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ORIGINAL PAPER

On the spectrum of isomorphisms defined on the space of smooth functions which are flat at 0

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Abstract
In this note we study the spectrum and the Waelbroeck spectrum of the derivative operator composed with isomorphic multiplication operators defined in the space of smooth functions in [0, 1] which are flat at 0.

Keywords Spectrum · Volterra operator · Differentiation operator

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1 Introduction and preliminaries

1.1 Introduction

The spectrum of a continuous linear operator T defined on a locally convex space X is defined in an analogous way as in the case when X is a Banach space. Given $T \in \mathcal{L}(X)$, (here $\mathcal{L}(X)$ stands for the continuous linear operators on X), the resolvent of T, denoted by $\varrho(T)$, is defined as the subset of $\mathbb{C}$ formed by those λ such that $\lambda I - T$ admits a continuous linear inverse $(\lambda I - T)^{-1}$. For $\lambda \in \varrho(T)$ we denote, as usual, $R(\lambda, T) = (\lambda I - T)^{-1} \in L(X)$. When X is a Fréchet space, $\lambda I - T$ is an isomorphism if and only if $\lambda I - T$ is bijective. The spectrum of T is defined as $\sigma(T) := \mathbb{C}\backslash\varrho(T)$. The point spectrum of T is defined as $\{\lambda \in \mathbb{C} : T(x) = \lambda x \text{ for some } x \neq 0\}$. Due to the open mapping theorem, when X is a Fréchet space, if $\lambda \in \mathbb{C}\backslash\sigma_p(T)$, then $\lambda \in \varrho(T)$ if and only if $\lambda I - T$ is surjective. Contrary to what happens on the setting of Banach spaces, the spectrum of an operator defined on a Fréchet space could be empty, or unbounded (see [1–3, 7, 10, 12, 13]). Several authors consider the Waelbroeck spectrum of the operator, as a natural way in order to get holomorphy in the resolvent map. The Waelbroeck resolvent $\varrho^*(T)$ is defined as the set formed for those $\lambda \in \varrho(T)$ such that there is a neighbourhood V_λ of λ contained in $\varrho(T)$ such that $\{R(\lambda, T) : \lambda \in V_\lambda\}$ is an equicontinuous subset of $\mathcal{L}(X)$. The Waelbroeck spectrum $\sigma^*(T)$ of T is defined as $\mathbb{C}\backslash\varrho^*(T)$ (see [18]). From the definition it follows immediately

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$\overline{\sigma(T)} \subseteq \sigma^*(T)$. The inclusion can be strict, as it can be checked in [2, Remark 3.5 (vi)]. The example, stated without proof in [16, Example 2], is the Volterra operator in the space

$$C_0^\infty([0,1]) := \{f \in C^\infty([0,1]) : \ f^{(k)}(0) = 0 \ \text{ for all } k \in \mathbb{N}_0\}.$$

This space is endowed with its natural topology, which is generated by the norms $\|f\|_n := \sup\{|f^{(j)}(x)| : \ x \in [0,1], \ 0 \le j \le n\}$. These family of norms makes $C_0^\infty([0,1])$ a Fréchet nuclear space. More explicitly, [2, Remark 3.5 (vi)] can be stated as follows:

Proposition 1 *The following operators are surjective isomorphisms in* $C_0^\infty([0,1])$.

(a) *The derivative operator* $D : C_0^\infty[0,1] \to C_0^\infty[0,1]$, $f \mapsto f'$, *is an isomorphism which satisfies* $\sigma(D) = \sigma^*(D) = \emptyset$.
(b) *The inverse of* D *is the Volterra operator* $V : C_0^\infty([0,1]) \to C_0^\infty([0,1])$, $f \mapsto V(f)(x) := \int_0^x f(t)dt$, $x \in [0,1]$, *which satisfies* $\sigma(V) = \emptyset$, $\sigma^*(V) = \{0\}$.

The study of the spectrum of operators defined on Fréchet spaces or more general locally convex spaces has been an object of research in the last years, see e.g. [2, 3, 5, 6, 10, 12, 13, 15, 17].

Several of the aforementioned references are devoted to the study of the Cesàro operator in spaces of functions. Our main motivation is [3, 4], where Albanese, Bonet and Ricker showed that the Cesàro operator C defined on $C^\infty(\mathbb{R}_+)$ satisfies $\sigma(C) = \sigma_p(C) = \{1/n : \ n \in \mathbb{N}\}$ and $\sigma^*(C) = \overline{\sigma(C)}$. We study a class of operators which includes the Cesàro operator defined in the space $C_0^\infty([0,1])$, whose spectrum has been recently characterized by Albanese in [1].

1.2 Spectrum of operators on locally convex spaces

In this note we are concerned with spectra of isomorphisms on Fréchet spaces. In the next proposition we include first a basic result which compares spectra and Waelbroeck spectra of T and T^{-1} defined on a locally convex space X. It is a particular case of [3, Theorem 1.1], due to Albanese, Bonet and Ricker.

