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Solving discrete first-order matrix linear control problems with general parametric uncertainties: A probability-density-based approach

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Abstract

This paper deals with the probabilistic analysis of discrete first-order linear control models with uncertainties. For the sake of generality in our stochastic analysis, we assume that all model parameters (the initial and final states, the matrix containing the free dynamics part, and the control's coefficient) are random variables with an arbitrary joint probability density function. We then combine some results from classical Control Theory with Probability Theory to obtain, under very general hypotheses, the first probability density function of the control and the solution, which are parametric stochastic processes. To illustrate our theoretical findings, we also show two numerical examples and a classical discrete macroeconomic model whose parameters are treated as random variables.

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1. Introduction and motivation

Mathematical control theory deals with the principles underlying the design and analysis of control systems which, broadly speaking, are defined as systems that involve functions designed to influence an object's behavior to reach a goal [1]. Traditionally such models have been formulated using different approaches that mainly include discrete and continuous linear and nonlinear equations depending on the target problem [2–5]. In both settings, the model parameters are often set after measuring or approximating physical quantities, which involve uncertainties coming from measurement errors and/or the lack of knowledge of the corresponding physical phenomenon. Consequently, it is more realistic to deal with control models, including uncertainties in their formulation [6–9].

In dealing with discrete and continuous control models with randomness, most of the contributions assume specific patterns for describing uncertainties, such as Gaussian noises [10,11], Poisson processes [12,13] and Lévy dynamics [14,15]. It is worth pointing out that combinations of the aforementioned stochastic patterns as, for instance, the Wiener process with exponential and compensated Poisson jumps have also been applied [16,17], respectively. Under these approaches, the success of the corresponding theoretical development often relies upon strong hypotheses, such as independence, stationarity, or on assuming specific probability distributions for the increments and/or jumps of the corresponding driving stochastic processes. This could limit the applicability of the obtained results to real-world problems. This fact has motivated alternative approaches to deal with discrete and continuous models, where uncertainties are treated in a broader sense via random difference and differential equations [18–20]. In this context, some contributions have focused on the analysis of robust stability [21,22].

The mathematical treatment of random difference and differential equations includes spectral approaches, such as the generalized polynomial chaos, the stochastic Galerkin technique, the collocation method, and the discrete projection method [19,23]; simulations, such as Monte Carlo [24]; perturbation methods [19,25]; etc. All these methods are mainly oriented to obtain approximations of the first statistics, such as the mean and the variance of the solution. Additionally, the so-called Random Variable Transformation method (also termed the Probability Transformation method) and the Liouville–Gibbs equation-based method have been considered to calculate, exactly or approximately, the distribution of the solution [18,26]. To a large extent, the above-mentioned techniques have been mainly applied in the setting of continuous models formulated via differential equations [27–31], while contributions dealing with discrete models are more scarce [32,33].

In the setting of control problems formulated via differential equations, some interesting contributions have been recently published based on the aforementioned methods. In [34,35] authors apply spectral techniques based on polynomial chaos to address a flight control problem and to determine the controller of a spring-mass-damper system subject to vibrations, respectively. In [36,37] authors introduce different Monte Carlo-based techniques for obtaining controller approximations of and stabilizing nonlinear stochastic optimal control and analyzing the solution of optimal control problems with a stochastic parameter, respectively. The authors have recently applied the above-mentioned Random Variable Transformation method to determine the distribution of the process stochastic solution of discrete first [38], and second [39], order linear control problems where some of the model parameters are random variables with arbitrary distributions. Motivated by these recent works and the scarce results obtained so far in the setting of control discrete models, in this paper, we aim to rigorously obtain analogous results for discrete first-order matrix linear control problems with general parametric uncertainties. The interest in dealing with the random discrete case is not only motivated by the intrinsic mathematical goal of extending the results from the continuous case, but also on the basis that the dynamics of many physical systems that operate at specific (discrete) time steps may be affected by typical inaccuracies in the mechanical devices that govern the process itself and/or the control in Engineering. Besides, when operating with remote systems, it might be more realistic to assume that both the initial and the final/target states involve uncertainties. Further motivating examples, based on first-order matrix linear control problems, can be found in other real-world settings such as Economics, where the sources of uncertainties assigned to the model parameters are naturally motivated due to the complexity of financial markets. Section 5 shows an example from Economics that illustrates these motivating comments. Also, when solving nonlinear discrete control models, one often applies linearization techniques to approximate the model, making linear models easier to deal with. This strategy can be applied, for example, when studying epidemiological models. In this framework, uncertainties can be directly assigned to the entries of the matrix system (since they represent the contagion and recovering parameters, which depend on complex factors, that rarely are known in a deterministic way but randomly); the coefficient of the actuator (since the effectiveness of epidemic control measures, such as vaccination, have certain errors); the initial state (since it represents the initial subpopulation of infected people, which is known after surveys, so involving statistical errors coming from sampling), and the final state (since the main goal after applying the control is reaching a target percentage or the number of susceptible within an acceptable range that is not exactly known because of the variability in the effectiveness of vaccination). And last but not least, it is important to point out that the theoretical results that will be established throughout the paper are also applicable in the case that some model parameters are deterministic (see later Remark 4).

As in these two previous contributions to deal with continuous control problems [38,39], we shall rely on the aforementioned Random Variable Transformation technique to conduct our analysis method. It must be underlined that applying this stochastic technique in the discrete setting involves more difficulties than in the continuous scenario. Indeed, as it shall be seen later, the multidimensional mapping required to apply the Random Variable Transformation method successfully is technically more difficult to analyze, particularly regarding its invertibility. This makes the mathematical analysis more challenging than its continuous counterpart.

The main novelty of this paper is the treatment of uncertainties in discrete first-order matrix linear control models under very general assumptions with respect to both, first, where randomness can be considered in the model data, and secondly, the type of probability distribution that can be considered in the study. This gives great versatility in applying the theoretical results in practical cases. To the best of our knowledge, the foregoing discrete linear control problems have not yet been analyzed under our approach. In this paper, we tackle the random analysis of this fundamental model as a first step.

The paper is structured as follows. In Section 2, we adapt some deterministic preliminaries that will be used in Section 3, where we randomize the deterministic model and give expressions of the solution and the control. Additionally, motivated by the deterministic theory, in this section, we also establish the random Kalman controllability condition. In Section 4, we give semi-explicit expressions (in terms of integrals) for the first probability density functions (1-PDFs) of the solution and the control. In Section 5, we apply our theoretical findings in two numerical examples and in the study of the random dynamics of the Samuelson macroeconomic model with uncertainties. Finally, the conclusions are drawn in Section 6.

2. Deterministic preliminaries

For the sake of convenience, we first introduce some deterministic results that will be required to develop our subsequent random analysis. Let the following discrete first-order linear control system

$$\vec{x}(k+1) = \mathbf{A}\vec{x}(k) + \vec{b}\,u(k), \quad k = 0, 1, 2, \dots, K-1, \tag{1}$$

where $\vec{x}(k+1) \in \mathbb{R}^n$ is the response of the system at the step $k+1$, $\mathbf{A}$ is a $n \times n$ matrix containing the free dynamics part, $\vec{b}$ is a $n \times 1$ real vector determining how the control, $u(k) \in \mathbb{R}$, affects the response. Given an initial state,

$$\vec{x}(0) = \vec{x}_0 \in \mathbb{R}^n, \tag{2}$$

we are interested in exact discrete controllable systems. That is, those systems in which a given final state $\vec{x}_1 \in \mathbb{R}^n$ can be reached from every initial state $\vec{x}_0$, in a finite number of steps, K, i.e., given any initial condition $\vec{x}_0$, one must satisfy that $\vec{x}(K) = \vec{x}_1$. In Eq. (1), K denotes the final step, i.e., the step by which the system reaches the final state $\vec{x}_1 \in \mathbb{R}^n$.

The following well-known result provides an explicit solution of the discrete first-order control system Eqs. (1) and (2). For the sake of completeness, we include its proof in Appendix I.

Theorem 1. *The solution of the discrete first-order control system Eqs. (1) and (2) at any step $k = 1, 2, \dots, K$, is*

$$\vec{x}(k) = \mathbf{A}^k\vec{x}_0 + \sum_{j=1}^{k} \mathbf{A}^{k-j}\vec{b}\,u(j-1) = \mathbf{A}^k\vec{x}_0 + \mathcal{U}_k\vec{u}_k, \tag{3}$$

where each component of the vector $\vec{u}_k$ is the scalar control evaluated at $0, 1, 2, \dots, k-1$, i.e., $\vec{u}_k = [u(0), u(1), \dots, u(k-1)]^\top$, where, as usual, the superscript $^\top$ denotes the transpose operator. $\mathcal{U}_k$ is a $n \times k$ matrix defined, by means of columns, as $\mathcal{U}_k = [\mathbf{A}^{k-1}\vec{b}\,|\dots|\,\mathbf{A}\vec{b}\,|\,\vec{b}\,]$.

As has been previously pointed out, we will deal with controllable discrete systems, and besides calculating the solution of the system, it is also necessary to know the conditions of controllability as well as determine an expression for the controller satisfying such conditions. Below, we state a characterization of controllability for the system Eq. (1). For further details, see references [40,42,41], and therein.

Theorem 2. *The discrete first-order system Eqs. (1) and (2) is exactly controllable for $K \in \mathbb{N}$ finite if, and only if,* $\operatorname{Rank}(\mathcal{U}_n) = n$ *and $K \geq n$, being $\mathcal{U}_n$ the Kalman's controllability matrix.*

Proof. ($\Longrightarrow$) Suppose that discrete first-order system Eqs. (1) and (2) is exactly controllable for some $K \in \mathbb{N}$. We shall prove that $K \geq n$, and $\operatorname{Rank}(\mathcal{U}_n) = n$, being $\mathcal{U}_n = [\mathbf{A}^{n-1}\vec{b}\,|\dots|\,\mathbf{A}\vec{b}\,|\,\vec{b}\,]$. Given an initial state $\vec{x}_0$, as the system is exactly controllable for a given K, there exists some $\vec{u}_K \in \mathbb{R}^K$ such that

$$\vec{x}_1 = \vec{x}(K) = \mathbf{A}^K\vec{x}_0 + \mathcal{U}_K\vec{u}_K = \mathbf{A}^K\vec{x}_0 + \sum_{j=1}^{K} \mathbf{A}^{K-j}\vec{b}\,u(j-1).$$

We can define an auxiliary n-dimensional vector $\phi := \vec{x}_1 - \mathbf{A}^K \vec{x}_0 \in \mathbb{R}^n$, then

$$\phi = \sum_{j=1}^{K} \mathbf{A}^{K-j} \vec{b}\, u(j-1).$$

The right-hand side of this expression is a linear combination of K vectors $\{\mathbf{A}^{K-1}\vec{b}, \dots, \mathbf{A}\vec{b}, \vec{b}\}$. Therefore, the last expression can be written as

$$\phi = \mathbf{A}^{K-1}\vec{b}C_K + \dots + \mathbf{A}\vec{b}C_2 + \vec{b}C_1, \tag{4}$$

with $C_1, C_2, \dots, C_K$ are functions of $u(0), u(1), \dots, u(K-1)$, which are considered the "constants" of the system. If $K < n$, the linear system Eq. (4) is, in general, overdetermined, and its solution (for an arbitrary ϕ) does not always exist. Therefore, a necessary condition for the solvability of Eq. (4) is $K \geq n$, and the first part of the theorem is proved. On the other hand, it is known that, by the Cayley-Hamilton theorem, the i-th power, $\mathbf{A}^i$, of a matrix $\mathbf{A}$ of size n, with $i \geq n$, can be expressed as a linear combination of matrices $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^{n-1}$, where $\mathbf{I}$ denotes the identity matrix of size n. If $K \geq n$, then the linear combination Eq. (4) of vectors $\{\mathbf{A}^{K-1}\vec{b}, \dots, \mathbf{A}\vec{b}, \vec{b}\}$ can be reduced to a linear combination of vectors $\{\mathbf{A}^{n-1}\vec{b}, \dots, \mathbf{A}\vec{b}, \vec{b}\}$. Thus, for any arbitrary ϕ, the system Eq. (4) of n linear equations is solvable if and only if $\det(\mathcal{U}_n) \neq 0$, i.e., $\mathrm{Rank}(\mathcal{U}_n) = n$.

