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ORIGINAL PAPER

Global multiplicity, special closure and non-degeneracy of gradient maps

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Abstract
Given a polynomial map $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ with finite zero set, $p \geqslant n$, we introduce the notion of global multiplicity $\mathrm{m}(F)$ associated to F, which is analogous to the multiplicity of ideals in Noetherian local rings. This notion allows to characterize numerically the Newton non-degeneracy at infinity of F. This fact motivates us to study a combinatorial inequality concerning the normalized volume of global Newton polyhedra and to characterize the corresponding equality using special closures. We also study the Newton non-degeneracy at infinity of gradient maps.

Keywords Complex polynomial map · Milnor number · Multiplicity · Newton polyhedron

Mathematics Subject Classification Primary 32S99; Secondary 14B05 · 13P99

1 Introduction

The study of algebraic, geometrical and topological aspects of polynomial maps is one of the cornerstones of singularity theory, as can be seen in a wide family of articles with different purposes, some of them are [1, 8, 10–12, 14–16, 18]. Related to this fact, we highlight the result of Kouchnirenko [15] where the author shows an estimation of the global Milnor number of a given polynomial function $f \in \mathbb{C}[x_1, \ldots, x_n]$ with finite singular set. This estimation is made by using the combinatorial information supplied by the global Newton polyhedron of f (see Theorem 2.12). Moreover, in [15] a sufficient condition for the corresponding equality

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is given via the Newton non-degeneracy at infinity of f, which is actually a condition on the map $(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. In [6] and [7] we explored this notion when extending it to arbitrary polynomial maps $\mathbb{C}^n \longrightarrow \mathbb{C}^p$ (see Definition 2.10 (h)).

In [6] we characterized the Newton non-degeneracy at infinity of a given polynomial map $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ by means of the maximality of the special closure of F, which is a notion introduced in [5] (see Definition 2.14) inspired by the integral closure of ideals. As we showed in [7], in the case $p = n$, the Newton non-degeneracy of F can be characterized numerically in terms of the colength in $\mathbb{C}[x_1, \dots, x_n]$ of the ideal generated by the components of F (see Theorem 3.8); this characterization does not hold in general if $p > n$. This fact motivated us to investigate which is the correct numerical invariant of F leading to the characterization of this important property of F.

Let us denote by $\mu(F)$ the colength in $\mathbb{C}[x_1, \dots, x_n]$ of the ideal generated by the components of F (see (1)). This number is the tool to define the global Milnor number $\mu_\infty(f)$ of any given polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$ with finite singular set (see (3)). The global Milnor number of f is also called the *total Milnor number* of f in [8] and [14], since it is equal to the sum of the (local) Milnor numbers of f at each isolated singular point of f. Given a polynomial map $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, the number $\mu(F)$ also plays an important role in the general problem of the effective computation of the zero set of F, which in turn is a problem with a wide variety of applications in engineering and other scientific disciplines (see for instance [25]).

Supported by the works [25] and [26] of Sommese and Wampler, in this article we introduce the notion of *global multiplicity* $\mathrm{m}(F)$ for any given polynomial map $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ with finite zero set and $p \geqslant n$. This number allows us to characterize numerically the Newton non-degeneracy at infinity of F. This characterization has also motivated us to show an easily computable lower bound for the normalized n-dimensional volume of a given Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. We characterize the corresponding equality by means of the notion of special closure, thus leading to a significant class of global Newton polyhedra. Given a polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$, the normalized n-dimensional volume of the global Newton polyhedron attached to the gradient of f (instead of the global Newton polyhedron of f) also encodes valuable information. In view of this fact and the relevance of having effective methods to compute global Milnor numbers, we also analyze the Newton non-degeneracy at infinity of the gradient map associated to any given $f \in \mathbb{C}[x_1, \dots, x_n]$ and compare it with the Newton non-degeneracy of the map $(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$.

The article is organized as follows. In Sect. 2, which in turn is subdivided into three subsections, we expose the fundamental results that we need in the subsequent sections. In particular, we define the above-mentioned notion of global multiplicity and we recall some basic facts regarding the notion of Newton non-degeneracy at infinity. In the same section we also review some fundamental properties of the notion of special closure (of a polynomial map) and relate it with the integral closure of ideals and the class of tame polynomial functions (in the sense of Broughton [8] and Némethi–Zaharia [14]).

