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Norm Attaining Elements of the Ball Algebra $H^\infty(B_N)$

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Abstract. Let B_N be the Euclidean ball of $\mathbb{C}^N$. The space $H^\infty(B_N)$ of bounded holomorphic functions on B_N is known to have a predual, denoted by $G^\infty(B_N)$. We study the functions in $H^\infty(B_N)$ that attain their norm as elements of the dual of $G^\infty(B_N)$. We also examine similar questions for the polydisc algebra $H^\infty(\mathbb{D}^N)$ and for the space of Dirichlet series $\mathcal{D}^\infty(\mathbb{C}_+)$.

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1. Introduction

Ando [1] proved that the Banach space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on the unit disc $\mathbb{D}$ has a unique isometric predual. Let us denote it by $G^\infty(\mathbb{D})$. By the Bishop-Phelps theorem, the set $NA(G^\infty(\mathbb{D}))$ of functions $f \in H^\infty(\mathbb{D})$ which attain their norm as elements of the dual of $G^\infty(\mathbb{D})$ is a norm-dense subset of $H^\infty(\mathbb{D})$. Fisher [6] showed that $f \in H^\infty(\mathbb{D}), ||f|| = 1$, attains its norm as an element of the dual of $G^\infty(\mathbb{D})$ if and only if the radial limits $f^*(w)$ of f in the torus $\mathbb{T}$ satisfy that the set $\{w \in \mathbb{T} : |f^*(w)| = 1\}$ has positive Lebesgue measure on $\mathbb{T}$. The aim of this article is to investigate versions of Fisher's result for the Banach space of bounded holomorphic functions on the N-dimensional ball and the N-dimensional polydisc. Our main results are Theorems 5 and 8 and Propositions 6 and 7 in the case of the ball.

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The case of the polydisc is treated in Sect. 3. The final section deals with the Banach space of bounded Dirichlet series.

Let X be a complex Banach space. Its open unit ball is denoted by B_X and its closed unit ball by U_X. The space of all holomorphic functions on B_X (i.e. the $\mathbb{C}-$Fréchet differentiable functions $f : B_X \to \mathbb{C}$) will be denoted $H(B_X)$. The Banach space $H^\infty(B_X)$ of all bounded holomorphic functions f in $H(B_X)$ is endowed with the supremum norm $\|f\|_\infty = \sup_{x\in B_X} |f(x)|$. We denote by τ_0 the compact-open topology on $H^\infty(B_X)$, that is, the topology of uniform convergence on compact subsets of B_X. Recall that τ_0 is Hausdorff and coarser than the norm topology. Let $U_{H^\infty(B_X)}$ denote the closed unit ball of $H^\infty(B_X)$. The vector space $G^\infty(B_X)$, given by

$$G^\infty(B_X) := \{\varphi \in H^\infty(B_X)^* \ : \ \varphi_{|U_{H^\infty(B_X)}} \text{ is } \tau_0\text{-continuous}\}$$

is a Banach space when endowed with the dual norm. By using the Ng-Dixmier Theorem [12], Mujica [11], proved that the topological dual of $G^\infty(B_X)$ is isometrically isomorphic to $H^\infty(B_X)$. We abbreviate this fact by

$$G^\infty(B_X)^* \stackrel{1}{=} H^\infty(B_X).$$

For each $x \in B_X$ we denote by $\delta_x : H^\infty(B_X) \to \mathbb{C}$ the evaluation $\delta_x(f) := f(x)$ at the point x. Clearly δ_x is τ_0 continuous. Moreover, the vector space $\text{span}\{\delta_x : \ x \in B_X\}$ is a norm-dense subset in $G^\infty(B_X)$. Indeed, $\{\delta_x : \ x \in B_X\}$ separates points of $H^\infty(B_X)$. Hence $\text{span}\{\delta_x \ : \ x \in B_X\}$ is a subspace of $G^\infty(B_X)$ that is $w(G^\infty(B_X), H^\infty(B_X))$-dense in $G^\infty(B_X)$. Thus it is is also norm-dense subset of $G^\infty(B_X)$. We collect the following consequence for reference later in the paper.

Lemma 1. *If $\mathcal{F}$ is a closed subspace of $G^\infty(B_X)$ containing $\{\delta_x : \ x \in B_X\}$, then $\mathcal{F} = G^\infty(B_X)$.*

Let Y be a Banach space. The set of norm attaining functionals is defined to be the following subset of Y^* :

$$NA(Y) := \{y^* \in Y^* : \ \text{there exists } y \in Y, \|y\| = 1 \text{ such that } \|y^*\| = y^*(y)\}$$

The Bishop–Phelps theorem (see, e.g., Theorem 8.11 in [2]) ensures that the set $NA(Y)$ of norm attaining functionals is a norm-dense subset of Y^*. As a consequence, for each non-trivial, complex Banach space X, there exists a norm-dense subset $NA(G^\infty(B_X))$ of $H^\infty(B_X)$, such that for every $f \in NA(G^\infty(B_X))$, there exists an element $\varphi \in G^\infty(B_X)$ with $\|\varphi\| = 1$ such that

$$\|f\|_\infty = \varphi(f).$$

The aim of this paper is to study those functions $f \in H^\infty(B_X)$ that attain their norm as elements of the dual of $G^\infty(B_X)$, that is, those $f \in NA(G^\infty(B_X))$. We mainly concentrate on the case $X = (\mathbb{C}^N, \|.\|_2)$ and hence, B_X is the N-dimensional Euclidean ball which henceforth will be denoted B_N.

In the one dimensional case, $B_N = \mathbb{D}$ and its boundary is the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In this case, by a result by Fatou, there is an isometric isomorphism between $H^\infty(\mathbb{D})$ and

$$H^\infty(\mathbb{T}) := \left\{ g \in L^\infty(\mathbb{T}) : \hat{g}(k) = \int_{\mathbb{T}} w^{-k} g(w) dm_1(w) = 0, \ k = -1, -2, \dots \right\}.$$

The isometric isomorphism $H^\infty(\mathbb{D}) \to H^\infty(\mathbb{T})$ is given by

$$\begin{aligned} H^\infty(\mathbb{D}) &\longrightarrow H^\infty(\mathbb{T}) \\ f &\longrightarrow f^* \end{aligned}$$

where the radial limit

$$f^*(w) := \lim_{r \to 1-} f(rw),$$

exists almost everywhere on $\mathbb{T}$ (with respect to the Lebesgue normalized measure on $\mathbb{T}$, denoted by $dm_1(w) = \frac{dt}{2\pi}$, where $w = e^{it}$.) From this point of view $H^\infty(\mathbb{D}) \stackrel{1}{=} H^\infty(\mathbb{T})$ is a closed subspace of $L^\infty(\mathbb{T})$, and hence it is a dual space. In fact, if $H_0^1(\mathbb{T})$ is the closed subspace of $L^1(\mathbb{T})$ given by

$$H_0^1(\mathbb{T}) = \left\{ f \in L_1(\mathbb{T}) : \hat{f}(-n) = \int_{\mathbb{T}} f(w) w^n dm_1(w) = 0, \ \text{for all } n = 0, 1, 2, \dots \right\},$$

then

$$H^\infty(\mathbb{T}) \stackrel{1}{=} \left(L^1(\mathbb{T}) / H_0^1(\mathbb{T}) \right)^*.$$