($\Longleftarrow$) First we shall proof that for any $N \geq n$, $\mathrm{Rank}(\mathcal{U}_N) = \mathrm{Rank}(\mathcal{U}_n)$. By the definition of $\mathcal{U}_k$, $\mathrm{Rank}(\mathcal{U}_N) \geq \mathrm{Rank}(\mathcal{U}_n)$ when $N \geq n$; and using the hypothesis $\mathrm{Rank}(\mathcal{U}_n) = n$, the inequality $\mathrm{Rank}(\mathcal{U}_N) \geq n$ is fulfilled. On the other hand, $\dim(\mathcal{U}_N) = n \times N$, therefore $\mathrm{Rank}(\mathcal{U}_N) \leq n$. Combining both inequalities yields the desired result, $\mathrm{Rank}(\mathcal{U}_N) = \mathrm{Rank}(\mathcal{U}_n)$, $N \geq n$. Now, suppose that $\mathrm{Rank}(\mathcal{U}_n) = n$, then $\mathrm{Rank}(\mathcal{U}_K) = n$. Consider $\vec{x} \in \mathbb{R}^n$ such that

$$\vec{x} = \vec{x}_1 - \mathbf{A}^K \vec{x}_0, \quad \vec{x}_0, \vec{x}_1 \in \mathbb{R}^n.$$

Then, there exists a control $\vec{u}_K \in \mathbb{R}^K$ such that $\mathcal{U}_K \vec{u}_K = \vec{x}$. Therefore,

$$\vec{x} = \vec{x}_1 - \mathbf{A}^K \vec{x}_0 = \mathcal{U}_K \vec{u}_K.$$

Hence,

$$\vec{x}_1 = \mathbf{A}^K \vec{x}_0 + \mathcal{U}_K \vec{u}_K.$$

So, we have obtained a solution $\vec{x}_1$ of the system Eq. (1) such that $\vec{x}(K) = \vec{x}_1$ and $\vec{x}(0) = \vec{x}_0$. Then, the system is exactly controllable. □

An explicit expression of the control $\vec{u}_K = [u(0), \dots, u(K-1)]^\top$ is established in the next lemma. In a general Hilbert space, details of the proof can be found in Leiva and Uzcategui [40]. Note that we have adapted the statement of the lemma to the problem we are dealing with.

Lemma 3 [40]**.** *The Eq. (1) is exactly controllable for some $K \in \mathbb{N}$ if, and only if, the discrete controllability Grammian, defined by $\mathbf{L} = \mathcal{U}_K \mathcal{U}_K^\top$, is invertible. Moreover, in this case $\mathbf{S} = \mathcal{U}_K^\top \mathbf{L}^{-1}$ is a right inverse of $\mathcal{U}_K$ and the control $\vec{u}$ steering an initial state $\vec{x}_0$ to a final state $\vec{x}_1$ is given by*

$$\vec{u} = \vec{u}_K = \mathbf{S}\big(\vec{x}_1 - \mathbf{A}^K \vec{x}_0\big). \tag{5}$$

Remark 1. In the particular case where the system is controllable in exactly n steps, that is $K = n$, the control $\vec{u}$ is

$$\vec{u} = \mathcal{U}_n^{-1}(\vec{x}_1 - \mathbf{A}^n\vec{x}_0).$$

Notice that in Lemma 3 the matrices $\mathbf{L}$ and $\mathbf{S}$, with dimension $n \times n$ and $K \times n$, respectively, and the control vector $\vec{u}$ depends on $K \in \mathbb{N}$. To simplify further calculations, we have removed this dependency in the notation.

3. The randomization of the discrete first-order control system

So far, we have introduced the discrete first-order problem based on a deterministic discrete control system, Eqs. (1) and (2). For this problem, we have presented the necessary and sufficient conditions to reach exact controllability. In addition, expressions for the solution and the control have been explicitly given. In this section, we take advantage of the previous results to study the randomized counterpart of problem Eqs. (1) and (2). For the sake of generality, we will assume that all model data are random quantities whose outcomes are denoted by ω. The problem Eqs. (1) and (2) then writes

$$\begin{aligned}\vec{x}(k+1,\omega) &= \mathbf{A}(\vec{\alpha}(\omega))\vec{x}(k,\omega) + \vec{b}(\vec{\beta}(\omega))\,u(k,\omega), \quad k = 0,1,2,\dots,K-1,\\ \vec{x}(0,\omega) &= \vec{x}_0(\omega).\end{aligned} \tag{6}$$

Here, we will assume that the matrix $\mathbf{A}$ and the vector $\vec{b}$ depend, respectively, on a finite number of random variables collected in the random vectors $\vec{\alpha}(\omega) = [\alpha_1(\omega),\dots,\alpha_s(\omega)]$ and $\vec{\beta}(\omega) = [\beta_1(\omega),\dots,\beta_m(\omega)]$. In our subsequent analysis, all the components of the random vectors $\vec{\alpha}(\omega)$, $\vec{\beta}(\omega)$, the initial state $\vec{x}_0(\omega) = [x_{01}(\omega),\dots,x_{0n}(\omega)]^\top$ and the final target $\vec{x}_1(\omega) = [x_{11}(\omega),\dots,x_{1n}(\omega)]^\top$ are assumed to be real-valued absolutely continuous random variables defined on a common complete probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$, where outcomes $\omega \in \Omega$. To keep our analysis as general as possible, instead of assuming that all the foregoing random data are independent, hereinafter we will assume that $f_0(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta})$ denotes the joint probability density function (PDF) of the random vector $[\vec{x}_0(\omega), \vec{x}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega)]$ that collects all random inputs. At this point, it is interesting to underline that we are implicitly assuming that the stochastic control model Eq. (6) has parametric uncertainties, i.e., uncertainties are introduced via a finite number (in our case $2n + s + m$) of random variables.

According to Theorem 1, the solution stochastic process of the randomized problem Eq. (6) is given by

$$\begin{aligned}\vec{x}(k,\omega) &= \mathbf{A}(\vec{\alpha}(\omega))^k\vec{x}_0(\omega) + \sum_{j=1}^{k}\mathbf{A}(\vec{\alpha}(\omega))^{k-j}\vec{b}(\vec{\beta}(\omega))u(j-1,\omega)\\ &= \mathbf{A}(\vec{\alpha}(\omega))^k\vec{x}_0(\omega) + \mathcal{U}_k(\vec{\alpha}(\omega),\vec{\beta}(\omega))\vec{u}_k(\omega), \quad k = 1,\dots,K,\end{aligned} \tag{7}$$

being $\vec{u}_k(\omega)$ a vector containing the first k components of the stochastic controller defined by

$$\vec{u}(\omega) = \mathbf{S}\Big(\vec{\alpha}(\omega),\vec{\beta}(\omega)\Big)\Big(\vec{x}_1(\omega) - \mathbf{A}(\vec{\alpha}(\omega))^K\vec{x}_0(\omega)\Big), \tag{8}$$

where $\mathbf{S}(\vec{\alpha}(\omega),\vec{\beta}(\omega)) = \mathcal{U}_K^\top(\vec{\alpha}(\omega),\vec{\beta}(\omega))\mathbf{L}^{-1}(\vec{\alpha}(\omega),\vec{\beta}(\omega))$, being

$$\mathbf{L}\Big(\vec{\alpha}(\omega),\vec{\beta}(\omega)\Big) = \mathcal{U}_K\Big(\vec{\alpha}(\omega),\vec{\beta}(\omega)\Big)\mathcal{U}_K^\top(\vec{\alpha}(\omega),\vec{\beta}(\omega))$$

the random controllability Grammian. Note that $\mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega))$ and $\mathbf{L}(\vec{\alpha}(\omega), \vec{\beta}(\omega))$ are matrices of sizes, $K \times n$ and $n \times n$, respectively. For the sake of clarity in the presentation of the randomized control model from its deterministic counterpart, we have explicitly indicated ω-dependence with respect to the random vectors $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$ in expressions Eqs. (7) and (8). Henceforth, we will simplify the notation to express this dependency by simply writing $\mathbf{A}(\omega) := \mathbf{A}(\vec{\alpha}(\omega))$.

The following result gives a characterization of the exact controllability of the random system Eq. (6). It is a direct consequence of Theorem 2 and that we are assuming that all the inputs are absolutely continuous random variables.

Theorem 4 Random Kalman controllability condition**.** *Let $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ be a complete probability space. Let $\mathbf{A}(\omega) := \mathbf{A}(\vec{\alpha}(\omega))$ and $\vec{b}(\omega) := \vec{b}(\vec{\beta}(\omega))$ be random matrices whose entries depend on absolutely continuous random vectors $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$, $\omega \in \Omega$. Then, necessary and sufficient conditions for the random system Eq. (6) to be controllable in a given step $K \in \mathbb{N}$ are*

$$K \geq n \quad \text{and} \quad \mathbb{P}[\{\omega \in \Omega : \text{Rank}(\mathcal{U}_n(\omega)) = n\}] = 1,$$

where $\mathcal{U}_n(\omega) = [\mathbf{A}^{n-1}(\omega)\vec{b}(\omega) \mid \ldots \mid \mathbf{A}(\omega)\vec{b}(\omega) \mid \vec{b}(\omega)]$ has dimension n.

Remark 2. Based on its deterministic counterpart, hereinafter, the matrix $\mathcal{U}_n(\omega)$ in Theorem 4 will be referred to as the random Kalman's controllability matrix.

Observe that condition

$$\mathbb{P}\left[\left\{\omega \in \Omega : \text{Rank}(\mathcal{U}_n(\omega)) = \text{Rank}\left(\left[\mathbf{A}^{n-1}(\omega)\vec{b}(\omega) \mid \ldots \mid \mathbf{A}(\omega)\vec{b}(\omega) \mid \vec{b}(\omega)\right]\right) = n\right\}\right] = 1,$$

is guaranteed when all elements of $\mathbf{A}(\omega)$ and $\vec{b}(\omega)$, $\omega \in \Omega$, are absolutely continuous random variables. As a direct consequence of Theorem 4, one gets the following result:

Proposition 1. *Let $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ be a complete probability space. If all elements of the random matrices $A(\omega), \vec{b}(\omega)$, $\omega \in \Omega$ are absolutely continuous random variables, then problem Eq. (6) is controllable for a given $K \in \mathbb{N}$ if, and only if $K \geq n$.*

Roughly speaking, solving the deterministic control problem Eq. (1) consists in obtaining the solution, $\vec{x}(k)$, and the control, $u(k)$, so that the solution is driven from a given initial state, $\vec{x}_0$ to a final state $\vec{x}_1$, which is also prefixed. In dealing with its random counterpart Eq. (6), the solution and the control are both stochastic processes, namely $\vec{x}(k, \omega)$ and $u(k, \omega)$, respectively. So, determining relevant statistical properties of these random functions is also a major goal. In the case of the n-dimensional stochastic process $\vec{x}(k, \omega)$ (analogously for $u(k, \omega)$) the most important moments are the mean, $\mu_{\vec{x}}(k) := \mathbb{E}[\vec{x}(k, \omega)]$ and the variance-covariance matrix, $\Sigma_{\vec{x}}(k) := \mathbb{E}[\vec{x}(k, \omega)\vec{x}(k, \omega)^\top]$, where $\mathbb{E}[\cdot]$ denotes the expectation operator. However, the computation of the so-called first probability density function (1-PDF) of $\vec{x}(k, \omega)$, $f_1(\vec{x}, k)$ ($f_1(u, k)$ for $u(k, \omega)$), is a more desirable goal since from this deterministic function one can calculate the foregoing moments

$$\mu_{\vec{x}}(k) = \int_{\mathbb{R}^n} \vec{x} f_1(\vec{x}, k) d\vec{x}, \quad \Sigma_{\vec{x}}(k) = \int_{\mathbb{R}^n} (\vec{x} - \mu_{\vec{x}}(k))(\vec{x} - \mu_{\vec{x}}(k))^\top f_1(\vec{x}, k) d\vec{x}, \tag{9}$$

as well as to construct confidence regions for k and $\alpha \in (0, 1)$ fixed ($1 - \alpha$ is referred to as the confidence level) by determining $y \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} (f_1(\vec{x}, k) - y) d\vec{x} = 1 - \alpha, \qquad f_1(\vec{x}, k) \geq y.$$

Furthermore, the 1-PDF permits computing the probability that the solution lies within any Borel set, say $\mathcal{A}$, of $\mathbb{R}^n$,

$$\mathbb{P}\big[\{\omega \in \Omega : \vec{x}(k,\omega) \in \mathcal{A}\}\big] = \int_{\mathcal{A}} f_1(\vec{x},k)d\vec{x},$$

for $k \in \mathbb{N}$ fixed.

In the next section, we will focus on the computation of the 1-PDF of the solution $\vec{x}(k,\omega)$ and of the control $u(k,\omega)$, for each $k = 1, 2, \ldots, K$. To attack this problem, we will take advantage of the so-called Random Variable Transformation (RVT) method that is stated in the following theorem.