Proposition 2 *Let* X *be a locally convex space and* $T \in \mathcal{L}(X)$ *be an isomorphism.* $\sigma(T^{-1}) = \{\lambda^{-1} : \ \lambda \in \sigma(T)\}$ *and* $\sigma^*(T^{-1})\backslash\{0\} = \{\lambda^{-1} : \ \lambda \in \sigma^*(T)\backslash\{0\}\}$.

As a consequence of Proposition 2, if T is an isomorphism and $\lambda \neq 0$, then λ is an accumulation point in $\sigma^*(T)\backslash\{0\}$ if and only if λ^{-1} is an accumulation point of $\sigma^*(T^{-1})\backslash\{0\}$. For $\lambda = 0$ we see below that nothing can be asserted. When X is a Banach space and T is an isomorphism on X then 0 is neither in the (Waelbroeck) spectrum of T nor in that of T^{-1}. In the case of Fréchet spaces we see that 0 can appear in the Waelbroeck spectrum of an isomorphism and in that of its inverse, and that when it appears it can be both, an isolated point or an accumulation point. In the next example, we consider the space $\omega = \mathbb{C}^\mathbb{N}$ of sequences of complex numbers endowed with the product topology. The proof relies on the fact that, if X, Y are locally convex spaces, $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, and we consider the direct sum $T \oplus S \in \mathcal{L}(X \oplus Y)$, then $\sigma_p(T \oplus S) = \sigma_p(T) \cup \sigma_p(S)$, $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ and $\sigma^*(T \oplus S) = \sigma^*(T) \cup \sigma^*(S)$.

Example 3 Let $T : \omega \to \omega$, $(x_n) \mapsto (nx_n)$. Then T is an isomorphism, $T^{-1} : \omega \to \omega$, $(x_n) \mapsto (\frac{1}{n}x_n)$, $\sigma(T) = \sigma_p(T) = \sigma^*(T) = \mathbb{N}$ and $\sigma(T^{-1}) = \sigma_p(T^{-1}) = \{\frac{1}{n} : \ n \in \mathbb{N}\}$ and $\sigma^*(T^{-1}) = \overline{\sigma(T^{-1})} = \sigma(T^{-1}) \cup \{0\}$.

The operators D and V defined on $C_0^\infty([0,1])$ are the same as in Proposition 1.

(a) $S := T \oplus T^{-1} \in \mathcal{L}(\omega \oplus \omega)$ satisfies $\sigma(S) = \sigma(S^{-1}) = \sigma_p(S) = \sigma_p(S^{-1}) = \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\sigma^*(S) = \sigma^*(S^{-1}) = \overline{\sigma(S)} = \sigma(S) \cup \{0\}$.
(b) $S := D \oplus V \in \mathcal{L}(C_0^\infty([0,1]) \oplus C_0^\infty([0,1]))$ satisfies $\sigma(S) = \sigma(S^{-1}) = \emptyset$ and $\sigma^*(S) = \sigma^*(S^{-1}) = \{0\}$.
(c) $S := D \oplus T \in \mathcal{L}(C_0^\infty([0,1]) \oplus \omega)$ satisfies $\sigma(S) = \sigma^*(S) = \mathbb{N}$, $\sigma(S^{-1}) = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\sigma^*(S^{-1}) = \overline{\sigma(S^{-1})} = \sigma(S^{-1}) \cup \{0\}$.
(d) $S := V \oplus T \in \mathcal{L}(C_0^\infty([0,1]) \oplus \omega)$ satisfies $\sigma(S) = \mathbb{N}$, $\sigma^*(S) = \mathbb{N}_0$, $\sigma(S^{-1}) = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\sigma^*(S^{-1}) = \overline{\sigma(S^{-1})} = \sigma(S^{-1}) \cup \{0\}$.

The next result is stated for Banach algebras in [11, Exercise 7.3.7]

Proposition 4 *Let X be a locally convex space and let $A, B \in \mathcal{L}(X)$. Then $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.*

Proof For $\lambda \in \varrho(AB)\backslash\{0\}$, set $T := \lambda^{-1}I + \lambda^{-1}B(\lambda I - AB)^{-1}A \in \mathcal{L}(X)$. A direct computation shows $T = (\lambda I - BA)^{-1}$. □

Proposition 5 *Let X be a locally convex space and let $A, B \in \mathcal{L}(X)$, B being an isomorphism. Then $\sigma_p(AB) = \sigma_p(BA)$, $\sigma(AB) = \sigma(BA)$ and $\sigma^*(AB) = \sigma * (BA)$.*

Proof If $\lambda \in \sigma_p(AB)$ and $x \in X\backslash\{0\}$ satisfies $ABx = \lambda x$, then $BA(Bx) = \lambda Bx$ and $Bx \neq 0$, and hence $\lambda \in \sigma_p(BA)$. Conversely, if $\lambda \in \sigma_p(BA)$ and $x \in X\backslash\{0\}$ satisfies $BAx = \lambda x$, then there is $y \in X$ such that $By = x$. Then $BABy = \lambda By$, and the injectivity of B yields $ABy = \lambda y$, and consequently $\lambda \in \sigma_p(AB)$.