Ando in [1] proved that $H^\infty(\mathbb{D})$ has a unique isometric predual. Accordingly, $L^1(\mathbb{T})/H_0^1(\mathbb{T}) \stackrel{1}{=} G^\infty(\mathbb{D})$. As far as we know, it is an open question for $N \geq 2$ whether there is a unique predual of the corresponding H^∞-spaces in the case of the N-dimensional ball and the N-polydisc. In this paper, we will introduce another natural predual and show, in Theorems 5 and 10, that it coincides with $G^\infty(B_X)$.

The characterization of norm attaining elements of $f \in H^\infty(\mathbb{D})$ was obtained by S. Fisher in 1969.

Theorem 2 (Fisher [6, Theorem 2]). *Let f be an element of norm one in $H^\infty(\mathbb{D})$. The function f attains its norm as an element of the dual of $L^1(\mathbb{T})/H_0^1(\mathbb{T}) = G^\infty(\mathbb{D})$ if and only if $f^*(w) = \lim_{r \to 1-} f(rw)$ (a.e. in $\mathbb{T}$) satisfies that*

$$\{w \in \mathbb{T} : |f^*(w)| = 1\}$$

has positive Lebesgue measure on $\mathbb{T}$.

In this paper, in Sect. 2, we explore several variable versions of Fisher's result. We also examine, in Sects. 3 and 4, similar questions for the polydisc algebra $H^\infty(\mathbb{D}^N)$ and for the space of Dirichlet series $\mathcal{D}^\infty(\mathbb{C}_+)$.

2. The Case of the Euclidean Ball

Recall that the Euclidean open unit ball in $\mathbb{C}^N$ is:

$$B_N := \left\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\|_N := \sqrt[2]{|z_1|^2 + \cdots + |z_N|^2} < 1 \right\}.$$

The unit sphere in $\mathbb{C}^N$ is:

$$S_N := \left\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\|_N := \sqrt[2]{|z_1|^2 + \cdots + |z_N|^2} = 1 \right\}.$$

(Observe that this is not completely standard notation since the usual notation for the N-dimensional real sphere in $\mathbb{R}^N$ is S_{N-1}.)

By σ_N we denote the unique rotation-invariant positive Borel measure on S_N for which

$$\sigma_N(S_N) = 1.$$

In other words, σ_N is the Haar measure of the N-dimensional sphere.

In [15, p.84], the space $H^\infty(B_N)$, is defined as

$$H^\infty(B_N) := \left\{ f \in H(B_N) : \|f\|_\infty := \sup_{z \in B_N} |f(z)| < \infty \right\}.$$

The ball algebra is the Banach subalgebra of $H^\infty(B_N)$ given by

$$A(B_N) := \{f : \overline{B}_N \to \mathbb{C} : \ f \text{ is continuous on } \overline{B}_N \text{ and holomorphic on } B_N\}.$$

Finally, by $A(S_N) = A(B_N) \cap C(S_N)$, we understand the restrictions of the elements of $A(B_N)$ to the sphere S_N, i.e.

$$A(S_N) := \{f_{|S_N} : \ f \in A(B_N)\}.$$

By the maximum modulus theorem, the mapping $\pi : A(B_N) \to A(S_N)$ defined by $\pi(f) := f_{|S_N}$ is an isometry.

Hardy spaces have a dual definition. The Hardy space $H^\infty(S_N)$ is the weak-star closure of $A(S_N)$ in $L^\infty(S_N, \sigma_N)$. i.e.

$$H^\infty(S_N) := \overline{A(S_N)}^{w(L_\infty(S_N), L_1(S_N))}.$$

As the polynomials are dense in $A(B_N)$ we have that $\operatorname{span}\{z^\beta : \beta \in \mathbb{N}_0^N\}$ is a $\|.\|_\infty$ dense subspace of $A(B_N)$. Hence, $\operatorname{span}\{w^\beta : \beta \in \mathbb{N}_0^N\}$ is $\|.\|_\infty$ dense in $A(S_N)$. Thus

$$H^\infty(S_N) = \overline{\operatorname{span}\{w^\beta : \beta \in \mathbb{N}_0^N\}}^{w(L_\infty(S_N), L_1(S_N))}.$$

At this point, we show that $H^\infty(S_N)$ and $H^\infty(B_N)$ are isometrically isomorphic. We need some notation and results that can be found, for example, in the books [15] and [16]. *The invariant Poisson kernel of* B_N is the kernel function $P_N : B_N \times S_N \to [0, +\infty[$

$$P_N(z, w) := \frac{(1 - |z|^2)^N}{|1 - < z, w > |^{2N}}.$$

The *Poisson integral* $P(g)$ of a function g in $L^1(S_N, \sigma_N)$ is defined, for $z \in B_N$, by

$$P_N(g)(z) := \int_{S_N} P(z,w)g(w)d\sigma_N(w).$$

We have that $P_N : H^\infty(S_N) \longrightarrow H^\infty(B_N)$ is a linear isometry onto.

To prove that this mapping is onto, the concept of Korányi, or K-limit, of a holomorphic function on B_N is needed. For $\alpha > 1$ and $w \in S_N$ we set

$$D_\alpha(w) := \left\{ z \in \mathbb{C}^N : |w - z| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

Clearly $D_\alpha(w) \subset B_N$. We say that a function $F : B_N \to \mathbb{C}$ has *K-limit* $\lambda \in \mathbb{C}$ at $w \in S_N$ if the following is true: For every $\alpha > 1$ and for every sequence (z_j) in $D_\alpha(w)$ that converges to a point $w \in S_N$, we have that $F(z_j)$ converges to λ and write

$$(K - \lim F)(w) = \lambda.$$

The following result (see e.g. [15, Section 5.4.]) is important and very useful for our paper.

Theorem 3. *If f is a function in $H^\infty(B_N)$ then f has finite K-limits f^* σ_N-almost everywhere on S_N. Moreover, $f^* \in H^\infty(S_N)$, $\|f^*\|_\infty = \|f\|_\infty$ and*

$$P_N(f^*) = f.$$

In other words, the mapping $f \to f^*$ is a linear isometry from $H^\infty(B_N)$ onto $H^\infty(S_N)$.

