$, then

$$\xi^* - \xi_n = \frac{(\xi^{**} - \xi^*)\theta^{2^n}}{1 - \theta^{2^n}}, \quad \text{where} \quad \theta = \frac{\xi^*}{\xi^{**}} < 1. \tag{19}$$

– If $\xi^* = \xi^{**}$, then

$$\xi^* - \xi_n = \frac{\xi^*}{2^n}. \tag{20}$$

First, from (17), it follows that $\{\xi_n\}$ is a majorizing sequence of $\{y_n\}$. Then, for $n \geq 1$ and $m \geq 1$, we have

$$\Vert y_{n+m} - y_n\Vert \leq \sum_{i=n}^{n+m-1} \Vert y_{i+1} - y_i\Vert \leq \sum_{i=n}^{n+m-1} (\xi_{i+1} - \xi_i) = \xi_{n+m} - \xi_n,$$

so that, if $m \to \infty$, from the convergence of $\{y_n\}$ and $\{\xi_n\}$, it follows

$$\|y^* - y_n\| \le \xi^* - \xi_n.$$

Therefore,
(a) If $\xi^* < \xi^{**}$, then

$$\|y^* - y_n\| \le \theta^{2^n} \frac{(\xi^{**} - \xi^*)}{1 - \theta^{2^n}}, \quad \text{where} \quad \theta = \frac{\xi^*}{\xi^{**}} < 1. \tag{21}$$

(b) If $\xi^* = \xi^{**}$, then

$$\|y^* - y_n\| \le \frac{1}{2^n} \xi^*. \tag{22}$$

Now, from (a) and (b), the result is proved. □

On the other hand, it is clear that both methods (2) and (6) need to perform an LU factorization and solve the corresponding triangular system at each step. Therefore, both methods have the same operational cost. Therefore, they also have the same value of q_2. Thus, Newton's method and method (6) have the same computational efficiency.

Now, we study the accessibility of the previous iterative processes (2) and (6) using similar procedures as in [18, 19]. For this purpose, we will analyze the dynamical behavior of both iterative processes. We will apply methods (2) and (6) to the complex polynomial equation $p(z) = 0$ where

$$p(z) = z^3 - z$$

with three different roots $z = \pm 1$ and $z = 0$. We paint in red, blue, and yellow the convergence after 100 iterations to the roots of the polynomial with a tolerance of 10^{-3}; in other cases, the point is painted in black.

As shown in Figure 2, the dynamical behavior of method (6) for different values of *Tol*, when the value of *Tol* is close to 0, is similar to the behavior of the Newton's method, see Figure 1.

Once the accessibility has been graphically analyzed and showing that method (6) has similar accessibility than method (2), we want to prove it in a numerical way, and for that purpose, we compute the percentage of points which converges after 200 iterations with a tolerance of 10^{-3} to any of the roots, and this information is tabulated in Table 7.

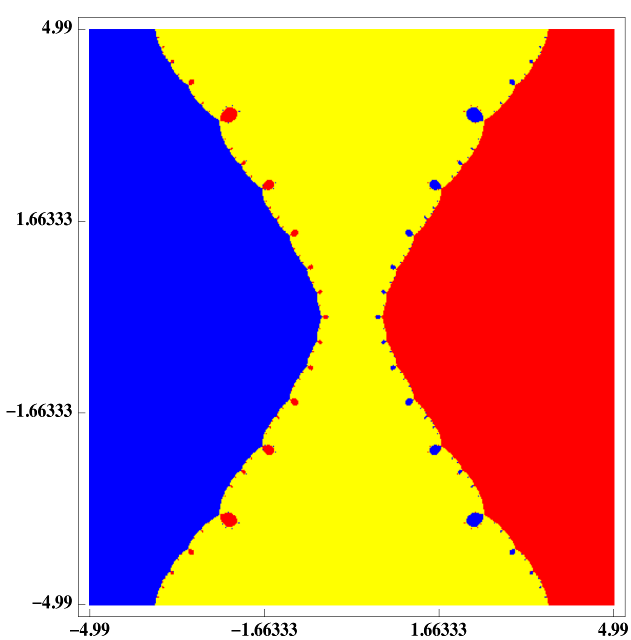
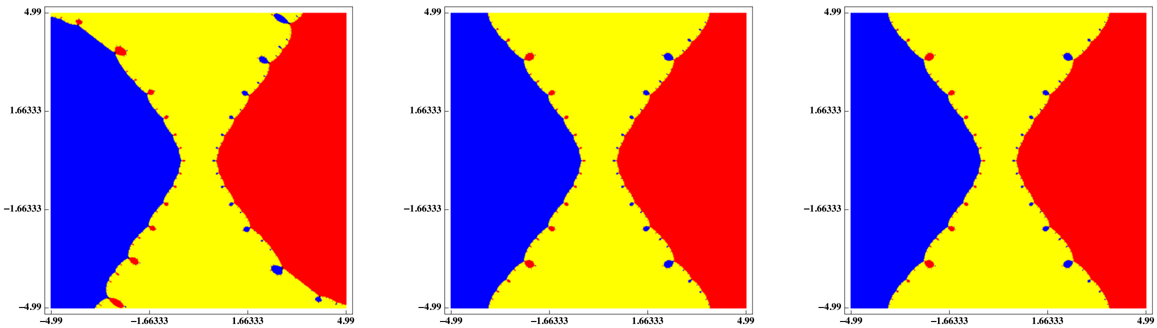


