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Local confluence of conditional and generalized term rewriting systems ☆

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ABSTRACT

Reduction-based systems are used as a basis for the implementation of programming languages, automated reasoning systems, mathematical analysis tools, etc. In such inherently non-deterministic systems, guaranteeing that diverging steps can be eventually rejoined is crucial for a faithful use in most applications. This property of reduction systems is called *local confluence*. In a landmark 1980 paper, Gérard Huet characterized local confluence of a *Term Rewriting System* as the joinability of all its *critical pairs*. In this paper, we characterize local confluence of *Conditional Term Rewriting Systems*, where reduction steps may depend on the satisfaction of specific conditions in rules: a conditional term rewriting system is locally confluent if and only if (i) all its *conditional critical pairs* and (ii) all its *conditional variable pairs* (which we introduce in this paper) are joinable. Furthermore, the logic-based approach we follow here is well-suited to analyze local confluence of more general reduction-based systems. We exemplify this by (i) including (context-sensitive) replacement restrictions in the arguments of function symbols, and (ii) allowing for more general conditions in rules. The obtained systems are called *Generalized Term Rewriting Systems*. A characterization of local confluence is also given for them.

1. Introduction

In the last 50 years, *conditional rules* $\ell \to r \Leftarrow c$, specifying a *reduction step* $\ell \to r$ which is triggered only if the *condition* c is 'satisfied', have been used in automated reasoning systems like Mathematica [58], REDUCE [25,26] or SCRATCHPAD [19,20]. They are central in the expressivity of rule-based programming languages like Haskell [27] or logic-based specification and verification systems like Maude [5,11]. They are also useful to reason about cryptographic protocols [1] and cyber-physical systems [6,53].

Reduction relations are pervasive in computer science as they are used to express computations in most computational systems and programming languages as the aforementioned ones. Confluence is a property of (abstract) reduction relations $\to$ guaranteeing that, for all abstract objects s (often called *expressions* without loss of generality) which can be reduced into two different reducts t and t', respectively (written $s \to^* t$ and $s \to^* t'$), there is some u to which both t and t' are reducible, i.e., both $t \to^* u$ and $t' \to^* u$ hold. A weaker property is *local* confluence, where only a *single* reduction step is allowed on s, i.e., $s \to t$ and $s \to t'$. As usual, they are defined by the commutation of the diagrams displayed in Fig. 1. Confluence guarantees that reductions reaching an end, i.e., leading to an *irreducible* expression, obtain one and the same expression, which can then be considered as the *result* of the computation or reasoning process.

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Fig. 1. Local confluence (left) and confluence (right).

In a landmark 1980 paper [28], Gérard Huet proved the following result for Term Rewriting Systems (TRSs [3]), see [28, Lemma 3.1] and [3, Theorem 6.2.4]:

A term rewriting system is locally confluent if and only if all its *critical pairs* are *joinable*.

Critical pairs $\langle s,t\rangle$ of a TRS $\mathcal{R}$ are obtained from *rules* $\ell \to r$ and $\ell' \to r'$ of $\mathcal{R}$ that *overlap* at a nonvariable position p of ℓ, i.e., $\ell|_p$ (which is *not* a variable) and ℓ' *unify* with *most general unifier* (*mgu*) substitution θ (i.e., $\theta(\ell|_p) = \theta(\ell')$, see Section 2 for details about the notion of *mgu*). Then, (i) $s = \theta(\ell[r']_p)$ is the result of replacing the subterm $\ell|_p$ at position p of the left-hand side ℓ of the first rule by the right-hand side r' of the second rule and then applying the substitution θ. Also, (ii) $t = \theta(r)$ is obtained by instantiating the right-hand side r of the first rule [32]. Pairs $\langle s,t\rangle$ are *joinable* if there is an expression u to which both s and t can be reduced in zero or more steps.

Kaplan pioneered the analysis of computational properties of *conditional TRSs* (CTRSs) $\mathcal{R}$ where, in contrast to TRSs, conditional rules $\ell \to r \Leftarrow c$ are also allowed [30]. Here, c is a sequence of conditions $s \approx t$ for terms s and t and a symbol '$\approx$' which is given the same *interpretation* in all rules of $\mathcal{R}$; for instance (see, e.g., [46, Definition 7.1.3]): (i) as *joinability* statements, where both s and t are expected to be reduced to the *same* term; or (ii) by imposing a left-to-right *orientation* to the condition so that t should be obtained after zero or more reduction steps on s; or (iii) by requiring the *conversion* of terms s and t by means of reduction sequences where not only 'direct' steps $s \to t$ but also *inverse* steps $s \leftarrow t$ (i.e., $t \to s$) are allowed. Depending on the chosen interpretation *Join*, *Oriented*, and *Semi-Equational* CTRSs are obtained. The choice may drastically change the confluence of the system.

Example 1 *(Semi-equational vs. join semantics).* Consider the following CTRS $\mathcal{R}$ [43, Example 1.1] and [44, Example 1.2]:

$$\mathsf{a} \to \mathsf{b} \tag{1}$$

$$\mathsf{a} \to \mathsf{c} \tag{2}$$

$$\mathsf{b} \to \mathsf{c} \Leftarrow \mathsf{b} \approx \mathsf{c} \tag{3}$$

There is a single critical pair $\langle \mathsf{b},\mathsf{c}\rangle$. As remarked by Middeldorp, as a semi-equational system, $\mathcal{R}$ is confluent. As a join system, though, it is *not* confluent.

Example 2 *(Oriented vs. join semantics).* With the following CTRS $\mathcal{R}$ [46, Example 7.3.3]:

$$\mathsf{a} \to \mathsf{b} \tag{4}$$

$$\mathsf{f}(x) \to \mathsf{c} \Leftarrow x \approx \mathsf{a} \tag{5}$$

- We can rewrite $\mathsf{f}(\mathsf{a})$ to $\mathsf{f}(\mathsf{b})$ by using rule (4) on a, written $\mathsf{f}(\underline{\mathsf{a}}) \to_{(4)} \mathsf{f}(\mathsf{b})$.
- We also have $\underline{\mathsf{f}(\mathsf{a})} \to_{(5)} \mathsf{c}$. This is because, when variable x in the left-hand side $\mathsf{f}(x)$ of rule (5) is instantiated to a, the corresponding instance $\mathsf{a} \approx \mathsf{a}$ of the condition $x \approx \mathsf{a}$ of the rule is trivially satisfied (in all aforementioned semantics for conditions).

Thus, we obtain the following *peak* [7, Section 4.1]

$$\mathsf{f}(\mathsf{b}) \leftarrow \underrightarrow{\mathsf{f}(\overleftarrow{\mathsf{a}})} \to \mathsf{c} \tag{6}$$

where the upper $\overleftarrow{\text{left arrow}}$ and lower $\underrightarrow{\text{right arrow}}$ highlight the specific reduction which is performed in the peak. Now, we have an interesting situation:

1. If $\mathcal{R}$ is viewed as an *oriented* CTRS, then, as pointed out by Ohlebusch, $\mathsf{f}(\mathsf{b})$ and c are *not* joinable as both $\mathsf{f}(\mathsf{b})$ and c are *irreducible*. Thus, as an *oriented CTRS*, $\mathcal{R}$ is not (locally) confluent.
2. If $\mathcal{R}$ is viewed as a *join* CTRS, then $\mathsf{f}(\mathsf{b})$ and c *are* joinable because $\mathsf{f}(\mathsf{b}) \to_{(5)} \mathsf{c}$ due to the possibility of joining the two components of the obtained instance $\mathsf{b} \approx \mathsf{a}$ of the condition by applying rule (4) to the second component a to obtain b. Actually, we will be able to prove that, as a *join CTRS*, $\mathcal{R}$ is locally confluent and also confluent.

Suitable *conditional* generalizations $\langle s,t\rangle \Leftarrow d$ of the notion of critical pair for conditional rules $\ell \to r \Leftarrow c$ and $\ell' \to r' \Leftarrow c'$ have been given [31,9,2], where $s = \theta(\ell[r']_p)$ and $t = \theta(r)$ are obtained as above, and d is $\theta(c), \theta(c')$. However, in view of $\mathcal{R}$ in Example 2, *without* such pairs, the following question arises:

Are there *hidden* 'pairs' which could be used to obtain a *Huet-like* characterization of local confluence of CTRSs?

In this paper, we provide a *positive* answer to this question. For instance, for $\mathcal{R}$ in Example 2, the *hidden* pair has the following shape:

$$\langle \mathsf{f}(x'), \mathsf{c}\rangle \Leftarrow x \to x', x \approx \mathsf{a} \tag{7}$$

It is an example of a *conditional variable pair*, a new class of conditional pairs that we introduce here. We prove (7) joinable if $\mathcal{R}$ is viewed as a join CTRS, but not joinable if $\mathcal{R}$ is viewed as an oriented CTRS (Examples 41 and 48).

We show how to obtain a complete set of *conditional pairs* whose joinability characterizes local confluence of CTRSs. Accordingly, $\mathcal{R}$ in Example 2 can be proved confluent as a Join CTRS and non-confluent as an Oriented CTRS, see Example 48 below. Similarly for $\mathcal{R}$ in Example 1, see Example 47. Actually, we provide a more general treatment of the problem by (i) *mixing* conditions with *different* semantic interpretations *in the same rule*, and also (ii) considering replacement restrictions on selected arguments of function symbols as in *context-sensitive rewriting* [34]. The main contributions of this paper are summarized as follows:

1. The introduction of a new class of conditional pairs, the *conditional variable pairs* which capture variable peaks investigated in [9] for CTRSs.
2. A characterization of local confluence of CTRSs as the *joinability* of conditional critical pairs *plus* conditional variable pairs. As a corollary of Newman's Lemma, we obtain a characterization of confluence of CTRSs for terminating CTRSs.
3. The definition of *Generalized Term Rewriting Systems* (GTRSs) as a proper extension of CTRSs including the aforementioned features. We extend our previous results to obtain a characterization of local confluence of GTRSs.

After some preliminaries in Section 2, Section 3 shows how to obtain a first-order theory $\overline{\mathcal{R}}$ from $\mathcal{R}$ which takes into account the chosen evaluation of conditions of rules. One-step and many-step rewriting $\to_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$ are defined as *deducibility* of goals $s \to t$ (resp. $s \to^* t$) in such a theory, i.e., $\overline{\mathcal{R}} \vdash s \to t$ (resp. $\overline{\mathcal{R}} \vdash s \to^* t$). In this way, techniques for proving *feasibility* of sequences of atoms (with respect to $\overline{\mathcal{R}}$, see [22]), can be used to prove interesting properties of CTRSs. In particular, they are heavily used to analyze (non)joinability of conditional pairs. Section 4 recalls the taxonomy of *peaks* in conditional rewriting investigated by Dershowitz, Okada, and Sivakumar [9] (distinguishing *disjoint*, *critical*, and *variable* peaks). Section 5 discusses the notion of *conditional* pair, which abstracts conditional critical pairs and conditional variable pairs, which are introduced in this section together with specific results to prove their (non)joinability. Section 6 shows that the *joinability* of the aforementioned classes of conditional pairs (which are called *extended conditional critical pairs* altogether, ECCP) characterizes local confluence of CTRSs. As a simple corollary of this result, a characterization of confluence of *terminating* CTRSs is obtained. In Section 7, we define *Generalized Term Rewriting Systems* GTRSs and extend our previous results to obtain a characterization of local confluence of GTRSs and confluence of terminating GTRSs. Section 8 discusses some related work. Section 9 concludes.

This paper is an extended and revised version of [37].

2. Preliminaries

In the following, we often write *iff* instead of *if and only if*. We assume some familiarity with the basic notions of term rewriting [3, 46,54] and first-order logic [15,42], where missing definitions can be found. For the sake of readability, though, here we summarize the main notions and notations we use.

Abstract reduction relations Given a binary relation $\mathsf{R} \subseteq A \times A$ on a set A, we often write $a \;\mathsf{R}\; b$ instead of $(a,b) \in \mathsf{R}$. The *transitive* closure of R is denoted by R^+, and its *reflexive and transitive* closure by R^*. An element $a \in A$ is *irreducible* (or an R-*normal form*), if there exists no b such that $a \;\mathsf{R}\; b$. Given $a \in A$, if there is no infinite sequence $a = a_1 \;\mathsf{R}\; a_2 \;\mathsf{R}\; \cdots \;\mathsf{R}\; a_n \;\mathsf{R}\; \cdots$, then a is R-*terminating* (or *well-founded*); R is *terminating* if a is R-terminating for all $a \in A$. We say that R is (locally) *confluent* if, for all $a,b,c \in A$, whenever $a \;\mathsf{R}^*\; b$ and $a \;\mathsf{R}^*\; c$ (resp. $a \;\mathsf{R}\; b$ and $a \;\mathsf{R}\; c$), there exists $d \in A$ such that $b \;\mathsf{R}^*\; d$ and $c \;\mathsf{R}^*\; d$.

Signatures, terms, positions In this paper, $\mathcal{X}$ denotes a countable set of *variables* and $\mathcal{F}$ denotes a *signature*, i.e., a set of *function symbols* $\{f, g, \ldots\}$, each with a fixed *arity* given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F},\mathcal{X})$ and $\mathcal{T}(\mathcal{F})$ is the set of *ground* terms, i.e., without variable occurrences. The set of variables occurring in t is $\mathcal{V}ar(t)$. Terms are viewed as labeled trees in the usual way. *Positions* p are represented by chains of positive natural numbers used to address subterms $t|_p$ of t. Positions are ordered by the *prefix ordering* $\leq$ on sequences: given positions p, q, we write $p \leq q$ iff p is a prefix of q. If $p \not\leq q$ and $q \not\leq p$, we say that p and q are *disjoint* (or *parallel*). The root position of a term is denoted as Λ. The *set of positions* of a term t is $\mathcal{P}os(t)$. The set of positions of a subterm s in t is denoted $\mathcal{P}os_s(t)$. The set of positions of non-variable symbols in t is denoted as $\mathcal{P}os_{\mathcal{F}}(t)$. A *context* is a term C with a 'hole' $\square$, often viewed as a fresh constant symbol. A context with a single hole is written $C[\,]$, or $C[\,]_p$ to make the position of $\square$ explicit. A binary relation R on terms is *closed under substitutions* iff for all substitutions σ and terms $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$, if $s \;\mathsf{R}\; t$, then $\sigma(s) \;\mathsf{R}\; \sigma(t)$; it is *closed under contexts* if for all contexts $C[\,]_p$ and terms s and t, if $s \;\mathsf{R}\; t$, then $C[s]_p \;\mathsf{R}\; C[t]_p$. Equivalently, for all k-ary symbols $f \in \mathcal{F}$, arguments $1 \leq i \leq k$, and terms $s_1, \ldots, s_k$ and t_i, if $s_i \;\mathsf{R}\; t_i$, then $f(s_1, \ldots, s_i, \ldots, s_k) \;\mathsf{R}\; f(s_1, \ldots, t_i, \ldots, s_k)$.

Table 1
Generic sentences of the FO-theory of CTRSs.

Label	Sentence
(Rf)	$(\forall x)\ x \to^* x$
(Co)	$(\forall x, y, z)\ x \to y \wedge y \to^* z \Rightarrow x \to^* z$
$(\mathrm{Pr})_{f,i}$	$(\forall x_1, \ldots, x_k, y_i)\ x_i \to y_i \Rightarrow f(x_1, \ldots, x_i, \ldots, x_k) \to f(x_1, \ldots, y_i, \ldots, x_k)$
$(\mathrm{Rl})_\alpha$	$(\forall x_1, \ldots, x_n)\ s_1 \approx t_1 \wedge \cdots \wedge s_n \approx t_n \Rightarrow \ell \to r$

Unification A renaming ρ is a bijection from $\mathcal{X}$ to $\mathcal{X}$. A substitution σ is a mapping $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ from variables into terms which is homomorphically extended to a mapping (also denoted σ) $\sigma : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. It is standard to assume that substitutions σ satisfy $\sigma(x) = x$ except for a *finite* set of variables. Thus, we often write $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ to denote a substitution. Terms s and t *unify* if there is a substitution σ (i.e., a *unifier*) such that $\sigma(s) = \sigma(t)$. If s and t unify, then there is a (unique, up to renaming) *most general unifier* (*mgu*) θ of s and t satisfying that, for any other unifier σ of s and t, there is a substitution τ such that, for all $x \in \mathcal{X}$, $\sigma(x) = \tau(\theta(x))$.

Overlapping terms A term s *overlaps* a term t if s is unifiable with a *nonvariable* subterm of t [46, Definition 4.3.3]. Terms s and t are *nonoverlapping* if neither s overlaps t nor t overlaps s.

First-order logic Besides the signature $\mathcal{F}$ of function symbols, we also consider a signature Π of *predicate symbols*. Atoms and first-order formulas are built using such function and predicate symbols, and also variables in $\mathcal{X}$, in the usual way. We often write $A[x]$ to make explicit that variable x occurs (possibly many times) in A. A first-order theory (FO-theory for short) Th is a set of sentences (formulas whose variables are all *quantified*). In the following, given an FO-theory Th and a formula φ, $\mathsf{Th} \vdash \varphi$ means that φ is *deducible* from (or a *logical consequence* of) Th by using a correct and complete deduction procedure (e.g., resolution [49]).