Using the results of Sect. 2 and [7], in Sect. 3 we show the characterization of the Newton non-degeneracy at infinity of a given polynomial map $\mathbb{C}^n \longrightarrow \mathbb{C}^p$, for $p \geqslant n$, in terms of its global multiplicity (see Theorem 3.8 and Corollary 3.9). In the same section we also introduce the notion of *global reduction*, with analogy with the usual notion of reduction of an ideal in a local ring (see for instance [17, §1]).

Section 4 is devoted to showing the said combinatorial inequality regarding the normalized volume of global Newton polyhedra. The corresponding equality can be characterized in terms of global reductions (see Theorem 4.2). In Sect. 5 we study the condition of Newton non-degeneracy at infinity on a given gradient map, thus leading to a substantial class of global

Newton polyhedra (characterized in Proposition 5.12) for which this condition is generic. We also characterize the class of global Newton polyhedra supporting a polynomial whose gradient is Newton non-degenerate at infinity (Proposition 5.15).

We highlight that, as we indicate in Remark 5.13 (c), the condition of Newton non-degeneracy at infinity of ∇f, for a given polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ with finite zero set, allows to compute effectively the Łojasiewicz exponent at infinity of ∇f (we refer to [16] for an introduction of Łojasiewicz exponent at infinity and their applications in many contexts, as the Jacobian conjecture or the effective Nullstellensatz).

2 Global multiplicity and the special closure of polynomial maps

2.1 Global multiplicity

Let us fix a polynomial map $F = (F_1, \dots, F_p) : \mathbb{C}^n \longrightarrow \mathbb{C}^p$. Given an integer $n \in \mathbb{Z}_{\geqslant 1}$, let us denote $\{1, \dots, n\}$ by $[n]$. If $A \in M_{p\times n}(\mathbb{C})$, $A = [a_{ij}]$, then we denote by F_A the polynomial map $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ whose i-th component function is equal to $F_1a_{1i} + F_2a_{2i} + \cdots + F_pa_{pi}$, for all $i \in [n]$. That is, considering F and F_A as row matrices, the map F_A is equal to the product FA. Equivalently, $F_A = L \circ F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, where L denotes the linear map $\mathbb{C}^p \longrightarrow \mathbb{C}^n$ determined by A. Clearly we have $F^{-1}(0) \subseteq F_A^{-1}(0)$. The relation between the sets $F^{-1}(0)$ and $F_A^{-1}(0)$, for a generic matrix $A \in M_{p\times n}(\mathbb{C})$, has been an object of study in the works [25, §13.5] and [26] of Sommese and Wampler. In these articles, the maps F_A are also called *randomizations* of F. We will be particularly interested in the case $p > n$.

Let us consider the number

$$\mu(F) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} \tag{1}$$

where $\mathbf{I}(F)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the components of F. When $p = n$, we refer to this number as the *multiplicity* of F (as in [7]). This denomination is motivated by the fact that, as explained in (4), in any given Cohen-Macaulay local ring R, the multiplicity (in the sense of Samuel) of any parameter ideal I of R equals the colength of I (see [21, Theorem 17.11]). The number $\mu(F)$ is also usually called the *degree* of F.

We recall the well-known fact that if $F^{-1}(0)$ is finite, then

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} = \sum_{x\in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)} \tag{2}$$

where $\mathcal{O}_{n,x}$ is the ring of analytic function germs $(\mathbb{C}^n, x) \longrightarrow \mathbb{C}$ and $\mathbf{I}_x(F)$ is the ideal of $\mathcal{O}_{n,x}$ generated by the germs at x of the components of F (see [9, p. 150]). We denote $\mathcal{O}_{n,0}$ simply by $\mathcal{O}_n$.

If $f \in \mathbb{C}[x_1, \dots, x_n]$, we denote by ∇f the gradient map of f. That is, the map $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ given by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

The zeros of ∇f are usually called *singular points* of f. If f has a finite number of singular points, then the *global Milnor number of* f, that we denote by $\mu_\infty(f)$, is defined as

$$\mu_\infty(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots x_n]}{\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle} = \mu(\nabla f). \tag{3}$$

Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map with $p > n$. As shown in [25, Theorem 13.5.1] or in [26, Proposition 2.2.1] (which, in turn, are direct applications of the techniques developed in [22]), there exists a non-empty Zariski open subset $U \subseteq M_{p\times n}(\mathbb{C})$ such that, for all $A \in U$, the sets $F_A^{-1}(0)$ and $F^{-1}(0)$ have the same irreducible components of positive dimension. As a direct consequence we obtain the following result.

