The Wigner global wave front set in spaces of tempered ultradistributions

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Abstract

In this paper we introduce and study global wave front sets in terms of the τ-Wigner transform in global ultradifferentiable classes of Beurling type modulated with weight functions in the sense of Braun, Meise, and Taylor, and we compare it with other wave front sets existing in the literature defined by different time-frequency analysis tools, such as the short-time Fourier transform or Gabor frames. Conditions for the equality of these wave front sets are provided and some examples are given.

Keywords Global wave front set · τ-Wigner transform · Global ultradifferentiable classes · Gabor frames

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1 Introduction

A global wave front set can be defined as follows: a point does not belong to the wave front set of a distribution if there exists an open conic set which contains the point such that the distribution satisfies regular enough properties in that conic set. Indeed, Hörmander [20] studied quadratic hyperbolic operators via two types of global wave front sets: the C^∞ wave front set and the analytic wave front set. While a global analytic wave front set was considered by, for instance, Nakamura [22] for the analysis of Schrödinger equations, being proved that it coincides with the analytic wave front set in [20] by Schulz and Wahlberg [28], global C^∞ wave front sets remained, as far as we know, almost forgotten in the literature. Rodino and Wahlberg [26] recovered

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them, using the fact that the definition of wave front set is about a simultaneous analysis of points and directions, which fits within the frame of time-frequency analysis (as pointed out in, for instance, [21, 23, 24]). Moreover, they defined a global wave front set based on the behaviour of the short-time Fourier transform of the distribution in an open conic set and on a suitable lattice at the same time, which is called the Gabor wave front set. They showed that the C^∞, the analytic, and the Gabor wave front set coincide in the space of tempered distributions. We refer to [25] for an overview about the topic.

Boiti, Jornet, and Oliaro, in [7], extended the results obtained in [26] to the space of ω-tempered ultradistributions of Beurling type, ω being a subadditive weight function in the sense of Braun, Meise, and Taylor [9]. Indeed, they showed the equality between the ultradifferentiable version of the analytic wave front set and that of the Gabor wave front set. See [10, 11] for wave front sets of Gel'fand–Shilov type. Recently, the author, Boiti, Jornet, and Oliaro [2], using the theory developed in [1], generalized the concept of C^∞ wave front set to spaces of ω-tempered ultradistributions following the ideas given in [20, 26], and provided conditions on the weight function under which this wave front set and the analytic wave front set in [7] are equal.

The Wigner transform is a tool originally devoted to the quantum mechanics field. For $f \in L^2(\mathbb{R})$, we define the Wigner transform of f by

$$\mathrm{Wig}(f, f)(x, \xi) = \int_{\mathbb{R}} f(x + y/2)\overline{f(x - y/2)}e^{-iy\cdot\xi}\,dy, \qquad (1.1)$$

and turns out to be a two-variable function depending on a position x and a momentum ξ. In the context of signals, the value of $\mathrm{Wig}(f, f)(x, \xi)$ provides, in principle, no information. Nonetheless, the Wigner transform is an example of quadratic time-frequency representation which enjoys interesting properties, such as that it is real-valued, or that it is covariant, in the sense that if $h(y) = e^{iy\cdot\eta}f(y - t)$, $t, \eta \in \mathbb{R}^d$, then

$$\mathrm{Wig}(h, h)(x, \xi) = \mathrm{Wig}(f, f)(x - t, \xi - \eta), \qquad x, \xi \in \mathbb{R}^d.$$

In this paper we work with an extension of the (cross-)Wigner transform of $f, g \in L^2(\mathbb{R}^d)$ (denoted by $\mathrm{Wig}(f, g)$ and called Wigner transform again for simplicity), which is defined replacing $\overline{f(x - y/2)}$ in (1.1) by $\overline{g(x - y/2)}$. It satisfies similar properties to the original Wigner transform. For instance,

$$|\mathrm{Wig}(\widehat{f}, \widehat{g})(x, \xi)| = C|\mathrm{Wig}(f, g)(-\xi, x)|, \qquad (1.2)$$

where $\widehat{f}$ denotes the Fourier transform of f. More precisely, we are mainly concerned about what it is called the τ-Wigner transform, where $\tau \in (0, 1)$, which recaptures the Wigner transform when $\tau = 1/2$. We recall that the τ-Wigner transform coincides with [5, Prop. 5.6]

$$\sigma * \mathrm{Wig}(f, g), \qquad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Hence, it belongs to the Cohen class, where σ is given in [5, (5.6)].

Very recently, Cordero and Rodino [14] defined a global wave front set in terms of the τ-Wigner transform in the space of square-integrable functions. They proved that this type of wave front sets contains a version of the one in [26], the converse inclusion could not be proven due to the presence of what in [5] call ghost frequencies that the Wigner transform detects. The ghost frequencies can be explained via the following example given in [5, Sect. 3]: let f be a signal in one dimension supported in $[a, b] \times [c, d]$ with $b < c$ and consider a time x near the middle of the interval $[b, c]$. It is clear that there is no frequency present in f. However, the term $f(x+y/2)\overline{f(x-y/2)}$ appearing in (1.1) accounts to a kind of 'folding' of the previous time of f onto the future time. Hence, by the choice of x, the intervals $[a, b]$ and $[b, c]$ may overlap, providing frequencies which in reality do not exist. The same may happen for some frequencies ξ, by formula (1.2).

We point out that the analytic wave front sets introduced in [7, 26] were given in terms of the short-time Fourier transform, since a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ respectively can be characterized in terms of the decay of the short-time Fourier transform [18, Theo. 2.7]. Therefore, we asked ourselves whether replacing the short-time Fourier transform by the τ-Wigner transform may provoke or not an alteration on the global wave front sets defined in global classes of ultradistributions in [7]. We prove that the wave front set given in [7] is related to the wave front set defined via the τ-Wigner transform, in the sense that the wave front set in [7] coincides with a linear transformation of the latter, the transformation being the identity when $\tau = 1/2$ (Theorem 16). We support this theorem with explicit examples, like Example 23, in which the Wigner wave front set rotates as the constant τ varies.

The paper is organized as follows: in Sect. 2 we introduce some preliminaries about the global spaces of ultradifferentiable functions $\mathcal{S}_\omega(\mathbb{R}^d)$ as well as new equivalent systems of seminorms for this space. Section 3 is devoted to introducing and studying the τ-Wigner wave front set (Definition 13). We prove that, for every ultradistribution in $\mathcal{S}'_\omega(\mathbb{R}^d)$, a point z_0 belongs to the wave front set given in [7] if and only if $\mathcal{J}_\tau(z_0)$ belongs to the τ-Wigner wave front set, where $\mathcal{J}_\tau$ is given in (3.3) (Theorem 16). As a consequence, the ω-wave front set coincides with the $(1/2)$-Wigner wave front set (Corollary 17). We compute the wave front set of some concrete ω-tempered ultradistributions in Sect. 4, making use of properties of this wave front set, such as the comparison between the Wigner wave front set of time-frequency shifts or that of the Fourier transform with the Wigner wave front set of the ultradistribution (Propositions 19 and 21). We observe that the τ-Wigner wave front set may depend on the constant τ in Example 23 and on the weight function ω in Example 24. Finally, in Sect. 5 we introduce the corresponding version of the wave front set described by Gabor frames in the ultradifferentiable setting, using the τ-Wigner transform, cf. [7, 26] (Definition 28). We take advantage of results existing in the literature, like [7, Theo. 3.17], to circunvent the use of modulation spaces of exponential type given by the τ-Wigner transform. In any case, when studying equalities between analytic and these wave front sets, we need ω to be subadditive in order to have a suitable definition of the modulation spaces (see [7, Sect. 3]). In Corollary 31 we obtain that for $\tau = 1/2$ and for any ω subadditive weight function, the wave front sets given in Definitions 12, 13, 27, and 28 coincide. Furthermore, for the weight functions given in

[2, Example 5.10] they all coincide with the Weyl wave front set defined in [2, Defin. 4.3] (see Remark 32).