Let assume now $\lambda \in \varrho(AB)$. We set $T_\lambda := B(\lambda I - AB)^{-1}B^{-1}$. It can be checked that $T_\lambda(\lambda I - BA) = (\lambda I - BA)T_\lambda = I$. Hence $\sigma(BA) \subseteq \sigma(AB)$. Conversely, if $\lambda \in \varrho(BA)$, one can check that $Q_\lambda := B^{-1}(\lambda I - BA)^{-1}B$ is the inverse of $(\lambda I - AB)$. Thus we have $\sigma(AB) = \sigma(BA)$. Moreover, for any compact set $K \subseteq \varrho(AB)$ we have

$$\{(\lambda I - BA)^{-1} : \lambda \in K\} = \{B(\lambda I - AB)^{-1}B^{-1} : \lambda \in K\}.$$

We conclude that $\{(\lambda I - BA)^{-1} : \lambda \in K\}$ is equicontinuous if and only if $\{(\lambda I - AB)^{-1} : \lambda \in K\}$ is so, which implies $\sigma^*(AB) = \sigma^*(BA)$. □

1.3 Representation of $C_0^\infty([0,1])$

It is well known that the space $C_0^\infty([0,1])$ is isomorphic to the space s of rapidly decreasing sequences. Bargetz has obtained in [9] an explicit isomorphism, which is used in [8] to obtain explicit representations as sequence spaces of important spaces of smooth functions appearing in functional analysis. We study in this note a wide class of isomorphisms defined on this space containing the differentiation operator, the Volterra operator and also the Cesàro operator. To do this, we need a representation of $C_0^\infty([0,1])$, as the one sided Schwartz space of rapidly decreasing smooth functions $S(\mathbb{R}^+)$. There is a natural representation for the one unit translate of this space

$$S([1,\infty)) := \{f \in C^\infty([1,\infty)) : \lim_{x\to\infty} x^n f^{(j)}(x) = 0 \text{ for all } j, n \in \mathbb{N}_0\}.$$

To get such representation, we need the well known Faà di Bruno formula, which we state below. Let $x \in \mathbb{R}$: if g is C^j, i.e f admits continuous derivatives up to order j, at x and f is C^j at $f(x)$ then

$$(f \circ g)^{(j)}(x) = \sum_{i=1}^{j} f^{(i)}(g(x)) B_{j,i}(g'(x), g''(x), \ldots, g^{(j-i+1)}(x)),$$

where $B_{j,i}$ are the Bell polynomials

$$B_{j,i}(x_1, x_2, \ldots, x_{j-i+1}) = \sum \frac{j!}{i_1! i_2! \cdots i_{j-i+1}!} \left(\frac{x_1}{1!}\right)^{i_1} \cdots \left(\frac{x_{j-i+1}}{(j-i+1)!}\right)^{i_{j-i+1}}, \quad (1.1)$$

$i_1 + \cdots + i_{j-i+1} = i, i_1 + 2i_2 + \cdots + (j-i+1)i_{j-i+1} = j$.

Remark 6 (a) From (1.1), it follows that $|B_{j,i}(x_1, x_2, \ldots, x_{j-i+1})| \leq B_{j,i}(|y_1|, |y_2|, \ldots, |y_{j-i+1}|)$ whenever $|x_l| \leq y_l$, $1 \leq l \leq j-i+1$.

(b) Let $i \leq j$ and let $(f_l(x))_{l=i}^{j-i+1}$ functions defined on $(0, 1]$ such that, there exists $t(i) \in \mathbb{R}$ such that $|f_l(x)| \leq x^{t(i)}$ for each $x \in (0, 1]$ and for each $1 \leq l \leq j-i+1$. Then there exists $M > 0$, $t \in \mathbb{R}$ such that $|B_{j,i}(f_1(x), f_2(x), \ldots, f_{j-i+1}(x))| \leq Mx^t$.

Proposition 7 *$C_0^\infty([0, 1]) = \{f \in C^\infty([0, 1]) : f^{(j)}(x) = o(x^n)$ as x approaches to 0 for all $j, n \in \mathbb{N}_0\}$.*

Proof Let $f \in C_0^\infty([0, 1])$. Without loss of generality, we assume f to be real valued. For any $x \in (0, 1]$, the mean value theorem implies $|f(x)| = |f'(t)|x$ for some $t \in (0, x)$. A reiteration of the argument produces $|f(x)| \leq \sup_{t\in[0,1]} |f^{(n)}(t)|x^n$ for all $n \in \mathbb{N}_0$. The condition $f^{(j)}(x) = o(x^n)$ as x approaches to 0 follows from the fact that $f^{(j)} \in C_0^\infty([0, 1])$ for all $j \in \mathbb{N}$. The other inclusion is trivial. □

From Proposition 7 and Leibnitz's formula, it follows immediately the following corollary.