Figure 1: Newton's method applied to $p(z) = z^3 - z$.



Method (6) with $Tol = 0.02$ | Method (6) with $Tol = 0.002$ | Method (6) with $Tol = 0.0002$

Figure 2: Basins of attraction associated to method (6) applied to polynomial $p(z) = z^3 - z$ with different values for tolerance.

Table 7: Percentage of convergence points for $p(z) = z^3 - z$.

Method	Percentage of convergent points
(6) with $Tol = 0.02$	99.97%
(6) with $Tol = 0.002$	99.99%
(6) with $Tol = 0.0002$	100.00%
(2)	100.00%

Therefore, we can conclude that the Newton's method and the iterative process given in (6) have similar accessibility, taking into account the variability of the parameter *Tol*.

3 The iterative method (6) for nondifferentiable systems of equations

Next, for the semilocal convergence of iterative process (6) for nondifferentiable systems of equations, we will give conditions for the starting point y_0 and the operator F in such a way that we can ensure the existence of a solution y^* of the system of Eq. (1), providing a ball of existence of solution of (1), $B(y_0, R)$, called the existence ball. We will also obtain a result of uniqueness of solution.

3.1 The semilocal convergence

In order to prove the semilocal convergence for the iterative process (6), we will denote $D_n = [y_n - \phi_n, y_n + \phi_n;\ F]$ and assume the following conditions

(I) There exists D_0^{-1}, for some $y_0 \in \Omega \subseteq \mathbb{R}^m$, with $\|D_0^{-1}\| \leq \beta$, $\|D_0^{-1}F(y_0)\| \leq \eta$ and $\|F(y_0)\| \leq \delta$.

(II) $\|[x, y;\ F] - [u, v;\ F]\| \leq L + K(\|x - u\|^p + \|y - v\|^p);\ L, K \geq 0;$ with $x, y, u, v \in \Omega;\ x \neq y;\ u \neq v$, and $p \in [0, 1]$.

First, we get a technical Lemma, the proof of which is immediate.

Lemma 8. *Let $\{y_n\}$ be the sequence generated by method* (6) *and $y_{n+1} \neq y_n$ with $y_n, y_{n+1} \in \Omega$. Then*

$$F(y_{n+1}) = \left([y_{n+1}, y_n; F] - D_n\right)\left(y_{n+1} - y_n\right). \tag{23}$$

□

Theorem 9. *Under the conditions (I)–(II), if the equation*

$$t = \frac{\beta\delta(1-\beta(L+K(2t^p+2\delta^p tol^p)))}{1-\beta(L+K(2t^p+2\delta^p tol^p))-M} + \delta tol, \tag{24}$$

where $M = \beta(L + K\delta^p(\beta^p + 2\mathrm{Tol}^p))$ has at least one positive real root, and the smallest positive real root denoted by R satisfies

$$M + \beta(L + K(2R^p + 2\delta^p tol^p)) < 1,$$

and $B(y_0, R) \subset \Omega$, then the sequence $\{y_n\}$ generated by (6) is well-defined and converges to y^*, a solution of $F(x) = 0$, with $y_n, y^* \in B(y_0, R)$. Moreover, y^* is unique in $B(y_0, R) \subset \Omega$.