2 Preliminaries

We consider weight functions in the sense of Braun, Meise, and Taylor [9]:

Definition 1 A weight function $\omega : [0, +\infty[\to [0, +\infty[$ is an increasing and continuous function satisfying:

- (α) There exists $L \geq 1$ such that $\omega(2t) \leq L\omega(t) + L$, $t \geq 0$;
- (β) $\omega(t) = o(t)$ as $t \to \infty$;
- (γ) There exist $a \in \mathbb{R}$ and $b > 0$ such that $\omega(t) \geq a + b \log(1 + t)$, $t \geq 0$;
- (δ) The function $\varphi(t) = \omega(e^t)$ is convex.

Weight functions can be extended to $\mathbb{C}^d$ as follows: $\omega(z) = \omega(|z|)$, $z \in \mathbb{C}^d$, where $|\cdot|$ stands for the Euclidean norm. Denoting $\langle z \rangle^2 = 1 + |z|^2$, it holds that $\omega(\langle z \rangle) \leq L\omega(z) + L$. Notice that condition (α) is weaker than subadditivity. The Young conjugate of φ is defined as $\varphi^*(t) = \sup_{s>0}\{st - \varphi(s)\}$. See [9, 16, 19] for more information of φ^*. There is an exhaustive list of properties [8, Appendix A] of the Young conjugate of φ. We state as a lemma some properties that we need and can be found in the references above.

Lemma 2 *For every $\lambda > 0$, $t \geq 0$, and $k, l \in \mathbb{N}_0$,*

(1) $t^k \leq \max\{1, e^{-\lambda\varphi^*(0)}\}e^{\lambda\varphi^*(\frac{k}{\lambda})}e^{\lambda\omega(t)}$,

(2) $\lambda\varphi^*\left(\dfrac{k}{\lambda}\right) + \lambda\varphi^*\left(\dfrac{l}{\lambda}\right) \leq \lambda\varphi^*\left(\dfrac{k+l}{\lambda}\right)$.

We use the following definition of Fourier transform of $f \in L^1(\mathbb{R}^d)$:

$$\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

In this work, we use global classes of ultradifferentiable functions in the Beurling setting modulated with weight functions, defined by Björck [4] for subadditive weight functions.

Definition 3 For a weight function ω, we define the space $\mathcal{S}_\omega(\mathbb{R}^d)$ as those $f \in L^1(\mathbb{R}^d)$ such that ($f, \widehat{f} \in C^\infty(\mathbb{R}^d)$ and) for all $\lambda > 0$ and $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x\in\mathbb{R}^d} |D^\alpha f(x)|e^{\lambda\omega(x)} < \infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^d} |D^\alpha \widehat{f}(\xi)|e^{\lambda\omega(\xi)} < \infty.$$

This is a Fréchet space, endowed with the natural topology. In [3, Lemma 2.11], [6, Theo. 4.8], [8, Theo. 2.5], and [18, Theo. 2.7] we can find other descriptions of $\mathcal{S}_\omega(\mathbb{R}^d)$ which provide different equivalent systems of seminorms for $\mathcal{S}_\omega(\mathbb{R}^d)$. The

strong dual of $\mathcal{S}_\omega(\mathbb{R}^d)$ is denoted by $\mathcal{S}'_\omega(\mathbb{R}^d)$ and it is called the space of ω-*tempered ultradistributions*.

For $f, g \in L^2(\mathbb{R}^d)$, we define the *(cross)-Wigner transform of f and g* by

$$\text{Wig}(f, g)(z) = \int_{\mathbb{R}^d} f(x + y/2)\overline{g(x - y/2)}e^{-iy\cdot\xi}dy, \qquad z = (x, \xi) \in \mathbb{R}^{2d}.$$

Here we introduce the main tool of this paper, which generalizes the definition of Wigner transform:

Definition 4 For two functions f and g and $0 \le \tau \le 1$, we define the *(cross-)τ-Wigner transform* $\text{Wig}(f, g)_\tau$ by

$$\text{Wig}(f, g)_\tau(z) = \int_{\mathbb{R}^d} f(x + \tau y)\overline{g(x - (1 - \tau)y)}e^{-iy\cdot\xi}dy, \qquad z = (x, \xi) \in \mathbb{R}^{2d}.$$

For our purposes we exclude from our discussion the cases $\tau = 0$ and $\tau = 1$, which represent the (cross)-Rihaczek distribution and the conjugate-(cross)-Rihaczek distribution (see [12]).

Another important tool in the theory of time-frequency analysis is the short-time Fourier transform, defined below (see for instance [17, Chapter 3] for the definition in the context of tempered distributions). Before that, we introduce the translation, modulation, and phase-shift operators (time-frequency shifts) as follows: for a function f in $\mathbb{R}^d$, and $x, y, \eta \in \mathbb{R}^d$,

$$T_y f(x) = f(x - y),$$
$$M_\eta f(x) = e^{ix\cdot\eta} f(x),$$
$$\Pi(y, \eta) f(x) = e^{ix\cdot\eta} f(x - y).$$

Definition 5 Given a window function $g \in \mathcal{S}_\omega(\mathbb{R}^d)\backslash\{0\}$, the *short-time Fourier transform* (STFT for short) of $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ is defined, for $z \in \mathbb{R}^{2d}$, by

$$V_g u(z) = \langle u, \Pi(z)g\rangle.$$

The bracket $\langle\cdot, \cdot\rangle$ is consistent with the inner product $\langle\cdot, \cdot\rangle_{L^2(\mathbb{R}^d)}$.

We introduce the following notation: for a given function g defined in $\mathbb{R}^d$ and $0 < \tau < 1$,

$$\mathcal{I}_\tau g(x) = g\left(-\frac{1 - \tau}{\tau}x\right), \qquad x \in \mathbb{R}^d.$$

For any function g, $\mathcal{I}_{1/2}g = \mathcal{I}g$ is the reflection operator, and

$$\mathcal{I}_\tau \mathcal{I}_{1-\tau} g = \mathcal{I}_{1-\tau}\mathcal{I}_\tau g = g. \tag{2.1}$$

This is the extension of Definition 4 to ultradistributions:

Definition 6 For $0 < \tau < 1$, $g \in \mathcal{S}_\omega(\mathbb{R}^d)\backslash\{0\}$ and $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, we define the τ-*Wigner transform of u* by

$$\text{Wig}(u, g)_\tau(x, \xi) = \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau}ix\cdot\xi}\left\langle u, \Pi\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\mathcal{I}_\tau g\right\rangle, \qquad (x, \xi) \in \mathbb{R}^{2d},$$

(2.2)

where $\Pi\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\mathcal{I}_\tau g(y) = g\left(-\frac{1-\tau}{\tau}y + \frac{1}{\tau}x\right)e^{\frac{1}{\tau}iy\cdot\xi}$, for $y \in \mathbb{R}^d$.