Corollary 8 *Let $f \in C_0^\infty([0, 1])$ and $t \in \mathbb{R}$. Then the function $g(x) := x^t f$, for $x \in (0, 1]$, and $g(0) = 0$, belongs to $C_0^\infty([0, 1])$.*

Theorem 9 *The map $T : C_0^\infty([0, 1]) \to S([1, \infty))$, $f \mapsto T(f)$, defined as $T(f)(x) = \tilde{f}(x) := f(1/x)$, $x \in [1, \infty)$, is an isomorphism.*

Proof For $n \in \mathbb{N}$ and $x \geq 1$, by the Faà di Bruno Fórmula we have

$$x^n \tilde{f}^{(j)}(x) = x^n \sum_{i=1}^{j} f^{(i)}(1/x) B_{j,i}(-x^{-2}, 2x^{-3}, \ldots, (-1)^{j-i+1}(j-i+1)!x^{-(j-i+2)}). \quad (1.2)$$

We get $M > 0$, such that, for all $1 \leq i \leq j$

$$|x^n B_{j,i}(-x^{-2}, \ldots, (-1)^{j-i+1}(j-i+1)!x^{-(j-i+2)})| \leq Mx^n. \quad (1.3)$$

Let $t := 1/x \in (0, 1]$. From (1.2), (1.3) and Proposition 7 we get

$$\lim_{x\to\infty} x^n |\tilde{f}^{(j)}(x)| \leq M \lim_{t\to 0^+} \sum_{i=1}^{j} |f^{(i)}(t)| t^{-n} = 0. \quad (1.4)$$

Then T is well defined. Clearly T is injective. Since T is obviously pointwise–pointwise continuous, its graph is closed and hence T is continuous. We see that T is also surjective. Given $g \in S([1, \infty))$, we make an abuse of notation to define $\tilde{g}(x) = g(1/x)$, $x \in (0, 1]$, $\tilde{g}(0) = 0$. For all $j, n \in \mathbb{N}$, a completely symmetric argument to that used for getting (1.4), using Remark 6 (b), gives

$$\lim_{x\to 0^+} x^{-n} \tilde{g}^{(j)}(x) = 0.$$

From Proposition 7, it follows $\tilde{g} \in C_0^\infty([0, 1])$. We conclude from $T(\tilde{g}) = \tilde{\tilde{g}} = g$. □

2 Spectrum of multipliers on $C_0^\infty([0, 1])$

In view of Theorem 9, the results given in this section are closely related to [7, Proposition 3.3, Remark 3.5].

Definition 10 The space of multipliers of $C_0^\infty([0, 1])$ is defined as

$$\begin{aligned}\mathcal{M} := \{\omega : (0, 1] \to \mathbb{C} : \forall f \in C_0^\infty([0, 1])\\ \omega f \text{ can be extended to 0 as a function in } C_0^\infty([0, 1])\}.\end{aligned}$$

For $\omega \in \mathcal{M}$, we denote by $M_\omega : C_0^\infty([0, 1]) \to C_0^\infty([0, 1]),\ f \mapsto \omega f$, the corresponding multiplication operator.

Lemma 11 *A function $\omega : (0, 1] \to \mathbb{C}$ satisfies $\omega \in \mathcal{M}$ if and only if $\omega \in C^\infty((0, 1])$ and, for each $j \in \mathbb{N}_0$, there is $n \in \mathbb{N}$ such that $\omega^{(j)}(x) = o(x^{-n})$ as x approaches to 0.*

Proof By the definition, it is immediate to show that $\omega \in C^\infty(0, 1]$ whenever $\omega \in \mathcal{M}$. Theorem 9 yields that $\omega \in \mathcal{M}$ if and only if $\tilde{\omega}(x) := \omega(1/x)$ is a multiplier of $S([1, \infty))$. By the standard proof characterizing the multipliers of $S(\mathbb{R})$ (see [14]), this is equivalent to $\tilde{\omega} \in C^\infty([1, \infty))$ and for all $j \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\tilde{\omega}^{(j)}(x) = o(x^n)$ as x goes to ∞. Let $k \in \mathbb{N}$ such that $\tilde{\omega}'(x) = o(x^k)$ as x goes to ∞. This is equivalent to $\omega'(x) = o(x^{-k+2})$ as x approaches 0. Using Faà di Bruno formula one gets inductively the statement. □

Proposition 12 *Let $\omega \in \mathcal{M}$. The multiplication operator $M_\omega : C_0^\infty([0, 1]) \to C_0^\infty([0, 1])$ is an isomorphism if and only if $\omega(x) \neq 0$ for all $x \in (0, 1]$ and $1/\omega \in \mathcal{M}$.*