Corollary 2.1 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map such that $F^{-1}(0)$ is finite. Then, there exists a non-empty Zariski open subset $U \subseteq M_{p\times n}(\mathbb{C})$ such that $F_A^{-1}(0)$ is finite, for all $A \in U$.*

Definition 2.2 Given a property or condition (P_A) depending on a matrix $A \in M_{p\times n}(\mathbb{C})$, for some $p, n \geqslant 1$, we say that (P_A) holds for a *generic* $A \in M_{p\times n}(\mathbb{C})$, when there exists a non-empty Zariski open set $U \subseteq M_{p\times n}(\mathbb{C})$ such that (P_A) holds, for all $A \in U$.

Analogously, if $(R, \mathbf{m})$ denotes a Noetherian local ring whose residue field $k = R/\mathbf{m}$ is infinite and I is an ideal of R of finite colength, then we say that a given property holds for *sufficiently general elements* $h_1, \dots, h_d \in I$ if there exists a generating system $g_1, \dots, g_r$ of I and a non-empty Zariski-open set U in k^{rd} such that the d-tuple $(h_1, \dots, h_d) \in I \oplus \cdots \oplus I$ satisfies the said property provided that

(a) for all $i \in [d]$: $h_i = \sum_j u_{ij} g_j$, where $u_{ij} \in R$, for all $j \in [r]$, and
(b) the image of $(u_{11}, \dots, u_{1r}, \dots, u_{d1}, \dots, u_{dr})$ in k^{rd} belongs to U.

As a consequence of the lower semicontinuity of the number defined in (1), Corollary 2.1 also tells us that, if $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is a polynomial map such that $F^{-1}(0)$ finite, then $\mu(F_A)$ remains constant, for a generic $A \in M_{p\times n}(\mathbb{C})$, and equal to the maximum possible value of $\mu(F_A)$ whenever $F_A^{-1}(0)$ is finite. Let us denote this constant by $\mathrm{m}(F)$. We will usually refer to $\mathrm{m}(F)$ as the *global multiplicity* of F. Obviously, we have $\mu(F) = \mathrm{m}(F)$ if $p = n$.

Example 2.3 Let us fix an integer $a \geqslant 2$. Let us consider the polynomial map $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ given by $F(x, y) = (x^a, xy, y^a)$. Then $\mu(F) = 2a - 1$ and $\mathrm{m}(F) = a^2$. When $a = 1$, then $\mu(F) = 1$ and $\mathrm{m}(F) = 2$.

If $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is a polynomial map, let us denote by $\mathbf{L}(F)$ the $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1, \dots, x_n]$ generated by the component functions of F.

Lemma 2.4 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map such that $F^{-1}(0)$ is finite. If* $\dim_{\mathbb{C}} \mathbf{L}(F) = n$, *then* $\mathrm{m}(F) = \mu(F)$.

Proof Since $F^{-1}(0)$ is finite, we have $p \geqslant n$. Let us suppose that $\dim_{\mathbb{C}} \mathbf{L}(F) = n$. Then for any matrix $A \in M_{p\times n}$ of rank n we have $\mathbf{L}(F) = \mathbf{L}(F_A)$. In particular $\mathbf{I}(F) = \mathbf{I}(F_A)$ and therefore $\mu(F) = \mu(F_A)$. Since the equality $\mathrm{m}(F) = \mu(F_A)$ holds, by definition, for any A in a non-empty Zariski-open subset of $M_{p\times n}(\mathbb{C})$, we conclude that $\mathrm{m}(F) = \mu(F)$. □

We characterize the equality $\mu(F) = \mathrm{m}(F)$ in Corollary 2.9.