Theorem 5 RVT method [18, page 25]**.** *Let us consider $\vec{u}(\omega) = (u_1(\omega), \ldots, u_D(\omega))$ and $\vec{v}(\omega) = (v_1(\omega), \ldots, v_D(\omega))$ two D-dimensional continuous random vectors defined on a complete probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$. Let $\vec{r} : \mathbb{R}^D \to \mathbb{R}^D$ be a one-to-one deterministic transformation of $\vec{u}$ into $\vec{v}$, i.e., $\vec{v} = \vec{r}(\vec{u})$. Assume that $\vec{r}$ is continuous in $\vec{u}$ and has continuous partial derivatives w.r.t. each u_i, $1 \le i \le D$. Then, if $f_{\vec{u}}(\vec{u})$ denotes the joint PDF of random vector $\vec{u}(\omega)$, and $\vec{s} = \vec{r}^{-1} = (s_1(v_1, \ldots, v_D), \ldots, s_n(v_1, \ldots, v_D))$ represents the inverse mapping of $\vec{r} = (r_1(u_1, \ldots, u_D), \ldots, r_n(u_1, \ldots, u_D))$, the joint PDF of random vector $\vec{v}(\omega)$ is given by*

$$f_{\vec{v}}(\vec{v}) = f_{\vec{u}}(\vec{s}(\vec{v}))|J|,$$

where $|J|$, which is assumed to be different from zero, is the absolute value of the Jacobian, which is defined by the determinant

$$J = \det\left[\frac{\partial \vec{s}}{\partial \vec{v}}\right] = \det\begin{bmatrix} \frac{\partial s_1(v_1,\ldots,v_D)}{\partial v_1} & \cdots & \frac{\partial s_n(v_1,\ldots,v_D)}{\partial v_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(v_1,\ldots,v_D)}{\partial v_D} & \cdots & \frac{\partial s_D(v_1,\ldots,v_D)}{\partial v_D} \end{bmatrix}.$$

4. Computing the 1-PDF of the solution and the control stochastic processes

Suppose that the system is controllable for a given $K \in \mathbb{N}$, which should be greater or equal than n, $K \ge n$. The main objective of this section is to obtain an expression for the 1-PDF of the solution $\vec{x}(k,\omega)$ and of each component, $u(j,\omega)$, $j = 0, \ldots, k-1$, $k = 1, 2, \ldots K$, of the control $\vec{u}_K(\omega)$. For this aim, as previously indicated, given a fixed step $k = 1, 2, \ldots, K$, the RVT method will conveniently be applied. To this end, we first rewrite the solution $\vec{x}(k,\omega)$ in Eq. (7), taking into account the expression of the control Eq. (8), as

$$\begin{aligned}\vec{x}(k,\omega) &= \mathbf{A}(\vec{\alpha}(\omega))^k \vec{x}_0(\omega) + \mathcal{U}_k(\vec{\alpha}(\omega), \vec{\beta}(\omega))\big[\mathbf{I}_k \ \mathbf{O}_{k,K-k}\big] \\ &\qquad \mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\big(\vec{x}_1(\omega) - \mathbf{A}(\vec{\alpha}(\omega))^K \vec{x}_0(\omega)\big) \\ &= \mathbf{A}(\vec{\alpha}(\omega))^k \vec{x}_0(\omega) + \mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\big(\vec{x}_1(\omega) - \mathbf{A}(\vec{\alpha}(\omega))^K \vec{x}_0(\omega)\big), \quad k = 1, 2, \ldots, K, \end{aligned} \tag{10}$$

being

$$\mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega)) = \mathcal{U}_k(\vec{\alpha}(\omega), \vec{\beta}(\omega))\big[\mathbf{I}_k \ \mathbf{O}_{k,K-k}\big]\mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega)), \tag{11}$$

a square random matrix of size n that depends on the random vectors $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$, the fixed step k and the final step K. In Eq. (10), $\mathbf{I}_k$ is the identity matrix of dimension k and

$\mathbf{O}_{k,K-k}$ is a null matrix of size $k \times (K-k)$. Now, for convenience when applying the RVT method, we rewrite the solution as a linear combination of the initial and final states

$$\vec{x}(k,\omega) = \mathbf{G}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\vec{x}_0(\omega) + \mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\vec{x}_1(\omega), \quad k = 1, 2, \ldots, K, \tag{12}$$

where

$$\mathbf{G}(\vec{\alpha}(\omega), \vec{\beta}(\omega)) = \mathbf{A}(\vec{\alpha}(\omega))^k - \mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\mathbf{A}(\vec{\alpha}(\omega))^K, \tag{13}$$

is a square matrix of dimension n.

For the value of $k = 1, 2, \ldots, K$, previously fixed, we apply the RVT method to obtain the PDF of the random vector $\vec{x}(k,\omega)$ in terms of the joint PDF $f_0(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta})$. As we have less equations (n) than random variables ($D := 2n + s + m$), we define the deterministic mapping $\vec{r}: \mathbb{R}^D \to \mathbb{R}^D$, whose components are defined by mappings $\vec{r}_1 : \mathbb{R}^D \to \mathbb{R}^n$, $\vec{r}_2 : \mathbb{R}^D \to \mathbb{R}^n$, $\vec{r}_3 : \mathbb{R}^D \to \mathbb{R}^s$, and $\vec{r}_4 : \mathbb{R}^D \to \mathbb{R}^m$, defined, respectively, as follows:

$$\begin{aligned}
\vec{z}_1 &= \vec{r}_1(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \mathbf{G}(\vec{\alpha}, \vec{\beta})\vec{x}_0 + \mathbf{H}(\vec{\alpha}, \vec{\beta})\vec{x}_1,\\
\vec{z}_2 &= \vec{r}_2(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{x}_1,\\
\vec{z}_3 &= \vec{r}_3(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\alpha},\\
\vec{z}_4 &= \vec{r}_4(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\beta}.
\end{aligned}$$

If the inverse of the deterministic matrix $\mathbf{G}(\vec{\alpha}, \vec{\beta})$ exists, we will be able to formally compute the inverse mapping of $\vec{r}$, $\vec{s}: \mathbb{R}^D \to \mathbb{R}^D$, as

$$\begin{aligned}
\vec{x}_0 &= \vec{s}_1(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \mathbf{G}^{-1}(\vec{z}_3, \vec{z}_4)(\vec{z}_1 - \mathbf{H}(\vec{z}_3, \vec{z}_4)\vec{z}_2),\\
\vec{x}_1 &= \vec{s}_2(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_2,\\
\vec{\alpha} &= \vec{s}_3(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_3,\\
\vec{\beta} &= \vec{s}_4(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_4,
\end{aligned}$$

being its Jacobian $J = \det(\mathbf{G}^{-1}(\vec{z}_3, \vec{z}_4)) \neq 0$. Then, according to the RVT technique, the joint PDF of the random vector $(\vec{x}(k,\omega), \vec{x}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega))$ is given by

$$f_{\vec{x}(k),\vec{x}_1,\vec{\alpha},\vec{\beta}}(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = f_0\big(\mathbf{G}^{-1}(\vec{z}_3, \vec{z}_4)(\vec{z}_1 - \mathbf{H}(\vec{z}_3, \vec{z}_4)\vec{z}_2), \vec{z}_2, \vec{z}_3, \vec{z}_4\big)\left|\det\big(\mathbf{G}^{-1}(\vec{z}_3, \vec{z}_4)\big)\right|,$$

where, $f_0 := f_0(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta})$ is the joint PDF of the input data. Finally, marginalizing the last expression with respect to the random vectors $\vec{z}_2(\omega) = \vec{x}_1(\omega)$, $\vec{z}_3(\omega) = \vec{\alpha}(\omega)$ and $\vec{z}_4(\omega) = \vec{\beta}(\omega)$, one obtains the 1-PDF of the response for each $k = 1, \ldots, K$,

$$f_1(\vec{x}; k) = \int_{\mathbb{R}^{n+s+m}} f_0\Big(\mathbf{G}^{-1}(\vec{\alpha}, \vec{\beta})\Big(\vec{x} - \mathbf{H}(\vec{\alpha}, \vec{\beta})\vec{x}_1\Big), \vec{x}_1, \vec{\alpha}, \vec{\beta}\Big)\left|\det(\mathbf{G}^{-1}(\vec{\alpha}, \vec{\beta}))\right| \mathrm{d}\vec{x}_1\, \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta}. \tag{14}$$

If the matrix $\mathbf{G}$ is not invertible, but the matrix $\mathbf{H}$ is invertible, the same above transformation $\vec{r}$ can be considered when applying the RVT method, but taking $\vec{z}_1 = \vec{x}_0$ and $\vec{z}_2 = \mathbf{G}(\vec{\alpha}, \vec{\beta})\vec{x}_0 + \mathbf{H}(\vec{\alpha}, \vec{\beta})\vec{x}_1$ and then isolating $\vec{x}_1$ from the latter expression, $\vec{x}_1 = \mathbf{H}^{-1}(\vec{z}_3, \vec{z}_4)(\vec{z}_2 - \mathbf{G}(\vec{z}_3, \vec{z}_4)\vec{z}_1)$. In this case, the 1-PDF of the response is given by

$$f_1(\vec{x}; k) = \int_{\mathbb{R}^{n+s+m}} f_0\Big(\vec{x}_0, \mathbf{H}^{-1}(\vec{\alpha}, \vec{\beta})\Big(\vec{x} - \mathbf{G}(\vec{\alpha}, \vec{\beta})\vec{x}_0\Big), \vec{\alpha}, \vec{\beta}\Big)\left|\det(\mathbf{H}^{-1}(\vec{\alpha}, \vec{\beta}))\right| \mathrm{d}\vec{x}_0\, \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta}. \tag{15}$$

At this point, recall that both matrices $\mathbf{G}$ and $\mathbf{H}$, that have size n, depend on k. Therefore, it may happen that there exists a step k for which neither of the two matrices is invertible. For example, if we observe the definition of matrix $\mathbf{H}$ in Eq. (11), if $k < n \leq K$, then $\operatorname{rank}(\mathbf{H}) < n$, and the matrix $\mathbf{H}$ is not invertible. In the case that both matrices are singular, next, we shall see that the RVT method can still be applied to obtain the 1-PDF of the solution. Next, we explain how to proceed with. Given any initial and final states, $\vec{x}_0$ and $\vec{x}_1$, we calculate the following vector

$$\mathbf{G}\vec{x}_0 + \mathbf{H}\vec{x}_1 = \left[\vec{g}_1 \dots \vec{g}_n\right]\begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix} + \left[\vec{h}_1 \dots \vec{h}_n\right]\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix} = \sum_{i=1}^{n}\left(x_{0i}\vec{g}_i + x_{1i}\vec{h}_i\right) := \vec{w} \in \mathbb{R}^n,$$

where $\vec{g}_i$ and $\vec{h}_i$ denote, respectively, the columns of matrices $\mathbf{G}$ and $\mathbf{H}$, and $x_{0,i}$, $x_{1,i}$, $i = 1, \dots, n$ are the components of $\vec{x}_0$ and $\vec{x}_1$, respectively. Therefore, given a vector $\vec{w}$, it can be obtained from a linear combination of elements of $\mathcal{G} = \{\vec{g}_1, \dots, \vec{g}_n, \vec{h}_1, \dots, \vec{h}_n\} \subset \mathbb{R}^n$. So, $\mathcal{G}$ is a generator set, and we can find a basis, thus

$$\mathrm{rank}(\mathbf{G}) \le n, \quad \mathrm{rank}(\mathbf{H}) \le n, \quad \mathrm{rank}(\mathbf{G}+\mathbf{H}) = n.$$

Suppose that $\mathrm{rank}(\mathbf{G}) = p$, then $\mathrm{rank}(\mathbf{H}) \ge n - p$, in order to fulfil the above equality. For the sake of simplicity in the notation when applying later the RVT method, we assume that the first columns of matrices $\mathbf{G}$ and $\mathbf{H}$ are linearly independent, i.e., the n vectors $\{\vec{g}_1, \dots, \vec{g}_p, \vec{h}_1, \dots, \vec{h}_{n-p}\}$ define a basis in $\mathbb{R}^n$. Therefore, the square matrix $\hat{\mathbf{M}} = [\vec{g}_1, \dots, \vec{g}_p, \vec{h}_1, \dots, \vec{h}_{n-p}]$ of size n has rank n and is also invertible. Now, let $\vec{\hat{x}}_0$ and $\vec{\hat{x}}_1$ be two vectors containing the first p components and $n - p$ components of $\vec{x}_0$ and $\vec{x}_1$, respectively, that is, $\vec{\hat{x}}_0 = [x_{01}, \dots, x_{0p}]^\top$ and $\vec{\hat{x}}_1 = [x_{11}, \dots, x_{1(n-p)}]^\top$. The vectors with the rest of the components of $\vec{x}_0$ and $\vec{x}_1$ will be denoted by $\vec{\tilde{x}}_0 = [x_{0(p+1)}, \dots, x_{0n}]^\top$ and $\vec{\tilde{x}}_1 = [x_{1(n-p+1)}, \dots, x_{1n}]^\top$, respectively.

With this notation, we can define the mapping $\vec{r}\colon \mathbb{R}^D \to \mathbb{R}^D$ as

$$\begin{aligned}
\vec{z}_1 &= \vec{r}_1(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \hat{\mathbf{M}}(\vec{\alpha}, \vec{\beta})\begin{bmatrix} \vec{\hat{x}}_0 \\ \vec{\hat{x}}_1 \end{bmatrix} + \tilde{\mathbf{M}}(\vec{\alpha}, \vec{\beta})\begin{bmatrix} \vec{\tilde{x}}_0 \\ \vec{\tilde{x}}_1 \end{bmatrix}, \\
\vec{z}_2 &= \vec{r}_2(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \begin{bmatrix} \vec{\tilde{x}}_0 \\ \vec{\tilde{x}}_1 \end{bmatrix}, \\
\vec{z}_3 &= \vec{r}_3(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\alpha}, \\
\vec{z}_4 &= \vec{r}_4(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\beta},
\end{aligned}$$

where $\tilde{\mathbf{M}} = [\vec{g}_{p+1}, \dots, \vec{g}_n, \vec{h}_{n-p+1}, \dots, \vec{h}_n]$. The inverse mapping $\vec{s}\colon \mathbb{R}^D \to \mathbb{R}^D$ and the Jacobian are

$$\begin{aligned}
\begin{bmatrix} \vec{\hat{x}}_0 \\ \vec{\hat{x}}_1 \end{bmatrix} &= \vec{s}_1(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \hat{\mathbf{M}}^{-1}(\vec{z}_3, \vec{z}_4)\left(\vec{z}_1 - \tilde{\mathbf{M}}(\vec{z}_3, \vec{z}_4)\vec{z}_2\right), \\
\begin{bmatrix} \vec{\tilde{x}}_0 \\ \vec{\tilde{x}}_1 \end{bmatrix} &= \vec{s}_2(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_2, \\
\vec{\alpha} &= \vec{s}_3(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_3, \\
\vec{\beta} &= \vec{s}_4(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_4,
\end{aligned}$$

and $J = \det(\hat{\mathbf{M}}^{-1}(\vec{z}_3, \vec{z}_4))$, respectively. Note that the Jacobian is well defined and non-zero since, by construction, matrix $\hat{\mathbf{M}}(\vec{z}_3, \vec{z}_4)$ is invertible.