Proof First we observe that if $\omega(x_0) = 0$ for some $x_0 \in (0, 1]$, then $M_\omega f(x_0) = 0$ for all $f \in C_0^\infty([0, 1])$. Hence M_ω is not surjective (observe, for instance $f(x) = e^{-1/x} \in C_0^\infty([0, 1])$). If M_ω is an isomorphism, then the inverse T satisfies $M_\omega T(f) = \omega T f = f$, hence $T(f)(x) = (1/\omega(x)) f(x)$ for all $x \in (0, 1]$. This means $1/\omega \in \mathcal{M}$ and $T = M_{1/\omega}$. The converse is trivial. □

Corollary 13 *Let $\omega \in \mathcal{M}$. The multiplication operator $M_\omega : C_0^\infty([0, 1]) \to C_0^\infty([0, 1])$ is an isomorphism if and only if $\omega(x) \neq 0$ for all $x \in (0, 1]$ and there is $m \in \mathbb{N}$ such that $(1/\omega(x)) = o(x^{-m})$ as x approaches 0. In particular, for every $p \in \mathbb{R}$, if we define $\omega_p(x) := x^p$, then M_{ω_p} is an isomorphism.*

Proof By Lemma 11 and Proposition 12, we only need to show the sufficiency of the condition. Assume that $\omega(x) \neq 0$ and there is $m \in \mathbb{N}$ such that $((1/\omega(x)) = o(x^{-m})$ as x approaches 0. We need to show that $1/\omega \in \mathcal{M}$. Let $j \in \mathbb{N}$. By applying Faà di Bruno formula we get

$$((\omega(x))^{-1})^{(j)} = \sum_{i=1}^{j} (-1)^i i! (\omega(x))^{-i-1} B_{j,i}(\omega'(x), \ldots, \omega^{(j-i+1)}(x)).$$

Therefore we apply the hypothesis to get $k \in \mathbb{N}$ such that $(\omega(x)^{-1})^{(j)} = o(x^{-k})$ as x approaches 0. The conclusion follows from Lemma 11 and Remark 6. □

Corollary 14 *If $\omega \in \mathcal{M}$ then the spectrum of M_ω is*

$$\sigma(M_\omega) = \omega((0, 1]) \cup \{\lambda \notin \omega((0, 1]) :\ x^n/(\lambda - \omega) \text{ unbounded in } (0, 1]\ \forall n \in \mathbb{N}\}.$$

Proof It follows from Corollary 13 applied to the multiplier $\lambda - \omega$, for any $\lambda \in \mathbb{C}\backslash\omega((0,1])$. □

Lemma 15 *Let $(\omega_i)_{i\in I} \subseteq \mathcal{M}$. Assume that for each $j \in \mathbb{N}$ there are $M_j > 0$ and $t(j) \in \mathbb{R}$ such that $|\omega_i^{(j)}(x)| \leq M_j x^{t(j)}$ for every $x \in (0,1]$, $i \in I$. Then the set of multiplication operators $(M_{\omega_i})_{i\in I} \subseteq \mathcal{L}(C_0^\infty([0,1]))$ is equicontinuous.*

Proof By the Banach Steinhauss theorem, we only need to show that, for every $f \in C_0^\infty([0,1])$, the set $\{M_{\omega_i}(f) : i \in I\}$ is bounded in $C_0^\infty([0,1])$. This happens when, for every $k \in \mathbb{N}_0$, $\{(M_{\omega_i} f)^{(k)}(x) : x \in [0,1], i \in I\}$ is bounded in $\mathbb{C}$. Let fix $k \in \mathbb{N}_0$. Let $M = \max\{M_j : 0 \leq j \leq k\}$, $t = \min\{t(j) : 0 \leq j \leq k\}$. Then

$$\max_{x\in[0,1]} |(\omega_i f)^{(k}(x)| = \sup_{x\in(0,1]} \left| \sum_{j=0}^{k} \binom{k}{j} \omega_i^{(j)}(x) f^{(k-j)}(x) \right| \leq M \sup_{x\in(0,1]} \sum_{j=0}^{k} x^t |f^{(k-j)}(x)|.$$

We conclude since $x^t \in \mathcal{M}$ by Corollary 12. □

The following proposition is a direct consequence of Lemma 15.

Proposition 16 *Let $K \subseteq \mathbb{C}$ be compact. The multiplication operators $\{M_{h_\lambda} : \lambda \in K\}$ form an equicontinuous subset of $\mathcal{L}(C_0^\infty([0,1]))$ in the following cases:*

(a) $h_\lambda(x) := f(\lambda)x^\lambda$, $f \in C(K)$.
(b) $h_\lambda(x) := f(\lambda)e^{g(\lambda)x^t}$, *if* $t \geq 0$ *and* $f, g \in C(K)$.