Remark 2.5 Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map with $p > n$ and let us take a matrix $A \in M_{p\times n}(\mathbb{C})$. Let us denote by B the $n \times n$ submatrix of A formed by the first n rows and first n columns of A and let us suppose that B is invertible. Let us consider the maps F and F_A as row matrices. Hence, multiplying both sides of the equality $FA = F_A$ by B^{-1} (on the right), we obtain a row matrix G of the form

$$\left[\, F_1 + c_{n+1,1}F_{n+1} + \cdots + c_{p,1}F_p \quad \cdots \quad F_n + c_{n+1,n}F_{n+1} + \cdots + c_{p,n}F_p \,\right]$$

for some matrix of coefficients $C = [c_{ij}] \in M_{(p-n)\times n}(\mathbb{C})$. Let us consider the above row matrix as a polynomial map $G : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Obviously, for a generic $A \in M_{p\times n}(\mathbb{C})$, we have $\mathbf{I}(G) = \mathbf{I}(F_A) \subseteq \mathbf{I}(F)$ and $\mu(F_A) = \mathrm{m}(F)$. In particular $\mu(F) \leqslant \mu(F_A) = \mu(G) = \mathrm{m}(F)$.

Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be an ideal of R of finite colength. Let d denote the dimension of R. We denote by $e(I)$ the multiplicity of I, in the sense of Samuel (see for instance [17, §11], [21, §14] or [29, §2]). By the theorem of existence of reductions (see [21, Theorem 14.14]), we have that $e(I) = e(h_1, \ldots, h_d)$, for sufficiently general elements $h_1, \ldots, h_d \in I$. We recall that if $J \subseteq I$ is an ideal of R generated by d elements and $e(I) = e(J)$, then J is called a *minimal reduction* of I. If in addition we assume that R is Cohen-Macaulay then, by [21, Theorem 17.11], we obtain that

$$e(I) = e(h_1, \ldots, h_d) = \ell(R/\langle h_1, \ldots, h_d\rangle) \geqslant \ell(R/I) \tag{4}$$

where $\ell(M)$ denotes the length of any given R-module M.

Therefore, we find that if $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is a polynomial map such that $F^{-1}(0)$ is finite and $A \in M_{p\times n}(\mathbb{C})$ verifies that $F_A^{-1}(0)$ is finite, then

$$\mu(F) = \sum_{x\in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)} \leqslant \sum_{x\in F^{-1}(0)} e\big(\mathbf{I}_x(F)\big) \tag{5}$$

$$\leqslant \sum_{x\in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F_A)} \leqslant \sum_{x\in F_A^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F_A)} = \mu(F_A) \leqslant \mathrm{m}(F). \tag{6}$$

If $f \in \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we denote by $\deg(f)$ the degree of f. If $F = (F_1, \ldots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, we define $\deg(F) = \max\{\deg(F_1), \ldots, \deg(F_p)\}$. We denote by $\mathrm{d}(F)$ the vector formed by the degrees of the components of F.

Example 2.6 Let us consider the map $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ given by

$$F(x, y) = \big(x^2 - y^2,\ xy(x - y),\ x^2 - 3y\big)$$

for all $(x, y) \in \mathbb{C}^2$. It is easy to see that $F^{-1}(0) = \{(0, 0), (3, 3)\}$. By using Singular [13], we have $\mu(F) = 3$. In order to obtain the value of $\mathrm{m}(F)$ we apply Remark 2.5. That is, let us consider the map $G : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by

$$G(x, y) = \big(x^2 - y^2 + \alpha(x^2 - 3y),\ xy(x - y) + \beta(x^2 - 3y)\big)$$

for generic $\alpha, \beta \in \mathbb{C}$. We observe that this map can be rewritten as the sum $H_1 + H_2$, where H_1 and H_2 are given by

$$H_1(x, y) = \big(x^2 - y^2 + \alpha x^2,\ xy(x - y)\big)$$
$$H_2(x, y) = \big(-3\alpha y,\ \beta(x^2 - 3y)\big).$$

The map H_1 is homogeneous with finite zero set, $\mathrm{d}(H_1) = (2, 3)$ and $\mathrm{d}(H_2) = (1, 2)$. Therefore $\mathrm{m}(F) = \mu(G) = 6$, by [7, Corollary 4.10].

Remark 2.7 Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a homogeneous map with $F(0) = 0$. We recall that if $F^{-1}(0)$ is finite, then the homogeneity of F forces that $F^{-1}(0) = \{0\}$. Therefore $\mu(F) = \dim_{\mathbb{C}} \mathcal{O}_n/I$, where I is the ideal of $\mathcal{O}_n$ generated by $F_1, \dots, F_n$.