Feasibility sequences An *f-condition* γ is an atom [22]. Sequences $\mathsf{F} = (\gamma_i)_{i=1}^n = (\gamma_1, \ldots, \gamma_n)$ of f-conditions are called *f-sequences*. We often drop 'f-' when no confusion arises. Empty sequences are written $()$. Given an FO-theory Th, a condition γ is Th-*feasible* (or just *feasible* if no confusion arises) if $\mathsf{Th} \vdash \sigma(\gamma)$ holds for some substitution σ; otherwise, it is *infeasible*. A sequence F is Th-feasible (or just *feasible*) iff there is a substitution σ such that, for all $\gamma \in \mathsf{F}$, γ is Th-feasible. Note that the empty f-sequence $()$ is trivially feasible.

Conditional term rewriting systems A CTRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where $\mathcal{F}$ is a signature and R is a set of rules $\ell \to r \Leftarrow c$, with c a sequence $s_1 \approx t_1, \cdots, s_n \approx t_n$ for some $n \geq 0$ and terms $\ell, r, s_1, \ldots, t_n$ such that $\ell \notin \mathcal{X}$. As usual, ℓ and r are called the *left-* and *right-hand sides* of the rule (*lhs* and *rhs*, respectively), and c is the *conditional part* of the rule. We often write $\gamma \in c$ to say that a condition γ as above is in c. *Labeled* rules are written $\alpha : \ell \to r \Leftarrow c$, where α is a *label*. In the following, given $\mathcal{R}$, we often write $\alpha \in \mathcal{R}$, instead of $\alpha \in R$, to say that α is a rule of $\mathcal{R}$. Whenever rules $\ell \to r \Leftarrow c$ of $\mathcal{R}$ are given numeric labels (n), as in Example 2 (with rules labeled (4) and (5)), we often write $\ell_{(n)}$, $r_{(n)}$, and $c_{(n)}$ to refer to the left- and right-hand sides, and also the conditional part of such rules. Conditional rules $\ell \to r \Leftarrow c$ are classified according to the distribution of variables among ℓ, r, and c [45, Definition 6.1]: *type 1*, if $\mathcal{V}ar(r) \cup \mathcal{V}ar(c) \subseteq \mathcal{V}ar(\ell)$; *type 2*, if $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$; *type 3*, if $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell) \cup \mathcal{V}ar(c)$; and *type 4*, otherwise. A rule of type n is often called an n-rule. An n-rule α is *proper* if for all $m < n$, α is *not* an m-rule. An n-CTRS contains only m-rules for some $m \leq n$ and at least a proper n-rule. A TRS is a 1-CTRS whose rules have no conditional part; we display them $\ell \to r$.

Grounding variables Let $\mathcal{F}$ be a signature and $\mathcal{X}$ be a set of variables such that $\mathcal{F} \cap \mathcal{X} = \emptyset$. Let $\mathcal{F}_{\mathcal{X}} = \mathcal{F} \cup C_{\mathcal{X}}$ where variables $x \in \mathcal{X}$ are considered as (different) *constant* symbols c_x of $C_{\mathcal{X}} = \{c_x \mid x \in \mathcal{X}\}$ and $\mathcal{F}$ and $C_{\mathcal{X}}$ are disjoint [23], see also [2, page 224]. Given a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, a ground term $t^{\downarrow} \in \mathcal{T}(\mathcal{F}_{\mathcal{X}})$ is obtained by replacing each occurrence of $x \in \mathcal{X}$ in t by c_x. Given a substitution $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, we define $\sigma^{\downarrow} = \{x_1 \mapsto t_1^{\downarrow}, \ldots, x_n \mapsto t_n^{\downarrow}\}$.

3. First-order theory of a CTRS. Rewriting as deduction

The presentation in this section is briefly anticipated in the second half of [23, Section 3.1]. In this paper, expressions $s \to t$ (intended to denote one-step reductions), $s \to^* t$ (zero or many-step reduction), $s \approx t$ (conditions in rules), etc., are viewed as *atoms* with (binary) predicate symbols $\to$, $\to^*$, $\approx$, etc. We collect all these predicate symbols in a set Π. The *meaning* of such expressions as *reduction relations* (e.g., $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}}^* t$) is given by deduction using an FO-theory $\overline{\mathcal{R}}$ associated to the conditional system $\mathcal{R}$ [33, Section 4.5]. Given a CTRS $\mathcal{R}$ over a signature $\mathcal{F}$, $\overline{\mathcal{R}}$ is obtained from the generic sentences in Table 1 where

- (Rf) expresses *reflexivity* of many-step rewriting;
- (Co) expresses *compatibility* of one-step and many-step rewriting;
- for each k-ary function symbol f, $1 \leq i \leq k$, and $x_1, \ldots, x_k$ and y_i distinct variables, $(\mathrm{Pr})_{f,i}$ enables the *propagation* of rewriting steps in the i-th immediate subterm of a term with root symbol f; finally,
- for each rule $\alpha : \ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ in $\mathcal{R}$, with variables $x_1, \ldots, x_n$, $(\mathrm{Rl})_\alpha$ expresses the application of a rewriting step $\sigma(\ell) \to \sigma(r)$ for some substitution σ, provided that, for all $1 \leq i \leq n$, $\sigma(s_i) \approx \sigma(t_i)$ can be proved.

$$\begin{array}{lll} \text{Join} & (\mathrm{J}) & (\forall x,y,z) \quad x \to^* z \wedge y \to^* z \Rightarrow x \approx y \\ \text{Oriented} & (\mathrm{O}) & (\forall x,y) \quad x \to^* y \Rightarrow x \approx y \\ & (\mathrm{SE}_1) & (\forall x) \quad x \approx x \\ \text{Semi-equational} & (\mathrm{SE}_2) & (\forall x,y,z) \quad x \to y \wedge y \approx z \Rightarrow x \approx z \\ & (\mathrm{SE}_3) & (\forall x,y,z) \quad y \to x \wedge y \approx z \Rightarrow x \approx z \end{array}$$

Fig. 2. Sentences for different semantics of CTRSs.

We also need sentences defining the *meaning* of the predicate $\approx$ used in the conditions of rules, see, e.g., [46, Definition 7.1.3]. Fig. 2 displays the appropriate set of sentences which should be added to obtain the theory $\overline{\mathcal{R}}_J$, $\overline{\mathcal{R}}_O$, or $\overline{\mathcal{R}}_{\mathrm{SE}}$ (or just $\overline{\mathcal{R}}$ if the semantics *Join*, *Oriented*, or *Semi-equational* of condition evaluation is clear from the context).

Definition 3. Let $\mathcal{R}$ be a CTRS and $\mathrm{CI} \in \{J, O, \mathrm{SE}\}$ denote a *computational interpretation* of symbol $\approx$ in the conditional part of rules $\alpha \in R$. Then,

$$\overline{\mathcal{R}}_{\mathrm{CI}} = \{(\mathrm{Rf}), (\mathrm{Co})\} \cup \{(\mathrm{Pr})_{f,i} \mid f \in \mathcal{F}, 1 \le i \le ar(f)\} \cup \{(\mathrm{Rl})_\alpha \mid \alpha \in R\} \cup \mathcal{S}_{\mathrm{CI}}$$

where

$$\mathcal{S}_{\mathrm{CI}} = \begin{cases} \{(\mathrm{J})\} & \text{if } \mathrm{CI} = \mathrm{J} \\ \{(\mathrm{O})\} & \text{if } \mathrm{CI} = \mathrm{O} \\ \{(\mathrm{SE}_1), \ldots, (\mathrm{SE}_3)\} & \text{if } \mathrm{CI} = \mathrm{SE} \end{cases}$$

We often write $\overline{\mathcal{R}}$ if CI is clear from the context.

Remark 4. In the following, given a CTRS $\mathcal{R}$, unless explicitly given otherwise, $\overline{\mathcal{R}}$ is the first order theory associated to $\mathcal{R}$ according to Definition 3.

Note that $\overline{\mathcal{R}}$ is a *Horn theory*. Thus, we often use *resolution* [49] as an appropriate deduction calculus for $\overline{\mathcal{R}}$.

Example 5. For $\mathcal{R}$ in Example 1,

$$\begin{array}{rcl} \overline{\mathcal{R}}_{\mathrm{J}} & = & \{(\mathrm{Rf}), (\mathrm{Co}), (8), (9), (10), (\mathrm{J})\} \\ \overline{\mathcal{R}}_{\mathrm{SE}} & = & \{(\mathrm{Rf}), (\mathrm{Co}), (8), (9), (10), (\mathrm{SE}_1), (\mathrm{SE}_2), (\mathrm{SE}_3)\} \end{array}$$

with rule sentences

$$\mathsf{a} \to \mathsf{b} \tag{8}$$

$$\mathsf{a} \to \mathsf{c} \tag{9}$$

$$\mathsf{b} \approx \mathsf{c} \Rightarrow \mathsf{b} \to \mathsf{c} \tag{10}$$

Example 6. For $\mathcal{R}$ in Example 2,

$$\begin{array}{rcl} \overline{\mathcal{R}}_J & = & \{(\mathrm{Rf}), (\mathrm{Co}), (11), (12), (13), (\mathrm{J})\} \\ \overline{\mathcal{R}}_O & = & \{(\mathrm{Rf}), (\mathrm{Co}), (11), (12), (13), (\mathrm{O})\} \end{array}$$

with propagation sentence

$$(\forall x_1, x_2)\ x_1 \to x_2 \Rightarrow \mathsf{f}(x_1) \to \mathsf{f}(x_2) \tag{11}$$

and rule sentences

$$\mathsf{a} \to \mathsf{b} \tag{12}$$

$$(\forall x)\ x \approx \mathsf{a} \Rightarrow \mathsf{f}(x) \to \mathsf{c} \tag{13}$$

Definition 7 *(Feasible rule)*. Let $\mathcal{R}$ be a CTRS. A rule $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$ is $\overline{\mathcal{R}}$-*(in)feasible* iff c is $\overline{\mathcal{R}}$-(in)feasible.

For instance, rule (3) of $\mathcal{R}$ in Example 1 is $\overline{\mathcal{R}}_{\mathrm{O}}$- and $\overline{\mathcal{R}}_{\mathrm{J}}$-infeasible. However, it is $\overline{\mathcal{R}}_{\mathrm{SE}}$-feasible.

Definition 8 *(Rewriting as deduction)*. Let $\mathcal{R}$ be a CTRS. For all terms s and t, we write $s \to_{\overline{\mathcal{R}}} t$ (resp. $s \to^*_{\overline{\mathcal{R}}} t$) iff $\overline{\mathcal{R}} \vdash s \to t$ (resp. $\overline{\mathcal{R}} \vdash s \to^* t$). We often just write $s \to_{\mathcal{R}} t$ and $s \to^*_{\mathcal{R}} t$ if no confusion arises.

The following result makes explicit well-known properties of unconditional rewriting which also hold for CTRSs and we use in the sequel.

Proposition 9. *Let $\mathcal{R}$ be a CTRS. Then, $\to_{\mathcal{R}}$ is closed under contexts and substitutions.*

As it is well-known, in term rewriting the variables occurring in subject terms are *never* instantiated. They behave as *constants* during rewriting sequences (see also [23, Proposition 6]).

Theorem 10. *Let $\mathcal{R}$ be a CTRS and s,t be terms. Then, $s \to_{\mathcal{R}} t$ iff $s^{\downarrow} \to_{\mathcal{R}} t^{\downarrow}$ and $s \to^*_{\mathcal{R}} t$ iff $s^{\downarrow} \to^*_{\mathcal{R}} t^{\downarrow}$.*

Proof. By definition, $s \to_{\mathcal{R}} t$ holds iff $\overline{\mathcal{R}} \vdash s \to t$ iff $\overline{\mathcal{R}} \vdash (\forall \vec{x})\, s \to t$, where $\vec{x}$ are the variables occurring in s or t, which is equivalent to the unsatisfiability of $\overline{\mathcal{R}} \cup \{\neg(s^{\downarrow} \to t^{\downarrow})\}$ because $\neg(s^{\downarrow} \to t^{\downarrow})$ can be seen as the skolemized version of $\neg(\forall \vec{x}) s \to t$, i.e., $(\exists \vec{x})\neg(s \to t)$. This shows that deducing $s \to t$ and deducing $s^{\downarrow} \to t^{\downarrow}$ from $\overline{\mathcal{R}}$ is essentially the same thing. Similarly for $s \to^* t$. $\square$

The following result makes explicit that each rewriting step $s \to_{\mathcal{R}} t$ involves a position $p \in \mathcal{P}os(s)$, a substitution σ, and a rule $\ell \to r \Leftarrow c \in \mathcal{R}$ such that $s|_p = \sigma(\ell)$ and $\sigma(c)$ holds.

Proposition 11. *Let $\mathcal{R}$ be a CTRS and $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$. Then, $s \to_{\mathcal{R}} t$ iff there is $p \in \mathcal{P}os(s)$ and $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$ such that (i) $s|_p = \sigma(\ell)$ for some substitution σ, (ii) for all $\gamma \in c$, $\overline{\mathcal{R}} \vdash \sigma(\gamma)$ holds, and (iii) $t = s[\sigma(r)]_p$.*

Proof. The only if part follows by induction on $p \in \mathcal{P}os(s)$. If $p = \Lambda$, then $s = \sigma(\ell)$ and, since $\overline{\mathcal{R}} \vdash \sigma(\gamma)$ holds for all $\gamma \in c$, by $(\mathrm{Rl})_\alpha$ we conclude $s = \sigma(\ell) \to_{\mathcal{R}} \sigma(r) = t$. If $p = i.p'$ and $root(s) = f$, we use $(\mathrm{Pr})_{f,i}$ and the induction hypothesis. As for the *if* part, since $s \to_{\mathcal{R}} t$, by Theorem 10, the goal $s^{\downarrow} \to t^{\downarrow}$ is deducible from $\overline{\mathcal{R}}$. Thus, we assume the use of resolution and proceed by induction on the number n of resolution steps used to obtain an empty set from $G = \{\neg(s^{\downarrow} \to t^{\downarrow})\}$ by applying resolution steps using clauses in $\overline{\mathcal{R}}$. *(Base: $m = 1$)* There is a clause $(\mathrm{Rl})_\alpha$ for some (unconditional) rule $\alpha : \ell \to r \in \mathcal{R}$ so that $s^{\downarrow} = \sigma(\ell)$ and $t^{\downarrow} = \sigma(r)$ for some substitution σ. Thus, the rewriting position is $p = \Lambda \in \mathcal{P}os(s)$. *(Induction step: $m > 1$)* We consider two cases:

1. There is a clause $(\mathrm{Rl})_\alpha$ for some rule $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$ with c consisting of $n > 0$ (atomic) conditions $\gamma_1, \ldots, \gamma_n$ so that $s = \sigma(\ell)$ for some substitution σ, and an empty set is obtained from $G' = \{\neg\sigma(\gamma_1), \ldots, \neg\sigma(\gamma_n)\}$ using $m-1$ resolution steps. In this case, we just need to take $p = \Lambda$ to fulfill the desired conclusion.
2. There is a clause $(\mathrm{Pr})_{f,i}$ for some $f \in \mathcal{F}$ and $1 \le i \le ar(f)$ so that (a) $s^{\downarrow} = f(s_1^{\downarrow}, \ldots, s_i^{\downarrow}, \ldots, s_k^{\downarrow})$ for some terms $s_1, \ldots, s_k$, (b) $t = f(s_1^{\downarrow}, \ldots, t_i^{\downarrow}, \ldots, s_k^{\downarrow})$, and an empty set is obtained from $G' = \{\neg(s_i^{\downarrow} \to t_i^{\downarrow})\}$ using $m-1$ resolution steps. By the induction hypothesis and Theorem 10, $s_i \to_{\mathcal{R}} t_i$ holds and there is $q \in \mathcal{P}os(s_i)$, $\ell \to r \Leftarrow c \in \mathcal{R}$ and substitution σ such that $s_i|_q = \sigma(\ell)$, $t_i = s_i[\sigma(r)]_q$, and each atom in $\sigma(c)$ is deducible from $\overline{\mathcal{R}}$. Since $s|_{i.q} = \sigma(\ell)$ and $t = s[\sigma(r)]_{i.q}$, the conclusion follows by letting $p = i.q$. $\square$

Definition 12 *(Confluence and termination of CTRSs).* A CTRS $\mathcal{R}$ is (locally) $\overline{\mathcal{R}}$-confluent (resp. $\overline{\mathcal{R}}$-terminating) iff $\to_{\overline{\mathcal{R}}}$ is (locally) confluent (resp. terminating).

If no confusion arises, we remove the prefix '$\overline{\mathcal{R}}$-' and just talk of confluence and termination of $\mathcal{R}$. In confluence analysis, testing joinability of terms is essential.

Definition 13 *(Joinable terms).* Given a CTRS $\mathcal{R}$, terms $s,t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ are *$\overline{\mathcal{R}}$-joinable* (written $s \downarrow_{\overline{\mathcal{R}}} t$) iff there is a term u such that $s \to^*_{\overline{\mathcal{R}}} u$ and $t \to^*_{\overline{\mathcal{R}}} u$. If no confusion arises, we use $\downarrow_{\mathcal{R}}$ or even $\downarrow$.

The following consequence of Theorem 10, is used below.

Corollary 14. *Let $\mathcal{R}$ be a CTRS. Two terms s and t are $\overline{\mathcal{R}}$-joinable iff $s^{\downarrow}$ and $t^{\downarrow}$ are $\overline{\mathcal{R}}$-joinable.*

The next result shows how joinability of terms can be proved as the feasibility of a sequence. This is used in the following.