Proof. First, we will prove that the sequence $\{y_n\}$ generated by (6) is well-defined and $y_n \in B(y_0, R)$. Note that the smallest positive real root R of (24) satisfies

$$R = \frac{\beta\delta}{1-S} + \delta tol, \tag{25}$$

where $S = \frac{M}{1-\beta(L+K(2t^p+2\delta^p Tol^p))} \in (0,1)$. If $y_0 \in \Omega$ satisfies condition (I), then y_1 is well-defined and $\|y_1 - y_0\| \leq \|D_0^{-1}\|\|F(y_0)\| \leq \beta\delta < R$. Therefore, $y_1 \in B(y_0, R)$. Using Lemma 8, we get

$$F(y_1) = \left([y_1, y_0; F] - [y_0 - \phi_0, y_0 + \phi_0; F]\right)(y_1 - y_0).$$

Taking norms on both sides, we get

$$\begin{aligned}\|F(y_1)\| &\leq \big\|[y_1, y_0; F] - [y_0 - \phi_0, y_0 + \phi_0; F]\big\|\, \|y_1 - y_0\| \\ &\leq (L + K(\|y_1 - y_0\|^p + \|\phi_0\|^p + \|\phi_0\|^p))\,\|y_1 - y_0\| \\ &\leq \left(L + K\left(\beta^p\delta^p + 2\delta^p Tol^p\right)\right)\beta\delta \\ &\leq M\delta,\end{aligned}$$

where $M = \beta\left(L + K(\beta^p\delta^p + 2\delta^p tol^p)\right)$. As $M < 1$, $\|F(y_1)\| < \delta$. Using (25), we get $\|y_1 \pm \phi_1 - y_0\| \leq \|y_1 - y_0\| + \|\phi_1\| \leq \beta\delta + \delta Tol < R$ and hence $y_1 \pm \phi_1 \in B(y_0, R)$. Again by using (II), we have

$$\begin{aligned}\|I - D_0^{-1}D_1\| &\leq \|D_0^{-1}\|\|D_1 - D_0\| \\ &\leq \beta\|[y_1 - \phi_1, y_1 + \phi_1; F] - [y_0 - \phi_0, y_0 + \phi_0; F]\| \\ &\leq \beta\,(L + K(\|y_1 - \phi_1 - y_0\|^p + \|\phi_0\|^p + \|y_1 + \phi_1 - y_0\|^p + \|\phi_0\|^p)) \\ &\leq \beta\left(L + K(2R^p + 2\delta^p tol^p)\right) < 1.\end{aligned}$$

Hence, by the Banach Lemma for inverse operators, D_1^{-1} exists and

$$\|D_1^{-1}\| \leq \frac{\beta}{1-\beta(L+K(2R^p+2\delta^p Tol^p))}.$$

Therefore,

$$\|y_2 - y_1\| \leq \|D_1^{-1}\|\|F(y_1)\| \leq \frac{\beta}{1-\beta(L+K(2R^p+2\delta^p Tol^p))} M\delta = S\beta\delta.$$

Since $S < 1$, $\|y_2 - y_1\| < \beta\delta$ and $\|y_2 - y_0\| \leq \|y_2 - y_1\| + \|y_1 - y_0\| \leq (1+S)\|y_1 - y_0\| < \frac{\beta\delta}{1-S} < R$. Therefore, $y_2 \in B(y_0, R)$. Using Lemma 8 and conditions (II), we have

$$\begin{aligned}\|F(y_2)\| &\leq \|[y_2, y_1; F] - [y_1 - \phi_1, y_1 + \phi_1; F]\|\|y_2 - y_1\| \\ &\leq (L + K(\|y_2 - y_1\|^p + 2\|\phi_1\|^p))\,\|y_2 - y_1\| \\ &\leq \left(L + K(R^p + 2\delta^p \mathrm{Tol}^p)\right)\|y_2 - y_1\|.\end{aligned}$$

Also, $\|y_2 \pm \phi_2 - y_0\| \le \|y_2 - y_0\| + \|\phi_2\| < \frac{\beta\delta}{1-S} + \delta Tol = R$ and $y_2 \pm \phi_2 \in B(y_0, R)$.
Now,

$$\begin{aligned}\|y_3 - y_2\| &\le \|D_2^{-1}\|\|F(y_2)\| \\ &\le \frac{\beta(L + K(R^p + 2\delta^p Tol^p))}{1 - \beta(L + K(2R^p + 2\delta^p Tol^p))}\|y_2 - y_1\| \\ &= S\|y_2 - y_1\|.\end{aligned}$$

Therefore, $\|y_3 - y_2\| \le S^2\|y_1 - y_0\| < \|y_1 - y_0\| < R$.