Example 2.8 Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map, $p \geqslant n$, and let $d \in \mathbb{Z}_{\geqslant 1}$. Let us suppose that F_i is a homogeneous polynomial with $\deg(F_i) = d$, for all $i \in [p]$. If $F^{-1}(0)$ is finite (which is equivalent to saying that $F^{-1}(0) = \{0\}$, since F is homogeneous), then $\mathrm{m}(F) = d^n$.

Moreover, let us assume that each F_i is a homogeneous polynomial with $d_i = \deg(F_i)$, for all $i \in [p]$, and $d_1 \geqslant \cdots \geqslant d_n > d_{n+1} \geqslant \cdots \geqslant d_p$. Let $G = (F_1, \dots, F_n)$, that is, G is the map formed by the first n components of F. If $G^{-1}(0)$ is finite, then $\mathrm{m}(F) = \mu(G) = d_1 \cdots d_n$, by [7, Corollary 4.10].

Corollary 2.9 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map such that $F^{-1}(0)$ is finite. Then $\mu(F) = \mathrm{m}(F)$ if and only if for any $A \in M_{p\times n}(\mathbb{C})$ such that $F_A^{-1}(0)$ is finite we have $\mathrm{m}(F) = \mu(F_A)$, $F^{-1}(0) = F_A^{-1}(0)$ and $\mathbf{I}_x(F) = \mathbf{I}_x(F_A)$, for all $x \in F^{-1}(0)$.*

Proof Let us suppose that $\mathrm{m}(F) = \mu(F)$ and let $A \in M_{p\times n}(\mathbb{C})$ such that $F_A^{-1}(0)$ is finite. Then all inequalities of (5) and (6) become equalities, in particular $\mu(F_A) = \mathrm{m}(F)$. Let us remark that $\mathbf{I}_x(F_A)$ is contained in the maximal ideal of $\mathcal{O}_{n,x}$ for all $x \in F_A^{-1}(0)$, hence $\dim_{\mathbb{C}} \mathcal{O}_{n,x}/\mathbf{I}_x(F_A) \geqslant 1$, for all $x \in F_A^{-1}(0)$. Therefore $F^{-1}(0) = F_A^{-1}(0)$ and moreover

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)} = e\big(\mathbf{I}_x(F)\big) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F_A)} \tag{7}$$

for any $x \in F^{-1}(0)$. In particular, $\mathbf{I}_x(F) = \mathbf{I}_x(F_A)$, for all $x \in F^{-1}(0)$.

Moreover, by the theorem of existence of reductions (see [21, Theorem 14.14]) and relation (4), we conclude that (7) implies that $\mathbf{I}_x(F)$ is equal to any of its minimal reductions, for all $x \in F^{-1}(0)$.

The converse follows easily by applying the chain of inequalities (5) and (6). $\square$

2.2 The Newton non-degeneracy at infinity

Let us fix coordinates $(x_1, \dots, x_n) \in \mathbb{K}^n$ and let $f \in \mathbb{K}[x_1, \dots, x_n]$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by ∇f the gradient map of f. That is, the map $\mathbb{K}^n \longrightarrow \mathbb{K}^n$ given by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

We also define the map $\mathrm{G}(f) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ given by

$$\mathrm{G}(f) = \left(x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n}\right). \tag{8}$$

Let us observe that

$$\begin{gathered}(\nabla f)^{-1}(0) \subseteq \mathrm{G}(f)^{-1}(0)\\ (\nabla f)^{-1}(0) \cap \big(\mathbb{C} \smallsetminus \{0\}\big)^n = \mathrm{G}(f)^{-1}(0) \cap \big(\mathbb{C} \smallsetminus \{0\}\big)^n.\end{gathered} \tag{9}$$

If $k = (k_1, \dots, k_n) \in \mathbb{Z}_{\geqslant 0}$, then we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ simply by x^k.

For the sake of completeness, here we recall some basic definitions from [6] and [7].

Definition 2.10 (a) Let $A \subseteq \mathbb{R}^n_{\geqslant 0}$ be a finite set. The *global Newton polyhedron determined by* A, denoted by $\widetilde{\Gamma}_+(A)$, is the convex hull of $A \cup \{0\}$. We say that a given subset $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is a *global Newton polyhedron* when there exists some finite $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(A)$.