Proposition 15. [23, Corollary 24] *Let $\mathcal{R}$ be a CTRS and s,t be terms. Then, s and t are $\overline{\mathcal{R}}$-joinable iff $s^{\downarrow} \to^* z, t^{\downarrow} \to^* z$, where z is a variable, is $\overline{\mathcal{R}}$-feasible.*

Proof. By Corollary 14, s and t are $\overline{\mathcal{R}}$-joinable iff $s^{\downarrow}$ and $t^{\downarrow}$ are $\overline{\mathcal{R}}$-joinable. Since $s^{\downarrow}$ and $t^{\downarrow}$ have no variable, the sequence $s^{\downarrow} \to^* z, t^{\downarrow} \to^* z$ is $\overline{\mathcal{R}}$-feasible iff there is a term u such that $s^{\downarrow} \to^*_{\overline{\mathcal{R}}} u$ and $t^{\downarrow} \to^*_{\overline{\mathcal{R}}} u$ hold, i.e., iff $s^{\downarrow}$ and $t^{\downarrow}$ are $\overline{\mathcal{R}}$-joinable. $\square$

In some cases, removing infeasible rules from a CTRS $\mathcal{R}$ may *alter* its computational properties. For instance, the oriented CTRS $\mathcal{R} = \{\mathsf{a} \to \mathsf{b} \Leftarrow \mathsf{a} \approx \mathsf{b}\}$ is not *operationally terminating* [39] because the attempt to reduce a using the only (conditional) rule in $\mathcal{R}$

launches infinitely many auxiliary attempts to evaluate the occurrence of a in the *condition* of the rule. However, the rule is clearly $\overline{\mathcal{R}}$-*infeasible* as b cannot be reached from a. The removal of the rule from $\mathcal{R}$, though, leaves an *empty* system which does *not* exhibit this problem anymore. See [40, Section 4] for a more detailed discussion. However, infeasible rules can be *removed* at once without modifying $\to_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$ as they cannot be used in any deduction to establish a rewriting step. Since confluence and termination of $\mathcal{R}$ only depend on $\to_{\mathcal{R}}$ and $\to^*_{\mathcal{R}}$, we can remove infeasible rules to obtain a simplified system whose confluence and termination is equivalent to that of $\mathcal{R}$.

4. Peaks and (local) confluence

Given a CTRS $\mathcal{R}$ and terms s, t, t', the situation

$$t\ {}_{\mathcal{R}}\!\leftarrow s \to_{\mathcal{R}} t'$$

is often called a *local peak*, or just a *peak*, if no confusion arises (see, e.g., [54, Section 1.2]). By Proposition 11, there are positions $\overline{p}, \overline{p'} \in \mathcal{P}os(s)$, rules $\alpha : \ell \to r \Leftarrow c$ and $\alpha' : \ell' \to r' \Leftarrow c'$, and substitutions σ and σ', such that (i) $s|_{\overline{p}} = \sigma(\ell)$ and $\sigma(c)$ hold in $\overline{\mathcal{R}}$; and (ii) $s|_{\overline{p'}} = \sigma'(\ell')$ and $\sigma'(c')$ hold in $\overline{\mathcal{R}}$. Thus, every peak is of the form

$$\overline{u} = s[\sigma'(r')]_{\overline{p'}}\ {}_{\mathcal{R}}\!\leftarrow s[\underleftarrow{\sigma'(\ell')}]_{\overline{p'}} = s = s[\underrightarrow{\sigma(\ell)}]_{\overline{p}} \to_{\mathcal{R}} s[\sigma(r)]_{\overline{p}} = \overline{v} \tag{14}$$

Remark 16 *(Structure of a peak).* Note that (14) pays no attention to the theory $\overline{\mathcal{R}}$ at stake beyond its use (through deduction) to establish the rewriting steps. This will be important in Section 7.5 below where the discussion in this section is applied to the more general setting of GTRSs. Regarding *positions*, a thorough consideration will be required, though.

Depending on the relative location of positions $\overline{p}$ and $\overline{p'}$ in $\sigma(\ell)$ and $\sigma'(\ell')$ in (14), different classes of peaks are usually distinguished: *disjoint*, *critical*, and *variable* peaks [9, Sections 2.1–2.3].

Disjoint peaks If $\overline{p}$ and $\overline{p'}$ in (14) are *disjoint*, then $s = s[\sigma(\ell)]_{\overline{p}}[\sigma'(\ell')]_{\overline{p'}} = s[\sigma'(\ell')]_{\overline{p'}}[\sigma(\ell)]_{\overline{p}}$. Accordingly, (14) can be written as follows:

$$\overline{u} = s[\sigma'(r')]_{\overline{p'}}[\sigma(\ell)]_{\overline{p}}\ {}_{\mathcal{R}}\!\leftarrow s[\underleftarrow{\sigma'(\ell')}]_{\overline{p'}}[\underrightarrow{\sigma(\ell)}]_{\overline{p}} \to_{\mathcal{R}} s[\sigma'(\ell')]_{\overline{p'}}[\sigma(r)]_{\overline{p}} = \overline{v} \tag{15}$$

Disjoint peaks are always *joinable*:

$$\overline{u} = s[\sigma'(r')]_{\overline{p'}}[\underrightarrow{\sigma(\ell)}]_{\overline{p}} \to_{\mathcal{R}} s[\sigma'(r')]_{\overline{p'}}[\sigma(r)]_{\overline{p}}\ {}_{\mathcal{R}}\!\leftarrow s[\underleftarrow{\sigma'(\ell')}]_{\overline{p'}}[\sigma(r)]_{\overline{p}} = \overline{v} \tag{16}$$

where the reduction step $\sigma(\ell) \to_{\mathcal{R}} \sigma(r)$ is possible in (16) because it is possible in (15). Similarly for $\sigma'(\ell') \to_{\mathcal{R}} \sigma'(r')$.

Non-disjoint peaks If $\overline{p}$ and $\overline{p'}$ in (14) are *not* disjoint, then we can write $s = s[\sigma(\ell)[\sigma'(\ell')]_p]_{\overline{p}}$, i.e., (without loss of generality) $\overline{p'} = \overline{p}.p$ for some $p \in \mathcal{P}os(\sigma(\ell))$. Hence, (14) can be written as follows:

$$\overline{u} = s[\sigma(\ell)[\sigma'(r')]_p]_{\overline{p}}\ {}_{\mathcal{R}}\!\leftarrow s[\underrightarrow{\sigma(\ell)[\overleftarrow{\sigma'(\ell')}]_p}]_{\overline{p}} \to_{\mathcal{R}} s[\sigma(r)]_{\overline{p}} = \overline{v}$$

By removing the untouched context $s[\ ]_{\overline{p}}$ around $\sigma(\ell)$, and assuming that α and α' share no variable (rename if necessary), we can use a single substitution σ to obtain:

$$u = \sigma(\ell)[\sigma(r')]_p\ {}_{\mathcal{R}}\!\leftarrow \underrightarrow{\sigma(\ell)[\overleftarrow{\sigma(\ell')}]_p} \to_{\mathcal{R}} \sigma(r) = v \tag{17}$$

Now we consider two cases for $p \in \mathcal{P}os(\sigma(\ell))$:

- $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$, which characterizes (17) as a *critical* peak; and
- $p \notin \mathcal{P}os_{\mathcal{F}}(\ell)$, which characterizes (17) as a *variable* peak.

Critical peaks (proper and improper) If the critical peak (17) is obtained from (i) a single rule $\alpha : \ell \to r \Leftarrow c$ and a renamed version α' of α, and (ii) $p = \Lambda$, then borrowing [2, Definition 4.2] we call it *improper*. Otherwise, we call it *proper*.

Proposition 17. *Every improper critical peak of a 2-CTRS is joinable.*

Proof. Since $p = \Lambda$ and $s = \sigma(\ell) = \sigma(\ell')$, improper critical peaks (17) become

$$\sigma(r')\ {}_{\mathcal{R}}\!\leftarrow \underleftarrow{\sigma(\ell')} = \underrightarrow{\sigma(\ell)} \to_{\mathcal{R}} \sigma(r) \tag{18}$$

Since α is a 2-rule, $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$, and then $\sigma(r) = \sigma(r')$. $\square$

Dealing with *proper* 3-rules α where $\mathcal{V}ar(r) \nsubseteq \mathcal{V}ar(\ell)$, this may *fail* to hold.

Example 18. For the following oriented 3-CTRS $\mathcal{R}$ [52, Introduction]:

$$\mathsf{g(b)} \to \mathsf{true} \tag{19}$$

$$\mathsf{g(c)} \to \mathsf{true} \tag{20}$$

$$\mathsf{a} \to \mathsf{f}(x) \Leftarrow \mathsf{g}(x) \approx \mathsf{true} \tag{21}$$

we have the following improper critical peak:

$$\mathsf{f(b)} \;{}_{\mathcal{R}}\!\leftarrow \underleftrightarrow{\mathsf{a}} \to_{\mathcal{R}} \mathsf{f(c)} \tag{22}$$

This peak is *not* $\overline{\mathcal{R}}_{\mathrm{O}}$-joinable as both $\mathsf{f(b)}$ and $\mathsf{f(c)}$ are irreducible terms.

Variable peaks If p in (17) satisfies $p \notin \mathcal{P}os_{\mathcal{F}}(\ell)$, then (17) can be written

$$u = \sigma(\ell)[C[\sigma(r')]_q]_{p_x} \;{}_{\mathcal{R}}\!\leftarrow \underrightarrow{\sigma(\ell)[C[\overleftarrow{\sigma(\ell')}]_q]_{p_x}} \to_{\mathcal{R}} \sigma(r) = v \tag{23}$$

for some context C, variable $x \in \mathcal{V}ar(\ell)$ and $p_x \in \mathcal{P}os_x(\ell)$ such that $p = p_x.q$ for some position q. Dershowitz et al. call (23) a *variable peak* [9]. As implicitly shown in Case 2a of the proof of [28, Lemma 3.1], variable peaks of TRSs are *always joinable*. For CTRSs, this is *not* true: if $\mathcal{R}$ in Example 2 is viewed as an O-CTRS, then (6), i.e., $\mathsf{f(b)} \leftarrow \underrightarrow{\mathsf{f}(\overleftarrow{\mathsf{a}})} \to \mathsf{c}$ is a *non-joinable* variable peak.

5. Conditional pairs in the analysis of local confluence of CTRSs

In order to give a homogeneous treatment to the peaks (17) and (23) we consider *conditional pairs*

$$\underbrace{\langle s,t\rangle}_{\text{peak}} \Leftarrow \underbrace{A_1,\ldots,A_n}_{\text{conditional part}} \tag{24}$$

where s,t are terms and A_i are atoms for all $1 \le i \le n$. In this way, conditional variable pairs like (7), i.e., $\langle \mathsf{f}(x'),\mathsf{c}\rangle \Leftarrow x \to x', x \approx \mathsf{a}$, are instances of (24) using atoms with *different* predicates ($\to$ and $\approx$) in the conditional part.

Definition 19 *(Feasible conditional pair).* Let $\mathcal{R}$ be a CTRS. A conditional pair $\langle s,t\rangle \Leftarrow c$ is $\overline{\mathcal{R}}$*-feasible* (or just *feasible* if $\overline{\mathcal{R}}$ is clear from the context) iff c is $\overline{\mathcal{R}}$-feasible.

Definition 20 *(Joinable conditional pair).* Let $\mathcal{R}$ be a CTRS. A conditional pair $\langle s,t\rangle \Leftarrow c$ is $\overline{\mathcal{R}}$*-joinable* (or just *joinable* if $\overline{\mathcal{R}}$ is clear from the context) iff for all substitutions σ, whenever $\overline{\mathcal{R}} \vdash \sigma(\gamma)$ holds for all $\gamma \in c$, terms $\sigma(s)$ and $\sigma(t)$ are $\overline{\mathcal{R}}$-joinable.

A conditional pair (24) is *trivial* if $s = t$. Trivial and infeasible conditional pairs are obviously joinable. This has been used for proving joinability of conditional critical pairs of CTRSs in, e.g., [2, Theorem 4.2], also with the sufficient criterion of *context-joinability* [2, Definition 4.4] which uses grounding of variables to provide a sufficient condition for joinability. As in [23, Section 6], we prove joinability of terms and critical pairs by proving the *(in)feasibility* of sequences [22] (see Section 2). However, in this paper we use the more general conditional pairs (24); and proofs of non-joinability of such pairs rely on the next result, whose proof is straightforward.

Proposition 21. *Let $\mathcal{R}$ be a CTRS and $\pi : \langle s,t\rangle \Leftarrow c$ be a conditional pair (24). If (i) $\sigma(c)$ is $\overline{\mathcal{R}}$-feasible for some substitution σ, and (ii) $\sigma(c), \sigma(s) \to^* z, \sigma(t) \to^* z$ is $\overline{\mathcal{R}}$-infeasible (for some $z \notin \mathcal{V}ar(\sigma(c),\sigma(s),\sigma(t))$), then π is not $\overline{\mathcal{R}}$-joinable.*

Remark 22. In order to use Proposition 21, we have to use a substitution σ. The following heuristics are useful.

1. The simplest choice is the empty substitution, i.e., $\sigma = \varepsilon$. This is easily mechanizable. Example 48 below illustrates this.
2. Choosing another substitution, usually trying to fulfill the conditions in the proposition. Example 49 illustrates this.

Now, we investigate how to represent critical and variable peaks of CTRSs by means of a *finite* number of *conditional pairs* (24).

5.1. Conditional critical pairs in CTRSs

The notion of *conditional critical pair* of a CTRS, see, e.g., [46, Definition 7.1.8(1)], is analogous to that of critical pairs of TRSs.

Definition 23 *(Conditional critical pair)*. Let $\mathcal{R}$ be a CTRS and $\alpha : \ell \to r \Leftarrow c$ and $\alpha' : \ell' \to r' \Leftarrow c'$ be ($\overline{\mathcal{R}}$-feasible) rules of $\mathcal{R}$ sharing no variable (rename if necessary). Let $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ be a nonvariable position of ℓ such that $\ell|_p$ and ℓ' unify with *mgu* θ. Then,

$$\langle \theta(\ell[r']_p), \theta(r) \rangle \Leftarrow \theta(c), \theta(c') \tag{25}$$

is a *conditional critical pair* (CCP) of $\mathcal{R}$ and p is called the *critical position*.

If the conditional part of (25) is empty, we just call it a *critical pair* and write $\langle s, t \rangle$, as usual. Example 18 shows that improper critical peaks can jeopardize (local) confluence of CTRSs. Thus, we introduce the following.

Definition 24 *(Proper and improper conditional critical pairs)*. Given a CTRS $\mathcal{R}$,

1. If α and α' in Definition 23 are renamed versions of the *same* rule and $p = \Lambda$, then (25) is called an *improper* conditional critical pair [2, Definition 4.2].
2. Conditional critical pairs not fitting case 1 are called *proper*.

Let $\mathsf{iCCP}(\overline{\mathcal{R}})$ (or just $\mathsf{iCCP}(\mathcal{R})$ if no confusion arises) be the set of $\overline{\mathcal{R}}$-feasible *improper conditional critical pairs* of proper 3- or 4-rules in $\mathcal{R}$. Let $\mathsf{pCCP}(\overline{\mathcal{R}})$, or just $\mathsf{pCCP}(\mathcal{R})$, be the set of $\overline{\mathcal{R}}$-feasible *proper conditional critical pairs* of $\mathcal{R}$.

Proposition 17 motivates restricting the attention to proper 3- or 4-rules in $\mathsf{iCCP}(\mathcal{R})$ above.

Remark 25 *(Root conditional critical pairs)*. In the context of this paper, we only need to consider *one* of the two conditional critical pairs obtained from (variable disjoint) rules $\alpha : \ell \to r \Leftarrow c$ and $\alpha' : \ell' \to r' \Leftarrow c'$ overlapping at the *root* critical position Λ when considering α as the "main" rule and α' as the "auxiliary" one, or vice versa (in the realm of TRSs, these are called *root* critical pairs, see [3, Exercise 6.19]). For θ an *mgu* of ℓ and ℓ', the corresponding *root* CCPs would be $\pi_1 : \langle \theta(r'), \theta(r) \rangle \Leftarrow \theta(c), \theta(c')$ and $\pi_2 : \langle \theta(r), \theta(r') \rangle \Leftarrow \theta(c'), \theta(c)$. Since the order of conditions does not preclude their satisfaction by a substitution and our notion of joinability treats both components s and t of a conditional pair $\langle s, t \rangle \Leftarrow d$ in the same way (requiring joinability, a symmetric relation), the joinability of π_1 and that of π_2 are equivalent.[1]

Remark 26 *(Conditional critical pairs in the literature)*. The notion of conditional critical pair as given in, e.g., [2, Definition 4.2] and [46, Definition 7.1.8(1)] only considers *proper* conditional critical pairs to formulate confluence results for CTRSs. With such a definition in mind, $\mathcal{R}$ in Example 18 has no conditional critical pair. In contrast, other authors do *not* dismiss improper critical pairs from their definition (see [8, paragraph above Theorem 3.3] or [13, Definition 6]) and associated results. Thus, in order to avoid confusion, we prefer to explicitly collect proper and improper conditional critical pairs in different sets $\mathsf{pCCP}(\mathcal{R})$ and $\mathsf{iCCP}(\mathcal{R})$ and give our results accordingly.

5.1.1. Joinability of conditional critical pairs and joinability of critical peaks

The following result shows that substitutions satisfying the conditional part of conditional critical pairs determine a critical peak.