In a similar manner, by using the principle of mathematical induction, we can establish the following recurrence relations.

$$\begin{aligned}\|D_n^{-1}\| &\le \frac{\beta}{1 - \beta(L + K(2R^p + 2\delta^p Tol^p))}, \\ \|F(y_n)\| &\le \left(L + K(R^p + 2\delta^p Tol^p)\right)\|y_n - y_{n-1}\|, \\ \|y_{n+1} - y_n\| &\le S\|y_n - y_{n-1}\| \le S^n\|y_1 - y_0\| < \beta\delta, \\ \|y_{n+1} - y_0\| &\le \frac{1 - S^{n+1}}{1 - S}\|y_1 - y_0\| < \frac{\beta\delta}{1 - S} < R, \\ \|y_{n+1} \pm \phi_{n+1} - y_0\| &< \frac{\beta\delta}{1 - S} + \delta Tol = R,\end{aligned}$$

Now, using $S < 1$, we have

$$\|y_{n+j} - y_n\| \le \sum_{i=1}^{j}\|y_{n+i} - y_{n+i-1}\| \le \sum_{i=1}^{j} S^{n+i-1}\|y_1 - y_0\| < \frac{S^n}{1 - S}\|y_1 - y_0\|. \tag{26}$$

Hence, $\{y_n\}$ is a Cauchy sequence that converges to y^*. Since

$$\|F(y_n)\| \le \left(L + K(R^p + 2\delta^p Tol^p)\right)\|y_n - y_{n-1}\|,$$

and $\|y_n - y_{n-1}\| \to 0$ as $n \to \infty$, then $F(y^*) = 0$ by using the continuity of F.

In order to prove the uniqueness part, suppose x^* is another solution of (1) in $B(y_0, R)$ and if $P = [x^*, y^*; F]$ is invertible, then $x^* = y^*$ since $P(x^* - y^*) = F(x^*) - F(y^*)$. But, $\|I - D_0^{-1}P\| \le \|D_0^{-1}\|\|D_0 - P\| < 1$, therefore $x^* = y^*$. □

3.2 On the accessibility

Now, we compare the accessibility of the previous iterative process (6) and different derivative-free point-to-point iterative processes such as the Steffensen method, Backward–Steffensen method, and Center–Steffensen method.

For this purpose, we will analyze the dynamical behavior of both iterative processes. We will apply the methods to the nondifferentiable complex polynomial

$$p(z) = z^3 + z|z| - 2z,$$

with three different roots $z = \pm 1$ and $z = 0$. We paint in red, blue, and yellow the convergence after 100 iterations to the roots of the polynomial with a tolerance of 10^{-3}; in other case, the point is painted in black.

As shown in Figures 3 and 4 the dynamical behavior of method (6) with $Tol = 0.02$ and the Steffensen, Backwards–Steffensen, and Center–Steffensen method, it can be seen that the new method is really better than the other ones.