We recall the relation between the short-time Fourier transform and the Wigner transform (cf. [17, Lemma 4.3.1]): for all $f, g \in L^2(\mathbb{R}^d)$, we have

$$\text{Wig}(f, g)(x, \xi) = 2^d e^{2ix\cdot\xi} V_{\mathcal{I}g}f(2x, 2\xi), \qquad x, \xi \in \mathbb{R}^d.$$

We can extend this result to ultradistributions and for arbitrary $\tau \in (0, 1)$, with a similar proof (cf.[13, Prop.1.1.30],):

Lemma 7 *Let* $0 < \tau < 1$. *For all* $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$, *we have*

$$\text{Wig}(u, g)_\tau(x, \xi) = \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau}ix\cdot\xi} V_{\mathcal{I}_\tau g}u\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right), \qquad x, \xi \in \mathbb{R}^d.$$

Proof According to [13, Prop. 1.3.30], the equality above holds for $u, g \in L^2(\mathbb{R}^d)$. For the general case, we have

$$\text{Wig}(u, g)_\tau(x, \xi) = \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau}ix\cdot\xi}\left\langle u, \Pi\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\mathcal{I}_\tau g\right\rangle$$

$$= \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau}ix\cdot\xi} V_{\mathcal{I}_\tau g}u\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right).$$

$\square$

Reading backwards Lemma 7 and using the fact that $\mathcal{I}_\tau\mathcal{I}_{1-\tau}g = g$, we obtain the STFT in terms of the τ-Wigner transform.

Lemma 8 *Let* $0 < \tau < 1$. *For all* $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$, *we have*

$$V_g u(x, \xi) = \tau^d e^{-i(1-\tau)x\cdot\xi} \text{Wig}(u, \mathcal{I}_{1-\tau}g)_\tau((1 - \tau)x, \tau\xi), \qquad x, \xi \in \mathbb{R}^d.$$

The relation between the τ-Wigner and the short-time Fourier transforms given in Lemma 7 yields the following equivalence:

Theorem 9 *Let* $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 < \tau < 1$. *Then,* $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ *if, and only if,*

$$given \ g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}, \ \forall\lambda > 0 \ \exists C_\lambda > 0 \ : \ \sup_{z\in\mathbb{R}^{2d}} |\text{Wig}(u, g)_\tau(z)|e^{\lambda\omega(z)} \leq C_\lambda.$$

(2.3)

Proof Fixed $0 < \tau < 1$, we observe that $g \in \mathcal{S}_\omega(\mathbb{R}^d)\backslash\{0\}$ if and only if $\mathcal{I}_\tau g \in \mathcal{S}_\omega(\mathbb{R}^d)\backslash\{0\}$.

If $u \in \mathcal{S}_\omega(\mathbb{R}^d)$, then by [18, Theo. 2.7], for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\sup_{w \in \mathbb{R}^{2d}} |V_{\mathcal{I}_\tau g} u(w)| e^{\lambda \omega(w)} \leq C_\lambda.$$

Thus, by Lemma 7 and the fact that ω is increasing (notice that $\frac{1}{1-\tau} > 1$ and $\frac{1}{\tau} > 1$), we obtain (2.3): for all $\lambda > 0$,

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} |\operatorname{Wig}(u,g)_\tau(x,\xi)| e^{\lambda\omega(x,\xi)}$$
$$\leq \left(\frac{1}{\tau}\right)^d \sup_{(x,\xi)\in\mathbb{R}^{2d}} |V_{\mathcal{I}_\tau g} u\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)| e^{\lambda\omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)}$$
$$= \left(\frac{1}{\tau}\right)^d \sup_{w \in \mathbb{R}^{2d}} |V_{\mathcal{I}_\tau g} u(w)| e^{\lambda\omega(w)} \leq \left(\frac{1}{\tau}\right)^d C_\lambda.$$

On the other hand, assume that (2.3) holds. Without losing generality, we assume that the function given in $\mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ is of the form $\mathcal{I}_{1-\tau} g$. We fix $q \in \mathbb{N}$ such that $2^q \geq \max\left\{\frac{1}{1-\tau}, \frac{1}{\tau}\right\}$. Then, since ω is increasing and by condition (α) of the weight,

$$\omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) \leq \omega\left(\max\left\{\frac{1}{1-\tau}, \frac{1}{\tau}\right\}(x,\xi)\right) \leq \omega(2^q(x,\xi))$$
$$\leq L^q \omega(x,\xi) + L^q + \cdots + L.$$

So, for all $\lambda > 0$ we have by Lemma 7

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} |V_g u(x,\xi)| e^{\lambda\omega(x,\xi)}$$
$$= \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left|V_{\mathcal{I}_\tau(\mathcal{I}_{1-\tau}g)} u\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right| e^{\lambda\omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)}$$
$$= \tau^d \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\operatorname{Wig}(u, \mathcal{I}_{1-\tau}g)(x,\xi)| e^{\lambda\omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)}$$
$$\leq \tau^d e^{\lambda L^q + \cdots + \lambda L} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\operatorname{Wig}(u, \mathcal{I}_{1-\tau}g)(x,\xi)| e^{\lambda L^q \omega(x,\xi)}$$
$$\leq \tau^d e^{\lambda L^q + \cdots + \lambda L} C_{\lambda L^q}.$$

$\square$

Therefore, given $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ and fixed $g \in \mathcal{S}_\omega(\mathbb{R}^d)\backslash\{0\}$,

$$\sup_{z\in\mathbb{R}^{2d}} |\operatorname{Wig}(u,g)_\tau(z)| e^{\lambda\omega(z)}, \qquad \lambda > 0, \tag{2.4}$$

defines another equivalent system of seminorms for the space $\mathcal{S}_\omega(\mathbb{R}^d)$. In fact, we can replace the $L^\infty(\mathbb{R}^{2d})$-norm in (2.4) by $L^{p,q}(\mathbb{R}^{2d})$-norms, $1 \leq p, q \leq +\infty$, as done in [8, Theo. 2.5]:

Corollary 10 *Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $0 < \tau < 1$, and $1 \leq p, q \leq +\infty$. Then, $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if*

$$\text{given } g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}, \ \forall \lambda > 0 \, \exists C_\lambda > 0 \ : \ \left\| \text{Wig}(u, g)_\tau(z) e^{\lambda\omega(z)} \right\|_{L^{p,q}(\mathbb{R}^{2d})} \leq C_\lambda. \tag{2.5}$$

Proof If u satisfies formula (2.5), then the result follows proceeding as in [8, Lemma 3.16]. On the other hand, if $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, then for all $\lambda > 0$

$$\left\| \text{Wig}(u, g)_\tau(z) e^{\lambda\omega(z)} \right\|_{L^{p,q}(\mathbb{R}^{2d})}$$
$$\leq \left\| \text{Wig}(u, g)_\tau(z) e^{(\lambda+s)\omega(z)} \right\|_{L^\infty(\mathbb{R}^{2d})} \left\| e^{-s\omega(z)} \right\|_{L^{p,q}(\mathbb{R}^{2d})}$$

where $s > 0$ is taken large enough (see [8, (2.3)]). We then obtain (2.5) by Theorem 9. $\square$

We finish this section completing the corresponding version of [18, Theo. 2.7] for the τ-Wigner transform. The proof follows by Theorem 9, [18, Theo. 2.7], and Lemma 7.

Proposition 11 *Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $0 < \tau < 1$, and $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then, $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if $\text{Wig}(u, g)_\tau \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.*

3 Global wave front sets

We recall the definition of global ω-wave front set introduced in [7, Defin. 3.1], inspired by [26, Defin. 3.1]:

Definition 12 Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$. We say that $0 \neq z_0 \in \mathbb{R}^{2d}$ does not belong to the *ω-wave front set of u*, $\text{WF}'_\omega(u)$, if there is an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing the point z_0 such that

$$\sup_{z \in \Gamma} |V_g u(z)| e^{\lambda\omega(z)} < +\infty, \qquad \lambda > 0.$$

This definition is based on the system of seminorms in [18, Theo. 2.7]. Thus, according to Theorem 9, it is natural to introduce the following notion:

Definition 13 (The τ, ω-Wigner global wave front set) Let $0 < \tau < 1$, $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, and $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$. We say that $0 \neq z_0 \in \mathbb{R}^{2d}$ does not belong to the *τ, ω-Wigner global wave front set of u*, $\text{WF}^\tau_\omega(u)$, if there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that

$$\sup_{z \in \Gamma} |\text{Wig}(u, g)_\tau(z)| e^{\lambda\omega(z)} < +\infty, \qquad \lambda > 0. \tag{3.1}$$

From now on, the set $\mathrm{WF}_\omega^\tau(u)$ will be called the Wigner wave front set of u for simplicity. As the ω-wave front set, $\mathrm{WF}_\omega^\tau(u)$ is a closed conic subset of $\mathbb{R}^{2d} \setminus \{0\}$. Now, we prove that this definition does not depend on the choice of the window function g (cf. [7, Prop. 3.2] for the ω-wave front set, and [26, Cor. 3.3]). We first reformulate [7, Prop. 2.12] using the equivalences between the STFT and τ-Wigner transform.