Now, we define the following square permutation matrix of dimension $2n$:

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_p & \mathbf{O}_{p,n-p} & \mathbf{O}_{p,n-p} & \mathbf{O}_p \\ \mathbf{O}_{n-p,p} & \mathbf{O}_{n-p} & \mathbf{I}_{n-p} & \mathbf{O}_{n-p,p} \\ \mathbf{O}_{n-p,p} & \mathbf{I}_{n-p} & \mathbf{O}_{n-p} & \mathbf{O}_{n-p,p} \\ \mathbf{O}_{p,p} & \mathbf{O}_{p,n-p} & \mathbf{O}_{p,n-p} & \mathbf{I}_p \end{bmatrix},$$

where $\mathbf{I}_k$ is the identity matrix, and $\mathbf{O}_k$ and $\mathbf{O}_{k,n-k}$ denote the null matrices of dimensions k and $k \times (n-k)$, respectively. Then, we can rewrite the inverse transformation $\vec{s}$ as

$$\begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \end{bmatrix} = \begin{bmatrix} \hat{\vec{x}}_0 \\ \tilde{\vec{x}}_0 \\ \hat{\vec{x}}_1 \\ \tilde{\vec{x}}_1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \hat{\vec{x}}_0 \\ \hat{\vec{x}}_1 \\ \tilde{\vec{x}}_0 \\ \tilde{\vec{x}}_1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \vec{s}_1(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) \\ \vec{s}_2(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \hat{\mathbf{M}}^{-1}(\vec{z}_3, \vec{z}_4)\Big(\vec{z}_1 - \tilde{\mathbf{M}}(\vec{z}_3, \vec{z}_4)\vec{z}_2\Big) \\ \vec{z}_2 \end{bmatrix},$$

$$\vec{\alpha} = \vec{s}_3(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_3,$$
$$\vec{\beta} = \vec{s}_4(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_4.$$

Applying the RVT method, the joint PDF of the random vector $(\vec{z}_1(\omega), \vec{z}_2(\omega), \vec{z}_3(\omega), \vec{z}_4(\omega))$ is

$$f_{\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4}(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = f_0\left(\mathbf{P}\begin{bmatrix} \hat{\mathbf{M}}^{-1}(\vec{z}_3, \vec{z}_4)\Big(\vec{z}_1 - \tilde{\mathbf{M}}(\vec{z}_3, \vec{z}_4)\vec{z}_2\Big) \\ \vec{z}_2 \end{bmatrix}, \vec{z}_3, \vec{z}_4\right) |\det(\hat{\mathbf{M}}^{-1}(\vec{z}_3, \vec{z}_4))|.$$

As $\vec{z}_1(\omega) = \vec{x}(t,\omega)$, marginalizing the joint PDF with respect to the random vector $(\vec{z}_2(\omega), \vec{z}_3(\omega), \vec{z}_4(\omega)) = (\tilde{\vec{x}}_0(\omega), \tilde{\vec{x}}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega))$, the PDF of the response is obtained for each $k = 1, \ldots, K$,

$$f_1(\vec{x}; k) = \int_{\mathbb{R}^{n+s+p}} f_0\left(\mathbf{P}\begin{bmatrix} \hat{\mathbf{M}}^{-1}(\vec{\alpha}, \vec{\beta})\left(\vec{x} - \tilde{\mathbf{M}}(\vec{\alpha}, \vec{\beta})\begin{bmatrix} \tilde{\vec{x}}_0 \\ \tilde{\vec{x}}_1 \end{bmatrix}\right) \\ \tilde{\vec{x}}_0 \\ \tilde{\vec{x}}_1 \end{bmatrix}, \vec{\alpha}, \vec{\beta}\right)$$
$$|\det(\hat{\mathbf{M}}^{-1}(\vec{\alpha}, \vec{\beta}))| \,\mathrm{d}\tilde{\vec{x}}_0 \,\mathrm{d}\tilde{\vec{x}}_1 \mathrm{d}\vec{\alpha}\,\mathrm{d}\vec{\beta}, \tag{16}$$

where

$$\tilde{\vec{x}}_0 \,\mathrm{d}\tilde{\vec{x}}_1 \mathrm{d}\vec{\alpha}\,\mathrm{d}\vec{\beta} = \left(\prod_{p+1 \le i \le n} \mathrm{d}x_{0i}\right)\left(\prod_{n-p+1 \le i \le n} \mathrm{d}x_{1i}\right)\left(\prod_{1 \le i \le s} \mathrm{d}\alpha_i\right)\left(\prod_{1 \le i \le m} \mathrm{d}\beta_i\right).$$

Now, we compute the PDF of each component of the random control $\vec{u}(\omega)$ given in Eq. (8), i.e., the PDF of $u(k,\omega)$, $k = 0, 1, \ldots, K-1$. For a fixed component k, let $u(k,\omega)$ be the k-th component of the control

$$u(k,\omega) = \vec{e}_{k+1}^{\top}\vec{u}(\omega) = \vec{e}_{k+1}^{\top}\mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\big(\vec{x}_1(\omega) - \mathbf{A}(\vec{\alpha}(\omega))^K \vec{x}_0(\omega)\big)$$
$$= \nu(\omega) - \sum_{i=1}^{n} \gamma_i(\omega) x_{0i}(\omega), \quad k = 0, 1, \ldots, K-1,$$

where $\vec{e}_i$ denotes the i-th column of identity matrix $\mathbf{I}_K$ and the random variables $\nu(\omega)$ and $\gamma_i(\omega)$, $i = 1, \ldots, n$, are $\nu(\omega) = \nu(k; \vec{x}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega)) = \vec{e}_{k+1}^{\top}\mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\vec{x}_1(\omega)$ and

$\gamma_i(\omega) = \gamma_i(k; \vec{\alpha}(\omega), \vec{\beta}(\omega)) = \vec{e}_{k+1}^{\top}\mathbf{S}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\mathbf{A}(\vec{\alpha}(\omega))^K\vec{e}_i$. Therefore, we define the deterministic mapping $\vec{r}: \mathbb{R}^D \to \mathbb{R}^D$ as

$$
\begin{aligned}
z_{11} &= r_{11}(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \nu(k; \vec{x}_1, \vec{\alpha}, \vec{\beta}) - \sum_{i=1}^{n} \gamma_i(k; \vec{\alpha}, \vec{\beta})x_{0i},\\
z_{1i} &= r_{1i}(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = x_{0i}, \quad i = 2, \dots, n,\\
\vec{z}_2 &= \vec{r}_2(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{x}_1,\\
\vec{z}_3 &= \vec{r}_3(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\alpha},\\
\vec{z}_4 &= \vec{r}_4(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta}) = \vec{\beta}.
\end{aligned}
$$

The inverse mapping $\vec{s} = \vec{r}^{-1}$, $\vec{s}: \mathbb{R}^D \to \mathbb{R}^D$,

$$
\begin{aligned}
x_{01} &= s_{11}(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \frac{1}{\gamma_1(k; \vec{z}_3, \vec{z}_4)}\left(\nu(k; \vec{z}_2, \vec{z}_3, \vec{z}_4) - z_{11} - \sum_{i=2}^{n}\gamma_i(k; \vec{z}_3, \vec{z}_4)z_{1i}\right),\\
x_{0i} &= s_{1i}(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = z_{1i}, \quad i = 2, \dots, n,\\
\vec{x}_1 &= \vec{s}_2(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_2,\\
\vec{\alpha} &= \vec{s}_3(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_3,\\
\vec{\beta} &= \vec{s}_4(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = \vec{z}_4.
\end{aligned}
$$

The absolute value of the Jacobian of $\vec{s}$ is $|J| = \left|\frac{1}{\gamma_1(k; \vec{z}_3, \vec{z}_4)}\right| \neq 0$ w.p. 1 (observe that, by its own definition, $\gamma_1(k; \vec{z}_3, \vec{z}_4)$ is an absolutely random variable since it is defined via a Borel measurable mapping that transforms absolutely random variables). Then, applying the RVT technique we can obtain the PDF of the random vector $(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4)$ in terms of the joint PDF of the random vector of input parameters $(\vec{x}_0, \vec{x}_1, \vec{\alpha}, \vec{\beta})$,

$$
\begin{aligned}
f_{\vec{z}_1,\vec{z}_2,\vec{z}_3,\vec{z}_4}(\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4) = f_0\Bigg(&\frac{1}{\gamma_1(k; \vec{z}_3, \vec{z}_4)}\left(\nu(k; \vec{z}_2, \vec{z}_3, \vec{z}_4) - z_{11} - \sum_{i=2}^{n}\gamma_i(k; \vec{z}_3, \vec{z}_4)z_{1i}\right),\\
&z_{12}, \dots, z_{1n}, \vec{z}_2, \vec{z}_3, \vec{z}_4\Bigg)\left|\frac{1}{\gamma_1(k; \vec{z}_3, \vec{z}_4)}\right|. \qquad (17)
\end{aligned}
$$

As $u(k) = z_{11}$, marginalizing Eq. (17) with respect to the random vectors $\vec{z}_2(\omega) = \vec{x}_1(\omega)$, $\vec{z}_3(\omega) = \vec{\alpha}(\omega)$ and $\vec{z}_4(\omega) = \vec{\beta}(\omega)$ and the random variables $z_{1i}(\omega) = x_{0i}(\omega), i = 2, \dots, n$, we obtain the 1-PDF of the k-th component of the controller

$$
\begin{aligned}
f_1(u, k) = \int_{\mathbb{R}^{2n-1+s+m}} f_0\Bigg(&\frac{1}{\gamma_1(k; \vec{\alpha}, \vec{\beta})}\left(\nu(k; \vec{x}_1, \vec{\alpha}, \vec{\beta}) - u - \sum_{i=2}^{n}\gamma_i(k; \vec{\alpha}, \vec{\beta})x_{0i}\right),\\
&x_{02}, \dots, x_{0n}, \vec{x}_1, \vec{\alpha}, \vec{\beta}\Bigg)\left|\frac{1}{\gamma_1(k; \vec{\alpha}, \vec{\beta})}\right| \mathrm{d}\vec{x}_{0^*}\, \mathrm{d}\vec{x}_1\, \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta}, \qquad (18)
\end{aligned}
$$

where

$$
\mathrm{d}\vec{x}_{0^*}\, \mathrm{d}\vec{x}_1\, \mathrm{d}\vec{\alpha}\, \mathrm{d}\vec{\beta} = \left(\prod_{2\le i\le n} \mathrm{d}x_{0i}\right)\left(\prod_{1\le i\le n} \mathrm{d}x_{1i}\right)\left(\prod_{1\le i\le s} \mathrm{d}\alpha_i\right)\left(\prod_{1\le i\le m} \mathrm{d}\beta_i\right). \qquad (19)
$$