We also need the following well known fact, a proof of which is given for the sake of completeness.

Lemma 4. *Let X be a Banach space and let Y be a weak-star closed subspace of X^*. The subspace*

$$Y_\perp := \{x \in X : y^*(x) = 0, \text{ for all } y^* \in Y\},$$

satisfies

$$Y_\perp^\perp := \{x^* \in X^* : x^*(x) = 0, \text{ for all } x \in Y_\perp\} = Y,$$

and Y is isometrically isomorphic to $(X/Y_\perp)^*$.

Proof. Clearly, by the definition, $Y \subset Y_\perp^\perp$. Assume that the reverse inclusion is not true. Hence there exists $x_0^* \in Y_\perp^\perp \backslash Y$.

Since Y is $w(X^*, X)$ closed and convex we can find $\varphi : X^* \to \mathbb{C}$, $w(X^*, X)$-continuous, such that

$$\varphi(x_0^*) = 1 \text{ and } \varphi(y^*) = 0,$$

for all $y^* \in Y$. Since φ is weak-star continuous, there exists $x_0 \in X$ such that

$$\varphi(x^*) = x^*(x_0),$$

for all $x^* \in X^*$. Thus, $x_0^*(x_0) = 1$ and $y^*(x_0) = 0$ for all $y^* \in Y$. Hence x_0 belongs $Y_\perp$. But, $x_0^* \in Y_\perp^\perp$, which, by definition implies

$$x_0^*(x_0) = 0.$$

This is a contradiction.

Finally, we have $(X/Y_\perp)^* \overset{1}{=} Y_\perp^\perp = Y$, as follows from [10, Theorem 1.10.17] for example. □

Now we define

$$H_0^1(S_N) = \left\{ g \in L_1(S_N) \, : \, \int_{S_N} g(w)f(w)d\sigma_N(w) = 0 \text{ for all } f \in A(S_N) \right\}.$$

Since

$$H^\infty(S_N) := \overline{A(S_N)}^{w(L_\infty(S_N),L_1(S_N))} = \overline{\operatorname{span}\{w^\beta : \beta \in \mathbb{N}_0^N\}}^{w(L_\infty(S_N),L_1(S_N))},$$

the subspace $H^\infty(S_N) \subset L_\infty(S_N)$ is $w(L_\infty(S_N), L_1(S_N))$-closed in $L_\infty(S_N)$ and

$$\begin{aligned} H_0^1(S_N) &= \left\{ g \in L_1(S_N) : \int_{S_N} g(w)f(w)d\sigma_N(w) = 0, \text{ for all } f \in H^\infty(S_N) \right\} \\ &= \left\{ g \in L_1(S_N) : \hat{g}(-\beta) := \int_{S_N} g(w)w^\beta d\sigma_N(w) = 0, \text{ for all } \beta \in \mathbb{N}_0^N \right\}. \end{aligned}$$

In the notation of Lemma 4, with $X = L_1(S_N)$, $X^* = L_\infty(S_N)$ and $Y = H^\infty(S_N)$ (which is weak-star closed in X^*), we have

$$\begin{aligned} Y_\perp &= H^\infty(S_N)_\perp = H_0^1(S_N), \\ Y_\perp^\perp &= H_0^1(S_N)^\perp = H^\infty(S_N). \end{aligned}$$

Lemma 4 implies the isometric isomorphism

$$H^\infty(S_N) \overset{1}{=} \left(L_1(S_N)/H_0^1(S_N)\right)^*.$$

Next we show that $G^\infty(B_N)$ and $L^1(S_N)/H_0^1(S_N)$ are isometrically isomorphic. Thus, these two natural preduals of $H^\infty(B_N)$ coincide, and so the extension of Ando's result on the uniqueness of the predual of $H^\infty(\mathbb{D})$ to several variables is still open.

Theorem 5. *For every $N \in \mathbb{N}$ we have that*

$$L^1(S_N)/H_0^1(S_N) = G^\infty(B_N)$$

isometrically.

Proof. First we prove that $L^1(S_N)/H_0^1(S_N) \subset G^\infty(B_N)$.

Let $[\varphi] \in L^1(S_N)/H_0^1(S_N)$ and $g \in H^\infty(S_N)$. The duality is given by

$$< [\varphi], g >= \int_{S_N} \varphi(w)g(w)d\sigma_N(w) = \int_{S_N} (\varphi(w) + \eta(w))g(w)d\sigma_N(w),$$

for every $\varphi \in L_1(S_N)$ and every $\eta \in H_0^1(S_N)$.

We identify $L^1(S_N)/H_0^1(S_N)$ as a subspace of the dual of $H^\infty(S_N)$ in the following natural way. Define $T_{[\varphi]} : H^\infty(B_N) \longrightarrow \mathbb{C}$ by

$$T_{[\varphi]}(f) :=< [\varphi], f^* >= \int_{S_N} \varphi(w) f^*(w) d\sigma_N(w).$$

We check that $T_{[\varphi]}$ belongs to $G^\infty(B_N)$ for every equivalence class $[\varphi] \in L^1(S_N)/H_0^1(S_N)$.

Clearly

$$|T_{[\varphi]}(f)| \leq \int_{S_N} |\varphi(w)| \|f^*\|_\infty d\sigma_N(w) = \|\varphi\|_1 \|f\|_\infty.$$

Hence, $T_{[\varphi]}$ belongs to $H^\infty(B_N)^*$. This fact and the equality $\|T_{[\varphi]}\| = \|[\varphi]\|$ are consequences of the isometric isomorphism $H^\infty(S_N) \stackrel{1}{=} \left(L_1(S_N)/H_0^1(S_N)\right)^*$ and Theorem 3.

Let us check that $T_{[\varphi]}$ is τ_0-continuous when restricted to the closed unit ball $U_{H^\infty(B_N)}$ of $H^\infty(B_N)$.

By Theorem 3, we know that if $f \in H^\infty(B_N)$ and $f^* \in H^\infty(S_N)$ is its K-limit that exists a.e. in S_N, then

$$f(z) = \int_{S_N} P_N(z, w) f^*(w) d\sigma_N(w)$$

for all $z \in B_N$. Conversely, if $h \in H^\infty(S_N)$, then $P_N(h) \in H^\infty(B_N)$ and we have

$$P_N(h)^*(w) = h(w)$$

a.e. on S_N.

For each $z \in B_N$ the mapping $P_N(z, .) : S_N \to]0, +\infty[$ is continuous on S_N. Hence $P_N(z, .) \in L^1(S_N)$.