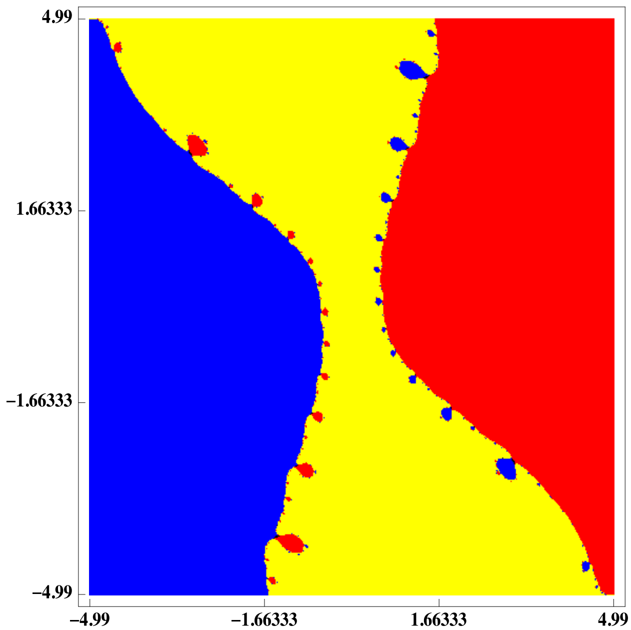
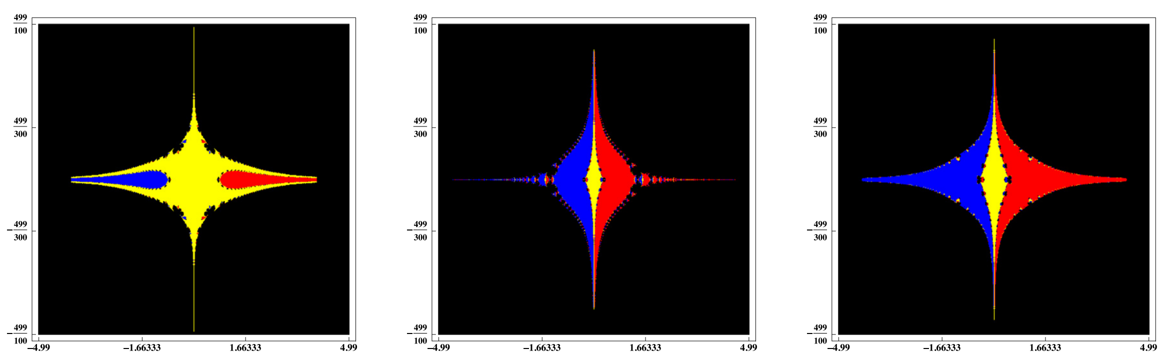


Figure 3: Method (6) with Tol = 0.02.



Steffensen's method Backward-Steffensen method Center-Steffensen method

Figure 4: Basins of attraction to polynomial $p(z) = z^3 + z|z| - 2z$.

Once the accessibility has been graphically analyzed, showing that method (6) is really better than the other ones, we have to prove it in a numerical way and, for that purpose, we compute the percentage of points which converges after 200 iterations with a tolerance of 10^{-3} to any of the roots. This information is tabulated in Table 8.

Table 8: Percentage of convergence points for $f(z) = z^3 + z|z| - 2z$.

Method	Percentage of convergent points
(6) with $Tol = 0.02$	99.97%
Steffensen	8.31%
Backward–Steffensen	7.10%
Center–Steffensen	9.71%

3.3 A numerical study

In this section, we consider a nondifferentiable problem. For this, we consider the nonlinear integral Eq. (18), where $f(x) = -1$, $\lambda \in \mathbb{R}$, $G(s,t)$ is the Green's function, $H(x(t)) = |x(t)|$, and $s, t \in [0,1]$.

By following the same process of discretization as in Section 2.2 and by using again the Gauss–Legendre quadrature, where w_j and t_j are the corresponding weights and nodes, respectively, and denoting the approximation $x(s_i)$ by x_i for $i = 1, \ldots, m$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, the nonlinear operator $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ modelizes our problem:

$$F_i(x) = x_i - 1 - \lambda \sum_{j=1}^{m} w_{ij}|x_j|, \quad i = 1, 2, .., m$$

where $w_{ij} = w_j G(t_i, t_j), i, j = 1, \ldots, m$.

Obviously, this nonlinear operator F is nondifferentiable, and the divided difference operator for approximating the Jacobian can be expressed as

$$[u, v, F] = I - \lambda C, \tag{27}$$

for $u, v \in C[0,1]$, where $C_{ij} = (w_{ij}) \frac{\|u_j\| - \|v j\|}{u_j - v_j}, i, j = 1, \ldots, m$. Therefore, by choosing $m = 8$, $\lambda = \frac{1}{2}$, and the starting guess $\mathbf{y}_0 = (0.5, \ldots, 0.5)$, in the domain $\Omega = B(0,3) \subseteq C[0,1]$. We have condition (I) in Section 3.2 and it is verified since $\|W\| \leq 0.1173$ and $\lambda\|C\| \leq 0.05865$. Then, by applying the Banach Lemma for inverse operators, D_0^{-1} exists and $\|D_0^{-1}\| \leq \beta = 1.0623$, so that $\|D_0^{-1}F(\mathbf{y}_0)\| \leq \eta = 1.6246$.