Remark 3. In practical cases, the semi-explicit representations of the 1-PDF, $f_1(\vec{x}; k)$, given in Eqs. (14)–(16), of the solution stochastic process of problem Eq. (6) may become computationally unaffordable for high-dimensional integration domains. In such situations, it is convenient to have alternative representations of the $f_1(\vec{x}; k)$. With this aim, we here note that in the particular case that the random vector $\vec{x}_0(\omega)$ is independent of the random vectors $\vec{x}_1(\omega)$, $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$, then $f_1(\vec{x}; k)$, given in Eq. (14), can be represented via the following expectation

$$f_1(\vec{x}; k) = \mathbb{E}_{\vec{x}_1,\vec{\alpha},\vec{\beta}}\Big[f_0\Big(\mathbf{G}^{-1}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\Big(\vec{x} - \mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\vec{x}_1(\omega)\Big), \vec{x}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega)\Big) \\ |\det(\mathbf{G}^{-1}(\vec{\alpha}(\omega), \vec{\beta}(\omega)))|\Big], \tag{20}$$

where $\mathbb{E}_{\vec{x}_1,\vec{\alpha},\vec{\beta}}[\cdot]$ stands for the computation of the expectation with respect to the random vectors $\vec{x}_1(\omega)$, $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$. Then, the 1-PDF $f_1(\vec{x}; k)$ can be approximated by applying Monte Carlo simulations. For fixed k, we first obtain a number of samples, say M, of the random vectors $\vec{x}_1(\omega)$, $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$, according to their respective assumed distributions. Secondly, we evaluate the term $f_0\Big(\mathbf{G}^{-1}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\Big(\vec{x} - \mathbf{H}(\vec{\alpha}(\omega), \vec{\beta}(\omega))\vec{x}_1(\omega)\Big), \vec{x}_1(\omega), \vec{\alpha}(\omega), \vec{\beta}(\omega)\Big)|\det(\mathbf{G}^{-1}(\vec{\alpha}(\omega), \vec{\beta}(\omega)))|$ for the M samples obtained in the first step. In the third, and final step, we average the M values obtained in the second step. Similar expressions to the one given in Eq. (20) can be easily given for the expressions Eqs. (15) and (16), as well as for the expression of the 1-PDF of the control, $u(k, \omega)$, given in Eqs. (18) and (19). Indeed, in this case, by assuming that $x_{01}(\omega)$ is independent of the rest of random variables, i.e., $x_{02}(\omega), \dots, x_{0n}(\omega)$, $\vec{x}_1(\omega)$, $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$ (notice that for the case of the control, this assumption is weaker than the one assumed for the case of the solution), the resulting expression for the 1-PDF of the control writes

$$f_1(u, k) = \mathbb{E}_{\vec{x}_{0*},\vec{x}_1,\vec{\alpha},\vec{\beta}}\Bigg[f_0\Bigg(\frac{1}{\gamma_1(k; \vec{\alpha}, \vec{\beta})}\Bigg(\nu(k; \vec{x}_1, \vec{\alpha}, \vec{\beta}) - u - \sum_{i=2}^{n} \gamma_i(k; \vec{\alpha}, \vec{\beta}) x_{0i}\Bigg), \\ x_{02}, \dots, x_{0n}, \vec{x}_1, \vec{\alpha}, \vec{\beta}\Bigg) \left|\frac{1}{\gamma_1(k; \vec{\alpha}, \vec{\beta})}\right|\Bigg], \tag{21}$$

where $\mathbb{E}_{\vec{x}_{0*},\vec{x}_1,\vec{\alpha},\vec{\beta}}[\cdot]$ denotes the expectation operator with respect to random vectors $\vec{x}_{0*}(\omega)$, $\vec{x}_1(\omega)$, $\vec{\alpha}(\omega)$ and $\vec{\beta}(\omega)$. Notice that, according to Eq. (19), $\vec{x}_{0*}(\omega) = (x_{02}(\omega), \dots, x_{0n}(\omega))$. The practical computation of expression Eq. (21) can be done using Monte Carlo simulations similarly as it has been indicated for Eq. (20). The error of Monte Carlo approximations decreases with the number of simulations according to $\mathcal{O}(M^{-1/2})$ [24].

Remark 4. We have studied the general problem in which all the inputs are considered random variables. If some of them are not random variables, but deterministic constants, then following an analogous process to the one described above, the corresponding 1-PDF can be obtained. Alternatively, one can directly exploit the explicit expressions for the 1-PDF of the solution and control obtained in this section by considering the non-random inputs (constants) as Dirac delta that, as it is well known, can be interpreted as the PDF of degenerate random variables, i.e., deterministic constants. For instance, if the control coefficient is deterministic, say $\vec{b}(\omega) = \vec{b}_0 \in \mathbb{R}^n$, and $\mathbf{G}$ is invertible (so, expression Eq. (14) applies), the 1-PDF of the solution stochastic process of problem Eq. (6) can be calculated as follows

$$f_1(\vec{x}; k) = \int_{\mathbb{R}^{n+s+m}} f_{\vec{x}_0,\vec{x}_1,\vec{\alpha}}\Big(\mathbf{G}^{-1}(\vec{\alpha}, \vec{\beta})(\vec{x} - \mathbf{H}(\vec{\alpha})\vec{x}_1), \vec{x}_1, \vec{\alpha}\Big)$$

$$|\det(\mathbf{G}^{-1}(\vec{\alpha},\vec{\beta}))\delta(\vec{\beta}-\vec{\beta}_0)|\,\mathrm{d}\vec{x}_1\,\mathrm{d}\vec{\alpha}\,\mathrm{d}\vec{\beta}, \tag{22}$$

where $f_{\vec{x}_0,\vec{x}_1,\vec{\alpha}}$ denotes the joint PDF of the random vector $(\vec{x}_0,\vec{x}_1,\vec{\alpha})$. Analogous expressions can be written when expressions Eqs. (15) and (16) apply.

5. Numerical examples and applications

In this section, we show three examples where the previous theoretical findings are applied. To better illustrate the applicability of our results, we will consider different scenarios with regard to the model parameters where randomness appears. In the Examples 1 and 2, we will assume that the initial and final vector states are, respectively, random. In Example 3, we will consider the randomization of an important macroeconomic model where all its parameters are assumed to be random variables. It is important to point out that a wide range of probability distributions have been considered to carry out computations in the three examples. This is a distinctive feature of the proposed approach since the results are established in many contributions by assuming specific probabilistic patterns, mainly of Gaussian type. In the subsequent examples, we shall see that our results apply to the case that uncertainties are not only of Gaussian (see Example 1) but also when model parameters may have other distributions like Beta, Triangular, etc. (see Examples 2 and 3). It aims at showing the generality of the results established in Section 4. Additionally, we will compare the numerical results obtained utilizing the new method proposed in Section 4 with the corresponding ones calculated via Monte Carlo simulations.

Example 1. Let us consider problem Eq. (6), where the following data are deterministic

$$\mathbf{A}=\begin{bmatrix}1/2 & 1\\ -1 & 1/4\end{bmatrix},\qquad \vec{b}=\begin{bmatrix}1/2\\ 1\end{bmatrix},\qquad \vec{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix},\qquad K=10, \tag{23}$$

while the initial state, $\vec{x}_0(\omega)$, is assumed to be a random vector with a multivariate Normal distribution with mean, $\vec{\mu}$, and variance-covariance matrix, Σ, given by

$$\vec{\mu}=[3,2]^\top,\qquad \Sigma=\begin{bmatrix}0.08 & 0.03\\ 0.03 & 0.03\end{bmatrix},\qquad \text{i.e.}\qquad \vec{x}_0(\omega)=[x_{01}(\omega),x_{02}(\omega)]^\top\sim \mathrm{N}(\vec{\mu},\Sigma). \tag{24}$$

Notice that

$$\operatorname{rank}(\mathcal{U}_2)=\operatorname{rank}\left(\mathbf{A}\vec{b}|\vec{b}\right)=\operatorname{rank}\left(\begin{bmatrix}5/4 & 1/2\\ -1/4 & 1\end{bmatrix}\right)=2,$$

so Kalman's controllability condition is fulfilled and problem Eq. (6) is exactly controllable for $K\geq 2$.

In Fig. 1, we show the 1-PDF of the solution stochastic process, which is bidimensional, at the beginning ($k=0$) and at the step $k=8$. In both time instants, we have highlighted two confidence regions at different confidence levels $1-\alpha=0.5$ (blue) and $1-\alpha=0.9$ (red). By observing the vertical scale and the support of the 1-PDF in both plots, we can observe as the 1-PDFs become leptokurtic (uncertainty decreases). In Fig. 2, we have graphically represented the portrait phase of the solution at the steps $k=0,1,\dots,10$. At each one of these values, we have also plotted confidence regions at the same confidence levels as in Fig. 1. We can see, in full agreement with Fig. 1, that uncertainty reduces as k increases until the system reaches the deterministic final state $\vec{x}_1=(1,1)^\top$, so its variability is zero. In particular, it is instructive to compare the confidence regions plotted in Fig. 2 for the steps $k=0$ and $k=8$

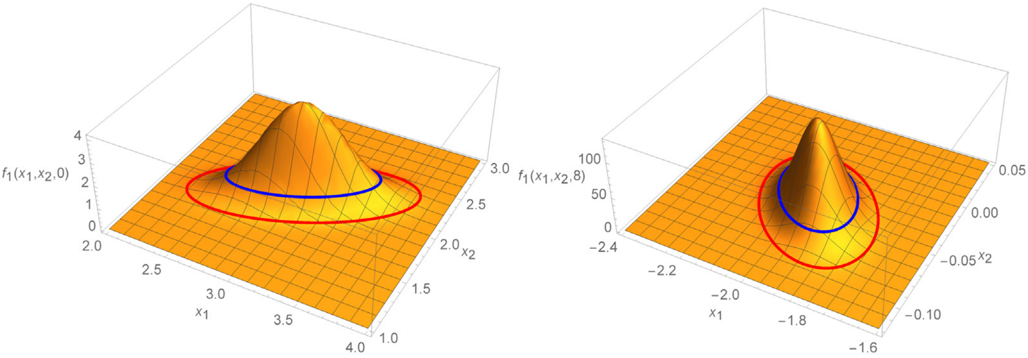

Fig. 1. 1-PDF of the solution stochastic process to the random control problem Eq. (6) at the steps $k = 0$ (left) and $k = 8$ (right). On the PDF's surface, we have highlighted confidence regions at different confidence levels $1 - \alpha = 0.5$ (blue) and $1 - \alpha = 0.9$ (red). Example 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

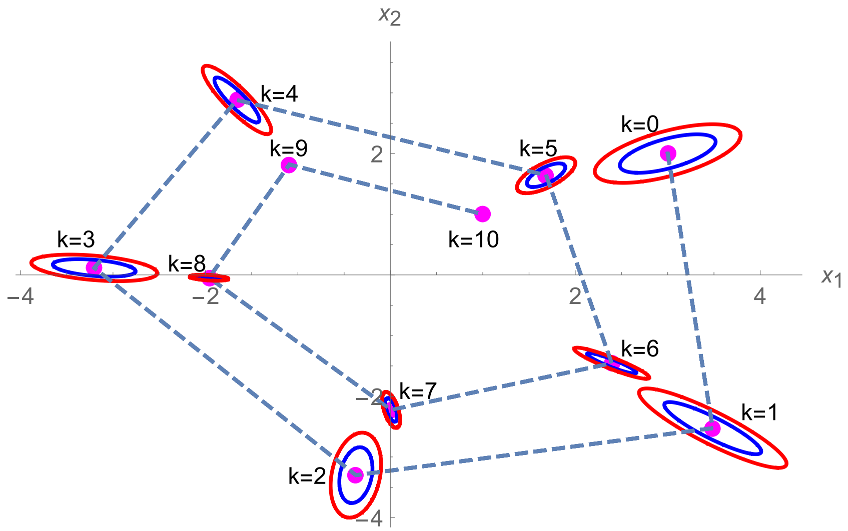

Fig. 2. Portrait phase for the random control problem Eq. (6). The dashed spiral line represents the expectation of the solution. We have also plotted two confidence regions at the confidence levels $1 - \alpha = 0.5$ (blue) and $1 - \alpha = 0.9$ (red) and at the steps $k = 0, 1, \ldots, 10$. Example 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with the graphical representation shown in Fig. 1. Finally, in Fig. 3, we have plotted the evolution of the PDF of the scalar control $u(k, \omega)$ as the step k changes from 0 to 9 (notice that, according to the model Eq. (6), $\vec{x}(k+1, \omega)$ depends on $u(k, \omega)$).

It is interesting to observe, from Fig. 2, that the random system is correctly guided from the random initial state $\vec{x}_0(\omega)$, given in Eq. (24), until it reaches the deterministic (with no variance) final state $\vec{x}_1 = [1, 1]^\top$ in exactly $K = 10$ steps. Since the control $u(t)(\omega)$ is stochastic, it is worth noticing that Fig. 3 provides a stochastic description of the dynamics of the control (via its PDF) at each step k to drive the random system accurately. Remarkably, the

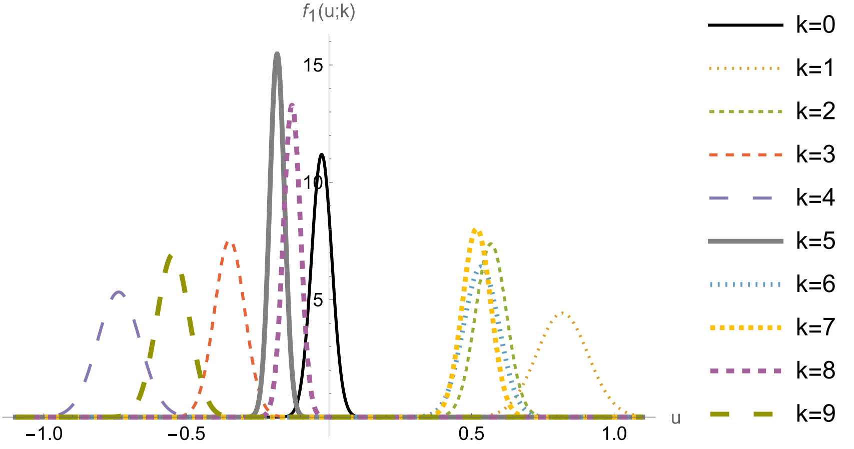

Fig. 3. PDF of the control stochastic process to the random control problem Eq. (6) at the steps $k = 0, 1, \dots, 9$. Example 1.