Proposition 27. *Let $\mathcal{R}$ be a CTRS and $\pi : \langle s, t \rangle \Leftarrow d$ be a conditional critical pair where (i) $s = \theta(\ell[r']_p)$ and $t = \theta(r)$ for some rules $\ell \to r \Leftarrow c$ and $\ell' \to r' \Leftarrow c'$, $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$, and θ an* mgu *of $\ell|_p$ and ℓ', and (ii) $d = \theta(c), \theta(c')$. Let σ be a substitution such that $\sigma(d)$ holds (i.e., π is $\overline{\mathcal{R}}$-feasible). Then, $\sigma(s) \,{}_{\mathcal{R}}\!\leftarrow \sigma(\ell) \to_{\mathcal{R}} \sigma(t)$ is a critical peak.*

Proof. Since $\sigma(d)$ holds, both $\sigma(\theta(c))$ and $\sigma(\theta(c'))$ hold as well. Since $\theta(\ell|_p) = \theta(\ell')$, we have

$$\sigma(\theta(\ell)) = \sigma(\theta(\ell)[\theta(\ell')]_p) = \sigma(\theta(\ell))[\sigma(\theta(\ell'))]_p \to_{\mathcal{R}} \sigma(\theta(\ell))[\sigma(\theta(r'))]_p = \sigma(s)$$

and $\sigma(\theta(\ell)) \to_{\mathcal{R}} \sigma(\theta(r)) = \sigma(t)$ as desired. $\square$

The following result establishes that every critical peak κ has an *associated* (feasible) conditional critical pair π.

Proposition 28 *(Critical peaks from conditional critical pairs). Let $\mathcal{R}$ be a CTRS and $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c'$ be rules sharing no variable (rename if necessary) and determining a critical peak (17) at position $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ with substitution σ. The ($\overline{\mathcal{R}}$-feasible) conditional critical pair $\pi : \langle \theta(\ell)[\theta(r')]_p, \theta(r) \rangle \Leftarrow \theta(c), \theta(c')$, where θ is the* mgu *of $\ell|_p$ and ℓ' satisfies $\sigma = \tau \circ \theta$ for some substitution τ.*

Proof. Since there is a critical peak κ determined by α and α' with substitution σ at position $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$, we have that $\sigma(\ell) = \sigma(\ell)[\sigma(\ell')]_p \to \sigma(\ell)[\sigma(r')]_p$ and $\sigma(\ell) = \sigma(\ell)[\sigma(\ell')]_p \to \sigma(r)$, and both $\sigma(c)$ and $\sigma(c')$ hold. Since $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ and $\sigma(\ell|_p) = \sigma(\ell')$, we have

[1] This could be different when *asymmetric* notions of joinability are considered (e.g., *parallel closedness*, see [28, page 815] and [3, Definition 6.4.5]).

that $\ell|_p$ and ℓ' unify with σ. Thus, consider the critical pair $\langle\theta(\ell)[\theta(r')]_p, \theta(r)\rangle \Leftarrow \theta(c), \theta(c')$, where θ is the *mgu* of $\ell|_p$ and ℓ' and, by definition of *mgu*, $\sigma = \tau\circ\theta$ for some substitution τ. Furthermore, since both $\sigma(c)$ and $\sigma(c')$ hold, π is feasible. □

Propositions 27 and 28 justify the dismissal of *infeasible* critical pairs from pCCP($\mathcal{R}$) and iCCP($\mathcal{R}$). By Proposition 28, there is a mapping ϖ such that for each critical peak κ, there is an *associated* critical pair $\pi = \varpi(\kappa)$. Joinability of critical pairs π implies joinability of critical peaks κ such that $\pi = \varpi(\kappa)$.

Proposition 29 *(Joinability of conditional critical pairs and peaks). Let $\mathcal{R}$ be a CTRS, κ be a critical peak (17), and $\pi = \varpi(\kappa)$. If π is $\overline{\mathcal{R}}$-joinable, then κ is $\overline{\mathcal{R}}$-joinable.*

Proof. In this proof, the symbols α, σ, etc., have the meaning established in the proof of Proposition 28. Note that both $\sigma(c)$ and $\sigma(c')$ hold. By Proposition 28, there is a substitution τ such that $\sigma = \tau\circ\theta$. Thus, $\sigma(c) = \tau(\theta(c))$ and $\sigma(c') = \tau(\theta(c'))$ hold. Since $\pi = \varpi(\kappa)$ and π is joinable, by definition of $\overline{\mathcal{R}}$-joinability (Definition 20), we have that $\sigma(\ell)[\sigma(r')]_p = \tau(\theta(\ell))[\tau(\theta(r'))]_p)$ and $\sigma(r) = \tau(\theta(r))$ are joinable. Hence, κ is joinable. □

5.2. Conditional variable pairs in CTRSs

Overlapping terms are not the only source of divergent peaks in conditional rewriting. Variable peaks do *not* involve overlapping terms, but can be harmful [9, Section 3]. Up to now, no corresponding notion of *conditional pair* capturing such divergencies has been proposed. The following definition fills this gap.

Definition 30 *(Conditional variable pair).* Let $\mathcal{R}$ be a CTRS, $\ell \to r \Leftarrow c \in \mathcal{R}$, $x \in \mathcal{V}ar(\ell)$, $p \in \mathcal{P}os_x(\ell)$ and x' be a fresh variable. Then,

$$\langle\ell[x']_p, r\rangle \Leftarrow x \to x', c \tag{26}$$

is a *conditional variable pair* (CVP). Variable x is called the *critical variable* of the pair, and p is called the *critical position*. Let CVP($\overline{\mathcal{R}}$) (or just CVP($\mathcal{R}$) if no confusion arises) be the set of all $\overline{\mathcal{R}}$*-feasible conditional variable pairs* $\langle\ell[x']_p, r\rangle \Leftarrow x \to x', c$ for rules $\ell \to r \Leftarrow c \in \mathcal{R}$, $x \in \mathcal{V}ar(\ell)$, and $p \in \mathcal{P}os_x(\ell)$.

The rule (5) of $\mathcal{R}$ in Example 2 defines the conditional variable pair (7).

5.2.1. Joinability of conditional variable pairs and joinability of variable peaks

As for conditional critical pairs, we have the following result.

Proposition 31. *Let $\mathcal{R}$ be a CTRS and $\pi : \langle s, t\rangle \Leftarrow x \to x', c$ be a conditional variable pair where $s = \ell[x']_p$ and $t = r$ for some rule $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$, $p \in \mathcal{P}os_x(\ell)$, and $x' \notin \mathcal{V}ar(\alpha)$. Let σ be a substitution such that both $\sigma(x) \to \sigma(x')$ and $\sigma(c)$ hold (hence π is feasible). Then, $\sigma(s)\ {}_{\mathcal{R}}\!\leftarrow \sigma(\ell) \to_{\mathcal{R}} \sigma(t)$ is a variable peak.*

Proof. Since $\sigma(x) \to_{\mathcal{R}} \sigma(x')$, we have

$$\sigma(\ell) = \sigma(\ell[x]_p) = \sigma(\ell)[\sigma(x)]_p \to_{\mathcal{R}} \sigma(\ell)[\sigma(x')]_p = \sigma(s)$$

Since $\sigma(c)$ holds, we also have $\sigma(\ell) \to_{\mathcal{R}} \sigma(r) = \sigma(t)$ as desired. □

As for conditional critical peaks, variable peaks are also *covered* by appropriate instantiations of (feasible) conditional variable pairs.

Proposition 32 *(Variable peaks as conditional variable pairs). Let $\mathcal{R}$ be a CTRS, and $\alpha : \ell \to r \Leftarrow c$, $\alpha' : \ell' \to r' \Leftarrow c'$, x, p_x, σ, and $C[\,]_q$ be as in a variable peak (23). Let $\pi : \langle\ell[x']_{p_x}, r\rangle \Leftarrow x \to x', c$, with $x' \notin \mathcal{V}ar(\alpha) \cup \mathcal{V}ar(\alpha')$, together with ς given by $\varsigma(y) = \sigma(y)$ for all $y \in \mathcal{V}ar(\alpha)$ (in particular $\varsigma(x) = \sigma(x) = C[\sigma(\ell')]_q$), $\varsigma(y) = \sigma(y)$ for all $y \in \mathcal{V}ar(\alpha')$, and $\varsigma(x') = C[\sigma(r')]_q$. Then, $\varsigma(c)$ and $\varsigma(c')$ hold, $u = \varsigma(\ell[x']_{p_x})$, and $v = \varsigma(r)$, and π is $\overline{\mathcal{R}}$-feasible.*

Again, by Proposition 32, there is a mapping ϖ such that for each variable peak κ, we have an *associated* conditional variable pair $\pi = \varpi(\kappa)$. The following result establishes that joinability of conditional variable pairs π implies joinability of variable peaks κ such that $\pi = \varpi(\kappa)$.

Proposition 33 *(Joinability of conditional variable pairs and peaks). Let $\mathcal{R}$ be a CTRS, κ be a variable peak as in (23), and $\pi = \varpi(\kappa)$. If π is $\overline{\mathcal{R}}$-joinable, then κ is $\overline{\mathcal{R}}$-joinable.*

Proof. By Proposition 32, there is a substitution ς such that both $\varsigma(c)$ and $\varsigma(c')$ hold, $u = \varsigma(\ell[x']_{p_x})$, and $v = \varsigma(r)$. By definition of ς, we have $\varsigma(x) = C[\sigma(\ell')]_q$ and $\varsigma(x') = C[\sigma(r')]_q$, i.e., ς satisfies the conditional part $x \to x', c$ of π. By joinability of π, $u = \varsigma(\ell[x']_{p_x})$ and $v = \varsigma(r)$ are joinable. Thus, κ is joinable. □

For TRSs $\mathcal{R}$, CVP($\mathcal{R}$) is in general *not empty* as rules $\ell \to r$ with non-ground left-hand sides ℓ produce a CVP $\langle \ell[x]_p, r\rangle \Leftarrow x \to x'$ for each $x \in \mathcal{V}ar(\ell)$ and $p \in \mathcal{P}os_x(\ell)$. In the following section we provide some sufficient conditions guaranteeing joinability of CVPs, thus dismissing many of them when joinability of CVPs is investigated.

5.2.2. Joinable conditional variable pairs

For rules $\ell \to r \Leftarrow c$ whose active variables x in ℓ are also active everywhere else in ℓ and r (and missing in c), we have the following.

Proposition 34. *Let $\mathcal{R}$) be a CTRS, $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$, and $x \in \mathcal{V}ar(\ell) - \mathcal{V}ar(c)$. Then, for all $p \in \mathcal{P}os_x(\ell)$, the conditional variable pair $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x', c$ is joinable.*

Proof. If $x \to x', c$ is not feasible, it is obvious. Otherwise, let σ be a substitution such that the conditional part of π holds. This means that (i) $\sigma(c)$ holds and also (ii) $\sigma(x) \to_{\mathcal{R}} \sigma(x')$. By (i) we have $\sigma(\ell) \to_{\mathcal{R}} \sigma(r)$. Let σ' be $\sigma'(x) = \sigma(x')$ and $\sigma'(y) = \sigma(y)$ if $y \neq x$. By (ii) and Proposition 9, $\sigma(\ell)[\sigma(x')]_p \to^*_{\mathcal{R}} \sigma'(\ell)$. Since $x \notin \mathcal{V}ar(c)$, $\sigma'(c) = \sigma(c)$ holds. Thus, $\sigma'(\ell) \to_{\mathcal{R}} \sigma'(r)$ and, by (ii) and Proposition 9, $\sigma(r) \to^*_{\mathcal{R}} \sigma'(r)$. Hence, π is joinable. □

Thus, we can dismiss conditional variable pairs obtained from rules satisfying Proposition 34 (in particular, unconditional rules) from proofs of local confluence of rewriting. For oriented CTRSs, we have the following refinement.

Proposition 35. *Let $\mathcal{R}$ be an oriented CTRS and $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x', c \in$ CVP($\mathcal{R}$) for a rule $\alpha : \ell \to r \Leftarrow c \in \mathcal{R}$, such that $x \in \mathcal{V}ar(\ell)$, and for all $s \approx t \in c$, $x \notin \mathcal{V}ar(s)$. Then, π is $\overline{\mathcal{R}}_O$-joinable.*

Proof. The proof is similar to that of Proposition 34. We use the same notation here and discuss the small differences after (i) and (ii). Since $\mathcal{R}$ is oriented, for all conditions $s \approx t \in c$, since $x \notin \mathcal{V}ar(s)$, we have $\sigma'(s) = \sigma(s) \to^*_{\mathcal{R}} \sigma(t)$. Since $\sigma(x) \to_{\mathcal{R}} \sigma'(x)$, by Proposition 9 we have $\sigma(t) \to^*_{\mathcal{R}} \sigma'(t)$. Thus, $\sigma'(c)$ holds, as required. The remainder of the proof does not change. □

For semi-equational CTRSs, we have the following:

Proposition 36. *Let $\mathcal{R}$ be a semi-equational CTRS and $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x', c \in$ CVP($\mathcal{R}$) for a rule $\alpha : \ell \to r \Leftarrow c$ such that $x \in \mathcal{V}ar(\ell)$. Then, π is $\overline{\mathcal{R}}_{SE}$-joinable.*

Proof. Let σ be a substitution such that $\sigma(x) \to_{\mathcal{R}} \sigma(x')$ and for all $u \approx v \in c$, $\overline{\mathcal{R}}_{\mathrm{SE}} \vdash \sigma(u) \approx \sigma(v)$. Let σ' be as follows: $\sigma'(x) = \sigma(x')$ and for all variables $y \neq x$, $\sigma'(y) = \sigma(y)$. Note that $\sigma(x) \to_{\mathcal{R}} \sigma'(x) = \sigma(x') = \sigma'(x')$. Then, (i) Since $\sigma(x) \to_{\mathcal{R}} \sigma(x')$, we have

$$\sigma(\ell[x']_p) = \sigma(\ell)[\sigma(x')]_p \to^*_{\mathcal{R}} \sigma'(\ell)[\sigma(x')]_p = \sigma'(\ell)[\sigma'(x')]_p = \sigma'(\ell)$$

(ii) Since $\sigma(x) \to_{\mathcal{R}} \sigma(x')$, we have $\sigma'(x) = \sigma(x') \leftrightarrow^*_{\mathcal{R}} \sigma(x)$. Thus, for all $u \approx v \in c$, $\sigma'(u) \leftrightarrow^*_{\mathcal{R}} \sigma(u)$. Since $\sigma(c)$ holds in $\overline{\mathcal{R}}_{\mathrm{SE}}$, $\sigma(u) \leftrightarrow^*_{\mathcal{R}} \sigma(v)$ and $\sigma(v) \leftrightarrow^*_{\mathcal{R}} \sigma'(v)$. Therefore, $\sigma'(u) \leftrightarrow^*_{\mathcal{R}} \sigma(u) \leftrightarrow^*_{\mathcal{R}} \sigma(v) \leftrightarrow^*_{\mathcal{R}} \sigma'(v)$, i.e., $\sigma'(c)$ holds in $\overline{\mathcal{R}}_{\mathrm{SE}}$. This means that $\sigma'(\ell) \to_{\mathcal{R}} \sigma'(r)$. (iii) By Proposition 9, $\sigma(r) \to^*_{\mathcal{R}} \sigma'(r)$. Thus, $\sigma(\ell[x']_p) \to^*_{\mathcal{R}} \sigma'(\ell) \to_{\mathcal{R}} \sigma'(r)$ and $\sigma(r) \to^*_{\mathcal{R}} \sigma'(r)$, i.e., π is $\overline{\mathcal{R}}_{\mathrm{SE}}$-joinable. □

Remark 37. In order to use Proposition 21 with conditional variable pairs π with critical variable x, the following heuristics are useful:

1. $\sigma(x) = \ell^{\downarrow}$, $\sigma(x') = r^{\downarrow}$ for some unconditional rule $\ell \to r \in \mathcal{R}$.
2. $\sigma(x) = \sigma(\ell)$, $\sigma(x') = \sigma(r)$ for some unconditional rule $\ell \to r \in \mathcal{R}$.

5.2.3. Specialization of conditional variable pairs

In this section, we provide a transformation of CVPs which preserves (non)joinability and, as shown below, it is useful in practice. First, we need the following

Definition 38 *(Specializing CVPs)*. Let $\mathcal{R}$ be a CTRS and $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x', c$ be a CVP for the rule $\alpha : \ell \to r \Leftarrow c$. Consider the following *specialized* conditional pairs:

- Given a (possibly renamed) rule $\alpha' : \ell' \to r' \Leftarrow c' \in \mathcal{R}$ sharing no variables with α

$$\pi_{\alpha'} = \langle \ell_{x \mapsto \ell'}[r']_p, r_{x \mapsto \ell'}\rangle \Leftarrow c_{x \mapsto \ell'}, c' \tag{27}$$

where $\ell_{x\mapsto\ell'}$, $r_{x\mapsto\ell'}$, and $c_{x\mapsto\ell'}$ are obtained by replacing all occurrences of x in ℓ, r, and c by ℓ'. We say that $\pi_{\alpha'}$ is a *rule* specialization of π.

- Given a k-ary symbol $f \in \mathcal{F}$ with $k > 0$, $1 \leq i \leq k$, and variables $x_1, \ldots, x_i, \ldots, x_k, x'_i$ not occurring in π,

$$\pi_{f,i} = \langle \ell_{f,i}[f(x_1, \ldots, x'_i, \ldots, x_k)]_p, r_{f,i}\rangle \Leftarrow x_i \to x'_i, c_{f,i} \tag{28}$$

where $\ell_{f,i}$, $r_{f,i}$, and $c_{f,i}$ are obtained by replacing all occurrences of x in ℓ, r, and c by $f(x_1, \ldots, x_i, \ldots, x_k)$. We say that $\pi_{f,i}$ is an *argument* specialization of π.