Given $(f_n) \cup \{f\} \subset U_{H^\infty(B_N)}$ such that (f_n) converges to f with respect to the compact-open topology on B_N, we have $(f_n^*) \cup \{f^*\} \subset U_{H^\infty(S_N)}$. But $U_{H^\infty(S_N)}$ is a weak-star closed subset of $U_{L^\infty(S_N)}$ which, in turn, is a $w(L^\infty(S_N), L^1(S_N))$-compact set. Since $L^1(S_N)$ is separable, it follows that $U_{H^\infty(S_N)}$ is a metrizable compact set with the weak-star topology. Consider now any subsequence $(f_{n_k}^*)$ that is $w(L^\infty(S_N), L^1(S_N))$-convergent to some $h \in U_{H^\infty(S_N)}$. We will have

$$\begin{aligned} P_N(h)(z) &= \int_{S_N} P_N(z, w) h(w) d\sigma_N(w) \\ &=< P_N(z, .), h >= \lim_{k\to\infty} < P_N(z, .), f_{n_k} > \\ &= \lim_{k\to\infty} \int_{S_N} P_N(z, w) f_{n_k}^*(w) d\sigma_N(w) \\ &= \lim_{k\to\infty} f_{n_k}(z) = f(z), \end{aligned}$$

for all $z \in B_N$. Hence,

$$h(w) = P_N(h)^*(w) = f^*(w)$$

a.e in S_N. We have just proved that the only weak-star adherent point of (f_n^*) is f^*. Thus (f_n^*) weak-star converges to f^*. In particular

$$\begin{aligned} T_{[\varphi]}(f) = \int_{S_N} f^*(w)\varphi(w)d\sigma_N(w) \\ =< [\varphi], f^* >= \lim_{n\to\infty} < [\varphi], f_n^* > \\ \lim_{n\to\infty} T_{[\varphi]}(f_n), \end{aligned}$$

and $T_{[\varphi]}$ is continuous with the compact-open topology when restricted to the closed unit ball of $H^\infty(B_N)$; i.e. $T_{[\varphi]} \in G^\infty(B_N)$.

For the other inclusion observe that

$$\delta_z(f) = P_N(f^*)(z) = \int_{S_N} P_N(z,w)f^*(w)d\sigma_N(w) = T_{[P_N(z,.)]}(f),$$

for every $z \in B_N$ and every $f \in H^\infty(B_N)$. Thus

$$\text{span}\{\delta_z : z \in B_N\} \subset L^1(S_N)/H_0^1(S_N).$$

The conclusion follows from Lemma 1. □

Theorem 5 permits us to get a sufficient condition for a function on $H^\infty(B_N)$ to attain the norm.

Proposition 6. *If f is an element of $H^\infty(B_N)$ of norm one such that the set*

$$E := \{w \in S_N : |f^*(w)| = 1\},$$

has positive σ_N measure in S_N, then f attains its norm as an element of the dual of $L^1(S_N)/H_0^1(S_N) = G^\infty(B_N)$.

Proof. Define $\varphi : S_N \longrightarrow \mathbb{C}$ by

$$\varphi(w) = \begin{cases} \frac{|f^*(w)|}{f^*(w)} \frac{1}{\sigma_N(E)}, & \text{if } w \in E \\ 0, & \text{otherwise.} \end{cases}$$

We have that φ is a bounded measurable function on S_N. Thus $\varphi \in L^1(S_N)$ and

$$\int_{S_N} |\varphi(w)|d\sigma_N(w) = \frac{1}{\sigma_N(E)} \int_E d\sigma_N(w) = 1.$$

Define $T_{[\varphi]} : H^\infty(B_N) \longrightarrow \mathbb{C}$ by

$$T_{[\varphi]}(g) :=< [\varphi], g^* >= \int_{S_N} \varphi(w)g^*(w)d\sigma_N(w).$$

By Theorem 5, $T_{[\varphi]} \in L^1(S_N)/H_0^1(S_N) = G^\infty(B_N)$ and

$$|T_{[\varphi]}(g)| \leq \|g^*\|_\infty \|\varphi\|_1 = \|g\|_\infty \|\varphi\|_1 = \|g\|_\infty,$$

for every $g \in H^\infty(B_N)$. Hence

$$\|T_{[\varphi]}\| \leq 1.$$

But

$$T_{[\varphi]}(f) = \int_{S_N} \varphi(w) f^*(w) d\sigma_N(w) = \frac{1}{\sigma_N(E)} \int_E |f^*(w)| d\sigma_N(w) = 1 = ||f||.$$

and f in the dual of $G^\infty(B_N)$ attains its norm at $T_{[\varphi]}$. $\square$

A partial converse to the above proposition is the following.

Proposition 7. *If f is an element of $H^\infty(B_N)$ of norm one such that there exists $\varphi \in L^1(S_N)$ with $\|\varphi\|_1 = 1$ and $T_{[\varphi]}(f) = 1$, then*

$$\sigma_N(\{w \in S_N : |f^*(w)| = 1\}) > 0.$$

Proof. We denote $E = \{w \in S_N : |f^*(w)| = 1\}$.

Assume that $\sigma_N(E) = 0$.

Let

$$K_n = \left\{ w \in S_N : |f^*(w)| < \frac{n-1}{n} \right\}.$$

Clearly $S_N \setminus E = \cup_{n=1}^\infty K_n$.

We have that $T_{[\varphi]} \in G^\infty(B_N)$ and is of norm one since

$$1 = T_{[\varphi]}(f) \leq \|[\varphi]\| \|f\|_\infty = \|[\varphi]\| \leq \|\varphi\|_1 = 1.$$

For each n, we get

$$\begin{aligned}
\int_{S_N \setminus K_n} |\varphi(w)| d\sigma_N(w) + \int_{K_n} |\varphi(w)| d\sigma_N(w) = 1 = \int_{S_N} f(w)\varphi(w) d\sigma_N(w) \\
= \int_{S_N \setminus K_n} f(w)\varphi(w) d\sigma_N(w) + \int_{K_n} f(w)\varphi(w) d\sigma_N(w) \\
\leq \int_{S_N \setminus K_n} |f(w)\varphi(w)| d\sigma_N(w) + \int_{K_n} |f(w)\varphi(w)| d\sigma_N(w) \\
\leq \int_{S_N \setminus K_n} |\varphi(w)| d\sigma_N(w) + \frac{n-1}{n} \int_{K_n} |\varphi(w)| d\sigma_N(w).
\end{aligned}$$

Thus, $\int_{K_n} |\varphi(w)| d\sigma_N(w) = 0$. Since n is arbitrary, we get

$$\int_{S_N \setminus E} |\varphi(w)| d\sigma_N(w) = 0.$$

But, by hypothesis $\sigma_N(E) = 0$ and finally we arrive at the contradiction

$$1 = \int_{S_N} |\varphi(w)| d\sigma_N(w) = 0.$$

$\square$

A subset E of S_N is called a *peak set* if there exists $f \in A(B_N)$ such that $f(z) = 1$ for every $z \in E$ and $|f(z)| < 1$ for every $z \in \overline{B}_N \setminus E$. Every peak set is a null set.