Proposition 17 *Let $\omega \in \mathcal{M}$. The Waelbroeck spectrum of M_ω is $\sigma^*(M_\omega) = \overline{\sigma(M_\omega)} = \overline{\omega((0,1])}$.*

Proof We first observe $\{\lambda \notin \omega((0,1]) : x^n/(\lambda - \omega) \text{ unbounded in } (0,1]\ \forall n \in \mathbb{N}\} \subseteq \overline{\omega((0,1])}$. Actually, for any such λ there must exist a sequence $(x_n) \subseteq (0,1]$ convergent to 0 such that $\lim \omega(x_n) = \lambda$. Hence, by Corollary 14, we have $\overline{\sigma(M_\omega)} = \overline{\omega((0,1])}$. We only need to show $\sigma^*(M_\omega) \subseteq \overline{\omega((0,1])}$. Let $\lambda_0 \in \mathbb{C}\backslash\overline{\omega((0,1])}$. We choose $r > 0$ such that there exists $c > 0$ satisfying

$$|\lambda - \omega(x)| > c \quad \forall \lambda \in B(\lambda_0, r),\ x \in (0,1].$$

From Lemma 11, given $j \in \mathbb{N}_0$ there are $k \in \mathbb{N}$ and $C > 0$ such that $|\omega^{(i)}(x)| \leq Cx^{-k}$ for $1 \leq i \leq j$. From this, Faà di Bruno formula and Remark 6 we get $M > 0$, $t \in \mathbb{R}$ such that, for every $\lambda \in B(\lambda_0, r)$, $x \in (0,1]$ we have

$$|((\lambda - \omega(x))^{-1})^{(j)}| = \left| \sum_{i=1}^{j} i!(\lambda - \omega(x))^{-i-1} B_{j,i}(\omega'(x)\cdots\omega^{(j-i+1)}(x)) \right| \leq Mx^t.$$

Lemma 15 gives the equicontinuity of $\{M_{(\lambda-\omega(x))^{-1}} : \lambda \in B(\lambda_0, r)\}$. Hence $\lambda_0 \in \varrho^*(M_\omega)$. □

From Corollary 14 and Proposition 17 we get the following:

Example 18 For $p \in \mathbb{R}$, let $\omega_p(x) = x^p \in \mathcal{M}$.

(i) If $p > 0$ then $\sigma(M_{\omega_p}) = (0,1]$ and $\sigma^*(M_{\omega_p}) = [0,1]$
(ii) If $p < 0$ then $\sigma(M_{\omega_p}) = \sigma^*(M_{\omega_p}) = [1,\infty)$.

3 Spectrum of Cesàro type operators on $C_0^\infty([0,1])$

In this section we study the spectrum of operators of type VM_{ω_p} and also of type $M_{\omega_p}V$, where $\omega_p = x^p$, $p \in \mathbb{R}$, M_{ω_p} is the multiplication operator and V is the Volterra operator. These operators are isomorphisms in view of Proposition 1 and Corollary 13. The relevant case $M_{\omega_{-1}}V$ gives the Cesàro operator. By Proposition 2, the results will determine the spectrum of DM_{ω_p} and $M_{\omega_p}D$.

Lemma 19 *Let $g \in C_0^\infty([0,1])$ and $q < 0$. For $c \in \mathbb{C}$ with $Re(c) > 0$ let $h_c(x) := \int_0^x e^{c(x^q - t^q)} g(t)dt$, for $x \in [0,1]$. Then $h_c \in C_0^\infty([0,1])$ for every $c \in \mathbb{C}$ with $Re(c) > 0$. Moreover $\{h_c : c \in K\}$ is a bounded subset of $C_0^\infty([0,1])$ for any compact set $K \subseteq \{z \in \mathbb{C} : Re(z) > 0\}$.*

Proof Since $g \in C_0^\infty([0,1])$, from the hypothesis combined with Proposition 7 it follows that, for each $n \in \mathbb{N}$ there is $M_0^n > 0$ such that, for each $c \in \mathbb{C}$ and $x \in [0,1]$, we have

$$|h_c(x)| \leq \int_0^x |g(t)|dt \leq M_0^n x^n.$$

The derivative satisfies $h_c'(x) = g(x) + cqx^{q-1}h_c(x)$. Inductively we get, for each $j \geq 2$, $0 \leq i \leq j-1$, polynomials P_i^j of three variables, with $0 \leq i \leq j$, such that

$$h_c^{(j)}(x) = g^{(j-1)}(x) + \sum_{i=0}^{j-2} P_i^j(x^{-1}, x^q, c)g^{(i)}(x) + P_{j-1}^j(x^{-1}, x^q, c)h_c(x).$$

Hence we conclude $h_c(x) \in C_0^\infty([0,1])$ from Proposition 7. The continuity of each $P_i(x^{-1}, x^q, c)$ with respect to c yields that, for each $j, n \in \mathbb{N}_0$, there exist constants $M_j^n > 0$ such that, for each $c \in K$, $x \in [0,1]$

$$|h_c^{(j)}(x)| \leq M_j^n x^n \leq M_j^n.$$

□

Theorem 20 *For $p \in \mathbb{R}$, consider the operator $T_p := VM_{\omega_p} : C_0^\infty([0,1]) \to C_0^\infty([0,1])$, $f \mapsto \int_0^x t^p f(t)dt$, (or $T_p := M_{\omega_p}V : C_0^\infty([0,1]) \to C_0^\infty([0,1])$, $f \mapsto x^p \int_0^x f(t)dt$).*