Proposition 14 *Let $0 \neq g, \psi, \gamma \in \mathcal{S}_\omega(\mathbb{R}^d)$ so that $\langle \gamma, \psi \rangle \neq 0$. Let $u \in \mathcal{S}_\omega'(\mathbb{R}^d)$ and $0 < \tau < 1$. Then:*

$$|\operatorname{Wig}(u, g)_\tau(x, \xi)| \leq \frac{1}{\tau^d (2\pi)^d |\langle \gamma, \psi \rangle|} (|V_\psi u| * |V_{\mathcal{I}_\tau g}\gamma|)\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right), \qquad x, \xi \in \mathbb{R}^d.$$

Proof Indeed, by [7, Prop. 2.12], we have

$$|V_{\mathcal{I}_\tau g} u(z)| \leq \frac{1}{(2\pi)^d |\langle \gamma, \psi \rangle|} (|V_\psi u| * |V_{\mathcal{I}_\tau g}\gamma|)(z). \tag{3.2}$$

By Lemma 8, the left-hand side of (3.2) is equal to $\tau^d |\operatorname{Wig}(u, g)_\tau((1 - \tau)x, \tau\xi)|$. With the change of variables $(1 - \tau)x = x'$ and $\tau\xi = \xi'$ we complete the proof. □

From Proposition 14 and Lemma 8, it easily follows that (cf. [14, Lemma 3.2]):

$$|\operatorname{Wig}(u, g)_\tau| \leq \frac{1}{(1 - \tau)^d (2\pi)^d |\langle \psi, \gamma \rangle|} (|\operatorname{Wig}(u, \mathcal{I}_{1-\tau}\psi)_\tau| * |\operatorname{Wig}(\gamma, g)_\tau|).$$

Proposition 15 *Let $0 < \tau < 1$, $u \in \mathcal{S}_\omega'(\mathbb{R}^d)$, $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $0 \neq z_0 \in \mathbb{R}^{2d}$. Assume that there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that (3.1) is satisfied. Then, for every $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and for any open conic set $\Gamma' \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 and such that $\overline{\Gamma'} \cap S_{2d-1} \subseteq \Gamma$, where S_{2d-1} is the unit sphere in $\mathbb{R}^{2d}$, we have*

$$\sup_{z \in \Gamma'} |\operatorname{Wig}(u, \psi)_\tau(z)| e^{\lambda\omega(z)} < +\infty, \qquad \lambda > 0.$$

Proof By Proposition 14 (where g, ψ, γ are replaced by $\psi, \mathcal{I}_\tau g, g$, and observe that $\langle \mathcal{I}_\tau g, g \rangle \neq 0$), we have for all $x, \xi \in \mathbb{R}^d$ that

$$|\operatorname{Wig}(u, \psi)_\tau(x, \xi)| \leq \frac{1}{\tau^d (2\pi)^d |\langle g, \mathcal{I}_\tau g \rangle|} (|V_{\mathcal{I}_\tau g} u| * |V_{\mathcal{I}_\tau \psi} g|)\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right).$$

Since $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$, by [18, Theo. 2.7], we obtain that for all $\mu > 0$ there exists $C_\mu > 0$ such that

$$|V_{\mathcal{I}_\tau \psi} g(z)| e^{\mu\omega(z)} \leq C_\mu, \qquad z \in \mathbb{R}^{2d}.$$

We have, by Lemma 8, for all $z = (x, \xi)$ and $z' = (z_1', z_2')$ in $\mathbb{R}^{2d}$, and every $\varepsilon > 0$,

$$(|V_{\mathcal{I}_\tau g}u| * |V_{\mathcal{I}_\tau \psi}g|)\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)$$

$$= \int_{\mathbb{R}^{2d}} \left|V_{\mathcal{I}_\tau g}u\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) - z'\right)\right| |V_{\mathcal{I}_\tau \psi}g(z')|dz'$$

$$= \tau^d \int_{\langle((1-\tau)z_1', \tau z_2')\rangle \leq \varepsilon\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\rangle} |\operatorname{Wig}(u,g)_\tau(x - (1-\tau)z_1', \xi - \tau z_2')|$$

$$\times |V_{\mathcal{I}_\tau \psi}g(z')|dz'$$

$$+ \int_{\langle((1-\tau)z_1', \tau z_2')\rangle > \varepsilon\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\rangle} \left|V_{\mathcal{I}_\tau g}u\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) - z'\right)\right| |V_{\mathcal{I}_\tau \psi}g(z')|dz'$$

$$= I_1(z) + I_2(z).$$

We fix $\varepsilon > 0$ small enough so that [7, (3.25)] (cf. [26])

$$z \in \Gamma', \;\; |z| \geq 1, \;\; \langle((1-\tau)z_1', \tau z_2')\rangle \left(\leq \varepsilon\langle z\rangle\right) \leq \varepsilon\left\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right\rangle$$

implies $z - ((1-\tau)z_1', \tau z_2') \in \Gamma$. By assumption, for all $\lambda > 0$ there exists $C_{\lambda L} > 0$ such that

$$I_1(z) \leq \tau^d C_{\lambda L} \int_{\langle((1-\tau)z_1', \tau z_2')\rangle \leq \varepsilon\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\rangle} e^{-\lambda L\omega(z - ((1-\tau)z_1', \tau z_2'))} |V_{\mathcal{I}_\tau \psi}g(z')|dz'$$

$$\leq \tau^d C_{\lambda L}e^{\lambda L}e^{-\lambda\omega(z)} \int_{\mathbb{R}^{2d}} e^{\lambda L\omega((1-\tau)z_1', \tau z_2')} |V_{\mathcal{I}_\tau \psi}g(z')|dz'$$

$$\leq \tau^d C_{\lambda L}e^{\lambda L}e^{-\lambda\omega(z)} \int_{\mathbb{R}^{2d}} e^{(\lambda L+m)\omega(z')} |V_{\mathcal{I}_\tau \psi}g(z')|e^{-m\omega(z')}dz'$$

$$\leq C_\lambda' e^{-\lambda\omega(z)}, \;\;\;\; z \in \Gamma', \;\; |z| \geq 1,$$

for some $C_\lambda' > 0$, where $m \geq (2d+1)/b$ ($b > 0$ being the constant in condition (γ) of the weight ω), to obtain that the integral is convergent.