Table 1
Comparison of the expectation of the control, $\mathbb{E}[u(k)(\omega)]$, for the random control problem Eq. (6) with data Eqs. (23) and (24), and the corresponding associate control, $u(k)$, for the deterministic control problem Eq. (2) with the data Eq. (23) and initial condition $\mathbb{E}[\vec{x}_0(\omega)] = [3, 2]^\top$. Example 1.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$u(k)$	-0.0255777	0.818269	0.568248	-0.348518	-0.737455
$\mathbb{E}[u(k)(\omega)]$	-0.025577	0.818269	0.568248	-0.348518	-0.737455
$\lvert u(k) - \mathbb{E}[u(k)(\omega)]\rvert$	5.2×10^{-17}	2.3×10^{-8}	1.6×10^{-14}	7.8×10^{-16}	1.8×10^{-11}
	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$
$u(k)$	-0.181843	0.534287	0.517829	-0.129702	-0.546761
$\mathbb{E}[u(k)(\omega)]$	-0.181843	0.534287	0.517829	-0.129702	-0.546761
$\lvert u(k) - \mathbb{E}[u(k)(\omega)]\rvert$	4.4×10^{-16}	8.9×10^{-16}	1.0×10^{-15}	3.1×10^{-16}	1.2×10^{-15}

control in the final step is random despite the final state is not. It is key information provided by our approach that is consistent with the own random nature of the control system.

To deeper interpret the results shown in Fig. 3, we have compared the expectation of the stochastic control (that can be easily calculated thanks to expression Eq. (18)) with the value of the corresponding associate deterministic control problem (consisting of taking as initial state $\mathbb{E}[\vec{x}_0(\omega)] = [\mathbb{E}[x_{01}(\omega)], \mathbb{E}[x_{02}(\omega)]]^\top = [3, 2]^\top$). This comparison has been made at each step $k = 0, 1, \dots, 9$. The results are collected in Table 1. These figures show that both approaches fully agree in the average (expectation) sense. It is worth pointing out that our random approach enables us to also compute the main probability characteristics of the stochastic control, such as the variance, asymmetry, kurtosis, etc., at each $k = 0, 1, \dots, 9$, since these statistics can be straightforwardly calculated from its 1-PDF represented in Fig. 3.

We conclude this example by comparing the results obtained for both the solution and the control, utilizing the method proposed in this paper and via Monte Carlo simulations. As it can be observed from Fig. 2, the variability of the systems reduces as the steps increase from $k = 0$ to $k = 10$. Therefore, to perform this comparison fairly, we have chosen the intermediate step $k = 3$ where the system still has variability (see Fig. 2), so it is expected that differences between both approaches can be better highlighted. Fig. 4 shows the 1-PDF of the solution

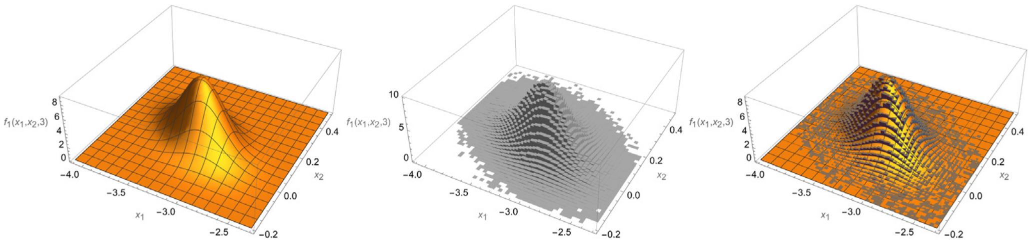

Fig. 4. 1-PDF of the solution stochastic process to the random control problem Eq. (6) at the step $k = 3$ using the newly proposed method (left), Monte Carlo with $M = 10^6$ simulations (center) and comparison between both methods (right). Example 1.

Table 2
Mean absolute error between the approximations of the 1-PDF of the solution stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the rectangular domain $\mathcal{D}_k$ divided into 50×50 cells for the steps $k = 3, 4, 5$. Example 1.

Error (solution)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{D}_k$
$k = 3$	0.183239	0.0581982	0.0189896	0.00594794	$[-4.1, -2.3] \times [-0.2, 0.5]$
$k = 4$	0.0959597	0.0321949	0.0108326	0.00574444	$[-2.1, -1.0] \times [2.0, 3.6]$
$k = 5$	0.235663	0.0846163	0.0257995	0.00902539	$[1.2, 2.1] \times [1.2, 2.1]$

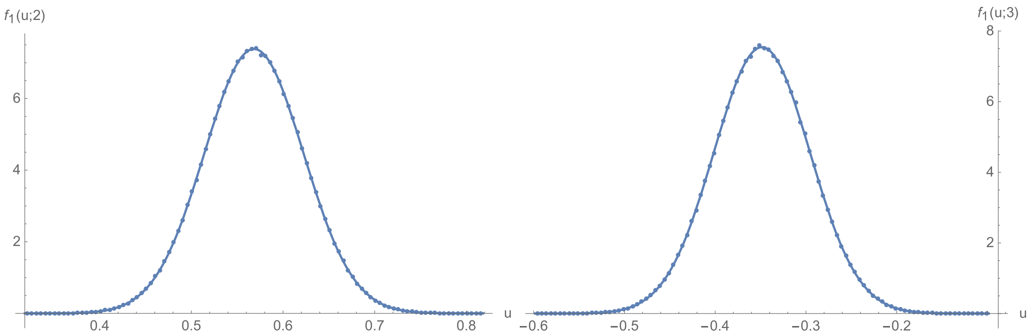

Fig. 5. PDF of the control stochastic process to the random control problem Eq. (6) at the steps $k = 2$ (left) and $k = 3$ (right) using the newly proposed method (solid curve) and Monte Carlo with $M = 10^6$ simulations (points). Example 1.

to the random control problem Eq. (6) at $k = 3$ using the corresponding expression given in Remark 4 for $\vec{x}_0$ deterministic (left), Monte Carlo with $M = 10^6$ simulations (center) and overlapping these two plots. To better compare these two approaches, in Table 2, we have calculated the mean absolute error between the approximations given by these two methods on the central point located at every cell of a mesh built on a rectangular domain $\mathcal{D}_k$ divided into 50×50 cells for the steps $k = 3, 4, 5$.

For the sake of completeness and consistency, what has been done before at step $k = 3$, in Fig. 5 and Table 3 we have performed an analogous comparison for the random control at the steps $k = 2$ and $k = 3$ to observe better its dynamics. Specifically, in Fig. 5 we compare the results obtained using the newly proposed method against Monte Carlo with $M = 10^6$

Table 3
Mean absolute error between the approximations of the PDF of the control stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the interval $\mathcal{I}_k$ divided into 100 pieces for the steps $k = 2, 3, 4$. Example 1.

Error (control)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{I}_k$
$k = 2$	0.107295	0.0402103	0.0101554	0.00404169	[0.32,0.82]
$k = 3$	0.100178	0.0422447	0.0119579	0.00353224	$[-0.6, -0.1]$
$k = 4$	0.111362	0.0301374	0.0104998	0.00334515	$[-1.04, -0.44]$

simulations. In Table 3, we collect the mean absolute error between both approximations. We can observe that both approximations agree at the expense of using many simulations via Monte Carlo.

Example 2. Let us consider problem Eq. (6), where the following data are deterministic

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \vec{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad K = 10, \tag{25}$$

while the final state is assumed to be a random vector, $\vec{x}_1(\omega) = [x_{11}(\omega), x_{12}(\omega)]^\top$, whose components are independent random variables with shifted Beta distribution and a Triangular distribution according to:

$$x_{11}(\omega) = v_{11}(\omega) - \frac{4}{3} \quad \text{where} \quad v_{11}(\omega) \sim \mathrm{Be}(4; 2) \tag{26}$$

and

$$x_{12}(\omega) \sim \mathrm{T}([14/9, 23/9]; 17/9). \tag{27}$$

Notice that

$$\operatorname{rank}(\mathcal{U}_2) = \operatorname{rank}\left(\mathbf{A}\vec{b}|\vec{b}\right) = \operatorname{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2,$$

so Kalman's controllability condition is fulfilled, and problem Eq. (6) is exactly controllable for $K \geq 2$.

Fig. 6 it is shown the 1-PDF of the solution to the random control problem Eq. (6) at the steps $k = 3$ (left) and $k = 10$ (right). On the surface with have highlighted the confidence regions at the confidence levels 0.5 (blue) and 0.9 (red). Looking at the vertical and domain scales, we can observe that the 1-PDF becomes platykurtic (uncertainty increases) as expected since the initial state $\vec{x}_0$ is deterministic while the final state $\vec{x}_1(\omega)$ is a random vector, so having variability. This behavior is in agreement with the results obtained in the portrait phase shown in Fig. 7. To complete the probabilistic analysis, in Fig. 8, we have graphically represented the evolution of the PDF of the control $u(k, \omega)$ for $k = 0, 1, \ldots, 9$.

Finally, as it has also been done in Example 1, in Table 4, we compare the expectation of the stochastic control against the corresponding values of its deterministic counterpart consisting of taking as final state $\mathbb{E}[\vec{x}_1(\omega)] = [\mathbb{E}[x_{11}(\omega)], \mathbb{E}[x_{12}(\omega)]]^\top = [-2/3, 2]^\top$). From the values collected in Table 4 we can clearly observe that both approaches show full agreement in the average or expectation sense. It is worth pointing out that our random approach also provides a full statistical description of the dynamics of the random control via the PDF represented in Fig. 8.

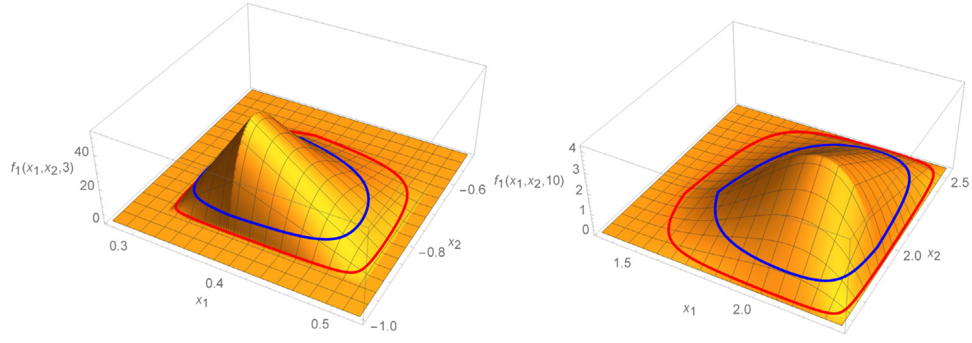

Fig. 6. 1-PDF of the solution stochastic process to the random control problem Eq. (6) at the steps $k = 3$ (left) and $k = 10$ (right). On the PDF's surface, we have highlighted confidence regions at different confidence levels $1 - \alpha = 0.5$ (blue) and $1 - \alpha = 0.9$ (red). Example 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

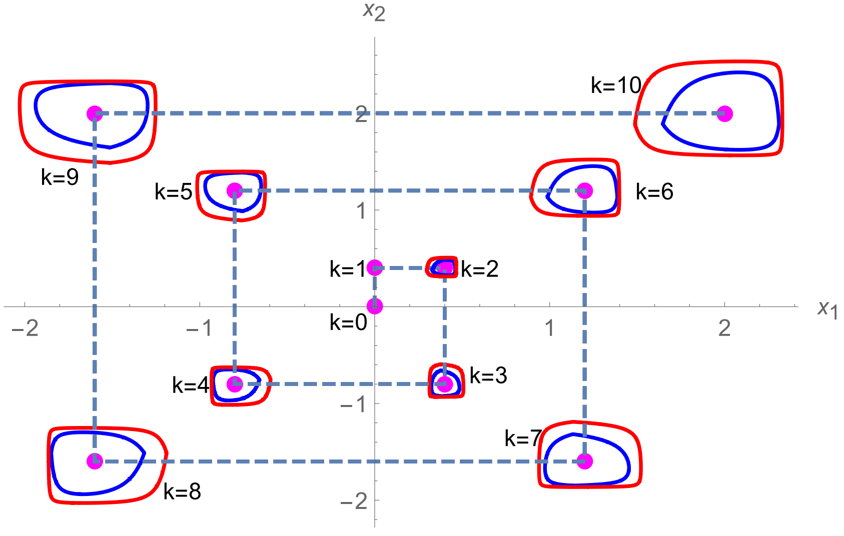

Fig. 7. Portrait phase for the random control problem Eq. (6). The dashed spiral line represents the expectation of the solution. We have also plotted two confidence regions at the confidence levels $1 - \alpha = 0.5$ (blue) and $1 - \alpha = 0.9$ (red) and at the steps $k = 0, 1, \ldots, 10$. Example 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Analogously, as it has been done in Example 1, we have performed a comparative analysis of the results obtained utilizing the proposed method against Monte Carlo with $M = 10^6$ simulations. The corresponding results are presented in Figs. 9 and 10, and Tables 5 and 6. We can observe that both approaches agree.