(i) *If $p \geq -1$ then $\sigma(T_p) = \emptyset$ and $\sigma^*(T_p) = \{0\}$.*
(ii) *If $p < -1$ then $\sigma(T_p) = \sigma_p(T_p) = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ and $\sigma^*(T_p) = \overline{\sigma(T_p)}$.*

Proof We study first the point spectrum in all cases. By Proposition 5, we only have to prove the statement for $T_p = VM_{\omega_p}$. Since T_p is an isomorphism (as a composition of isomorphisms), then $0 \in \varrho(T_p)$ for each $p \in \mathbb{R}$. For $\lambda \neq 0$, $T_p(f) = \lambda f$ for some $f \neq 0$ if and only if

$$\int_0^x t^p f dt = \lambda f(x). \tag{3.1}$$

For $p \neq -1$, since $x^p f = \lambda f'$ for every is equivalent to (3.1) for every $f \in C_0^\infty([0,1])$, a solution of (3.1) is of the form $f(x) = h_{\lambda,p}(x) := e^{\frac{x^{p+1}}{\lambda(p+1)}}$, and for $p = -1$ we get $h_{\lambda,-1}(x) := \omega_{\frac{1}{\lambda}}(x) = x^{\frac{1}{\lambda}}$. Therefore $\lambda \in \sigma_p(T_p)$ if and only if $h_{\lambda,p} \in C_0^\infty([0,1])$. Hence we have $\sigma_p(T_p) = \emptyset$ for $p \geq -1$ and $\sigma_p(T_p) = \{\lambda : \mathrm{Re}((\lambda(1+p))^{-1}) < 0\} = \{\lambda : \mathrm{Re}(\lambda) > 0\}$ for $p < -1$.

For any $p \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \sigma_p(T_p)$, $\lambda \neq 0$, and $g \in C_0^\infty([0,1])$, any solution $f \in C^\infty([0,1])$ of the equation $T_p(f) - \lambda f = g$ is a solution of the differential equation

$$y' - \frac{x^p}{\lambda} y = -\frac{1}{\lambda} g'.$$

If we assume also $f(0) = 0$ then f has the form

$$f(x) := \frac{1}{\lambda} h_{\lambda,p}(x) \int_0^x (-g'(t)) h_{-\lambda,p}(t) dt, \tag{3.2}$$

In case $h_{\lambda,p} \in \mathcal{M}$, then $\lambda \in \varrho(T_p)$ and (3.2) gives the resolvent formula

$$R(\lambda, T_p)(g) := \frac{1}{\lambda} M_{h_{\lambda,p}} V M_{h_{-\lambda,p}}(-D)(g), \; g \in C_0^\infty([0,1]). \tag{3.3}$$

(a) Case $p \geq -1$. In this case $h_{\lambda,p} \in \mathcal{M}$ for $p \geq -1$ and $\lambda \neq 0$, due to Lemma 11. Hence in this case, $\sigma(T) = \sigma(T_p) = \emptyset$. We show now $\sigma^*(T_p) = \{0\}$. Let $K \subset \mathbb{C} \setminus \{0\}$ compact and $f \in C_0^\infty([0,1])$. The resolvent formula (3.3) applies here. We define

$$B_p(K, f) := \{R(\lambda, C) f : \lambda \in K\} = \left\{ \frac{1}{\lambda} M_{h_{\lambda,p}} V M_{h_{-\lambda,p}}(-D)(f) : \lambda \in K \right\}$$

Proposition 16 together with the fact that equicontinuous sets are equibounded, yields the boundedness of $B(K, f)$ for each $p \geq -1$. Hence $\mathbb{C} \setminus \{0\} \subseteq \varrho^*(T_p)$, and consequently $\sigma^*(T_p) \subseteq \{0\}$. Let $f \in C_0^\infty([0,1])$, such that $f' \leq 0$ and $\int_0^1 f'(t) dt = -1$. For every $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$R(\lambda, T_p)(f)(x) = \frac{1}{\lambda} M_{h_{\lambda,p}} V M_{h_{-\lambda,p}}(-D(f))(x) = \frac{1}{\lambda} \int_0^x h_{\lambda,p}(x) h_{-\lambda,p}(t)(-f'(t)) dt.$$

Now we observe that, for every $0 \leq t \leq x \leq 1$, $0 < \lambda < 1$, $h_{\lambda,p}(x) h_{-\lambda,p}(t) = e^{\frac{x^{p+1} - t^{p+1}}{\lambda(p+1)}} \geq 1$ for any $p > -1$, and $h_{\lambda,-1}(x) h_{-\lambda,-1}(t) = \left(\frac{x}{t}\right)^{\frac{1}{\lambda}} \geq 1$. Altogether leads to

$$\left\langle \delta_1, R(\lambda, T_p)(f) \right\rangle = \frac{1}{\lambda} \int_0^1 h_{\lambda,p}(1) h_{-\lambda,p}(t)(-f'(t)) dt \geq \frac{1}{\lambda}.$$

From this we conclude$\{R(\lambda, T_p)(f) : 0 < \lambda < 1\}$ is not bounded. Hence $0 \notin \varrho^*(T_p)$, i.e. $\sigma^*(T_p) = \{0\}$.