(b) Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global Newton polyhedron. If $\langle , \rangle$ denotes the standard scalar product in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then we define $\ell(v, \widetilde{\Gamma}_+) = \min\{\langle v, k\rangle : k \in \widetilde{\Gamma}_+\}$. Let $\Delta(v, \widetilde{\Gamma}_+)$ denote the set of points of $\widetilde{\Gamma}_+$ where the minimum $\ell(v, \widetilde{\Gamma}_+)$ is attained. A *face* of $\widetilde{\Gamma}_+$ is any subset of $\widetilde{\Gamma}_+$ of the form $\Delta(v, \widetilde{\Gamma}_+)$, where $v \in \mathbb{R}^n \setminus \{0\}$. If Δ is a face of $\widetilde{\Gamma}_+$, then the *dimension* of Δ is defined as the minimum among the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. Hence the *dimension* of $\widetilde{\Gamma}_+$, denoted by $\dim(\widetilde{\Gamma}_+)$, is the maximum of the dimensions of the faces of $\widetilde{\Gamma}_+$ not passing through the origin.

(c) The *global boundary of* $\widetilde{\Gamma}_+$, denoted by $\widetilde{\Gamma}$, is the union of the faces of $\widetilde{\Gamma}_+$ not containing the origin.

(d) The 0-dimensional faces of $\widetilde{\Gamma}_+$ are called *vertices* of $\widetilde{\Gamma}_+$ and the faces of dimension $n-1$ of $\widetilde{\Gamma}_+$ are the *facets* of $\widetilde{\Gamma}_+$. We denote by $\mathbf{v}(\widetilde{\Gamma}_+)$ the set of vertices of $\widetilde{\Gamma}_+$.

(e) We say that $\widetilde{\Gamma}_+$ is *convenient* when for any $i \in [n]$, there exists some $r > 0$ such that $re_i \in \widetilde{\Gamma}_+$, where $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$.

(f) Let $f \in \mathbb{K}[x_1, \dots, x_n]$, let us write $f = \sum_k a_k x^k$. The *support* of f is $\mathrm{supp}(f) = \{k : a_k \neq 0\}$. If A denotes any compact subset of $\mathbb{R}^n_{\geqslant 0}$, then we denote by f_A the sum of all terms $a_k x^k$ such that $k \in A$, whenever $\mathrm{supp}(f) \cap A \neq \emptyset$. If $\mathrm{supp}(f) \cap A = \emptyset$, then we set $f_A = 0$. The *global Newton polyhedron of* f, also called *Newton polyhedron at infinity of* f, is $\widetilde{\Gamma}_+(\mathrm{supp}(f))$, which we will also denote simply by $\widetilde{\Gamma}_+(f)$.

(g) If $F = (F_1, \dots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, the *support of* F is defined as $\mathrm{supp}(F) = \mathrm{supp}(F_1) \cup \cdots \cup \mathrm{supp}(F_p)$. The *global Newton polyhedron of* F, denoted by $\widetilde{\Gamma}_+(F)$, is the convex hull of $\mathrm{supp}(F) \cup \{0\}$, which in turn is equal to the convex hull of $\widetilde{\Gamma}_+(F_1) \cup \cdots \cup \widetilde{\Gamma}_+(F_p)$. If A denotes any compact subset of $\mathbb{R}^n_{\geqslant 0}$, then we denote the map $((F_1)_A, \dots, (F_p)_A) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ by F_A.

(h) Let $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ be a polynomial map, $p \geqslant 2$. We say that F is *Newton non-degenerate at infinity*, when for any face Δ of $\widetilde{\Gamma}_+(F)$ not passing through the origin we have $F_\Delta^{-1}(0) \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}$.

(i) Given a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$, we say that f is *Newton non-degenerate at infinity* when the map $\mathrm{G}(f) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ is Newton non-degenerate at infinity. We remark that $\widetilde{\Gamma}_+(\mathrm{G}(f)) = \widetilde{\Gamma}_+(f)$.

(j) If V denotes a finite-dimensional vector subspace of $\mathbb{K}[x_1, \dots, x_n]$ and $F : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is a polynomial map whose component functions generate V, then the *global Newton polyhedron of* V is defined as $\widetilde{\Gamma}_+(V) = \widetilde{\Gamma}_+(F)$. It is immediate to see that the definition of $\widetilde{\Gamma}_+(V)$ does not depend on the chosen generating system of V.