Example 3. In this last example, we illustrate the theoretical results previously established within the setting of Macroeconomics utilizing the celebrated multiplier-accelerator model proposed by Samuelson, [43], and that has been extensively considered in Economics. Let $Y(k)$, $C(k)$, $I(k)$, and $g(k)$ be the national income, the consumption expenditure, the private investment, and the government expenditure at period k, respectively. Then, according to

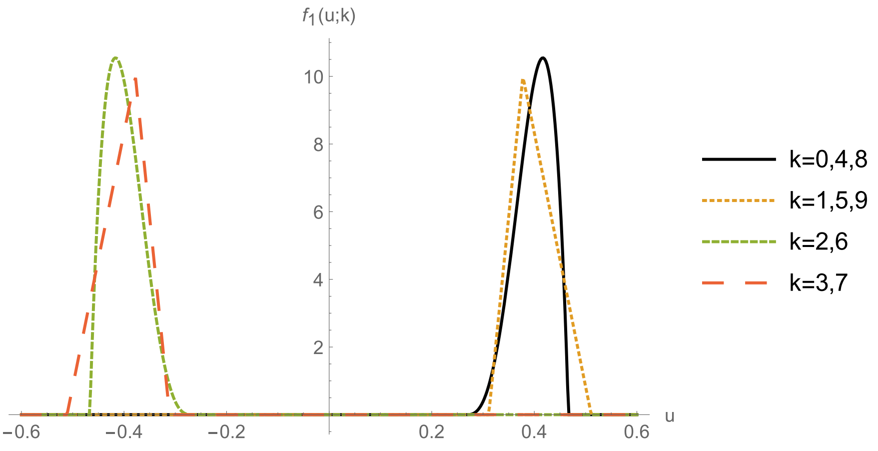

Fig. 8. PDF of the control stochastic process to the random control problem Eq. (6) at the steps $k = 0, 1, \ldots, 9$. Example 2.

Table 4
Comparison of the expectation of the control, $\mathbb{E}[u(k)(\omega)]$, for the random control problem Eq. (6) with data Eqs. (25)–(27), and the corresponding associate control, $u(k)$, for the deterministic control problem Eq. (2) with the data Eq. (25) and final state $\mathbb{E}[\vec{x}_1(\omega)] = [-2/3, 2]^\top$. Example 2.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$u(k)$	$2/5$	$2/5$	$-2/5$	$-2/5$	$2/5$
$\mathbb{E}[u(k)(\omega)]$	0.40	0.40	−0.40	−0.40	0.40
$\|u(k) - \mathbb{E}[u(k)(\omega)]\|$	1.5×10^{-8}	1.3×10^{-8}	1.5×10^{-8}	1.3×10^{-8}	1.5×10^{-8}
	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$u(k)$	$2/5$	$-2/5$	$-2/5$	$2/5$	$2/5$
$\mathbb{E}[u(k)(\omega)]$	0.40	−0.40	−0.40	0.40	0.4
$\|u(k) - \mathbb{E}[u(k)(\omega)]\|$	1.3×10^{-8}	1.5×10^{-8}	1.3×10^{-8}	1.5×10^{-8}	1.3×10^{-8}

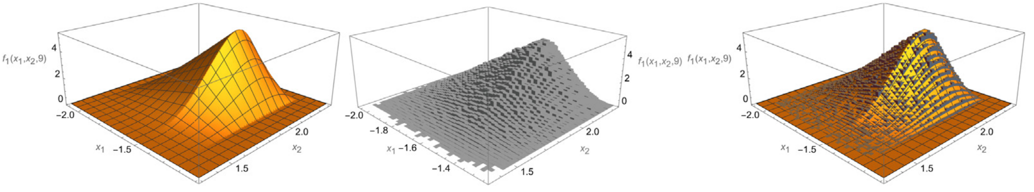

Fig. 9. 1-PDF of the solution stochastic process to the random control problem Eq. (6) at the step $k = 9$ using the newly proposed method (left), Monte Carlo with $M = 10^6$ simulations (center) and comparison between both methods (right). Example 2.

Table 5
Mean absolute error between the approximations of the 1-PDF of the solution stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the rectangular domain $\mathcal{D}_k$ divided into 50×50 cells for the steps $k = 7, 8, 9$. Example 2.

Error (solution)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{D}_k$
$k=7$	0.54694	0.168004	0.0551315	0.0178488	$[0.9, 1.6] \times [-1.9, -1.1]$
$k=8$	0.347771	0.117053	0.0365718	0.0117636	$[-1.9, -1.0] \times [-2.1, -1.2]$
$k=9$	0.244787	0.0804762	0.0255711	0.00874402	$[-2.1, -1.1] \times [1.3, 2.4]$

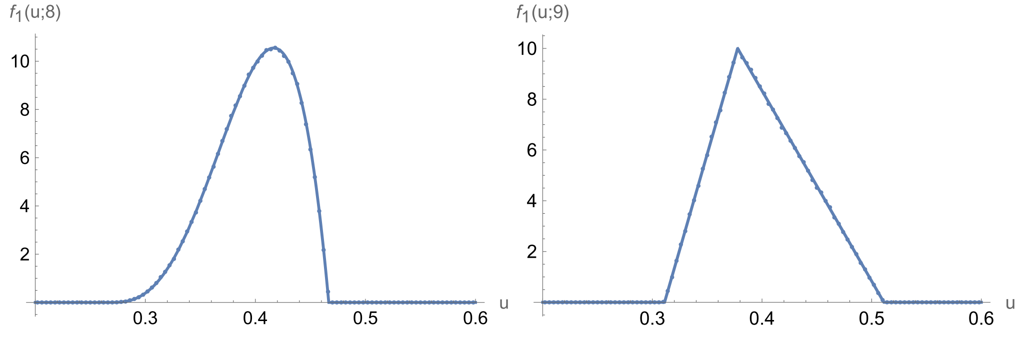

Fig. 10. PDF of the control stochastic process to the random control problem Eq. (6) at the steps $k = 8$ (left) and $k = 9$ (right) using the newly proposed method (solid curve) and Monte Carlo with $M = 10^6$ simulations (points). Example 2.

Table 6
Mean absolute error between the approximations of the PDF of the control stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the interval $\mathcal{I}_k$ divided into 100 pieces for the steps $k = 6, 7, 8$. Example 2.

Error (control)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{I}_k$
$k = 6$	0.120381	0.0458477	0.0138592	0.00515293	$[-0.6, -0.2]$
$k = 7$	0.1405	0.0340833	0.0111517	0.00530117	$[-0.6, -0.2]$
$k = 8$	0.135596	0.0406236	0.0123871	0.00464571	[0.2,0.6]

Samuelson's model

$$Y(k) = C(k) + I(k) + g(k), \tag{28}$$

where

$$\begin{aligned} C(k) &= \alpha Y(k-1), \\ I(k) &= \beta(C(k) - C(k-1)) = \alpha\beta Y(k-1) - \alpha\beta Y(k-2), \\ g(k) &= 1, \end{aligned} \tag{29}$$

being $0 < \alpha < 1$ the marginal propensity to consume and $\beta > 0$ stands for the accelerator coefficient, that is, the investment acceleration factor in terms of observed consumption change between consecutive periods, Kevin [44]. In this contribution, we assume that government expenditure is a control variable to make economic policy. Specifically, we will assume that it is modeled via a control variable, say u, which is introduced at period k with regard to its previous period $k-1$, i.e., $g(k) = u(k-1)$. Substituting the Hansen assumptions Eq. (29) with $g(k) = u(k-1)$, the Samuelson's model Eq. (28) can be written as a second-order difference equation, namely,

$$Y(k+1) - \alpha(1+\beta)Y(k) + \alpha\beta Y(k-1) = u(k), \quad k = 1, 2, \ldots, K. \tag{30}$$

As this model is based on a second-order recurrence, two values for both the initial and the final states are required to formulate it as a discrete control model. As a consequence, it will be a controllable problem if a given national income final state $\{Y(K), Y(K+1)\}$ can be reached from every initial state of the national income $\{Y(0), Y(1)\}$ in a finite number of steps, K. To take advantage of the theoretical results obtained in Section 4, in our subsequent analysis, we first perform the following change of variable: $\vec{x}(k) = (x_1(k), x_2(k))^\top := (Y(k-1), Y(k))^\top$, then the above equivalent shifted model can be written in the form Eq. (1)

$$\vec{x}(k+1) = \mathbf{A}\vec{x}(k) + \vec{b}u(k) = \begin{bmatrix} 0 & 1 \\ -\alpha\beta & \alpha(1+\beta) \end{bmatrix}\vec{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k), \quad k = 1, 2, \dots, K. \tag{31}$$

Note that, according to the above change of variable, the solution of Samuelson's model Eq. (30) is given by $x_2(k) = Y(k)$, To formulate the controllable problem according to Eq. (31), we fix K and assume that both, the initial state $\vec{x}(1) = (x_1(1), x_2(1)) = (Y(0), Y(1))$ (that corresponds to $k = 1$) and the final target $\vec{x}(K+1) = (x_1(K+1), x_2(K+1)) = (Y(K), Y(K+1))$ (that corresponds to $k = K$ when iterating the recurrence Eq. (31)), are known.

To carry out stochastic simulations on the randomized Samuelson's model, we will take $K = 10$, so, it shall be assumed that the probability distributions of the final national incomes $Y(10) = Y(10, \omega)$ and $Y(11) = Y(11, \omega)$ are known, as well as of the initial national incomes $Y(0) = Y(0, \omega)$ and $Y(1) = Y(1, \omega)$. Specifically, hereinafter we will assume the following distributions for the model parameters:

- $\vec{x}(1, \omega) = (Y(0, \omega), Y(1, \omega))$ is distributed according to a multivariate Uniform on the rectangle $[0.995, 1.005] \times [1.005, 1.015]$, i.e., $\vec{x}(1, \omega) \sim \mathrm{U}([0.995, 1.005] \times [1.005, 1.015])$.
- $\vec{x}(11, \omega) = (Y(10, \omega), Y(11, \omega))$ is distributed according to a multivariate Uniform on the rectangle $[1.85, 1.95] \times [1.95, 2.05]$, i.e., $\vec{x}(11, \omega) \sim \mathrm{U}([1.85, 1.95] \times [1.95, 2.05])$.
- $\alpha(\omega)$ is distributed according to a Uniform random variable on the interval [0.93,0.97], i.e., $\alpha(\omega) \sim \mathrm{U}([0.93, 0.97])$.
- $\beta(\omega)$ is distributed according to a Gaussian random variable with mean 1.6 and standard deviation 0.1 truncated on the interval [1.3,1.9], i.e., $\beta(\omega) \sim \mathrm{N}_{[1.3,1.9]}(1.6; 0.1)$.

As $\alpha(\omega)$ and $\beta(\omega)$ are absolutely continuous random variables, according to Proposition 1, the randomized Samuelson's model is controllable. Then, by applying the results derived in Section 4, we can calculate the 1-PDF, $f_1(x_1, x_2, k)$, of extended random control system Eq. (31). As $x_2(k, \omega) := Y(k, \omega)$, the 1-PDF, $f_1(y, k)$, of the Samuelson model Eq. (30) is obtained by marginalizing $f_1(x_1, x_2, k)$ with respect to x_1. In Fig. 11, we show the 1-PDF, $f_1(y, k)$, of the solution of the randomized Samuelson's model Eq. (30) at the steps $k = 0, 1, \dots, 10, 11$. It is interesting to observe in Fig. 11 how to change the shape of the PDFs from $k = 0$ and $k = 1$ (that correspond to Uniform distributions on the intervals [0.995,1.005] and [1.005,1.015], respectively) to $k = 10$ and $k = 11$ (that correspond to Uniform distributions on the intervals [1.85,1.95] and [1.95,2.05], respectively). In Fig. 12, we show, with further detail, the PDF of the solution at step $k = 9$, $f_1(y, 9)$, previous at the two final given steps $k = 10, 11$. We have included the approximation of $f_1(y, 9)$ via Monte Carlo simulations in this graphical representation. To carry out these simulations, we have considered the following range for y: $1.644 < y < 1.887$, and this interval has been divided into 100 sub-intervals. Then we generate an approximation in the middle point within each sub-interval averaging 10^6 simulations via Monte Carlo. We can see that both representations agree. In Fig. 13, we show the PDF, $f_1(u, k)$, of the stochastic control at the steps $k = 1, \dots, 10$. For consistency with Examples 1 and 2, in Table 7, we compare the expectation of the stochastic control with the values corresponding to the deterministic formulation of the control problem whose parameters are obtained as the expecta-

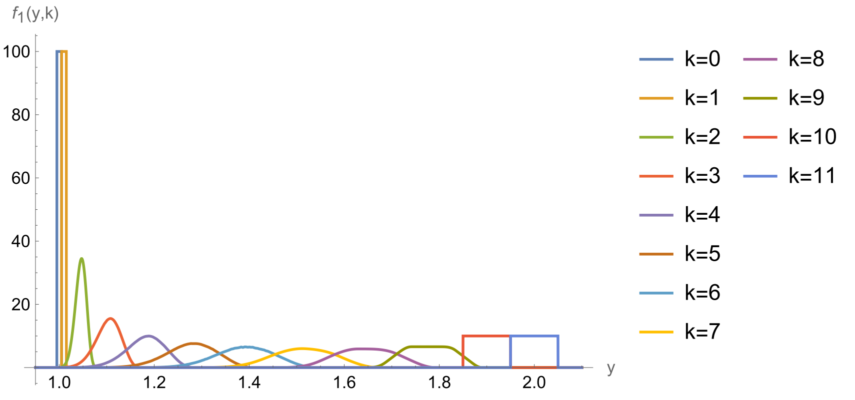

Fig. 11. 1-PDF of the solution stochastic process to the random control problem Eq. (30) at the steps $k = 0, 1, \dots, 11$. Example 3.