Let $\mathsf{CVP}(\mathcal{R}, \pi)$ be the subset of $\overline{\mathcal{R}}$*-feasible, non-trivial* conditional pairs in $\{\pi_{\alpha'} \mid \alpha' \in \mathcal{R}\} \cup \{\pi_{f,i} \mid f \in \mathcal{F}, 1 \leq i \leq ar(f)\}$.

Example 39. For the CVP (7) for $\mathcal{R}$ in Example 2, i.e.,

$$\langle \mathsf{f}(x'), \mathsf{c}\rangle \Leftarrow x \to x', x \approx \mathsf{a}$$

by applying the specializations in Definition 38, we obtain the following *rule* specializations (29), (30), and *argument* specialization (31)

$$\langle \mathsf{f}(\mathsf{b}), \mathsf{c}\rangle \Leftarrow \mathsf{a} \approx \mathsf{a} \tag{29}$$

$$\langle \mathsf{f}(\mathsf{c}), \mathsf{c}\rangle \Leftarrow \mathsf{f}(x'') \approx \mathsf{a}, x'' \approx \mathsf{a} \tag{30}$$

$$\langle \mathsf{f}(\mathsf{f}(y')), \mathsf{c}\rangle \Leftarrow y \to y', \mathsf{f}(y) \approx \mathsf{a} \tag{31}$$

Since the conditional part of (29) is trivial, we can simplify it into the pair

$$\langle \mathsf{f}(\mathsf{b}), \mathsf{c}\rangle \tag{32}$$

Both (30) and (31) are infeasible (disregarding the J-, O-, or SE-based theory $\overline{\mathcal{R}}$ for $\mathcal{R}$). In particular, regarding joinability and reachability evaluation of conditions, $\mathsf{f}(x) \approx \mathsf{a}$ cannot be satisfied because for all terms t, reductions on $\mathsf{f}(t)$ would eventually remove symbol f only to obtain c using rule (5); on the other hand, a is reducible to b only (which is irreducible). Similarly, $\mathsf{f}(t) \leftrightarrow^*_{\mathcal{R}} \mathsf{a}$ does not hold for any term t. Therefore, $\mathsf{CVP}(\mathcal{R}, (7)) = \{(32)\}$.

Proposition 40. *Let $\mathcal{R}$ be a CTRS and $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x', c$ be a CVP for the rule $\alpha : \ell \to r \Leftarrow c$. Then, π is $\overline{\mathcal{R}}$-joinable if and only if for all $\pi' \in \mathsf{CVP}(\mathcal{R}, \pi)$ π' is $\overline{\mathcal{R}}$-joinable.*

Proof. As for the *if* part, we proceed by contradiction. Assume that π' is $\overline{\mathcal{R}}$-joinable for all $\pi' \in \mathsf{CVP}(\mathcal{R}, \pi)$, but π is not $\overline{\mathcal{R}}$-joinable. Then, there is a substitution σ such that $s = \sigma(x) \to \sigma(x') = t$ and $\sigma(c)$ holds, but $\sigma(\ell[x']_p)$ and $\sigma(r)$ are not $\overline{\mathcal{R}}$-joinable. By Proposition 11, there is $q \in \mathcal{P}os(s)$ and $\alpha' : \ell' \to r' \Leftarrow c' \in \mathcal{R}$ such that (since we can assume that $\mathcal{V}ar(\alpha) \cap \mathcal{V}ar(\alpha') = \emptyset$) $s|_q = \sigma(\ell')$, $\sigma(c')$ holds, and $t = s[\sigma(r')]_q$. We consider the two possible cases for q.

1. If $q = \Lambda$, then, $s = \sigma(x) = \sigma(\ell')$, $\sigma(c')$ holds, and $t = \sigma(x') = \sigma(r')$. We assume that $\pi_{\alpha'} \in \mathsf{CVP}(\mathcal{R}, \pi)$, i.e., $\langle \ell[r']_p, r\rangle \Leftarrow c_{x\mapsto\ell'}, c'$ is $\overline{\mathcal{R}}$-joinable. Thus, if both $\sigma(c)$ and $\sigma(c')$ hold, then, since $\sigma(c) = \sigma(c_{x\mapsto\ell'})$, we know that $\sigma(\ell[r']_p) = \sigma(\ell)[\sigma(r')]_p = \sigma(\ell)[\sigma(x')]_p = \sigma(\ell[x']_p)$ and $\sigma(r)$ are $\overline{\mathcal{R}}$-joinable, thus leading to a contradiction.
2. If $q = i.q'$ for some $i \in \mathbb{N}$ and position q', then there is a k-ary symbol f and terms $s_1, \ldots, s_k$ such that $s = \sigma(x) = f(s_1, \ldots, s_i, \ldots, s_k)$, $i \in \mathcal{P}os(s)$, and $s_i \to s'_i$ for some term s'_i, and $t = \sigma(x') = f(s_1, \ldots, s'_i, \ldots, s_k)$. We assume that $\pi_{f,i} \in \mathsf{CVP}(\mathcal{R}, \pi)$, i.e., $\langle \ell[f(x_1, \ldots, x'_i, \ldots, x_k)]_p, r\rangle \Leftarrow x_i \to x'_i, c_{f,i}$ is joinable. Since $x_1, \ldots, x_k$ and x'_i do not occur in α, we can let $\sigma(x_i) = s_i$ for all $1 \leq i \leq n$, and $\sigma(x'_i) = t_i$. Therefore, $\sigma(x_i) = s_i \to t_i = \sigma(x'_i)$ holds. Since, by the previous extension of σ to variables $x_1, \ldots, x_k$ and x'_i, we have that $\sigma(c) = \sigma(c_{f,i})$ holds, and

$$\begin{array}{rcl} \sigma(\ell[f(x_1, \ldots, x'_i, \ldots, x_k)]_p) & = & \sigma(\ell)[\sigma(f(x_1, \ldots, x'_i, \ldots, x_k))]_p \\ & = & \sigma(\ell)[f(s_1, \ldots, s'_i, \ldots, s_k))]_p \\ & = & \sigma(\ell[x']_p) \end{array}$$

by $\overline{\mathcal{R}}$-joinability of $\pi_{f,i}$, $\sigma(\ell[x']_p)$ and $\sigma(r)$ are $\overline{\mathcal{R}}$-joinable, a contradiction.

As for the *only if* part, if π is not $\overline{\mathcal{R}}$-joinable, we prove the existence of a non-$\overline{\mathcal{R}}$-joinable conditional pair in $\mathsf{CVP}(\mathcal{R}, \pi)$. Let σ be a substitution such that $s = \sigma(x) \to_{\mathcal{R}} \sigma(x') = t$ and $\sigma(c)$ hold, but $\sigma(\ell[x']_p)$ and $\sigma(r)$ are not $\overline{\mathcal{R}}$-joinable. By Proposition 11, there is $q \in \mathcal{P}os(s)$ and $\alpha' : \ell' \to r' \Leftarrow c' \in \mathcal{R}$ such that $s|_q = \sigma(\ell')$, $\sigma(c')$ holds, and $t = s[\sigma(r')]_q$. Consider two cases for q.

- If $q = \Lambda$, then, $s = \sigma(x) = \sigma(\ell')$ and $t = \sigma(x') = \sigma(r')$ and we consider $\pi_{\alpha'} \in \mathsf{CVP}(\mathcal{R}, \pi)$, i.e., $\langle \ell[r']_p, r\rangle \Leftarrow c_{x\mapsto\ell'}, c'$. Since $\sigma(c) = \sigma(c_{x\mapsto\ell'})$ and $\sigma(c')$ hold and $\sigma(\ell[r']_p) = \sigma(\ell)[\sigma(r')]_p) = \sigma(\ell)[\sigma(x')]_p = \sigma(\ell[x']_p)$ and $\sigma(r)$ are not joinable, then $\pi_{\alpha'} \in \mathsf{CVP}(\mathcal{R}, \pi)$ is not $\overline{\mathcal{R}}$-joinable either.
- If $q = i.q'$, then we consider $\pi_{f,i} \in \mathsf{CVP}(\mathcal{R}, \pi)$ and proceed similarly to prove that it is not $\overline{\mathcal{R}}$-joinable. $\square$

Example 41. For $\mathcal{R}$ in Example 2, the only pair (32) in $\mathsf{CVP}(\mathcal{R},(7))$, i.e.,

$$\langle \mathsf{f}(\mathsf{b}), \mathsf{c}\rangle$$

is $\overline{\mathcal{R}}_J$-joinable: $\mathsf{f}(\mathsf{b}) \to_{(5)} \mathsf{c}$ because the corresponding instance $\mathsf{b} \approx \mathsf{a}$ of the condition $x \approx \mathsf{a}$ holds under a joinability semantics for $\approx$. By Proposition 40, (7) is joinable. However, (32) is *not* $\overline{\mathcal{R}}_O$-joinable (see Example 2).

6. Characterization of local confluence of CTRSs

We collect proper and improper conditional critical pairs together with conditional variable pairs in a set of *extended* conditional critical pairs.

Definition 42 *(Extended conditional critical pairs).* Let $\mathcal{R}$ be a CTRS. The set $\mathsf{ECCP}(\mathcal{R}) = \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R})$ is the set of *extended conditional critical pairs* of $\mathcal{R}$.

The main result of this paper is the following.

Theorem 43 *(Local confluence of CTRSs). A CTRS $\mathcal{R}$ is locally $\overline{\mathcal{R}}$-confluent iff for all $\pi \in \mathsf{ECCP}(\mathcal{R})$, π is $\overline{\mathcal{R}}$-joinable.*

Proof. For the *only if* part, consider $\pi : \langle s,t\rangle \Leftarrow c \in \mathsf{ECCP}(\mathcal{R})$ and a substitution σ such that $\sigma(c)$ holds. By Propositions 27 and 31, there is a term u such that $u \to_{\mathcal{R}} \sigma(s)$ and $u \to_{\mathcal{R}} \sigma(t)$. By local $\overline{\mathcal{R}}$-confluence of $\mathcal{R}$, $\sigma(s)$ and $\sigma(t)$ are $\overline{\mathcal{R}}$-joinable. Hence, π is $\overline{\mathcal{R}}$-joinable.

For the *if* part, $\mathcal{R}$ is locally $\overline{\mathcal{R}}$-confluent if and only if for all terms s, $\overline{u}$, $\overline{v}$ defining a peak (14), $\overline{u}$ and $\overline{v}$ are $\overline{\mathcal{R}}$-joinable. Consider a peak κ of the form (14). If κ is a disjoint peak, then it is $\overline{\mathcal{R}}$-joinable. Otherwise, κ is either a critical or a variable peak. By Proposition 28 (resp. Proposition 32), there is a corresponding $\pi \in \mathsf{ECCP}(\mathcal{R})$ representing κ. Since π is $\overline{\mathcal{R}}$-joinable, by Proposition 29 (resp. Proposition 33), κ is $\overline{\mathcal{R}}$-joinable. Thus, $\mathcal{R}$ is locally $\overline{\mathcal{R}}$-confluent. $\square$

Using Proposition 36, we have the following.

Corollary 44 *(Local confluence of SE-CTRSs). An SE-CTRS $\mathcal{R}$ is locally confluent iff for all $\pi \in \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R})$, π is $\overline{\mathcal{R}}$-joinable.*

As a corollary of Theorem 43 and Newman's Lemma, we have the following.

Theorem 45. *A CTRS $\mathcal{R}$ which is $\overline{\mathcal{R}}$-terminating is $\overline{\mathcal{R}}$-confluent iff for all $\pi \in \mathsf{ECCP}(\mathcal{R})$, π is $\overline{\mathcal{R}}$-joinable.*

The following corollary of Theorem 45 and Proposition 36, was given as a sufficient condition for confluence of SE-CTRSs in [8, Theorem 3.3].

Corollary 46 *(Confluence of terminating SE-CTRSs). A terminating SE-CTRS $\mathcal{R}$ is confluent iff for all $\pi \in \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R})$, π is joinable.*

In proofs of *termination* of CTRSs $\mathcal{R}$ with FO-theory $\overline{\mathcal{R}}$, it is often useful to consider the rewriting part $\ell \to r$ of conditional rules $\ell \to r \Leftarrow c$ only. For 2-CTRSs $\mathcal{R}$, a TRS, often called *underlying TRS*, $\mathcal{R}_u$, is obtained. Disregarding the underlying theory $\overline{\mathcal{R}}$, termination of $\mathcal{R}_u$ implies $\overline{\mathcal{R}}$-termination of $\mathcal{R}$.

Example 47. For $\mathcal{R} = \{\mathsf{a} \to \mathsf{b}, \mathsf{a} \to \mathsf{c}, \mathsf{b} \to \mathsf{c} \Leftarrow \mathsf{b} \approx \mathsf{c}\}$ in Example 1, we have $\mathsf{ECCP}(\mathcal{R}) = \mathsf{pCCP}(\mathcal{R}) = \{(33)\}$, with

$$\langle \mathsf{b}, \mathsf{c}\rangle \tag{33}$$

- As a J-CTRS. Since b and c are $\overline{\mathcal{R}}_J$-irreducible, (33) is not $\overline{\mathcal{R}}_J$-joinable. By Theorem 43, $\mathcal{R}$ is not locally $\overline{\mathcal{R}}_J$-confluent nor $\overline{\mathcal{R}}_J$-confluent.
- As an SE-CTRS. Since $\mathsf{b}\ {}_{(1)}\!\!\leftarrow \mathsf{a} \to_{(2)} \mathsf{c}$, we have $\mathsf{b} \to_{(3)} \mathsf{c}$, i.e., (33) is $\overline{\mathcal{R}}_{SE}$-joinable. By Theorem 43, $\mathcal{R}$ is locally $\overline{\mathcal{R}}_{SE}$-confluent. Since $\mathcal{R}_u = \{\mathsf{a} \to \mathsf{b}, \mathsf{a} \to \mathsf{c}, \mathsf{b} \to \mathsf{c}\}$ is clearly terminating, $\mathcal{R}$ is $\overline{\mathcal{R}}_{SE}$-terminating and, by Theorem 45, $\mathcal{R}$ is $\overline{\mathcal{R}}_{SE}$-confluent.

Example 48. For $\mathcal{R} = \{\mathsf{a} \to \mathsf{b}, \mathsf{f}(x) \to \mathsf{c} \Leftarrow x \approx \mathsf{a}\}$ in Example 2, $\mathsf{ECCP}(\mathcal{R}) = \mathsf{CVP}(\mathcal{R}) = \{(7)\}$, where (7) is $\langle \mathsf{f}(x'), \mathsf{c}\rangle \Leftarrow x \to x', x \approx \mathsf{a}$.

- As a J-CTRS. Since (7) is $\overline{\mathcal{R}}_J$-joinable (Example 41), by Theorem 43, $\mathcal{R}$ is locally $\overline{\mathcal{R}}_J$-confluent. Since $\mathcal{R}_u = \{\mathsf{a} \to \mathsf{b}, \mathsf{f}(x) \to \mathsf{c}\}$ is terminating, $\mathcal{R}$ is $\overline{\mathcal{R}}_J$-terminating and, by Theorem 45, is $\overline{\mathcal{R}}_J$-confluent.
- As an O-CTRS. The sequence $x \to x', x \approx \mathsf{a}$ is clearly $\overline{\mathcal{R}}_O$*-feasible* (with $\sigma = \{x \mapsto \mathsf{a}, x' \mapsto \mathsf{b}\}$, for instance). However, $x \to x', x \approx \mathsf{a}, \mathsf{f}(x') \to^* z, \mathsf{c} \to^* z$ is $\overline{\mathcal{R}}_O$*-infeasible*: σ above is the *only* substitution satisfying $x \to x', x \approx \mathsf{a}$ in $\overline{\mathcal{R}}_O$, but both $\sigma(\mathsf{f}(x')) = \mathsf{f}(\mathsf{b})$ and c are $\overline{\mathcal{R}}_O$-irreducible. By Proposition 21, (7) is not $\overline{\mathcal{R}}_O$-joinable. By Theorem 43, $\mathcal{R}$ is not locally $\overline{\mathcal{R}}_O$-confluent nor $\overline{\mathcal{R}}_O$-confluent.