Given a subset $\mathtt{I} \subseteq [n]$, we define $\mathbb{R}^n_{\mathtt{I}} = \{x \in \mathbb{R}^n : x_i = 0, \text{ for all } i \notin \mathtt{I}\}$. Hence, if $S \subseteq \mathbb{R}^n$, we set $S_{\mathtt{I}} = S \cap \mathbb{R}^n_{\mathtt{I}}$. In particular $S_{[n]} = S$ and $S_\emptyset = S \cap \{0\}$. Depending on the notation involved, we will also denote $S_{\mathtt{I}}$ by $S^{\mathtt{I}}$, as can be seen in (10) and in Corollary 4.4 (b).

If $P \subseteq \mathbb{R}^n$, then we denote by $\mathrm{V}_n(P)$ the n-dimensional volume of P. Given a finite set X, we denote the cardinal of X by $|X|$.

Definition 2.11 [15, p. 4] Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global convenient Newton polyhedron. The *Newton number of* $\widetilde{\Gamma}_+$ is defined as

$$\nu(\widetilde{\Gamma}_+) = \sum_{r=0}^{n}(-1)^{n-r} r! \sum_{\substack{\mathtt{I}\subseteq[n]\\ |\mathtt{I}|=r}} \mathrm{V}_r(\widetilde{\Gamma}_+^{\mathtt{I}}). \tag{10}$$

For instance, let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^3_{\geqslant 0}$ be the global Newton polyhedron given by

$$\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^a, y^b, z^c, x^a y^b, x^a z^c, y^b z^c, x^a y^b z^c),$$

for some $a, b, c \in \mathbb{Z}_{\geqslant 1}$. That is, $\widetilde{\Gamma}_+$ is the rectangular cuboid determined by the points ae_1, be_2 and ce_3. Then $\nu(\widetilde{\Gamma}_+) = 6abc - 2(ab + ac + bc) + a + b + c - 1$. If we take $\widetilde{\Gamma}'_+ = \widetilde{\Gamma}_+(x^a, y^b, z^c, x^a y^b z^c)$, then $\nu(\widetilde{\Gamma}'_+) = 3abc - ab - ac - bc + a + b + c - 1$.

Theorem 2.12 [15, 1.15] *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ with a finite number of singular points and such that $\widetilde{\Gamma}_+(f)$ is convenient. Then $\mu_\infty(f) \leqslant \nu(\widetilde{\Gamma}_+(f))$ and equality holds if f is Newton non-degenerate at infinity.*

If $\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^p_+$ are global Newton polyhedra in $\mathbb{R}^n_{\geqslant 0}$, then we denote by $\mathbf{P}(\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^p_+)$ the set of polynomial maps $\mathbb{C}^n \longrightarrow \mathbb{C}^p$ for which $\widetilde{\Gamma}_+(F_i) = \widetilde{\Gamma}^i_+$, for all $i \in [p]$. In particular, for a fixed global Newton polyhedron $\widetilde{\Gamma}_+$, we have $\mathbf{P}(\widetilde{\Gamma}_+) = \{f \in \mathbb{C}[x_1, \dots, x_n] : \widetilde{\Gamma}_+(f) = \widetilde{\Gamma}_+\}$.

Let $P_1, \dots, P_n$ be a collection of polytopes of $\mathbb{R}^n$. We denote by $\mathrm{MV}_n(P_1, \dots, P_n)$ the mixed volume of $P_1, \dots, P_n$ (see for instance [9, p. 337]). Let us recall that if $\lambda_1, \dots, \lambda_n \in [0, +\infty[$, then $\mathrm{V}_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a homogeneous polynomial function in $\lambda_1, \dots, \lambda_n$ and that $\mathrm{MV}_n(P_1, \dots, P_n)$ is defined as the coefficient of $\lambda_1 \cdots \lambda_n$ in this polynomial (see [9, p. 337]). The mixed volume $\mathrm{MV}_n(P_1, \dots, P_n)$ admits the following expression:

$$\mathrm{MV}_n(P_1, \dots, P_n) = \sum_{r=1}^{n}(-1)^{n-r} \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} \mathrm{V}_n(P_{i_1} + \cdots + P_{i_r}).$$

(see for instance [9, p. 338]). In particular, when $P_1 = \cdots = P_n$, then $\mathrm{MV}_n(P_1, \dots, P_n) = n!\mathrm{V}_n(P_1)$. If P is a polytope, then we refer to $n!\mathrm{V}_n(P)$ as the *normalized n-dimensional volume* of $\widetilde{\Gamma}_+$, which is always an integer (see [9, p. 336]).