A result by Fatou states that every compact subset of $\mathbb{T}$ of Lebesgue measure zero is a peak set of $A(\mathbb{D})$, a fact which is instrumental in the proof of Fisher's Theorem 2. On the other hand, there are null sets on S_N (respectively in $\mathbb{T}^N$), which are not peak sets [15, 10.1.1 and 11.2.5] (respectively [14, Theorem 6.3.4, p. 149-150]). We do not know if the converse of Proposition 6 is true or not. But, if we restrict ourselves to functions in $A(B_N)$ that attain their norm, we get the following characterization in terms of peak sets.

Theorem 8. *Let f be an element of $A(B_N)$ of norm one. The function f attains its norm as an element of $H^\infty(B_N)$ if and only if the set*

$$E(f) = \{w \in S_N : |f(w)| = 1\}$$

is not a peak set.

Before presenting the proof we need some notation and a lemma.

We recall that a complex Borel measure μ on S_N is a *Henkin measure* (See [15, 9.1.5, p. 186]) if

$$\lim_{n\to\infty} \int_{S_N} f_n(w) d\mu(w) = 0,$$

for every sequence (f_n) contained in the closed unit ball $U_{A(B_N)}$ of $A(B_N)$ that converges uniformly to 0 on the compact subsets of B_N, that is, converges to 0 in the τ_0 topology in B_N. (By the Montel theorem, a sequence (f_n) contained in $U_{A(B_N)}$ converges to 0 in τ_0 if and only if converges to 0 pointwise on B_N).

Lemma 9. (1) *For every Henkin measure μ there is $T \in G^\infty(B_N)$ such that*

$$T(f) = \int_{S_N} f(w) d\mu(w)$$

for each $f \in A(B_N)$, and $\|\mu\| \geq \|T\|$.

(2) *If $T \in G^\infty(B_N)$, then there is a Henkin measure μ on S_N such that*

$$T(f) = \int_{S_N} f(w) d\mu(w)$$

for each $f \in A(B_N)$, and $\|\mu\| = \|T\|$.

Proof. (1) Define $T_1 : A(B_N) \longrightarrow \mathbb{C}$ by

$$T_1(g) := \int_{S_N} g(w) d\mu(w).$$

Clearly, T_1 is a continuous linear form on $A(B_N)$ which is τ_0-continuous on $U_{A(B_N)}$ and

$$\|T_1\| \leq \|\mu\|.$$

Given $f \in H^\infty(B_N)$, the function $f_r(z) := f(rz)$, $0 \le r < 1$, belongs to $A(B_N)$. In addition, (f_r) converges to f uniformly on the compact subsets of B_N and

$$\|f_r\| \le \|f\|, \quad \|f\| = \sup_r \|f_r\|. \tag{1}$$

By [15, 11.3.1], since μ is a Henkin measure, the limit

$$\lim_{r\to 1-} \int_{S_N} f_r(w) d\mu(w) = \lim_{r\to 1-} T_1(f_r) \in \mathbb{C},$$

exists for every $f \in H^\infty(B_N)$.
We define $T : H^\infty(B_N) \longrightarrow \mathbb{C}$, by

$$T(f) := \lim_{r\to 1-} T_1(f_r).$$

T is linear and $T \in (H^\infty(B_N))^*$, since

$$|T(f)| \le \sup_r |T_1(f_r)| \le \|T_1\|\|f\|,$$

for every $f \in H^\infty(B_N)$. Moreover, $\|T\| = \|T_1\|$.

We claim that the restriction of T_1 to $U_{A(B_N)}$ is τ_0-uniformly continuous. Indeed, given $\varepsilon > 0$ there are a compact subset K of B_N and $\delta > 0$ such that $|T_1(g)| < \varepsilon$ if $g \in U_{A(B_N)}$ and $\sup_{z\in K} |g(z)| < \delta$. Hence, if $g, h \in U_{A(B_N)}$ and $\sup_{z\in K} |g(z) - h(z)| < \delta$, then

$$|T_1(g) - T_1(h))| = |2T_1(\frac{g-h}{2})| < 2\varepsilon.$$

Since $U_{A(B_N)}$ is τ_0-dense in $U_{H^\infty(B_N)}$, there exists a unique $\widetilde{T_1} : U_{H^\infty(B_N)} \longrightarrow \mathbb{C}$ that is τ_0-continuous and such that

$$\widetilde{T_1}(g) = T_1(g),$$

for all $g \in U_{A(B_N)}$. Given $f \in U_{H^\infty(B_N)}$, then $(f_r) \subset U_{A(B_N)}$, and (f_r) converges to f in τ_0 as $r \to 1-$. Thus

$$\widetilde{T_1}(f_r) \to \widetilde{T_1}(f),$$

in $\mathbb{C}$. But $\widetilde{T_1}(f_r) = T_1(f_r)$ for each $r \in [0, 1[$ and $T_1(f_r)$ converges to $T(f)$ by definition. This implies $\widetilde{T_1}(f) = T(f)$ for each $f \in U_{H^\infty(B_N)}$. We have obtained that the restriction of T to $U_{H^\infty(B_N)}$ is τ_0 continuous. This, by definition, implies that T belongs to $G^\infty(B_N)$.

(2) If $T \in G^\infty(B_N)$, the restriction of T to $U_{H^\infty(B_N)}$ is τ_0 continuous. If we define $T_1 : A(B_N) \longrightarrow \mathbb{C}$, by $T_1(g) := T(g)$, then T_1 is continuous for the sup-norm, $\|T_1\| \le \|T\|$ and T_1 is τ_0-continuous on $U_{A(B_N)}$. By (1), we have

$$|T(f)| = \lim_{r\to 1-} |T(f_r)| = \lim_{r\to 1-} |T_1(f_r)| \le \|T_1\| \sup_{r<1} \|f_r\| = \|T_1\|\|f\|,$$

for every $f \in H^\infty(B_N)$. Thus, $\|T_1\| = \|T\|$. We can consider $T_1 : A(S_N) \longrightarrow \mathbb{C}$. By the Riesz theorem, there is a complex Borel measure μ on S_N such that

$$T_1(h) := \int_{S_n} h(w)d\mu(w),$$

for every $h \in A(B_N)$ with $\|T_1\| = |\mu|(S_N) = \|\mu\|$.
The properties of T_1 imply that μ is a Henkin measure. □

Now we give the proof of Theorem 8:

Proof. Assume that $f \in A(B_N)$, $\|f\| = 1$ and that $E(f)$ is not a peak set. By [15, 10.1.1] there exists a Borel measure ρ, such that $\rho(E(f)) \neq 0$ such that

$$h(0) = \int_{S_N} h(w)d\rho(w), \tag{2}$$

for every $h \in A(B_N)$.