On the other hand, we know by [17] that there exist $c, \mu > 0$ such that

$$\left|V_{\mathcal{I}_\tau g}u\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) - z'\right)\right| \leq ce^{\mu\omega\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) - z'\right)}, \;\;\;\; x, \xi \in \mathbb{R}^d, \; z' \in \mathbb{R}^{2d}.$$

For $\langle((1-\tau)z_1', \tau z_2')\rangle > \varepsilon\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\rangle$, we proceed similarly as in the proof of [2, Lemma 5.3]: fix $q \in \mathbb{N}_0$ such that $2^q > \varepsilon^{-1}$. Then, denoting $z = (x, \xi)$, for all $\langle((1-\tau)z_1', \tau z_2')\rangle > \varepsilon\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\rangle$,

$$\omega(z) \leq \omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) \leq \omega\left(\left\langle\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right\rangle\right)$$

$$\leq \omega(\varepsilon^{-1}\langle((1-\tau)z_1', \tau z_2')\rangle)$$

$$\leq L^q \omega(\langle((1-\tau)z_1', \tau z_2')\rangle) + L^q + \cdots + L$$
$$\leq L^{q+1} \omega(z') + L^{q+1} + \cdots + L,$$

hence we deduce

$$-2L^{q+1}\omega(z') \leq -\omega(z) - \omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) + 2(L^{q+1} + \cdots + L),$$

and in particular, for all $\lambda, \mu > 0$,

$$-2(\lambda + \mu L)L^{q+1}\omega(z') \leq -\lambda\omega(z) - \mu L\omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)$$
$$+ 2(\lambda + \mu L)(L^{q+1} + \cdots + L).$$

Therefore,

$$
\begin{aligned}
I_2(z) &\leq c \int_{\langle(1-\tau)z_1', \tau z_2'\rangle > \varepsilon\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right)} e^{\mu\omega\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) - z'\right)} |V_{\mathcal{I}_\tau \psi} g(z')| dz' \\
&\leq c e^{\mu L} e^{\mu L \omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)} \int_{\langle(1-\tau)z_1', \tau z_2'\rangle > \varepsilon\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right)} e^{\mu L \omega(z')} |V_{\mathcal{I}_\tau \psi} g(z')| dz' \\
&= c e^{\mu L} e^{\mu L \omega\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)} \int_{\langle(1-\tau)z_1', \tau z_2'\rangle > \varepsilon\left(\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right)} e^{-2(\lambda+\mu L)L^{q+1}\omega(z')} \\
&\quad \times \left[|V_{\mathcal{I}_\tau \psi} g(z')|e^{2(\lambda+\mu L)L^{q+1}\omega(z')} e^{(m+\mu L)\omega(z')}\right] e^{-m\omega(z')} dz' \\
&\leq c e^{\mu L} e^{2(\lambda+\mu L)(L^{q+1}+\cdots+L)} e^{-\lambda\omega(z)} \\
&\quad \times \int_{\mathbb{R}^{2d}} \left[|V_{\mathcal{I}_\tau \psi} g(z')|e^{(2(\lambda+\mu L)L^{q+1}+m+\mu L)\omega(z')}\right] e^{-m\omega(z')} dz',
\end{aligned}
$$

where the constant $m > 0$ is taken as before. From this, we obtain that for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$I_2(z) \leq C_\lambda e^{-\lambda\omega(z)}, \qquad z \in \mathbb{R}^{2d}.$$

The proof is complete.   □

Now we proceed to study the relation of the global wave front sets introduced in Definitions 12 and 13, which is the most important result in this paper. To that aim, we denote

$$\mathcal{J}_\tau(x, \xi) = \left(x, \frac{1-\tau}{\tau}\xi\right), \quad \text{for } x, \xi \in \mathbb{R}^d. \tag{3.3}$$

Theorem 16 *For every ω weight function and $0 < \tau < 1$, we have*

$$WF_\omega'(u) = \mathcal{J}_\tau(WF_\omega^\tau(u)), \qquad u \in \mathcal{S}_\omega'(\mathbb{R}^d).$$

Proof Fix $g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and take $0 \neq z_0 = (x_0, \xi_0) \notin \mathrm{WF}'_\omega(u)$. Then, there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that

$$\sup_{z \in \Gamma} |V_g u(z)| e^{\lambda \omega(z)} < +\infty, \qquad \lambda > 0.$$

We recall that $\mathcal{I}_\tau g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. By [7, Prop. 3.2], for any open conic set $\Gamma' \subseteq \mathbb{R}^{2d} \setminus \{0\}$ satisfying that $z_0 \in \Gamma'$ and $\overline{\Gamma'} \cap S_{2d-1} \subseteq \Gamma$, we have

$$\sup_{z \in \Gamma'} |V_{\mathcal{I}_\tau g} u(z)| e^{\lambda \omega(z)} < +\infty, \qquad \lambda > 0.$$

By Lemma 7 and since $\left(\frac{1}{1-\tau} x, \frac{1}{\tau} \xi\right) = \mathcal{J}_\tau\left(\frac{1}{1-\tau}(x, \xi)\right)$, it follows that for all $z = (x, \xi) \in \mathbb{R}^{2d}$,

$$
\begin{aligned}
|\mathrm{Wig}(u, g)_\tau(z)| e^{\lambda \omega(z)} &= \left(\frac{1}{\tau}\right)^d \left| V_{\mathcal{I}_\tau g} u\left(\mathcal{J}_\tau\left(\frac{1}{1-\tau} z\right)\right) \right| e^{\lambda \omega(z)} \\
&\leq \left(\frac{1}{\tau}\right)^d \left| V_{\mathcal{I}_\tau g} u\left(\mathcal{J}_\tau\left(\frac{1}{1-\tau} z\right)\right) \right| e^{\lambda \omega\left(\mathcal{J}_\tau\left(\frac{1}{1-\tau} z\right)\right)},
\end{aligned}
\tag{3.4}
$$

or, equivalently,

$$|\mathrm{Wig}(u, g)_\tau(\mathcal{J}_\tau^{-1}((1-\tau)z))| e^{\lambda \omega(\mathcal{J}_\tau^{-1}((1-\tau)z))} \leq \left(\frac{1}{\tau}\right)^d |V_{\mathcal{I}_\tau g} u(z)| e^{\lambda \omega(z)}. \tag{3.5}$$

It is easy to see that $\overline{\Gamma} := \mathcal{J}_\tau^{-1}(\Gamma')$ is an open conic set in $\mathbb{R}^{2d} \setminus \{0\}$ which contains the point $\mathcal{J}_\tau^{-1}(z_0)$, since $z_0 \in \Gamma'$. For every $z \in \Gamma'$ we have that $\mathcal{J}_\tau^{-1}((1-\tau)z) \in \overline{\Gamma}$ and therefore the left-hand side in (3.5) is bounded for all $\lambda > 0$, hence

$$\sup_{z \in \overline{\Gamma}} |\mathrm{Wig}(u, g)_\tau(z)| e^{\lambda \omega(z)} < +\infty, \qquad \lambda > 0.$$

Then, $\mathcal{J}_\tau^{-1}(z_0) \notin \mathrm{WF}_\omega^\tau(u)$.

On the other hand, we fix $q \in \mathbb{N}_0$ such that $\max\left\{\frac{1}{\tau}, \frac{1}{1-\tau}\right\} \leq 2^q$. If $\mathcal{J}_\tau^{-1}(z_0) \notin \mathrm{WF}_\omega^\tau(u)$, then there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$, $\mathcal{J}_\tau^{-1}(z_0) \in \Gamma$, such that

$$\sup_{z \in \Gamma} |\mathrm{Wig}(u, g)_\tau(z)| e^{\lambda L^q \omega(z)} < +\infty, \qquad \lambda > 0.$$

We recall that $\mathcal{I}_{1-\tau} g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. By Proposition 15, for any $\Gamma' \subseteq \mathbb{R}^{2d} \setminus \{0\}$ satisfying $\overline{\Gamma'} \cap S_{2d-1} \subseteq \Gamma$ and $\mathcal{J}_\tau^{-1}(z_0) \in \Gamma'$ it holds that

$$\sup_{z \in \Gamma'} |\mathrm{Wig}(u, \mathcal{I}_{1-\tau} g)_\tau(z)| e^{\lambda L^q \omega(z)} < +\infty, \qquad \lambda > 0.$$