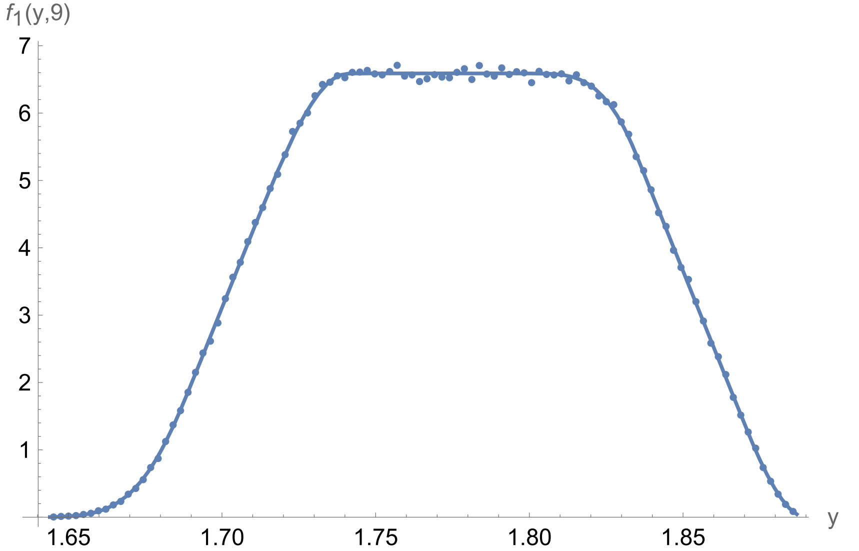

Fig. 12. 1-PDF of the solution stochastic process to the random control problem Eq. (30) at step $k = 9$ (solid line) and using 10^6 Monte Carlo simulations (dots). Example 3.

tion of each random model parameters: $\mathbb{E}[\vec{x}(1, \omega)] = [\mathbb{E}[Y(0, \omega)], \mathbb{E}[Y(1, \omega)]]^\top = [1, 1.01]^\top$, $\mathbb{E}[\vec{x}(11, \omega)] = [\mathbb{E}[Y(10, \omega)], \mathbb{E}[Y(11, \omega)]]^\top = [1.90, 2]^\top$, $\mathbb{E}[\alpha(\omega)] = 0.95$ and $\mathbb{E}[\beta(\omega)] = 1.6$. As in Examples 1 and 2, from the values collected in Table 7, we can observe that the results completely agree.

We finish this example showing a comparative analysis of our results for the approximations of the solution and the control against Monte Carlo. The results are shown both graphically (see Fig. 14) and numerically with the mean absolute errors at different steps (see Tables 8 and 9). We can observe from the plots and the error values that the results obtained via both approaches are consistent.

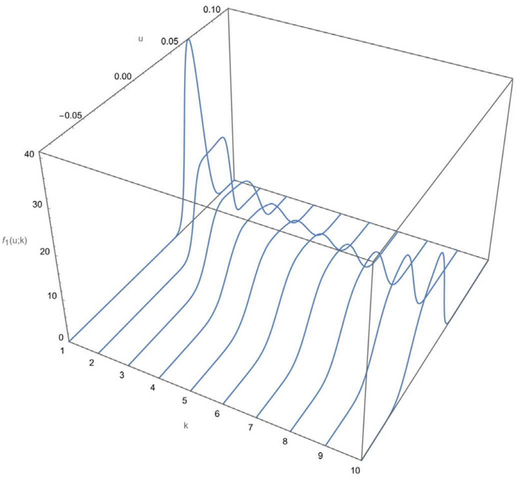

Fig. 13. 1-PDF of the control stochastic process to the random control problem Eq. (30) at the steps $k = 1, \ldots, 10$. Example 3.

Table 7
Comparison of the expectation of the control, $\mathbb{E}[u(k)(\omega)]$, for the random control problem Eq. (6) with data $\vec{x}(1,\omega) \sim \mathrm{U}([0.995, 1.005] \times [1.005, 1.015])$, $\vec{x}(11,\omega) \sim \mathrm{U}([1.85, 1.95] \times [1.95, 2.05])$, $\beta(\omega) \sim \mathrm{N}_{[1.3,1.9]}(1.6; 0.1)$ and $\beta(\omega) \sim \mathrm{N}_{[1.3,1.9]}(1.6; 0.1)$, and the corresponding associate control, $u(k)$, for the deterministic control problem Eq. (2) with the data $\mathbb{E}[\vec{x}(1,\omega)] = [\mathbb{E}[Y(0,\omega)], \mathbb{E}[Y(1,\omega)]]^\top = [1, 1.01]^\top$, $\mathbb{E}[\vec{x}(11,\omega)] = [\mathbb{E}[Y(10,\omega)], \mathbb{E}[Y(11,\omega)]]^\top = [1.90, 2]^\top$, $\mathbb{E}[\alpha(\omega)] = 0.95$ and $\mathbb{E}[\beta(\omega)] = 1.6$. Example 3.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$u(k)$	0.0713475	0.0565039	0.0448796	0.0357558	0.0285771
$\mathbb{E}[u(k)(\omega)]$	0.0701855	0.0558641	0.044837	0.0361572	0.0291058
$\lvert u(k) - \mathbb{E}[u(k)(\omega)]\rvert$	1.1×10^{-3}	6.4×10^{-4}	4.3×10^{-5}	4.0×10^{-4}	5.3×10^{-4}
	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
$u(k)$	0.0229142	0.0184349	0.0148815	0.0120543	0.00979774
$\mathbb{E}[u(k)(\omega)]$	0.0233346	0.018517	0.0145216	0.0111647	0.00836374
$\lvert u(k) - \mathbb{E}[u(k)(\omega)]\rvert$	4.2×10^{-4}	8.2×10^{-5}	3.6×10^{-4}	8.9×10^{-4}	1.5×10^{-3}

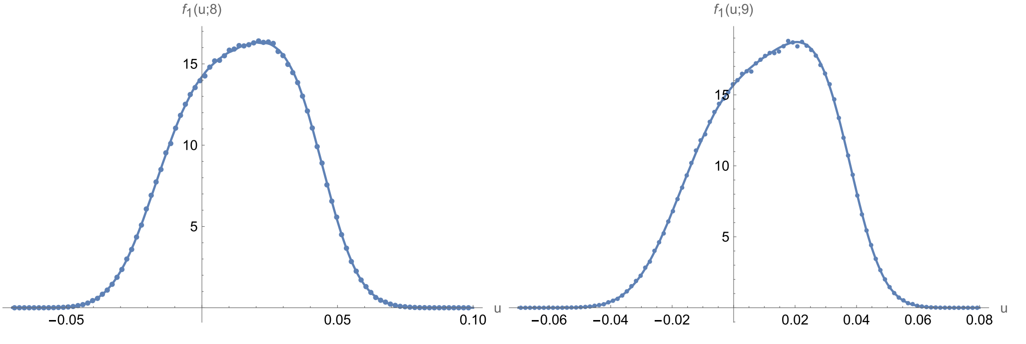

Fig. 14. PDF of the control stochastic process to the random control problem Eq. (6) at the steps $k = 8$ (left) and $k = 9$ (right) using the newly proposed method (solid curve) and Monte Carlo with $M = 10^6$ simulations (points). Example 3.

Table 8
Mean absolute error between the approximations of the 1-PDF of the solution stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the interval $\mathcal{I}_k$ divided into 100 pieces for steps $k = 6, 7, 8$. Example 3.

Error (solution)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{I}_k$
$k = 6$	0.227135	0.0654861	0.0249257	0.0152618	[1.190, 1.535]
$k = 7$	0.17502	0.0519701	0.0229236	0.00699532	[1.30,1.67]
$k = 8$	0.239276	0.0566746	0.0205628	0.00682887	[1.46,1.79]

Table 9
Mean absolute error between the approximations of the PDF of the control stochastic process to the random control problem Eq. (6) using the newly proposed method and Monte Carlo with M simulations on the central point located at every cell of a mesh built on the interval $\mathcal{I}_k$ divided into 100 pieces for the steps $k = 5, 6, 7$. Example 3.

Error (control)	$M = 10^4$	$M = 10^5$	$M = 10^6$	$M = 10^7$	$\mathcal{I}_k$
$k = 5$	0.399155	0.115524	0.0345608	0.0168062	$[-0.06, 0.12]$
$k = 6$	0.239157	0.0809467	0.0286726	0.00980201	$[-0.08, 0.12]$
$k = 7$	0.304388	0.0976728	0.0345355	0.0144451	$[-0.08, 0.11]$

6. Conclusions

In this paper, we have solved, from a probabilistic standpoint, full randomized first-order linear control models under very general assumptions. The term *solving* here means computing the first probability density function of the solution and of the control of the problem. We have seen that this is more advantageous than computing the first statistics, such as mean, variance, etc., as it is usually done. It has been shown that the so-called Random Variable Transformation method is an effective approach to obtaining semi-explicit representations (in terms of multidimensional integrals) of the aforementioned densities. Moreover, under mild conditions on the model parameters, these integral representations allow us to express the densities of the solution and control as expectations, which are very useful for computations using Monte Carlo simulations. A main advantage of the proposed approach is its wide range of applications with regard to the family of probability distributions that model parameters may have, including the general case where they might be statistically dependent. Although our study has focused on the linear case, we think that the ideas exhibited throughout our study can be useful to deal with more complex cases in forthcoming contributions as well as to open new avenues in the real-world applications of Control Theory with Uncertainties beyond the usual case where the stochasticity (noise) is of Gaussian type. Particularly, in our next step, we plan to extend the study performed in the paper to the case of discrete first-order control systems with delay. We will consider two different cases depending on the term of the equation where the control appears, first, when it acts on the unknown itself and, secondly, when the delay affects the control. In both scenarios, we will focus on computing explicit (or semi-explicit) formulas for the 1-PDF of the solution and of the control, which are stochastic processes. This analysis will be conducted under the general assumption that all model parameters and initial and final states are dependent random variables with a joint probability density.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 1

It is done using induction.

- If $k=1$, then by Eq. (1)

  $$\vec{x}(1)=\mathbf{A}\vec{x}(0)+\vec{b}\,u(0)\overset{\text{Eq. (2)}}{=}\mathbf{A}\vec{x}_0+\vec{b}\,u(0)=\mathbf{A}^1\vec{x}_0+\mathcal{U}_1\vec{u}_1.$$

- If $k=2$, then by Eq. (1)

  $$\begin{aligned}\vec{x}(2)&=\mathbf{A}\vec{x}(1)+\vec{b}\,u(1)\overset{\text{Step } k=1}{=}\mathbf{A}\Big(\mathbf{A}\vec{x}_0+\vec{b}\,u(0)\Big)+\vec{b}\,u(1)\\&=\mathbf{A}^2\vec{x}_0+\mathbf{A}\vec{b}\,u(0)+\vec{b}\,u(1)=\mathbf{A}^2\vec{x}_0+\Big[\mathbf{A}\vec{b}\,|\,\vec{b}\Big]\begin{bmatrix}u(0)\\u(1)\end{bmatrix}=\mathbf{A}^2\vec{x}_0+\mathcal{U}_2\vec{u}_2.\end{aligned}$$

- Let us assume, by the induction hypothesis, that this is true for k. Let us prove it for $k+1$

  $$\begin{aligned}\vec{x}(k+1)&=\mathbf{A}\vec{x}(k)+\vec{b}\,u(k)\\&\overset{\text{Eq. (3)}}{=}\mathbf{A}\big(\mathbf{A}^k\vec{x}_0+\mathcal{U}_k\vec{u}_k\big)+\vec{b}\,u(k)\\&=\mathbf{A}\left(\mathbf{A}^k\vec{x}_0+\Big[\mathbf{A}^{k-1}\vec{b}\,|\ldots|\,\mathbf{A}\vec{b}\,|\,\vec{b}\,\Big]\begin{bmatrix}u(0)\\u(1)\\\vdots\\u(k-1)\end{bmatrix}\right)+\vec{b}\,u(k)\\&=\mathbf{A}^{k+1}\vec{x}_0+\Big[\mathbf{A}^k\vec{b}\,|\ldots|\,\mathbf{A}^2\vec{b}\,|\,\mathbf{A}\vec{b}\,\Big]\begin{bmatrix}u(0)\\u(1)\\\vdots\\u(k-1)\end{bmatrix}+\vec{b}\,u(k)\\&=\mathbf{A}^{k+1}\vec{x}_0+\Big[\mathbf{A}^k\vec{b}\,|\ldots|\,\mathbf{A}^2\vec{b}\,|\,\mathbf{A}\vec{b}\,|\,\vec{b}\,\Big]\begin{bmatrix}u(0)\\u(1)\\\vdots\\u(k-1)\\u(k)\end{bmatrix}=A^{k+1}\vec{x}_0+\mathcal{U}_{k+1}\vec{u}_{k+1}.\end{aligned}$$

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