Define

$$g(z) = \begin{cases} 0 & \text{if } z \in S_N \setminus E(f) \\ \frac{\overline{f(z)}}{|\rho|(E(f))} & \text{if } z \in E(f) \end{cases}.$$

Since g is bounded and measurable, $g \in L^1(|\rho|)$. Hence, the measure $g|\rho|$ defined by $g|\rho|(M) = \int_M gd|\rho|$ for Borel measurable sets M is absolutely continuous with respect to ρ. The measure ρ is Henkin (this fact is a direct consequence of (2) and the definition of Henkin measure as given in [15, p. 187]), and so $g|\rho|$ is also a Henkin measure by [15, 9.3.1]. We set $T_1 : A(B_N) \longrightarrow \mathbb{C}$.

$$T_1(h) = \int_{S_N} h(w)g(w)d|\rho|(w).$$

We have

$$T_1(f) = \int_{S_N} f(w)g(w)d|\rho|(w) = 1,$$

and

$$|T_1(h)| \leq \int_{S_N} |h(w)||g(w)|d|\rho|(w) \leq \frac{|\rho|(E(f))}{|\rho|(E(f))} = 1,$$

for every $h \in A(B_N)$ with $\|h\| \leq 1$, and we have $g|\rho|$ a Henkin measure such that

$$T_1(f) = 1 \quad \text{and} \quad \|T_1\| = 1.$$

By Lemma 9.(1), there is $T \in G^\infty(B_N)$ with $\|T\| = \|T_1\| = 1$ such that

$$T(h) = T_1(h),$$

for every $h \in A(B_N)$.

In particular, $T(f) = 1$ and f attains its norm on $G^\infty(B_N)$.

Suppose now that $f \in A(B_N)$, $\|f\| = 1$, satisfies that $E(f)$ is a peak set and that there is $T \in G^\infty(B_N)$, $\|T\| = 1$ with $T(f) = 1$. To get a contradiction we are going to give an argument that follows closely the one given in Proposition 7:

By Lemma 9.(2) there exists μ a Henkin measure such that

$$T(h) = \int_{S_N} h(w) d\mu(w),$$

for every $h \in A(B_N)$ and

$$\|T\| = \|T_{|A(B_N)}\| = |\mu|(S_N) = 1.$$

By [15, 9.3.1] $|\mu|$ is also a Henkin measure. Hence, by [15, Lemma 11.3.3] (see also [15, Lemma 11.3.1]), $|\mu|(E(f)) = 0$. Let

$$K_n = \left\{ w \in S_N : |f(w)| < \frac{n-1}{n} \right\}.$$

Clearly $S_N \setminus E(f) = \cup_{n=1}^{\infty} K_n$.

We have that, for each n,

$$\begin{aligned} |\mu|(S_N \setminus K_n) + |\mu|(K_n) = |\mu|(S_N) = 1 = T(f) &= \int_{S_N} |f(w)| d|\mu|(w) \\ &= \int_{S_N \setminus K_n} |f(w)| d|\mu|(w) + \int_{K_n} |f(w)| d|\mu|(w) \\ &\leq \int_{S_N \setminus K_n} d|\mu|(w) + \frac{n-1}{n} \int_{K_n} d|\mu|(w) \\ &= |\mu|(S_N \setminus K_n) + \frac{n-1}{n} |\mu|(K_n). \end{aligned}$$

This implies $|\mu|(K_n) = 0$ for each n and $|\mu|(S_N \setminus E(f)) = 0$.

Therefore, $1 = |\mu|(S_N) = |\mu|(S_N \setminus E(f)) + |\mu|(E(f)) = 0$, a contradiction. □

3. The Case of the Polydisc

For a fixed $N \in \mathbb{N}$, the N-dimensional Poisson kernel [14, p. 17] $P_N : \mathbb{D}^N \times \mathbb{T}^N \to (0, \infty)$ is defined as

$$P_N(z, w) := \prod_{j=1}^{N} P_1(z_j, w_j) = \prod_{j=1}^{N} \frac{1 - |z_j|^2}{|1 - z_j \overline{w}_j|^2}.$$

It is well known ([14, Theorem 3.3.3, p.45]) that if $f \in H^\infty(\mathbb{D}^N)$ then the limit

$$f^*(w) = \lim_{r \to 1-} f(rw)$$

exists almost everywhere in $\mathbb{T}^N$, and

$$f(z) = \int_{\mathbb{T}^N} P_N(z,w) f^*(w) dm_N(w) \tag{3}$$

for all $z \in \mathbb{D}^N$. As a consequence there exists an isometric isomorphism

$$\begin{aligned} &H^\infty(\mathbb{D}^N) \longrightarrow H^\infty(\mathbb{T}^N) \\ &f \longrightarrow f^* \end{aligned}$$

where $H^\infty(\mathbb{T}^N) := \overline{A(\mathbb{T}^N)}^{w(L_\infty(\mathbb{T}^N), L_1(\mathbb{T}^N))}$,

$$A(\mathbb{D}^N) = \{f : \overline{\mathbb{D}}^N \to \mathbb{C} : \ f \text{ is continuous on } \overline{\mathbb{D}}^N \text{ and holomorphic on } \mathbb{D}^N\}$$

and

$$A(\mathbb{T}^N) := \{f_{|\mathbb{T}^N} : \ f \in A(\mathbb{D}^N)\}.$$

By the maximum modulus theorem $A(\mathbb{D}^N)$ and $A(\mathbb{T}^N)$ are isometrically isomorphic. By Fejer's theory for the polydisc we have

$$\begin{aligned} H^\infty(\mathbb{T}^N) := \Bigg\{ &g \in L^\infty(\mathbb{T}) : \ \hat{g}(\alpha) \\ &= \int_{\mathbb{T}^N} w^{-\alpha} g(w) dm_N(w) = 0, \ \text{for all } \alpha \in \mathbb{Z}^N \setminus \mathbb{N}_0^N \Bigg\}. \end{aligned}$$

On the other hand, by applying Lemma 4,

$$H^\infty(\mathbb{T}^N) \overset{1}{=} \left(L^1(\mathbb{T}^N)/H_0^1(\mathbb{T}^N)\right)^*,$$

where

$$\begin{aligned} H_0^1(\mathbb{T}^N) &= \left\{ f \in L_1(\mathbb{T}^N) : \ \hat{f}(-\beta) = \int_{\mathbb{T}} f(w) w^\beta dm_N(w) = 0, \ \text{for all } \beta \in \mathbb{N}_0^N \right\} \\ &= \overline{\operatorname{span}\{w^{-\alpha} : \ \text{ for all } \alpha \in \mathbb{Z}^N \setminus \mathbb{N}_0^N}\}. \end{aligned}$$

Very similar arguments to the ones given for the N-dimensional Euclidean ball can be given for the N-polydisc to obtain the following results.