As in (3.4), from the choice of $q \in \mathbb{N}_0$ it follows that for all $z \in \mathbb{R}^{2d}$ and $\lambda > 0$

$$
\begin{aligned}
|\operatorname{Wig}(u, \mathcal{I}_{1-\tau}g)_\tau(z)|e^{\lambda L^q \omega(z)} &= \left| V_g u \left(\mathcal{J}_\tau \left(\frac{1}{1-\tau} z \right) \right) \right| e^{\lambda L^q \omega(z)}. \\
&\geq \left| V_g u \left(\mathcal{J}_\tau \left(\frac{1}{1-\tau} z \right) \right) \right| e^{\lambda \omega \left(\mathcal{J}_\tau \left(\frac{1}{1-\tau} z \right) \right)} e^{-\lambda(1+\cdots+L^q)},
\end{aligned}
$$

which is bounded for all $z \in \Gamma'$, and therefore, for the open conic set $\overline{\Gamma} := \mathcal{J}_\tau(\Gamma')$ which contains z_0, we have

$$
\sup_{z \in \overline{\Gamma}} |V_g u(z)| e^{\lambda \omega(z)} < +\infty, \qquad \lambda > 0.
$$

Thus, $z_0 \notin \operatorname{WF}'_\omega(u)$, and the proof is complete. □

We therefore obtain a condition under which these wave front sets coincide (compare it with [14, Theo. 5.5]):

Corollary 17 *For every weight function ω,*

$$
\operatorname{WF}'_\omega(u) = \operatorname{WF}^{1/2}_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).
$$

By Theorem 16 and [7, Prop. 3.18], it is easy to characterize when the Wigner wave front set of an ultradistribution is empty.

Theorem 18 *Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$. Then $\operatorname{WF}^\tau_\omega(u) = \emptyset$ for all (or some) $0 < \tau < 1$ if and only if $u \in \mathcal{S}_\omega(\mathbb{R}^d)$.*

4 Examples

We remark that the ω-wave front set is invariant under time-frequency shifts [7, Prop. 3.19]. By Theorem 16, this holds true for the Wigner wave front set:

Proposition 19 *Let $0 < \tau < 1$ and $z \in \mathbb{R}^{2d}$. We have*

$$
\operatorname{WF}^\tau_\omega(\Pi(z)u) = \operatorname{WF}^\tau_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).
$$

Now, we compute explicitly the Wigner wave front set of some concrete $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, as done in [7, Sect. 5] and [26, Sect. 6].

Example 20 Let $u = \delta$ be the Dirac distribution, which belongs to $\mathcal{S}'_\omega(\mathbb{R}^d)$ for every weight function ω (since δ has compact support), and fix $\alpha \in \mathbb{N}_0^d$. It is easy to check that, for any $g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$,

$$
V_g D^\alpha \delta(x, \xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \xi^\beta \overline{D^{\alpha-\beta} g(-x)}, \qquad x, \xi \in \mathbb{R}^d.
$$

Then, by Lemma 7, we have

$$
\begin{aligned}
\mathrm{Wig}(D^\alpha \delta, g)_\tau(x, \xi) &= \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau} i x \cdot \xi} V_{\mathcal{I}_\tau g} D^\alpha \delta\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) \\
&= \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau} i x \cdot \xi} \sum_{\beta \le \alpha} \binom{\alpha}{\beta}\left(\frac{1}{\tau}\xi\right)^\beta \overline{D^{\alpha-\beta}\mathcal{I}_\tau g\left(-\frac{1}{1-\tau}x\right)} \\
&= \left(\frac{1}{\tau}\right)^d e^{\frac{1}{\tau} i x \cdot \xi} \sum_{\beta \le \alpha} \binom{\alpha}{\beta}\left(\frac{1}{\tau}\xi\right)^\beta \overline{D^{\alpha-\beta} g\left(\frac{1}{\tau}x\right)} \qquad (4.1)
\end{aligned}
$$

for all $(x, \xi) \in \mathbb{R}^{2d}$. In particular,

$$
\mathrm{Wig}(D^\alpha \delta, g)_\tau(0, \xi) = \left(\frac{1}{\tau}\right)^d \sum_{\beta \le \alpha} \binom{\alpha}{\beta}\left(\frac{1}{\tau}\xi\right)^\beta \overline{D^{\alpha-\beta} g(0)}. \qquad (4.2)
$$

We take $g \in \mathcal{S}_\omega(\mathbb{R}^d)$ with $\overline{g(0)} = 1$ and $D^\gamma g(0) = 0$ for all $0 \ne \gamma \in \mathbb{N}_0^d$ and we check that

$$
\{0\} \times (\mathbb{R}^d \setminus \{0\}) \subseteq \mathrm{WF}_\omega^\tau(D^\alpha \delta).
$$

Indeed, for every point of the form $(0, \xi_0)$, $\xi_0 \ne 0$ we have that all open conic set $\Gamma \subseteq \mathbb{R}^{2d}\setminus\{0\}$ containing the point $(0, \xi_0)$ there exists $\xi \in \mathbb{R}^d$ such that

$$
\left|\sum_{\beta \le \alpha} \binom{\alpha}{\beta}\left(\frac{1}{\tau}\xi\right)^\beta \overline{D^{\alpha-\beta} g(0)}\right| = \left|\left(\frac{1}{\tau}\xi\right)^\alpha\right| \ge 1.
$$

Hence, by (4.2),

$$
\sup_{z \in \Gamma} |\mathrm{Wig}(D^\alpha \delta, g)_\tau(z)| e^{\lambda \omega(z)} \ge \left(\frac{1}{\tau}\right)^d \sup_{(0,\xi)\in\Gamma} e^{\lambda \omega(\xi)} = +\infty, \qquad \lambda > 0.
$$

Now, we claim that

$$
\mathrm{WF}_\omega^\tau(D^\alpha \delta) \subseteq \{0\} \times (\mathbb{R}^d \setminus \{0\}).
$$

To this, take $(x_0, \xi_0) \in \mathbb{R}^{2d} \setminus \{0\}$ such that $x_0 \ne 0$. Consider an open conic set containing the point (x_0, ξ_0) of the form

$$
\Gamma = \{(x, \xi) \in \mathbb{R}^{2d} \ |\xi| < C|x|\},
$$

for some $C > 0$. By condition (α) of the weight, there exists $C' > 0$ such that, for $z = (x, \xi) \in \Gamma$,

$$
\omega(z) \le \omega(|x| + |\xi|) \le \omega((1+C)|x|) \le C'\omega(x) + C' \le C'\omega\left(\frac{1}{\tau}x\right) + C'.
$$

Therefore, using (4.1), for all $\lambda > 0$ and $z = (x, \xi) \in \Gamma$,

$$
\begin{aligned}
| \operatorname{Wig}&(D^\alpha \delta, g)_\tau(z) | e^{\lambda \omega(z)} \\
&\leq \left(\frac{1}{\tau}\right)^d e^{\lambda \omega(x,\xi)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left|\frac{1}{\tau}\xi\right|^{|\beta|} \overline{\left|D^{\alpha-\beta} g\left(\frac{1}{\tau}x\right)\right|} \\
&\leq \left(\frac{1}{\tau}\right)^d e^{\lambda C'} e^{\lambda C' \omega(\frac{1}{\tau}x)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C^{|\beta|} \left|\frac{1}{\tau}x\right|^{|\beta|} \overline{\left|D^{\alpha-\beta} g\left(\frac{1}{\tau}x\right)\right|}.
\end{aligned}
$$

For all $\lambda > 0$ and $\beta \leq \alpha$, we have by Lemma 2,

$$
\left|\frac{1}{\tau}x\right|^{|\beta|} \leq e^{\lambda \varphi^*\left(\frac{|\beta|}{\lambda}\right)} e^{\lambda \omega(\frac{1}{\tau}x)} \leq e^{\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)} e^{-\lambda \varphi^*\left(\frac{|\alpha-\beta|}{\lambda}\right)} e^{\lambda \omega(\frac{1}{\tau}x)}.
$$