Moreover, by (27), we deduce that

$$\|[x, y, F] - [u, v, F]\| \leq 2\|\lambda\|\|W\|,$$

and then, in Section 3.2 condition (II) with $p = 1$, it is verified for $L = 2\|\lambda\|\|W\| = 0.1173$ and $K = 0$. Thus, the parameter M defined in Theorem 9 for all values of tol is $M = 0.1246$, and by applying the theoretical results, we obtain the semilocal convergence radius for the nondifferentiable case. The values are shown in Table 9.

It can be observed that for this particular problem, in which $K = 0$ and taking into account (24) and (25), as tol increases, the semilocal convegence radius also increases, as shown in Table 9.

Finally, we approximate the solution of this nonlinear system by using the iterative methods mentioned with conditions expressed in Section 2.2. In Tables 10–12, we can observe the results showing very good behavior of the iterative process given in (6); for almost all values of tol, the stopping criteria is reached in two iterations. The solution of the problem with four decimal digits is

$$\mathbf{y}_{n+1} = (1.0022, 1.0203, 1.0621, 1.1304, 1.1744, 1.1336, 1.2227, 1.1671).$$

Table 9: Semilocal convergence radius of New Steffensen-type for.

Method	**(6) tol = 0.4**	**(6) tol = 0.3**	**(6) tol = 0.2**	**(6) tol = 0.1**	**(6) tol = 0.01**
Radius	1.2825	1.4354	1.5883	1.7413	1.8789

Table 10: Numerical results with starting guess $\mathbf{y}_0 = (0.5, 0.5, \ldots, 0.5)^T$.

Method	**(6) tol = 0.4**	**(6) tol = 0.3**	**(6) tol = 0.2**	**(6) tol = 0.1**	**(6) tol = 0.01**
k	3	2	2	2	2
$\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$	3.4064×10^{-108}	5.9000×10^{-108}	8.3439×10^{-108}	5.9000×10^{-108}	5.1322×10^{-107}
$\|F(\mathbf{y}_{n+1})\|$	2.7874×10^{-108}	5.7270×10^{-108}	6.9946×10^{-108}	4.4831×10^{-108}	4.9335×10^{-107}

Table 11: Numerical results with starting guess $\mathbf{y}_0 = (1, 1, \ldots, 1)^T$.

Method	(6) tol = 0.4	(6) tol = 0.3	(6) tol = 0.2	(6) tol = 0.1	(6) tol = 0.01
k	2	2	2	2	2
$\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$	3.4064×10^{-108}	3.4064×10^{-108}	3.4064×10^{-108}	4.8173×10^{-108}	1.4452×10^{-107}
$\|F(\mathbf{y}_{n+1})\|$	2.7874×10^{-108}	2.7874×10^{-108}	2.7874×10^{-108}	3.1247×10^{-108}	1.4156×10^{-107}

Table 12: Numerical results with bigger *tol* and $\mathbf{y}_0 = (1, 1, \ldots, 1)^T$.

Method	(6) tol = 3	(6) tol = 2	(6) tol = 1	(6) tol = 0.75	(6) tol = 0.5
k	3	2	2	2	2
$\|\mathbf{y}_{n+1} - \mathbf{y}_n\|$	0	0	3.4064×10^{-108}	3.4064×10^{-108}	3.4064×10^{-108}
$\|F(\mathbf{y}_{n+1})\|$	2.7874×10^{-108}	2.7874×10^{-108}	2.7874×10^{-108}	2.7874×10^{-108}	2.7874×10^{-108}

4 Conclusions

This work is devoted to present a derivative-free point-to-point iterative process that allow us to obtain approximated solutions for nonlinear systems with similar efficiency and characteristics as Newton's method. Semilocal convergence results are proved under suitable conditions of applicability in both cases, that is, the differentiable and nondifferentiable. Theoretical results are contrasted with dynamical study and numerical results for all cases.

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