Theorem 10. *For every $N \in \mathbb{N}$ we have*

$$L^1(\mathbb{T}^N)/H_0^1(\mathbb{T}^N) \overset{1}{=} G^\infty(\mathbb{D}^N).$$

Proposition 11. *Let f be an element of $H^\infty(\mathbb{D}^N)$ of norm one such that the set*

$$E := \{w \in \mathbb{T}^N : |f^*(w)| = 1\},$$

has positive Lebesgue measure (in $\mathbb{T}^N$). Then f attains its norm as an element of the dual of $L^1(\mathbb{T}^N)/H_0^1(\mathbb{T}^N)$.

Proposition 12. *If f is an element of $H^\infty(\mathbb{D}^N)$ of norm one such that there exists $\varphi \in L^1(\mathbb{T}^N)$ with $\|\varphi\|_1 = 1$ and $T_{[\varphi]}(f) = 1$, then the set*

$$\{w \in \mathbb{T}^N : |f^*(w)| = 1\},$$

has positive Lebesgue measure in the polytorus $\mathbb{T}^N$.

Example 13. The following example, which is inspired by [3, Theorem 3.1], shows that a polydisc (for $N > 1$) version of Theorem 8 does not hold. Let $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be the function $f(z, w) := (1/2)(1 + z)$, which belongs to $A(\mathbb{D} \times \mathbb{D})$. This function does not attain its norm on $G^\infty(\mathbb{D} \times \mathbb{D})$. Indeed, if it did, the function $g(z) = (1/2)(1 + z)$, as an element of $H^\infty(\mathbb{D})$, would attain its norm on $G^\infty(\mathbb{D})$, because $H^\infty(\mathbb{D})$ is canonically isometrically contained in $H^\infty(\mathbb{D} \times \mathbb{D})$. But the function g does not attain its norm on $G^\infty(\mathbb{D})$ by Fisher's Theorem 2, because $\{z \in \mathbb{T} \mid |g(z)| = 1\} = \{1\}$. On the other hand, $E(f) = \{(z, w) \in \mathbb{T} \times \mathbb{T} \mid |f(z, w)| = 1\} = \{1\} \times \mathbb{T}$, as it is easy to check. The set $E(f)$ is not a peak set of $A(\mathbb{D} \times \mathbb{D})$. Otherwise, it would be a zero set; see [14, 6.1.2. Theorem, p.132]. But if a function $h \in A(\mathbb{D} \times \mathbb{D})$ vanishes on $E(f)$, then $h(1, w) \in A(\mathbb{D})$ vanishes on $\mathbb{T}$. The maximum principle then implies that h vanishes on $\{1\} \times \mathbb{D}$, and therefore, there is no function $h \in A(\mathbb{D} \times \mathbb{D})$ vanishing only in $E(f)$. Observe that $E(f)$ is a null set in $\mathbb{T} \times \mathbb{T}$ which is not a peak set.

4. The Case of the Space of Dirichelt Series $\mathcal{D}^\infty(\mathbb{C}_+)$

Let $\mathcal{D}^\infty(\mathbb{C}_+)$ denote the Banach space of the Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ convergent and bounded on the right half plane $\mathbb{C}_+$ endowed with the supremum norm. We refer the reader to [4] and [13] for detailed information about this space.

The space $\mathcal{D}^\infty(\mathbb{C}_+)$ is a closed subspace of the Banach space $H^\infty(\mathbb{C}_+)$ of all bounded holomorphic functions in the right half plane $\mathbb{C}_+$ endowed with the supremum norm. Since, by the Montel theorem, the closed unit ball of $H^\infty(\mathbb{C}_+)$ is τ_0-compact, we can apply the Dixmier-Ng theorem [12] to obtain that

$$G^\infty(\mathbb{C}_+) := \{R \in H^\infty(\mathbb{C}_+)^* : \text{the restriction of } R \text{ to } U_{H^\infty(\mathbb{C}_+)} \text{ is } \tau_0 \text{ continuous}\},$$

is a predual of $H^\infty(\mathbb{C}_+)$.

It is well-known that the spaces $H^\infty(\mathbb{C}_+)$ and $H^\infty(\mathbb{D})$ are isometrically isomorphic. We are going to show that their preduals are also isometrically isomorphic.

Proposition 14. *$H^\infty(\mathbb{C}_+)$ is isometrically isomorphic to $H^\infty(\mathbb{D})$, and $G^\infty(\mathbb{C}_+)$ is isometrically isomorphic to $G^\infty(\mathbb{D})$.*

Proof. It is enough to consider the Cayley transformation $\varphi : \mathbb{C} \setminus \{1\} \to \mathbb{C} \setminus \{-1\}$ defined by

$$\varphi(z) = \frac{1+z}{1-z}$$

The Cayley transformation is a biholomorphic mapping with inverse

$$\varphi^{-1}(s) = \frac{s-1}{s+1}.$$

Actually it is also biholomorphic from $\mathbb{D}$ onto $\mathbb{C}_+$, and it is a homeomorphism from $\mathbb{T} \setminus \{1\}$ onto $\{ti : \ t \in \mathbb{R}\}$. Clearly the composition operator $T_\varphi : H^\infty(\mathbb{C}_+) \to H^\infty(\mathbb{D})$ defined by

$$T_\varphi(g) = g \circ \varphi,$$

for $g \in H^\infty(\mathbb{C}_+)$ is an isometry with inverse $(T_\varphi)^{-1} = T_{\varphi^{-1}}$. Its adjoint $T_\varphi^* : H^\infty(\mathbb{D})^* \to H^\infty(\mathbb{C}_+)^*$ is also an isometric isomorphism with

$$(T_\varphi^*)^{-1} = T_{\varphi^{-1}}^*.$$

It is enough to check that $T_\varphi^*(G^\infty(\mathbb{D})) = G^\infty(\mathbb{C}_+)$ to prove that $G^\infty(\mathbb{D})$ and $G^\infty(\mathbb{C}_+)$ are isometrically isomorphic.