Then,

$$
\begin{aligned}
\sup_{z \in \Gamma} &| \operatorname{Wig}(D^\alpha \delta, g)_\tau(z) | e^{\lambda \omega(z)} \\
&\leq \left(\frac{1}{\tau}\right)^d e^{\lambda C'} C^{|\alpha|} e^{\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \\
&\quad \times \sup_{x \in \mathbb{R}^d} e^{-\lambda \varphi^*\left(\frac{|\alpha-\beta|}{\lambda}\right)} e^{(\lambda + \lambda C')\omega(\frac{1}{\tau}x)} \left|D^{\alpha-\beta} g\left(\frac{1}{\tau}x\right)\right| \\
&= \left(\frac{1}{\tau}\right)^d e^{\lambda C'} C^{|\alpha|} e^{\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \\
&\quad \times \sup_{x \in \mathbb{R}^d} e^{-\lambda \varphi^*\left(\frac{|\alpha-\beta|}{\lambda}\right)} e^{(\lambda + \lambda C')\omega(x)} |D^{\alpha-\beta} g(x)|,
\end{aligned}
$$

which is finite, using the seminorms given in [6, Theo. 4.8] (as $g \in \mathcal{S}_\omega(\mathbb{R}^d)$). This shows that $(x_0, \xi_0) \notin \operatorname{WF}^\tau_\omega(D^\alpha \delta)$, and thus

$$
\operatorname{WF}^\tau_\omega(D^\alpha \delta) = \{0\} \times (\mathbb{R}^d \setminus \{0\}).
$$

Moreover, since the Wigner wave front set is invariant under the translation operator (Proposition 19), for the Dirac distribution δ_x centered at the point x we have

$$
\operatorname{WF}^\tau_\omega(D^\alpha \delta_x) = \{0\} \times (\mathbb{R}^d \setminus \{0\}).
$$

We now study how the Wigner wave front set interacts with the Fourier transform (cf. [27, Prop. 4.3]):

Proposition 21 *For all $0 < \tau < 1$, we have*

$$
\operatorname{WF}^\tau_\omega(\widehat{u}) = \mathcal{J} \operatorname{WF}^{1-\tau}_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d),
$$

where $\mathcal{J}(x, \xi) := (-\xi, x)$ for all $x, \xi \in \mathbb{R}^d$.

Proof Fix $g \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. By Lemma 7 and [17, Lemma 3.1.1], we have

$$|\operatorname{Wig}(\widehat{u}, \mathcal{I}_{1-\tau}\widehat{g})_\tau(x, \xi)| = \left(\frac{1}{\tau}\right)^d \left|V_{\widehat{g}}\widehat{u}\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)\right| = \left(\frac{1}{\tau}\right)^d \left|V_g u\left(-\frac{1}{\tau}\xi, \frac{1}{1-\tau}x\right)\right|.$$

By Lemma 8 we deduce

$$V_g u(x, \xi) = (1 - \tau)^d e^{-i\tau x \cdot \xi} \operatorname{Wig}(u, \mathcal{I}_\tau g)_{1-\tau}(\tau x, (1-\tau)\xi),$$

therefore

$$|\operatorname{Wig}(\widehat{u}, \mathcal{I}_{1-\tau}\widehat{g})_\tau(x, \xi)| = \left(\frac{1-\tau}{\tau}\right)^d |\operatorname{Wig}(u, \mathcal{I}_\tau g)_{1-\tau}(-\xi, x)|.$$

Hence, by Proposition 15, $(x, \xi) \notin \operatorname{WF}_\omega^\tau(\widehat{u})$ if and only if $(-\xi, x) \notin \operatorname{WF}_\omega^{1-\tau}(u)$. □

Example 22 Since $\widehat{D^\alpha \delta}(\xi) = \xi^\alpha$ for all $\xi \in \mathbb{R}^d$, it follows from Example 20 and Proposition 21 that

$$\operatorname{WF}_\omega^\tau(x^\alpha) = (\mathbb{R}^d \setminus \{0\}) \times \{0\},$$

which as a particular case includes the equality of sets for the distribution $u = 1$. Moreover, since the modulation operator does not affect the Wigner wave front set (Proposition 19), we obtain the analogous of [7, Example 5.2]: for $\xi \in \mathbb{R}^d$,

$$\operatorname{WF}_\omega^\tau(M_\xi 1) = (\mathbb{R}^d \setminus \{0\}) \times \{0\}.$$

Example 23 Consider $u(x) = e^{icx^2/2}$, where $x \in \mathbb{R}$ and $c \in \mathbb{R}\setminus\{0\}$. Observe that $u \in \mathcal{S}_\omega'(\mathbb{R})$ for every weight function ω. The ω-wave front set of u is equal to [7, (5.3)]

$$\operatorname{WF}_\omega'(u) = \{(x, cx) : x \in \mathbb{R} \setminus \{0\}\}, \qquad \text{for every weight } \omega.$$

By Theorem 16 we obtain

$$\operatorname{WF}_\omega^\tau(u) = \left\{\left(x, c\frac{\tau}{1-\tau}x\right) \ x \in \mathbb{R} \setminus \{0\}\right\}.$$

Notice that in this case

$$\operatorname{WF}_\omega^\tau(u) = \operatorname{WF}_\omega'(u) \quad \Leftrightarrow \quad \tau = 1/2.$$

Furthermore, if $\tau \neq 1/2$, we see that none of the wave front sets is contained in the other.

Example 23 illustrates that, in general, if $0 < \tau < \rho < 1$, it may fail that

$$\mathrm{WF}^{\tau}_{\omega}(u) \subseteq \mathrm{WF}^{\rho}_{\omega}(u) \ \text{ and } \ \mathrm{WF}^{\rho}_{\omega}(u) \subseteq \mathrm{WF}^{\tau}_{\omega}(u), \qquad u \in \mathcal{S}'_{\omega}(\mathbb{R}^d).$$

Moreover, it may happen that the intersection of $\mathrm{WF}^{\tau}_{\omega}(u)$ and $\mathrm{WF}^{\rho}_{\omega}(u)$ is empty.

Let us now turn our attention into weight functions. It is clear by definition that if $\omega \leq \sigma$ are two weight functions, then

$$\mathrm{WF}^{\tau}_{\omega}(u) \subseteq \mathrm{WF}^{\tau}_{\sigma}(u), \quad u \in \mathcal{S}'_{\omega}(\mathbb{R}^d).$$

We see that the Wigner wave front set also may depend on the weight function, like the ω-wave front set [7, Example 5.4].

Example 24 Take two (non-quasianalytic) weight functions $\omega \leq \sigma$ satisfying

$$\mathcal{S}_{\sigma}(\mathbb{R}^d) \cap \mathcal{D}(\mathbb{R}^d) \subsetneq \mathcal{S}_{\omega}(\mathbb{R}^d) \cap \mathcal{D}(\mathbb{R}^d),$$

where $\mathcal{D}(\mathbb{R}^d)$ is the space of smooth functions with compact support. By [7, Example 5.4], there exists a compactly supported, non-trivial function $f \in \mathcal{S}_{\omega}(\mathbb{R}^d)\backslash\mathcal{S}_{\sigma}(\mathbb{R}^d)$ such that

$$\mathrm{WF}'_{\omega}(f) = \emptyset \quad \text{ and } \quad \mathrm{WF}'_{\sigma}(f) = \{0\} \times (\mathbb{R}^d \setminus \{0\}).$$

Thus, by Theorem 16, we obtain, for $0 < \tau < 1$,

$$\mathrm{WF}^{\tau}_{\omega}(f) = \emptyset \quad \text{ and } \quad \mathrm{WF}^{\tau}_{\sigma}(f) = \{0\} \times (\mathbb{R}^d \setminus \{0\}).$$

Remark 25 We point out that the information that provides the Wigner wave front set and the ω-wave front set is the same in Examples 20, 22, and 24, independently of the τ chosen, while in Example 23, the Wigner wave front set depends on τ.