Example 49. For the O-CTRS $\mathcal{R} = \{\mathsf{g}(\mathsf{b}) \to \mathsf{true}, \mathsf{g}(\mathsf{c}) \to \mathsf{true}, \mathsf{a} \to \mathsf{f}(x) \Leftarrow \mathsf{g}(x) \approx \mathsf{true}\}$ in Example 18, with 3-rule (21), i.e., $\mathsf{a} \to \mathsf{f}(x) \Leftarrow \mathsf{g}(x) \approx \mathsf{true}$, and a renamed copy (21)' of it, we obtain the improper conditional critical pair

$$\langle \mathsf{f}(x'), \mathsf{f}(x)\rangle \Leftarrow \mathsf{g}(x) \approx \mathsf{true}, \mathsf{g}(x') \approx \mathsf{true} \tag{34}$$

Hence, $\mathsf{ECCP}(\mathcal{R}) = \mathsf{iCCP}(\mathcal{R}) = \{(34)\}$. With $\sigma = \{x \mapsto \mathsf{b}, x' \mapsto \mathsf{c}\}$, (i) the instantiated condition of (34), i.e., the sequence $\mathsf{g}(\mathsf{b}) \approx \mathsf{true}, \mathsf{g}(\mathsf{c}) \approx \mathsf{true}$ is $\overline{\mathcal{R}}_{\mathrm{O}}$-*feasible* (just use the two unconditional rules of $\mathcal{R}$); and (ii) the sequence $\mathsf{g}(\mathsf{b}) \approx \mathsf{true}, \mathsf{g}(\mathsf{c}) \approx \mathsf{true}, \mathsf{f}(\mathsf{c}) \to^* z, \mathsf{f}(\mathsf{b}) \to^* z$ is $\overline{\mathcal{R}}_{\mathrm{O}}$-*infeasible* because both $\mathsf{f}(\mathsf{c})$ and $\mathsf{f}(\mathsf{b})$ are $\overline{\mathcal{R}}_{\mathrm{O}}$-irreducible. By Proposition 21, (34) is not $\overline{\mathcal{R}}_{\mathrm{O}}$-joinable. By Theorem 43, $\mathcal{R}$ is not locally $\overline{\mathcal{R}}_{\mathrm{O}}$-confluent nor $\overline{\mathcal{R}}_{\mathrm{O}}$-confluent.

7. Generalized term rewriting systems

In this section we consider an extension of CTRSs and show that our results can be adapted to characterize local confluence of the corresponding reduction relation. Our extension is twofold:

1. *Signature level.* We restrict the arguments of function symbols on which reductions are allowed, as in *context-sensitive rewriting* (CSR, [34]).
2. *Rule level.* We permit the use of arbitrary atoms in the conditional part of rules. Such atoms are defined by means of definite Horn clauses.

Before presenting the notion of a generalized term rewriting system, we recall some notions from CSR.

7.1. Context-sensitive rewriting

Given a signature $\mathcal{F}$, a *replacement map* is a mapping μ satisfying that, for all symbols f in $\mathcal{F}$, $\mu(f) \subseteq \{1, \ldots, ar(f)\}$ [34]. The set of replacement maps for the signature $\mathcal{F}$ is $M_{\mathcal{F}}$. Extreme cases are $\mu_\bot$, disallowing replacements in all arguments of function symbols: $\mu_\bot(f) = \emptyset$ for all $f \in \mathcal{F}$, and $\mu_\top$, restricting no replacement: $\mu_\top(f) = \{1, \ldots, k\}$ for all k-ary $f \in \mathcal{F}$. The set $\mathcal{P}os^\mu(t)$ of μ-*replacing (or* active*) positions* of t is $\mathcal{P}os^\mu(t) = \{\Lambda\}$, if $t \in \mathcal{X}$, and $\mathcal{P}os^\mu(t) = \{\Lambda\} \cup \{i.p \mid i \in \mu(f), p \in \mathcal{P}os^\mu(t_i)\}$, if $t = f(t_1, \ldots, t_k)$. Positions of *active* non-variable symbols in t are denoted as $\mathcal{P}os^\mu_{\mathcal{F}}(t)$. Given a term t, $\mathcal{V}ar^\mu(t)$ (resp. $\mathcal{V}ar^{\not\mu}(t)$) is the set of variables occurring at active (resp. frozen) positions in t: $\mathcal{V}ar^\mu(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^\mu(t), x = t|_p\}$ and $\mathcal{V}ar^{\not\mu}(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \overline{\mathcal{P}os^\mu}(t), x = t|_p\}$. In general, $\mathcal{V}ar^\mu(t)$ and $\mathcal{V}ar^{\not\mu}(t)$ are not disjoint: $x \in \mathcal{V}ar(t)$ may occur active and also frozen in t. The strict prefix $\mathit{sprefix}_t(p)$ of a position p in a term t, i.e., the (possibly empty) sequence of symbols traversed when going from the root of t to position p (excluding p itself), determines the active/frozen status of p in t.

Proposition 50. [34, Proposition 3.3] *Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\mu \in M_{\mathcal{F}}$. If $p \in \mathcal{P}os(s) \cap \mathcal{P}os(t)$ and $\mathit{sprefix}_s(p) = \mathit{sprefix}_t(p)$, then $p \in \mathcal{P}os^\mu(s) \Leftrightarrow p \in \mathcal{P}os^\mu(t)$.*

A context $C[\,]_p$ is μ-*active* (or just active) iff $p \in \mathcal{P}os^\mu(C)$; equivalently (by Proposition 50), iff either $p = \Lambda$ or $C[\,]_p = f(t_1, \ldots, C_i[\,]_q, \ldots, t_k)$ for some $f \in \mathcal{F}$, terms $t_1, \ldots, t_k$, and active context $C_i[\,]_q$ such that $p = i.q$ for some $i \in \mu(f)$. A CS-TRS (resp. CS-CTRS) $(\mathcal{R}, \mu)$ consists of a TRS (resp. CTRS) $\mathcal{R}$ together with a replacement map μ.

7.2. Syntax of generalized term rewriting systems

We consider definite Horn clauses $\alpha : A \Leftarrow c$ (with label α) where c is a sequence $A_1, \ldots, A_n$ of atoms. If $n = 0$, then α is written A rather than $A \Leftarrow$.

Definition 51 *(Generalized term rewriting system).* Let $\mathcal{F}$ be a signature of function symbols, Π be a signature of predicate symbols, $\mu \in M_{\mathcal{F}}$ be a replacement map, H be a set of clauses $A \Leftarrow c$ where $root(A) \notin \{\to, \to^*\}$, and R be a set of *rewrite rules* $\ell \to r \Leftarrow c$ such that ℓ is not a variable (in both cases, c is a sequence $A_1, \ldots, A_n$ of atoms). The tuple $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ is called a *Generalized Term Rewriting System (GTRS)*.

The usual definition of *type* (1, 2, 3, or 4) of rules and CTRSs (see Section 2) also apply to GTRSs $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ in the obvious way. The rules in R also permit the usual distinction of function symbols $f \in \mathcal{F}$ as *defined* symbols (if $f = root(\ell)$ for some $\ell \to r \Leftarrow c \in R$) or *constructor* symbols (otherwise). We often denote as $\mathcal{D}_{\mathcal{R}}$ or $\mathcal{D}$ if no confusion arises (resp. $\mathcal{C}_{\mathcal{R}}$ or $\mathcal{C}$) the signature of *defined* (resp. constructor) *symbols* of $\mathcal{R}$.

7.3. First-order theory of a generalized term rewriting system

In order to define the FO-theory of a GTRS, besides the generic sentences in Table 1, we also consider the following one for definite Horn clauses α:

$(\mathrm{HC})_\alpha \quad (\forall x_1,\ldots,x_n)\ A_1 \wedge \cdots \wedge A_n \Rightarrow A$

Note that $(\mathrm{Rl})_\alpha$ can be seen as a particular case of $(\mathrm{HC})_\alpha$. Actually, since rewrite rules of GTRSs are more general than rules of CTRSs, we use $(\mathrm{HC})_\alpha$ both for clauses in H and rules in R.

Definition 52 *(FO-theory of a GTRS).* Let $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ be a GTRS. Then,

$$\overline{\mathcal{R}} = \{(\mathrm{Rf}), (\mathrm{Co})\} \cup \{(\mathrm{Pr})_{f,i} \mid f \in \mathcal{F}, i \in \mu(f)\} \cup \{(\mathrm{HC})_\alpha \mid \alpha \in H \cup R\} \tag{35}$$

is the FO-theory of $\mathcal{R}$.

Compared with Definition 3 for CTRSs, the following differences are noticeable:

1. The use of propagation rules $(\mathrm{Pr})_{f,i}$ is restricted to arguments $i \in \mu(f)$ rather than to *all* arguments $i \in \{1,\ldots,ar(f)\}$ of f. This formalizes the use of (context-sensitive) replacement restrictions in reductions.
2. No specific sentence in the FO-theory $\overline{\mathcal{R}}$ in Definition 52 permits rewritings on arguments s_i as part of proofs of atoms $P(s_1,\ldots,s_n)$. If necessary, this can be achieved by just adding clauses

$$P(x_1,\ldots,x_i,\ldots,x_n) \Leftarrow x_i \to y_i, P(x_1,\ldots,y_i,\ldots,x_n) \tag{36}$$

   to H for all desired n-ary predicates P and arguments $1 \le i \le n$, where $x_1,\ldots,x_n,y_i$ are distinct variables.
3. In GTRSs, the effect of $\mathcal{S}_{\mathrm{CI}}$ is made *explicit* by including auxiliary clauses for (J), (O), (SE), see Fig. 2, in H. For instance, $\alpha_{\mathrm{JO}} : x \approx y \Leftarrow x \to^* z, y \to^* z$ would be included in H to obtain (J), as $(\mathrm{HC})_{\alpha_{\mathrm{JO}}}$ is (J). Sometimes $\approx$ is *not* necessary to specify conditions in rules. For instance, if a reachability semantics is required for a condition, we can just write $s \to^* t$ instead of $s \approx t$ in the rule. Since $\to^*$ is already defined by rules (Rf) and (Co), which are included in $\overline{\mathcal{R}}$, we can let $H = \emptyset$. In this way, each condition in the conditional part of a GTRS rule can have *its own* semantics, which is defined by specific clauses in H, if necessary (see Example 54 below).

Remark 53 *(A GTRS determines its FO-theory).* In contrast to CTRSs $\mathcal{R} = (\mathcal{F}, R)$ whose theory $\overline{\mathcal{R}}$ cannot be obtained from $\mathcal{F}$ and R only (as rules (J), (O), etc., are required to describe the semantics of conditions $s \approx t$ in rules), the components $\mathcal{F}$, Π, μ, H, and R of a GTRS $\mathcal{R}$ determine the theory $\overline{\mathcal{R}}$ as in (35). Accordingly, in the following we do not explicitly mention $\overline{\mathcal{R}}$ unless it is technically required, as it can be obtained from $\mathcal{R}$.

Given a GTRS $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ with $H = \emptyset$, a number of well-known classes of rule-based systems is obtained: if only unconditional rules $\ell \to r$ are allowed in R, then we obtain a TRS (if $\mu = \mu_\top$) or a CS-TRS [34] (no restriction on μ); besides, if some rules $\ell \to r$ contain extra variables in r, we obtain an eTRS [14] (if $\mu = \mu_\top$), or a CS-eTRS; finally, if only conditional rules $\ell \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ for a given symbol $\approx$ with a given (J-, O-, or SE-) interpretation are included in R (and the appropriate clauses in H), then we obtain a CTRS (if $\mu = \mu_\top$), or a CS-CTRS [36]. The following example shows how *different* evaluation semantics in the conditional part of rules of GTRSs can be used.

Example 54. Consider the GTRS $\mathcal{R} = (\mathcal{F}, \Pi, \mu_\top, \emptyset, \{(37),(38)\})$ obtained as a variant of the CTRS in Example 2, with

$$\mathsf{a} \to \mathsf{b} \tag{37}$$

$$\mathsf{f}(x) \to \mathsf{c} \Leftarrow x \to y, y \to^* \mathsf{a} \tag{38}$$

Note that $x \to y, y \to^* \mathsf{a}$ encodes reachability of a from (instances of) x by using *at least one* rewriting step. This makes (38) infeasible, leading to a trivial proof of (local) confluence of the system using the results in Section 7.5 below. Alternatively, (38) could be written as follows:

$$\mathsf{f}(x) \to \mathsf{c} \Leftarrow x \approx_1 y, y \approx_2 \mathsf{a} \tag{39}$$

and clauses $x \approx_1 y \Leftarrow x \to y$ and $x \approx_2 y \Leftarrow x \to^* y$ would be included in H. This presentation highlights the idea that different semantics for conditions can be used in rules provided that appropriate clauses defining the predicates representing the conditions are given in H.

As for CTRSs, we can talk of J-, O-, or SE-GTRSs $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ provided that $\approx \in \Pi$ and the conditional part c of each rule $\ell \to r \Leftarrow c \in R$ consists of a mix of conditions $s \approx t$ for terms s and t, and atomic conditions A where $root(A) \in \Pi - \{\to, \to^*\}$. Besides, H should contain the appropriate clauses (in Fig. 2) to give the desired meaning to conditions $s \approx t \in c$ and *no other clause* $A \Leftarrow c$ with $root(A) = \approx$ so that the meaning of $\approx$ is given by the sentences in Fig. 2 only. For GTRSs we also have the following.

Proposition 55. *Let $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ be a GTRS. Then, $\to_{\mathcal{R}}$ is closed under substitutions and μ-active contexts.*

Proof. Remind that $\overline{\mathcal{R}}$ consists of sentences obtained from Horn clauses. Then, closedness under substitutions follows because deduction of goals $s \to t$ involves deductions $\overline{\mathcal{R}} \vdash \sigma(A)$ of instances of atoms A for some substitution σ. Since $\overline{\mathcal{R}} \vdash \sigma(A)$ is equivalent to $\overline{\mathcal{R}} \vdash (\forall\vec{x})\sigma(A)$ for all variables $\vec{x} = x_1, \ldots, x_n$ occurring in $\sigma(A)$, we also have $\overline{\mathcal{R}} \vdash \tau(\sigma(A))$ for all substitutions τ. Hence, $\overline{\mathcal{R}} \vdash \tau(s) \to \tau(t)$ and hence $\tau(s) \to_{\mathcal{R}} \tau(t)$ hold.

Regarding closedness under μ-active contexts, note that for all terms s_i and t_i, if $s_i \to_{\mathcal{R}} t_i$, then $(\mathrm{Pr})_{f,i}$ in $\overline{\mathcal{R}}$ enables the rewriting of $f(s_1, \ldots, s_i, \ldots, s_n)$ into $f(s_1, \ldots, t_i, \ldots, s_n)$ if $i \in \mu(f)$. $\square$

7.4. Examples of GTRSs

In the following, in order to keep a close connection with the original sources, rather than call them CS-CTRSs, we use TRS, CTRS, etc., when citing external examples. If no replacement map is explicitly given (as in TRSs and CTRSs), then $\mu_\top$ is assumed in the corresponding GTRS. In order to improve the understanding of replacement restrictions, in the next examples, frozen subterms in rules are often written in red.

Example 56 *(Use of replacement restrictions).* Consider the O-CTRS $\mathcal{R}$ in [17, Example 10] (COPS/387.trs[2])

$$\mathsf{g}(\mathsf{s}(x)) \to \mathsf{g}(x) \tag{40}$$

$$\mathsf{f}(\mathsf{g}(x)) \to x \Leftarrow x \approx \mathsf{s}(0) \tag{41}$$

We consider two replacement maps to restrict reductions on the arguments of function symbols: $\mu_\bot$ and the *canonical* replacement map $\mu^{can}_{\mathcal{R}}$, which is the most restrictive replacement map making all nonvariable positions in left-hand sides ℓ of rules $\ell \to r \Leftarrow c$ *active* [34, Section 5]. For $\mu_\bot$ and $\mu^{can}_{\mathcal{R}}$ (where $\mu^{can}_{\mathcal{R}}(\mathsf{f}) = \mu^{can}_{\mathcal{R}}(\mathsf{g}) = \{1\}$ and $\mu^{can}_{\mathcal{R}}(\mathsf{s}) = \emptyset$), we have the following O-CS-CTRSs:

$x \approx y \Leftarrow x \to^* y$	$x \approx y \Leftarrow x \to^* y$
$\mathsf{g}(\mathsf{s}(x)) \to \mathsf{g}(x)$	$\mathsf{g}(\mathsf{s}(x)) \to \mathsf{g}(x)$
$\mathsf{f}(\mathsf{g}(x)) \to x \Leftarrow x \approx \mathsf{s}(0)$	$\mathsf{f}(\mathsf{g}(x)) \to x \Leftarrow x \approx \mathsf{s}(0)$
$\mathcal{R}_\bot = (\mathcal{F}, \Pi, \mu_\bot, H, R)$ (COPS/1554.trs)	$\mathcal{R}_{\mathrm{can}} = (\mathcal{F}, \Pi, \mu^{can}_{\mathcal{R}}, H, R)$ (COPS/1555.trs)

For $\mathcal{R}_\bot$, we have: $\overline{\mathcal{R}}_\bot = \{(\mathrm{Rf}), (\mathrm{Co}), (\mathrm{O}), (42), (43)\}$, with

$$(\forall x)\ \mathsf{g}(\mathsf{s}(x)) \to \mathsf{g}(x) \tag{42}$$

$$(\forall x)\ x \approx \mathsf{s}(0) \Rightarrow \mathsf{f}(\mathsf{g}(x)) \to x \tag{43}$$

Note the absence of *propagation sentences* due to $\mu_\bot(f) = \emptyset$ for all $f \in \mathcal{F}$. For $\mathcal{R}_{\mathrm{can}}$, we have $\overline{\mathcal{R}}_{\mathrm{can}} = \{(\mathrm{Rf}), (\mathrm{Co}), (\mathrm{O}), (44), (45), (42), (43)\}$ with propagation sentences $(\mathrm{Pr})_{\mathsf{f},1}$ and $(\mathrm{Pr})_{\mathsf{g},1}$ as follows:

$$(\forall x_1, x_1')\ x_1 \to x_1' \Rightarrow \mathsf{f}(x_1) \to \mathsf{f}(x_1') \tag{44}$$

$$(\forall x_1, x_1')\ x_1 \to x_1' \Rightarrow \mathsf{g}(x_1) \to \mathsf{g}(x_1') \tag{45}$$

Note the absence of $(\mathrm{Pr})_{\mathsf{s},1}$; this is consistent with $\mu^{can}_{\mathcal{R}}(\mathsf{s}) = \emptyset$.