Let $R \in G^\infty(\mathbb{D})$. We have

$$T_\varphi^*(R)(g) = R(T_\varphi(g)) = R(g \circ \varphi)$$

for all $g \in H^\infty(\mathbb{C}_+)$. Let $K \subset \mathbb{D}$ be a compact set. The set $\varphi(K)$ is a compact subset of $\mathbb{C}_+$. Take (g_n) and g in the closed unit ball of $H^\infty(\mathbb{C}_+)$ such that (g_n) converges with respect to the compact open topology on $\mathbb{C}_+$ to g. Since (g_n) converges to g uniformly on $\varphi(K)$, we have $(g_n \circ \varphi)$ converges to $g \circ \varphi$ uniformly on $K = \varphi^{-1}(\varphi(K))$, for every K. Thus, $(g_n \circ \varphi)$ converges to $g \circ \varphi$ with respect to the compact open topology on $\mathbb{D}$. Hence, $(R(g_n \circ \varphi))$ converges to $R(g \circ \varphi)$ and we get

$$T_\varphi^*(R) \in G^\infty(\mathbb{C}_+).$$

Analogously we obtain $T_{\varphi^{-1}}^*(G^\infty(\mathbb{C}_+)) \subset G^\infty(\mathbb{D})$, from which it follows that

$$G^\infty(\mathbb{C}_+) = T_\varphi^* \circ T_{\varphi^{-1}}^*(G^\infty(\mathbb{C}_+)) \subset T_\varphi^*(G^\infty(\mathbb{D})).$$

□

Remark 15. Recall that for any fixed $\alpha > 1$ and $w \in \mathbb{T}$ the Stolz region is $S(\alpha, w) = \{z \in \mathbb{D} : |z - w| < \alpha(|1 - |z|)\}$ ([9, Definition 8.1.9.]). Since w is an accumulation point of $S(\alpha, w)$ it makes sense to speak about the limit at w of any function $f : S(\alpha, w) \to \mathbb{C}$. Actually, in [9, Theorem 8.1.11], it is proved that if $f \in H^\infty(\mathbb{D})$ the following equality holds on $\mathbb{T}$

$$f^*(w) = \lim_{z \in S(\alpha,w) \to w} f(z),$$

almost everywhere with respect to the Lebesgue measure.

In [4, p. 286 and 287] it is observed that if $g \in H^\infty(\mathbb{C}_+)$, then there exists a Lebesgue null set $A \subset \mathbb{R}$ such that the limit

$$\lim_{r\to 0+} g(r+it) := g^*(it)$$

exists for every $t \in \mathbb{R}\setminus A$ and actually that

$$g^*(it) = \lim_{z\in S(\alpha,\varphi^{-1}(it))\to\varphi^{-1}(it)} T_\varphi(g)(z).$$

In other words, the "horizontal" limits of g exist a.e. and coincide with the Fatou radial limits of its associated function $T_\varphi(g)$ belonging to $H^\infty(\mathbb{D})$.

We can now get the following consequence of Ando's Theorem [1] and Fisher's Theorem 2.

Corollary 16. *The space $H^\infty(\mathbb{C}_+)$ has a unique predual. Moreover, $g \in H^\infty(\mathbb{C}_+)$ with $\|g\|_{\mathbb{C}_+} = 1$ is norm attaining if and only if the set*

$$E := \{t \in \mathbb{R} : \, |g^*(it)| = 1\}$$

has positive (including $+\infty$) Lebesgue measure.

Proposition 17. *$\mathcal{D}^\infty(\mathbb{C}_+)$ is a dual space.*

Proof. By a result of F. Bayart (see e.g. [4, Theorem 3.11]), it is known that if (D_n) is a bounded sequence in $\mathcal{D}^\infty(\mathbb{C}_+)$ then there exists a subsequence (D_{n_k}) and a Dirichlet series $D \in \mathcal{D}^\infty(\mathbb{C}_+)$ such that for every $\sigma > 0$ the sequence (D_{n_k}) converges to D uniformly on $\mathbb{C}_\sigma := \{s \in \mathbb{C}; \operatorname{Re} s \geq \sigma\}$. Thus, if we denote by τ_+ the topology of uniform convergence on these half planes $\mathbb{C}_\sigma$, Bayart's result says that the closed unit ball of $\mathcal{D}^\infty(\mathbb{C}_+)$ is a compact set. Now the Dixmier-Ng theorem [12] implies that

$$\mathcal{G}^\infty(\mathbb{C}_+) := \{R \in \mathcal{D}^\infty(\mathbb{C}_+)^* : \text{the restriction of } R \text{ to } U_{\mathcal{D}^\infty(\mathbb{C}_+)} \text{ is } \tau_+ \text{ continuous}\}, \quad (4)$$

endowed with the topology induced by $\mathcal{D}^\infty(\mathbb{C}_+)^*$ is a predual of $\mathcal{D}^\infty(\mathbb{C}_+)$. □

We can now get a positive result about norm attaining elements of $\mathcal{D}^\infty(\mathbb{C}_+)$ with respect to that predual.

Proposition 18. *Consider the space $\mathcal{D}^\infty(\mathbb{C}_+)$ as the dual of $\mathcal{G}^\infty(\mathbb{C}_+)$. Given $D \in \mathcal{D}^\infty(\mathbb{C}_+)$ of norm one, if the set*

$$E := \{t \in \mathbb{R} : \, |D^*(it)| = 1\}$$

has positive (including $+\infty$) Lebesgue measure, then D is norm attaining.

Proof. As $\mathcal{D}^\infty(\mathbb{C}_+)$ is a closed subspace of $H^\infty(\mathbb{C}_+)$, we can consider $D \in H^\infty(\mathbb{C}_+)$. By Corollary 16, we know that there exists $R \in G^\infty(\mathbb{C}_+)$ such that

$$\|R\| = 1 = R(D).$$

Recall that $R \in H^\infty(\mathbb{C}_+)^*$ and satisfies that the restriction of R to $U_{H^\infty(\mathbb{C}_+)}$ is τ_0 continuous. We denote by S the restriction of R to $\mathcal{D}^\infty(\mathbb{C}_+)$. Since $U_{\mathcal{D}^\infty(\mathbb{C}_+)} \subset U_{H^\infty(\mathbb{C}_+)}$ we have that S is τ_0 continuous when restricted to $U_{\mathcal{D}^\infty(\mathbb{C}_+)}$. The theorem of Bayart [4, Theorem 3.11] implies that $U_{\mathcal{D}^\infty(\mathbb{C}_+)}$ is a compact set with respect to τ_+. The compact open topology τ_0 on $\mathbb{C}_+$ is Hausdorff and weaker than τ_+ on that ball. Hence both topologies coincide on $U_{\mathcal{D}^\infty(\mathbb{C}_+)}$ and $S \in \mathcal{G}^\infty(\mathbb{C}_+)$. Moreover

$$1 = \|R\| \geq \|S\| \geq |S(D)| = S(D) = R(D) = 1,$$

and D attains its norm. □

It is natural to ask whether the converse of Proposition 18 holds. Actually, by the Hahn-Banach theorem one can extend R in $\mathcal{G}^\infty(\mathbb{C}_+)$ to an element T belonging to $H^\infty(\mathbb{C}_+)^*$ with the same norm. But we don't know if it is possible to choose an extension T in $G^\infty(\mathbb{C}_+)$.

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