On the other hand, for the wave front set defined in [27, Defin. 4.1], Examples 20 and 22 are studied in [27, Prop. 5.3], obtaining the same equality for their wave front set.

We also remark that the Wigner wave front set provides different information than the classical Hörmander wave front set [20] for Dirac delta distributions centered at $\overline{x} \neq 0$. Indeed, he obtained that the wave front set of $\delta_{\overline{x}}$ is equal to $\{\overline{x}\} \times (\mathbb{R}^d\backslash\{0\})$, $\overline{x} \in \mathbb{R}^d$, coinciding with the case of Coriasco and Maniccia [15]. Nonetheless, the τ-Wigner transform provides more information for $u = 1$ in Example 22 than in Hörmander's case, as the wave front set is empty (since $u \in C^{\infty}(\mathbb{R}^d)$), while the wave front set of Coriasco and Maniccia detects the ray $(\mathbb{R}^d\backslash\{0\}) \times \{\overline{\xi}\}$ for $u = M_{\overline{\xi}}1$, $\overline{\xi} \in \mathbb{R}^d$.

Remark 26 Cordero and Rodino [14, (158)] give an equivalent definition of the wave front set defined by Rodino and Wahlberg (Definition 12 with $\omega(t) = \log(1 + t)$), denoted by $\mathrm{WF}_G(u)$. In [14, Theo. 5.5] it is shown that it is included in the wave front set defined in [14, Defin. 5.3] on the space $L^2(\mathbb{R}^d)$, the converse inclusion could not be proven because of the presence of ghost frequencies.

For $\omega(t) = \log(1+t)$, we observe by Hölder's inequality that

$$\mathrm{WF}_G(u) \subseteq \mathrm{WF}'_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).$$

Thus, by Corollary 17 we extend the inclusion in [14, Theo. 5.5] to the space of tempered distributions for $\tau = 1/2$. Moreover, this inclusion holds true if we replace the L^2-norm given in $\mathrm{WF}_G(u)$ by the L^p-norm, $1 \leq p \leq +\infty$, and also for the corresponding definition of [14, (158)] in $\mathcal{S}'_\omega(\mathbb{R}^d)$, for every weight function ω in the sense of Definition 1. On the other hand, we do not know whether the reverse inclusion can be satisfied for some $1 \leq p < +\infty$ and some weight function.

5 The Gabor–Wigner wave front set

For $\alpha, \beta > 0$, we consider the lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d \subseteq \mathbb{R}^{2d}$. For a window function $0 \neq g \in L^2(\mathbb{R}^d)$ we define a *Gabor frame for* $L^2(\mathbb{R}^d)$, denoted by $\{\Pi(\sigma)g\}_{\sigma \in \Lambda}$, if there exist two constants $A, B > 0$ such that

$$A \|f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{\sigma \in \Lambda} |\langle f, \Pi(\sigma)g\rangle|^2 \leq B \|f\|^2_{L^2(\mathbb{R}^d)}, \qquad f \in L^2(\mathbb{R}^d).$$

See [17] for conditions on α and β for which $\{\Pi(\sigma)g\}_{\sigma \in \Lambda}$ is a Gabor frame.

Rodino and Wahlberg [26] have shown that the information provided by the decay of the Gabor coefficients was necessary and sufficient to describe the micro-regularity of the tempered distribution. We state [7, Defin. 3.4], generalizing that idea to ω-tempered ultradistributions.

Definition 27 Let $0 \neq g \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ be a lattice, where $\alpha, \beta > 0$ are sufficiently small (so that $\{\Pi(\sigma)g\}_{\sigma \in \Lambda}$ is a Gabor frame). For $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, we say that $0 \neq z_0 \in \mathbb{R}^{2d}$ is not in the ω-*Gabor wave front set of* u, $\mathrm{WF}^G_\omega(u)$, if there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that

$$\sup_{\sigma \in \Lambda \cap \Gamma} |V_g u(\sigma)| e^{\lambda \omega(\sigma)} < +\infty, \qquad \lambda > 0.$$

We analogously define a global wave front set in open conic sets intersected with appropriate lattices, but replacing the STFT by the τ-Wigner transform:

Definition 28 (The τ, ω-Gabor–Wigner global wave front set of u) Let g and Λ be as in Definition 27. For $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $0 < \tau < 1$, we say that $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ is not in the τ, ω-*Gabor–Wigner global wave front set of* u, $\mathrm{WF}^{\tau,G}_\omega(u)$, if there exists an open conic set $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ containing z_0 such that

$$\sup_{\sigma \in \Lambda \cap \Gamma} |\mathrm{Wig}(u,g)_\tau(\sigma)| e^{\lambda \omega(\sigma)} < +\infty, \qquad \lambda > 0.$$

Analogously to Definition 13, we call $\mathrm{WF}^{\tau,G}_\omega(u)$ the Gabor–Wigner wave front set of u. The main aim of this section is to discuss the equality between the wave front sets

in Definitions 13 and 28. Contrary to [7, 26], we do not need to develop a theory of modulation spaces of exponential type using the Wigner transform. In fact, proceeding as in Theorem 16 it is easy to see that Definitions 27 and 28 satisfy a similar relation as that of Definitions 12 and 13:

Theorem 29 *For every ω weight function and $0 < \tau < 1$, we have*

$$\mathrm{WF}_\omega^G(u) = \mathcal{J}_\tau(\mathrm{WF}_\omega^{\tau,G}(u)), \qquad u \in \mathcal{S}_\omega'(\mathbb{R}^d).$$

In particular, these wave front sets coincide in $\mathcal{S}_\omega'(\mathbb{R}^d)$ for $\tau = 1/2$. Furthermore, from [7, Theo. 3.17] and Theorem 16, it follows that

Theorem 30 *Let ω be a subadditive weight function. For every $0 < \tau < 1$,*

$$\mathcal{J}_\tau(\mathrm{WF}_\omega^{\tau,G}(u)) = \mathrm{WF}_\omega^G(u) = \mathrm{WF}_\omega'(u) = \mathcal{J}_\tau(\mathrm{WF}_\omega^\tau(u)), \qquad u \in \mathcal{S}_\omega'(\mathbb{R}^d).$$

This shows that for any $0 < \tau < 1$, the Wigner wave front set coincides with the Gabor–Wigner wave front set in $\mathcal{S}_\omega'(\mathbb{R}^d)$, provided that ω is a subadditive weight function. Moreover,

Corollary 31 (Equalities of wave front sets) *If ω is subadditive, then*

$$\mathrm{WF}_\omega^{1/2,G}(u) = \mathrm{WF}_\omega^G(u) = \mathrm{WF}_\omega'(u) = \mathrm{WF}_\omega^{1/2}(u), \qquad u \in \mathcal{S}_\omega'(\mathbb{R}^d).$$

Remark 32 Notice that if ω is a subadditive weight function satisfying the hypotheses in [2, Cor. 5.9] and $0 < \tau < 1$, then the chain of equalities in Theorem 30 is extended to the global wave front set defined in [2, Defin. 4.3] for all $u \in \mathcal{S}_\omega'(\mathbb{R}^d)$. In particular, this whole chain holds for Gevrey weight functions $\omega(t) = t^a$ with $0 < a < 1$ small enough (see [2, Example 5.10]).

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Declarations

Conflict of interest Not applicable.

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