(b) Case $p < -1$. For $p < -1$ and $\mathrm{Re}(\lambda) \leq 0$, $\lambda \neq 0$, it follows form Lemma 11 that both $h_{\lambda,p}$ and $h_{-\lambda,p}$ belong to $\mathcal{M}$, and the resolvent formula (3.3) applies. Therefore, when $p < -1$, $\{\lambda : \mathrm{Re}(\lambda) \leq 0\} \subseteq \varrho(T_p)$. Hence we conclude $\sigma(T_p) = \sigma_p(T_p) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Since $\overline{\sigma(T_p)} \subseteq \sigma^*(T_p)$, we only lack to show $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \subseteq \varrho^*(T_p)$. When $\mathrm{Re}(\lambda) < 0$ then from (3.2) and (3.3) we get

$$R(\lambda, T_p)(g)(x) := -\frac{1}{\lambda} \int_0^x e^{\frac{x^{p+1} - t^{p+1}}{\lambda(p+1)}} g'(t) dt, \; g \in C_0^\infty([0,1])$$

Now, since D is an isomorphism, Lemma 19 yields

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} = \{\lambda \in \mathbb{C} : \mathrm{Re}(1/(\lambda(p+1))) > 0\} \subseteq \varrho^*(T_p).$$

□

Theorem 21 *For $p \in \mathbb{R}$, let $T_p := M_{\omega_p} D : C_0^\infty([0,1]) \to C_0^\infty([0,1])$, $f(x) \mapsto x^p f'(x) dt$, (or $T_p := D M_{\omega_p} : C_0^\infty([0,1]) \to C_0^\infty([0,1])$, $f(x) \mapsto (x^p f(x))' dt$).*

(i) *If* $p \leq 1$ *then* $\sigma(T_p) = \sigma^*(T_p) = \emptyset$.
(ii) *If* $p > 1$ *then* $\sigma(T_p) = \sigma_p(T_p) = \{\lambda \in \mathbb{C} : \ Re(\lambda) > 0\}$ *and* $\sigma^*(T_p) = \overline{\sigma(T_p)}$.

Proof Combining Proposition 2 and Theorem 20 we obtain all the statements except $0 \in \varrho^*(T_p)$ when $p \leq 1$. For $p < 1$, the resolvent $R(\lambda, T_p)$ can be directly computed:

$$R(\lambda, T_p)(f)(x) = -\int_0^x t^{-p} e^{\frac{\lambda(x^{-p+1}-t^{-p+1})}{-p+1}} f(t)dt = -M_{h_{\lambda,-p}} V M_{\omega_{-p}} M_{h_{-\lambda,-p}}(f),$$

for $h_{\lambda,-p}(x) := e^{\frac{\lambda x^{-p+1}}{-p+1}}$, $\omega_p(x) := x^p$. The equicontinuity of $\{R(\lambda, T_p)(f) : \ \lambda \in K\}$ for each $f \in C_0^\infty([0, 1])$ and $K \subseteq \mathbb{C}$ compact follows from Proposition 16 (b) and the fact that equicontinuous sets are equibounded.

For $p = 1$, we have

$$R(\lambda, T_1)(f)(x) = -x^\lambda \int_0^x \frac{f(t)}{t^{\lambda+1}} dt = -M_{\omega_\lambda} V M_{\omega_{-\lambda-1}}(f),$$

and we conclude in an analogous way using Proposition 16 (a). □

We finish with a description of the spectrum of the differentiation operator in the one sided Schwartz class, and we observe that the composition of these operators with multiplication by monomials (or other powers of x) can be described with the same arguments.

Theorem 22 *Let consider the differentiaton operator* $D : \ S([1, \infty)) \to S([1, \infty))$, $f \mapsto f'$ *and its inverse* $I : \ S([1, \infty)) \to S([1, \infty))$, $f \mapsto -\int_x^\infty f(t)dt$.

(i) $\sigma(D) = \sigma_p(D) = \{\lambda \in \mathbb{C} : \ Re(\lambda) < 0\}$ *and* $\sigma^*(D) = \overline{\sigma(D)}$.
(ii) $\sigma(I) = \sigma_p(I) = \{\lambda \in \mathbb{C} : \ Re(\lambda) < 0\}$ *and* $\sigma^*(I) = \overline{\sigma(I)}$.

Proof Observe that, by means of the isomorphism defined in Proposition 7, D is equivalent (making an abuse of notation) to $-M_{\omega_2} D : C_0^\infty([0, 1]) \to C_0^\infty([0, 1])$, $f \mapsto -x^2 f'(x)$. Now (i) follows from Theorem 21 (ii). Statement (ii) is a consequence of Proposition 2 and (i). □

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