Example 57. Consider the following O-GTRS:

$$x \approx y \Leftarrow x \to^* y \tag{46}$$

$$x \geq 0 \tag{47}$$

$$\mathsf{s}(x) \geq \mathsf{s}(y) \Leftarrow x \geq y \tag{48}$$

$$\mathsf{s}(\mathsf{s}(x)) \to x \Leftarrow x \geq \mathsf{s}(0) \tag{49}$$

$$\mathsf{test}(x) \to \mathsf{pev}(x) \Leftarrow x \approx \mathsf{s}(\mathsf{s}(0)) \tag{50}$$

$$\mathsf{test}(x) \to \mathsf{odd}(x) \Leftarrow x \approx \mathsf{s}(0) \tag{51}$$

$$\mathsf{test}(x) \to \mathsf{zero}(x) \Leftarrow x \approx 0 \tag{52}$$

where $\geq$ is defined by the Horn clauses (47) and (48). Rules (50), (51), and (52) define tests to check whether a number (in Peano notation) is *positive and even*, *odd* or *zero*. Note that Peano numbers keep its positive even/odd character after each application of rule (49). Furthermore, $\mathsf{s}(\mathsf{s}(0))$, $\mathsf{s}(0)$, and 0 are irreducible.

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[2] *Confluence Problems Data Base*: http://cops.uibk.ac.at/.

7.5. Local confluence of generalized term rewriting systems

The main concepts defined in Section 3: feasible rule (Definition 7), rewriting as deduction (Definition 8 and Theorem 10), confluence and termination (Definition 12), joinable terms (Definition 13, Corollary 14 and Proposition 15) remain unchanged as they are *relative* to the underlying FO-theory $\overline{\mathcal{R}}$ at stake. Proposition 11 requires a small adaptation to cope with context-sensitive replacement restrictions. We have the following.

Proposition 58. *Let $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ be a GTRS and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Then, $s \to_{\mathcal{R}} t$ iff there is $p \in \mathcal{P}os^{\mu}(s)$ and $\ell \to r \Leftarrow c \in \mathcal{R}$ such that (i) $s|_p = \sigma(\ell)$ for some substitution σ, (ii) for all $\gamma \in c$, $\overline{\mathcal{R}} \vdash \sigma(\gamma)$ holds, and (iii) $t = s[\sigma(r)]_p$.*

The proof of this result is obtained from the proof of Proposition 11 by using the version of Theorem 10 for GTRSs and taking into account, in the induction step, that the position p where the rewriting step is performed (by the use of a propagation rule $(\mathrm{Pr})_{f,i}$ for some $f \in \mathcal{F}$) is *active* (i.e., $i \in \mu(f)$), and then using Proposition 50 in the remainder of the induction step. Since the proof of Proposition 11 does *not* refer to the shape of conditions in rules, having more general conditions in the rules of GTRSs is irrelevant.

Proposition 58 is essential to generalize the taxonomy of *peaks* discussed in Section 4 to GTRSs. Accordingly, positions $\overline{p}, \overline{p'} \in \mathcal{P}os(s)$ in a peak (14) must be *active*, i.e., $\overline{p}, \overline{p'} \in \mathcal{P}os^{\mu}(s)$. By using again Proposition 50, we conclude that (i) disjoint peaks (15) are also joinable; (ii) position $p \in \mathcal{P}os(\ell)$ in critical peaks (17) is active, i.e., $p \in \mathcal{P}os^{\mu}(\ell)$; and (iii) position $p_x \in \mathcal{P}os(\ell)$ in variable peaks (23) is active, i.e., $p_x \in \mathcal{P}os^{\mu}(\ell)$. Then, all results in Section 4 hold for GTRSs as they pay no attention to the structure of $\mathcal{R}$ besides the role of replacement restrictions just discussed above.

Conditional pairs (24) can be used to represent conditional critical and variable pairs of GTRSs and Definition 19 (feasibility), Definition 20 (joinability), and Proposition 21 also work for GTRSs. The definitions of conditional critical pair and conditional variable pair must be slightly adapted to require that critical positions be active. The following definition provides the necessary adaptation of these essential definitions.

Definition 59 *(Conditional critical and variable pairs of a GTRS).* Let $\mathcal{R} = (\Omega, \mu, H, R)$ be a GTRS, and $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c'$ be feasible rules of $\mathcal{R}$ sharing no variable (rename if necessary).

- Let $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\ell)$ be a nonvariable position of ℓ such that $\ell|_p$ and ℓ' unify with *mgu* θ. Then, (25), i.e., $\langle \theta(\ell[r']_p), \theta(r) \rangle \Leftarrow \theta(c), \theta(c')$, is a *conditional critical pair* (CCP) of $\mathcal{R}$.
- Let $x \in \mathcal{V}ar^{\mu}(\ell)$, $p \in \mathcal{P}os^{\mu}_x(\ell)$ and x' be a fresh variable. Then, (26), i.e., $\langle \ell[x']_p, r \rangle \Leftarrow x \to x', c$ is a *conditional variable pair* (CVP). Variable x is called the *critical variable* of the pair.

In both cases, p is called the *critical position*.

We keep the distinction between *proper* and *improper* conditional critical pairs, and the notations $\mathsf{pCCP}(\mathcal{R})$, $\mathsf{iCCP}(\mathcal{R})$, and $\mathsf{CVP}(\mathcal{R})$ along the lines of those given in previous sections. Accordingly, we have the following.

Definition 60 *(Extended conditional critical pairs of a GTRS).* Let $\mathcal{R}$ be a GTRS. The set $\mathsf{ECCP}(\mathcal{R}) = \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R})$ is the set of *extended conditional critical pairs* of $\mathcal{R}$.

With these provisos, the results guaranteeing the correspondence between critical (resp. variable) peaks and conditional critical pairs (resp. conditional variable pairs), i.e., Propositions 27, 28, 29, 31, 32, and 33, also hold for GTRSs. The results about joinability of CVPs in Section 5.2.2 require some attention due to the use of replacement restrictions. Our next result provides the necessary reformulation. Given a theory Th and a binary relation R on terms, we say that an atom $A[x]$ is $(\mathsf{Th}, \mathsf{R}, x)$-preserving if for all terms s, t, and substitutions σ and σ' such that $\sigma(x) = s$, $\sigma'(x) = t$ and $\sigma(y) = \sigma'(y)$ for all $y \neq x$, if $s \; \mathsf{R} \; t$, then $\mathsf{Th} \vdash \sigma(A[x])$ implies $\mathsf{Th} \vdash \sigma'(A[x])$.

Proposition 61. *Let $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$ be a GTRS, $\alpha : \ell \to r \Leftarrow c \in R$, and $x \in \mathcal{V}ar^{\mu}(\ell)$ be such that $x \notin \mathcal{V}ar^{\not\mu}(\ell) \cup \mathcal{V}ar^{\not\mu}(r)$, $p \in \mathcal{P}os^{\mu}_x(\ell)$, and $\pi : \langle \ell[x']_p, r \rangle \Leftarrow x \to x', c$*

1. *If $x \notin \mathcal{V}ar(c)$, then, π is joinable.*
2. *If all conditions in c are of the form (i) $s \to^* t$ and such that $x \notin \mathcal{V}ar(s)$, and $x \notin \mathcal{V}ar^{\not\mu}(t)$, or (ii) an atom A which is $\to^*_{\mathcal{R}}$-preserving in all occurrences of x, then π is joinable.*
3. *If all conditions in c are of the form (i) $s \leftrightarrow^* t$ with $\leftrightarrow^*$ defined in H by modified versions of (SE_1), (SE_2), and (SE_3) where $\approx$ has been replaced by $\leftrightarrow^*$, and such that $x \notin \mathcal{V}ar^{\not\mu}(s) \cup \mathcal{V}ar^{\not\mu}(t)$, or (ii) an atom A which is $\leftrightarrow^*_{\mathcal{R}}$-preserving in all occurrences of x, then π is joinable.*

Proof.

1. In the proof of Proposition 34 use Proposition 55 instead of Proposition 9.

2. Analogous to Proposition 35, by using Proposition 55 instead of Proposition 9. The main difference with respect to O-CTRSs is that rules in GTRSs may include atoms A which are not reachability conditions $s \to^* t$. Requiring $\to^*_{\mathcal{R}}$-preservingness on all occurrences of x guarantees that $\sigma(A[x'])$ holds if $\sigma(A[x])$ holds.
3. Analogous to Proposition 36, taking into account $\leftrightarrow^*_{\mathcal{R}}$-preservingness in the previous proof. □

Specializing conditional variable pairs π of GTRSs is also possible along the lines of Definition 38. Now, only indices $i \in \mu(f)$ need to be considered in argument specializations $\pi_{f,i}$. Regarding the correctness result, Proposition 40 needs to be adapted to consider the use of *active* positions $p \in \mathcal{P}os^{\mu}_x(\ell)$ of variables x in the left-hand side ℓ of rules: since Proposition 58 should be used in the proof instead of Proposition 11, we have $q \in \mathcal{P}os^{\mu}(s)$. Thus, in the second case for q considered in the proof (item 2), i.e., $q = i.q'$, we have $i \in \mu(f)$ and (by Proposition 50) also $i \in \mathcal{P}os^{\mu}(s)$ and $q' \in \mathcal{P}os^{\mu}(s_i)$ for s and s_i as in the proof of Proposition 40. This enables the reduction $s_i \to_{\mathcal{R}} t_i$ which is required to finish the proof.

Since the proof of Theorem 43 for CTRSs in Section 6 relies on these results, which have been extended to GTRSs, we have the following.

Theorem 62 *(Local confluence of GTRSs). A GTRS $\mathcal{R}$ is locally confluent iff each $\pi \in \mathsf{ECCP}(\mathcal{R})$ is joinable.*

Corollary 63 *(Local confluence of SE-GTRSs). Let $\mathcal{R}$ be an SE-GTRS such that for all $\langle s,t\rangle \Leftarrow x \to x', c \in \mathsf{CVP}(\mathcal{R})$, (i) for all $u \leftrightarrow^* v \in c$, $x \notin \mathcal{V}ar^{\not\mu}(u) \cup \mathcal{V}ar^{\not\mu}(v)$ and (ii) all other atoms $A \in c$ are $(\overline{\mathcal{R}}, \leftrightarrow^*_{\mathcal{R}}, x)$-preserving. Then, $\mathcal{R}$ is locally confluent iff each $\pi \in \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R})$ is joinable.*

7.6. Underlying (extended) CS-TRS of a GTRS

For 3-GTRSs $\mathcal{R}$, the computation of $\mathcal{R}_u$ produces rules with *extra variables*, thus leading to an *eTRS* (extended TRS, see, e.g., [14]).

Example 64. For $\mathcal{R}$ in Example 18, the eTRS $\mathcal{R}_u$ consists of the rules:

$$\mathsf{g}(\mathsf{b}) \to \mathsf{true} \tag{53}$$

$$\mathsf{g}(\mathsf{c}) \to \mathsf{true} \tag{54}$$

$$\mathsf{a} \to \mathsf{f}(x) \tag{55}$$

Although eTRSs are nonterminating, they can be μ-terminating. For instance, the one rule system $\mathcal{R} = \{\mathsf{a} \to \mathsf{c}(x)\}$ is $\mu_\perp$-terminating, as no reduction is possible below c, disregarding the term we put there as an instance of x. For this reason, we give the following definition mimicking [46, Definition 7.1.2].

Definition 65 *(Underlying eTRS).* Given a GTRS $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$, the system $\mathcal{R}_u = (\mathcal{F}, \mu, R_u)$ with $R_u = \{\ell \to r \mid \ell \to r \Leftarrow c \in R\}$ is the *underlying* CS-eTRS of $\mathcal{R}$.

Example 66. For $\mathcal{R}$ in Example 56,

$$(\mathcal{R}_\perp)_u = (\mathcal{F}, \mu_\perp, \{(56),(57)\}) \text{ and } (\mathcal{R}_{\mathsf{can}})_u = (\mathcal{F}, \mu^{can}_{\mathcal{R}}, \{(56),(57)\})$$

with

$$\mathsf{g}(\mathsf{s}(x)) \to \mathsf{g}(x) \tag{56}$$

$$\mathsf{f}(\mathsf{g}(x)) \to x \tag{57}$$

are clearly terminating. Thus, both $\overline{\mathcal{R}}_\perp$ and $\overline{\mathcal{R}}_{\mathsf{can}}$ are terminating.

The following obvious result is used in this paper.

Proposition 67. *A GTRS $\mathcal{R}$ is terminating if $\mathcal{R}_u$ is terminating.*

Termination of particular classes of GTRSs (TRSs, CS-TRSs, CTRSs,...) can be investigated by using well-known existing methods and tools for them. The underlying CS-eTRS of a CTRS $\mathcal{R}$ is useful to investigate termination of $\mathcal{R}$. Alternatively, termination of GTRSs can be investigated by using the model-theoretical approach in [38, Sections 5.1 & 8.3.1] and [35, Section 5.5]. The tool AGES [21] provides some support for this.

7.7. Proving and disproving confluence of GTRSs

The following result generalizes to GTRSs the other main result of the first part of this paper.

Theorem 68 *(Confluence of GTRSs). A terminating GTRS $\mathcal{R}$ is confluent iff each $\pi \in \mathsf{ECCP}(\mathcal{R})$ is joinable.*

Corollary 69 *(Confluence of SE-GTRSs). Let $\mathcal{R}$ be a terminating SE-GTRS such that for all $\langle s,t\rangle \Leftarrow x \to x', c \in \mathsf{CVP}(\mathcal{R})$, (i) for all $u \leftrightarrow^* v \in c$, $x \notin \mathcal{V}ar^{\nmid}(u) \cup \mathcal{V}ar^{\nmid}(v)$ and (ii) all other atoms $A \in c$ are $(\overline{\mathcal{R}}, \leftrightarrow^*_{\mathcal{R}}, x)$-preserving. Then, $\mathcal{R}$ is confluent iff each $\pi \in \mathsf{pCCP}(\mathcal{R}) \cup \mathsf{iCCP}(\mathcal{R})$ is joinable.*

Example 70. For $\mathcal{R}_\perp$ in Example 56 there is no conditional critical pair because the only active position of the left-hand sides $\ell_{(40)}$ and $\ell_{(41)}$ of rules (40) and (41) is Λ. However, $\ell_{(40)}$ and $\ell_{(41)}$ do *not* unify. Also, $\mathsf{CVP}(\mathcal{R}_\perp) = \emptyset$ because all variables in the left-hand sides of the rules in $\mathcal{R}_\perp$ are *frozen*. By Theorem 62, $\mathcal{R}_\perp$ is locally confluent. Since it is terminating, by Theorem 68, $\mathcal{R}_\perp$ is confluent.

Example 71. For $\mathcal{R}_{\text{can}}$ in Example 56, with (41), i.e., $\mathsf{f}(\mathsf{g}(x)) \to x \Leftarrow x \approx \mathsf{s}(0)$ $1 \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\ell_{(41)})$ and (40)', i.e., $\mathsf{g}(\mathsf{s}(x')) \to \mathsf{g}(x')$, since $\mathsf{f}(\mathsf{g}(x))|_1 = \mathsf{g}(x)$ and $\mathsf{g}(\mathsf{s}(x'))$ unify with $\theta = \{x \mapsto \mathsf{s}(x')$, we obtain the conditional critical pair

$$\langle \mathsf{f}(\mathsf{g}(x')), \mathsf{s}(x')\rangle \Leftarrow \mathsf{s}(x') \to^* \mathsf{s}(0) \tag{58}$$

This pair is not joinable: the sequence $\mathsf{s}(x') \to^* \mathsf{s}(0)$ is clearly feasible, but the sequence

$$\mathsf{s}(x') \to^* \mathsf{s}(0), \mathsf{f}(\mathsf{g}(x')) \to^* z, \mathsf{s}(x') \to^* z$$

is infeasible: since the argument of s is frozen, the only way to satisfy the first condition $\mathsf{s}(x') \to^* \mathsf{s}(0)$ is defining $\sigma = \{x' \mapsto 0\}$; furthermore, in order to satisfy the last condition $\mathsf{s}(x') \to^* z$ we need $\sigma = \{x' \mapsto 0, z \mapsto 0\}$; however, $\sigma(\mathsf{f}(\mathsf{g}(x'))) = \mathsf{f}(\mathsf{g}(0))$ is irreducible; thus $\mathsf{f}(\mathsf{g}(0)) \to^* 0$ is *not* satisfied. By Proposition 21, it is not joinable. By Theorem 62, $\mathcal{R}_{\text{can}}$ is not locally confluent nor confluent.

Example 72. For $\mathcal{R}$ in Example 57, we have the following *proper* CCP*s* (see Remark 25):

$$\langle \mathsf{s}(x'), \mathsf{s}(x')\rangle \Leftarrow \mathsf{s}(x') \geq \mathsf{s}(0), x' \geq \mathsf{s}(0) \tag{59}$$

$$\langle \mathsf{odd}(x), \mathsf{pev}(x)\rangle \Leftarrow x \approx \mathsf{s}(\mathsf{s}(0)), x \approx \mathsf{s}(0) \tag{60}$$

$$\langle \mathsf{zero}(x), \mathsf{pev}(x)\rangle \Leftarrow x \approx \mathsf{s}(\mathsf{s}(0)), x \approx 0 \tag{61}$$

$$\langle \mathsf{zero}(x), \mathsf{odd}(x)\rangle \Leftarrow x \approx \mathsf{s}(0), x \approx 0 \tag{62}$$

Note that (59) is trivial, and (60), (61), and (62) are all infeasible: (60) due to need of rewriting an instance of x to $\mathsf{s}(\mathsf{s}(0))$ and also to $\mathsf{s}(0)$, which is not possible by using (50), the only rule that would be able to produce the expected result; (61) and (62) because the only way to rewrite an instance $\sigma(x)$ of x to 0 (as required by the second condition) is if $\sigma(x)$ is already 0, but then $\sigma(x)$ cannot be rewritten to any other term whether $\mathsf{s}(\mathsf{s}(0))$, as in (61), or $\mathsf{s}(0)$, as in (62). Thus, $\mathsf{pCCP}(\mathcal{R}) = \{(59)\}$ containing a trivially $\overline{\mathcal{R}}$-joinable conditional pair. Since all rules in $\mathcal{R}$ are of type 1, $\mathsf{iCCP}(\mathcal{R}) = \emptyset$. There are four conditional variable pairs, one per each rewrite rule:

$$\langle \mathsf{s}(\mathsf{s}(x')), x\rangle \Leftarrow x \to x', x \geq \mathsf{s}(0) \tag{63}$$

$$\langle \mathsf{test}(x'), \mathsf{pev}(x)\rangle \Leftarrow x \to x', x \approx \mathsf{s}(\mathsf{s}(0)) \tag{64}$$

$$\langle \mathsf{test}(x'), \mathsf{odd}(x)\rangle \Leftarrow x \to x', x \approx \mathsf{s}(0) \tag{65}$$

$$\langle \mathsf{test}(x'), \mathsf{zero}(x)\rangle \Leftarrow x \to x', x \approx 0 \tag{66}$$

Again, they are all joinable:

- Regarding (63), if $\sigma(x)$ rewrites to $\sigma(x')$ for some substitution σ and $\sigma(x) \geq \mathsf{s}(0)$ holds, then $\sigma(x) = \mathsf{s}(t)$ for some term t. Hence, only rule (49) applies to $\sigma(x)$. By definition of the rule, $\sigma(x) = \mathsf{s}(\mathsf{s}(\sigma(x')))$. Hence, the two components of the instance of the peak of (63) are identical and it is $\overline{\mathcal{R}}$-joinable.
- Regarding (64), if $\sigma(x)$ rewrites to $\sigma(x')$ for some substitution σ and $\sigma(x) \to^*_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$ holds, then (by irreducibility of $\mathsf{s}(\mathsf{s}(0))$, $\mathsf{s}(0)$, and 0 and because reduction preserves evenness) $\sigma(x) = \mathsf{s}^{2n}(0)$ for some $n > 1$. Since $2(n-1) \geq 2$, $\sigma(x') \to^*_{\mathcal{R}} \mathsf{s}(\mathsf{s}(0))$ holds we have that $\sigma(\mathsf{test}(x')) = \mathsf{test}(\sigma(x')) \to_{\mathcal{R}} \mathsf{pev}(\sigma(x'))$, and hence $\sigma(\mathsf{pev}(x)) = \mathsf{pev}(\mathsf{s}(\mathsf{s}(\sigma(x')))) \to_{\mathcal{R}} \mathsf{pev}(\sigma(x'))$. Therefore, (64) is joinable.
- (65) and (66) are similar to (64).

Since all conditional pairs in $\mathsf{ECCP}(\mathcal{R})$ are joinable, by Theorem 62, $\mathcal{R}$ is locally confluent. Since

$$\mathcal{R}_u = \{\mathsf{s}(\mathsf{s}(x)) \to x, \mathsf{test}(x) \to \mathsf{pev}(x), \mathsf{test}(x) \to \mathsf{odd}(x), \mathsf{test}(x) \to \mathsf{zero}(x)\}$$

is terminating, by Theorem 68, $\mathcal{R}$ is confluent.

8. Related work

Conditional variable pairs We are not aware of any similar proposal in the literature on confluence of CTRSs. However, in his analysis of *relative termination*, i.e., termination of the relation $\to^*_{\mathcal{R}'} \circ \to_{\mathcal{R}} \circ \to^*_{\mathcal{R}'}$ where $\mathcal{R}$ and $\mathcal{R}'$ are TRSs, Geser used *variable critical pair* to refer to (unconditional) pairs $\langle s,t\rangle$ which are obtained as usual by considering 'overlaps' between a nonvariable position $\ell|_p$ of the left-hand side ℓ of a rule $\ell \to r \in \mathcal{R}$ and the left-hand side ℓ' of a rule $\ell' \to r' \in \mathcal{R}'$ where ℓ' is a *variable* [16, page 44]; see also [29, page 1166, footnote 1] for a similar idea in the realm of the analysis of confluence of *equational* term rewriting systems.

Struth used *variable critical pair* to refer to (unconditional) critical pairs obtained by considering variables occurring more than once in a given rule to produce critical pairs representing peaks [51].

Conditional variable pairs are related to LH_μ-critical pairs for proving confluence of CS-TRSs $(\mathcal{R},\mu)$ [41]. However, LH_μ-critical pairs $\pi : \langle \ell[x']_p, r\rangle \Leftarrow x \to x'$ for a rule $\ell \to r \in \mathcal{R}$, $x \in \mathcal{V}ar^\mu(\ell)$ and $p \in \mathcal{P}os^\mu_x(\ell)$ are *different* from CVPs as x must be *frozen* in ℓ or r [41, Definition 20]; otherwise, π is joinable. CTRSs would have *no* LH_μ*-critical pair* as no replacement map is considered. For CS-TRSs, LH_μ-critical pairs are conditional variable pairs but not vice versa. For instance, for the CS-TRS $(\mathcal{R}_{\text{can}})_u$ in Example 66, rule (57) produces the conditional variable pair

$$\langle \mathsf{f}(\mathsf{g}(x')), x\rangle \Leftarrow x \to x' \tag{67}$$

However, (67) is *not* an LH_μ-critical pair because the critical variable x is *not* frozen anywhere in (57). Fortunately, Proposition 61.(1) guarantees joinability of (67), thus avoiding overloads in confluence proofs. Also, the new results about specialization and joinability of conditional (variable) pairs (Propositions 21 and 40) are also useful in proofs of confluence of CS-TRSs.

Local confluence of CTRSs Kaplan characterized local confluence of *simplifying* J-CTRSs as the joinability of (proper and improper) conditional critical pairs [31, Theorem 5.3]. A Join CTRS $\mathcal{R}$ is *simplifying* if all rules $\ell \to r \Leftarrow c \in \mathcal{R}$ satisfy (i) $\ell > r$ and (ii) for all $s \approx t \in c$, $\ell > s$ and $r > t$ for some *simplification ordering* $>$ (i.e., a well-founded and monotone ordering where $s > t$ for all terms s and strict subterms t of s, see [31, Definition 1.5]). Simplifyingness implies termination of CTRSs (but not vice versa). Note the use of termination already to prove local confluence, in contrast to Huet's work for TRSs.

In their work about confluence of J-CTRSs, Dershowitz, Okada, and Sivakumar observe that joinability of conditional critical pairs does *not* suffice to guarantee local confluence of J-CTRSs [9, top of page 37]. However, all their results are given as *sufficient conditions* for *confluence* with no explicit mention to local confluence. Furthermore, besides joinability of conditional critical pairs (proper and improper, no distinction is made) they require some *termination property* (decreasingness [9, Definition 7], as in their Theorem 3, or noetherianity, i.e., termination, as in Theorems 2 and 4). At the end of the paper, they write

> *Our proofs show that, for conditional systems, the notions of confluence, local-confluence, and joinable critical pairs can not be neatly disentangled. In particular, the noetherian condition was needed to show that a system is locally confluent if all critical pairs are shallow joinable.* [9, first paragraph of Section 6]

Once termination has been identified (and assumed) as *necessary* to show local confluence by (shallow) joinability of conditional critical pairs, Newman's Lemma makes confluence and local confluence identical. As a matter of fact, the research about confluence of CTRSs by joinability of conditional critical pairs in these and ensuing works directly addressed *confluence*, as some property implying termination was assumed: simplifyingness in [31], decreasingness in [9], quasi-reductiveness in [2], quasi-decreasingness in [13], etc., see also the summary of results provided in [46, Section 7.3]. In contrast, our Theorem 43 shows that conditional variable pairs (not used in any of the aforementioned works) *improve* on this situation, as local confluence of CTRSs $\mathcal{R}$ can be proved *à la Huet* as the joinability of extended conditional critical pairs in $\mathsf{ECCP}(\mathcal{R})$, imposing no additional requirement on $\mathcal{R}$ and using no termination property.

Generalized term rewriting systems CTRSs with more general rules as the ones proposed in Section 7 have been previously investigated in the literature. Early examples are, e.g., [48,10].

Generalized Rewrite Theories (GRTs [4]) are tuples $\mathcal{R} = (\Sigma, \phi, E, R)$ where (i) Σ is an *order-sorted* signature of function symbols, (ii) ϕ is a mapping establishing *frozen* arguments of k-ary function symbols f, (iii) E is a set of Σ*-sentences* $A \Leftarrow c$, where (iii.1) A is either an equation $u = v$ or a *membership* statement $t : s$ for some term t and sort symbol s and (iii.2) c is a sequence of equations $u = v$ and membership conditions $t : s$; finally, (iv) R is a set of rules $\ell \to r \Leftarrow c$, where c is a sequence of equalities $u = v$, membership statements $t : s$, and reachability tests $w \to^* w'$ (written $w \to w'$ in the rules) [4, Definition 2.4]. It is not difficult to see that a GRT can be seen as a GTRS $(\mathcal{F}, \Pi, \mu, H, R)$. As done in [12], the correspondence is as follows (by lack of space we omit the technical details):

1. Sorts s can be treated in unsorted first-order logic [57] by using monadic predicates. $\_ : s$ for each considered sort s, which are added to Π; then, each sorted symbol $f : s_1 \cdots s_k \to s$ in Σ for sorts $s_1, \ldots, s_k, s$ is viewed as an unsorted k-ary symbol $f \in \mathcal{F}$ and 'typing' clauses $f(x_1, \ldots, x_k) : s \Leftarrow x_1 : s_1, \ldots, x_k : s_k$ for distinct variables $x_1, \ldots, x_k$ are added to H.
2. For each k-ary function symbol f, we let $\mu(f) = \{1, \ldots, k\} - \phi(f)$.
3. Equalities $u = v$ and memberships $t : s$ for *sorted* terms u, v, and t, occurring in E are atoms of the equality predicates '$=$' and $\_ : s$ for each considered sort s. The Σ-sentences in E are treated as Horn clauses which take into account the sort information in terms u, v, and t. Such clauses are then included in H together with the usual clauses for equality.

4. The rules in the GRT can be seen as rules in R after some adaptations to deal with sorted variables.

Confluence of a subclass of GRTs (which properly include CS-TRSs and *oriented* (CS-)CTRSs), has been investigated in [13]. They prove that *strongly deterministic* [13, Definition 1] and *quasi-decreasing* [13, Definition 2] GRTs are confluent iff all (proper and improper) conditional critical pairs are joinable [13, Theorem 2]. Here, $\mathcal{R}$ is *strongly deterministic* if for all rules $\ell \to r \Leftarrow c$ of $\mathcal{R}$ variables in c are 'sequentially' introduced from ℓ or from a *previous* condition; furthermore, for all $s \approx t$ and substitutions σ, $\sigma(t)$ is irreducible [13, Definition 1]. This result does *not* apply to disprove confluence of $\mathcal{R}$ in Example 2 (which is not strongly deterministic, as the right-hand side a of the conditional part of rule (5) is not irreducible). For the same reason, the confluence of $\mathcal{R}$ (as a join system) could not be proved either. Similarly, confluence of $\mathcal{R}$ in Example 18 cannot be *disproved* as the system is *not* strongly deterministic (variable x occurs in the left-hand side $\mathsf{g}(x)$ of the condition of rule (21), but it does *not* occur in the left-hand side a of the rule).

9. Conclusions and future work

Analysis of (local) confluence of CTRSs We have introduced *conditional variable pairs* (Definition 30) which, together with proper and improper conditional critical pairs, provide a new characterization of local confluence of CTRSs:

> A CTRS $\mathcal{R}$ with FO-theory $\overline{\mathcal{R}}$ is locally $\overline{\mathcal{R}}$-confluent if and only if all (proper and improper) conditional critical pairs and all conditional variable pairs are $\overline{\mathcal{R}}$*-joinable*. (Theorem 43)

This characterization is valid for all usually considered classes of CTRSs (Join, Oriented, or Semi-Equational) according to the evaluation of their conditions, and for any type (1, 2, 3, or 4) of systems according to the distribution of variables in rules. Also, the following corollary is obtained from Theorem 43:

> A CTRS $\mathcal{R}$ with FO-theory $\overline{\mathcal{R}}$ which is $\overline{\mathcal{R}}$-terminating is $\overline{\mathcal{R}}$-confluent if and only if all (proper and improper) conditional critical pairs and all conditional variable pairs are $\overline{\mathcal{R}}$*-joinable*. (Theorem 45)

For semi-equational CTRSs conditional variable pairs can be dismissed (Corollaries 44 and 46). In this setting, proofs of (local) $\overline{\mathcal{R}}$-confluence heavily rely on the ability to (dis)prove $\overline{\mathcal{R}}$-joinability of conditional pairs, for which we have provided a number of new results, most of them for the new conditional variable pairs: Corollary 14, and Propositions 21, 34, 35, and also Proposition 40, which formalizes the use of a transformation of conditional variable pairs. As discussed in Section 8, most results in the literature obtain *sufficient conditions* for confluence of CTRSs by (i) requiring joinability of *proper* conditional critical pairs, (ii) additionally imposing syntactical restrictions on the rules (left-linearity, strong determinism, etc.) and (iii) requiring termination properties like simplifyingness, reductiveness, decreasingness, or quasi-decreasingness, which imply termination, but which are usually *stronger*, see [46, Section 7.2] for details of each of them, and [46, Lemma 7.2.20] for a hierarchy. In contrast, we provide a *characterization of local confluence* requiring *no termination property*; only joinability of proper and improper conditional critical pairs, and also of the new conditional variable pairs is required. Termination is only used to apply Newman's Lemma to obtain the usual characterization of confluence as local confluence for terminating systems. This generalizes Huet's result to CTRSs.

Our results for CTRSs were implemented as part of the 2022 and 2023 versions of the tool CONFident [24], which can be used to prove and disprove confluence of TRSs, CS-TRSs, CTRSs, and CS-CTRSs. The following table obtained from CONFident's results in the *full-run* of the International Confluence Competition CoCo 2023 (http://cops.uibk.ac.at/results/?y=2023-full-run&c=CTRS), shows that, in general, the use of CVPs in proofs of confluence of CTRSs leads to better results in the last version of CONFident with respect to the 2021 version (without CVPs):

Competition (confluence of CTRSs)	Yes	No	Maybe	Solved	Total
CoCo 2021 (no CVPs)	63	40	57	103	161
CoCo 2023 (with CVPs)	81	38	42	119	161

Generalized term rewriting systems We have introduced *Generalized Term Rewriting Systems* (GTRSs, Definition 51), where (i) replacement restrictions on specific arguments of function symbols are introduced by means of a replacement map *à la CSR*, and (ii) besides rewriting-based conditions $s \approx t$ (including joinability, reachability, and conversion conditions), which can now be used independently and mixed in the same rule, see Example 54, also (iii) more general, *atomic* conditions can be included in rules provided that (iv) they are defined by means of definite Horn clauses, see Section 7.4 for some examples. The obtained results for CTRSs smoothly extend to GTRSs due to the use of an appropriate FO-theory $\overline{\mathcal{R}}$ to describe computations with GTRSs (Definition 52) thus obtaining the corresponding characterizations of (local) confluence of a GTRS $\mathcal{R}$ by joinability of all pairs in $\mathsf{ECCP}(\mathcal{R})$ (Theorems 62 and 68). Since CS-CTRSs $\mathcal{R}$ are particular cases of GTRSs, our results improve on existing results for confluence of CS-CTRSs [36, Section 8.1.2], as we obtain a new characterization of (local) confluence of CS-CTRSs.

Our results for GTRSs were implemented (for CS-CTRSs only) as part of CONFident and tested in the CoCo 2023 category for proving confluence of context-sensitive rewriting. Confluence of CS-TRSs was proved using the results in [41]. Confluence of CS-CTRSs (examples 1362 to 1648 from COPS) was proved using the results in this paper. The following table shows the obtained results (extracted from CoCo 2023 *full-run*, http://cops.uibk.ac.at/results/?y=2023-full-run&c=CSR):

Yes	No	Maybe	Solved	Total
94	47	146	141	287

Future work An interesting subject for future work is providing an appropriate notion of *orthogonality* of CTRSs and GTRSs enabling a generalization of the well-known result for TRSs establishing that left-linearity and the absence of (proper) critical pairs guarantee confluence (see [50] and also [3, Corollary 6.3.11]), which does not hold for CTRSs, see Examples 2 and 18. Also, devising confluence criteria for GTRSs through more specific joinability criteria for ECCPs, as already done for TRSs by Huet (e.g., [28, Lemma 3.2]) and other researchers [18,47,56,55] is an interesting topic of research. Another subject for future work is developing criteria and techniques for proving *termination* of GTRSs which can be used in proofs of confluence of GTRSs.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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