Theorem 2.13 [24] *Let $\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^n_+$ be a family of n global Newton polyhedra in $\mathbb{R}^n_{\geqslant 0}$ and let $\widetilde{\Gamma}_+$ be the convex hull of $\widetilde{\Gamma}^1_+ \cup \cdots \cup \widetilde{\Gamma}^n_+$. Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a map belonging to $\mathbf{P}(\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^n_+)$ such that $F^{-1}(0)$ is finite. Then*

$$\mu(F) \leqslant \mathrm{MV}_n(\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^n_+) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+).$$

As a direct consequence of the previous result we have that if $f \in \mathbb{C}[x_1, \dots, x_n]$ has a finite number of singular points, then

$$\mu_\infty(f) \leqslant \mathrm{MV}_n\left(\widetilde{\Gamma}_+\left(\frac{\partial f}{\partial x_1}\right), \dots, \widetilde{\Gamma}_+\left(\frac{\partial f}{\partial x_n}\right)\right) \leqslant n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big).$$

Comparing the above relation and Theorem 2.12 there arises the problem of comparing the numbers $n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big)$ and $\nu(\widetilde{\Gamma}_+(f))$. We analyze the relations between them in Sect. 5.

2.3 The special closure

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If (P_x) denotes a condition depending on $x \in \mathbb{K}^n$, then we say that (P_x) *holds for all* $\|x\| \gg 1$ when there exists a constant $M > 0$ such that (P_x) holds for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Analogously, we say that (P_x) *holds for all* $\|x\| \ll 1$ when there exists some open neighbourhood of $0 \in \mathbb{K}^n$ such that (P_x) holds for all $x \in U$. Along this subsection we fix coordinates $(x_1, \dots, x_n)$ in $\mathbb{K}^n$. The following definition was introduced by the authors in [5] motivated by the notion of integral closure of ideals and its characterization in terms of analytic inequalities proven by Lejeune and Teissier in [19] (see also [20]).

Definition 2.14 Given a polynomial map $F = (F_1, \dots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ and $h \in \mathbb{K}[x_1, \dots, x_n]$, we say that h is *special with respect to* F when

$$|h(x)| \leqslant C\|F(x)\| \tag{11}$$

for all $\|x\| \gg 1$ and some constant $C > 0$.

We denote by $\mathrm{Sp}(F)$, or by $\mathrm{Sp}(F_1, \dots, F_p)$, the set of all polynomials $h \in \mathbb{K}[x_1, \dots, x_n]$ such that h is special with respect to F. We will refer to $\mathrm{Sp}(F)$ as the *special closure of* F. We remark that $\mathrm{Sp}(F)$ is a $\mathbb{K}$-vector subspace of $\mathbb{K}[x_1, \dots, x_n]$ containing $\mathbf{L}(F)$. Analogous with the usual notion of integrally closed ideal, if $\mathbf{L}(F) = \mathrm{Sp}(F)$ then we say that F is *specially closed*.

More generally, if V denotes any $\mathbb{K}$-vector subspace of $\mathbb{K}[x_1, \dots, x_n]$ and $F_1, \dots, F_p$ is any generating system of V, then we define the *special closure of* V as $\mathrm{Sp}(V) = \mathrm{Sp}(F)$, where F denotes the map $F = (F_1, \dots, F_p)$. If $G : \mathbb{K}^n \longrightarrow \mathbb{K}^q$ is any other polynomial map such that $\mathbf{L}(G) = V$, then it is easy to prove that there exist constants $C, D > 0$ such that $\|F(x)\| \leqslant C\|G(x)\| \leqslant D\|F(x)\|$, for all $x \in \mathbb{K}^n$. Therefore the definition of $\mathrm{Sp}(V)$ does not depend on the chosen generating system of V. It is immediate to check that $\mathrm{Sp}(V)$ is a $\mathbb{K}$-vector subspace of $\mathbb{K}[x_1, \dots, x_n]$ containing V. We say that V is *specially closed* when $V = \mathrm{Sp}(V)$.

Obviously, if V and V' are $\mathbb{K}$-vector subspaces of $\mathbb{K}[x_1, \dots, x_n]$ such that $V \subseteq V'$, then $\mathrm{Sp}(V) \subseteq \mathrm{Sp}(V')$. Moreover, if $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, then $\mathrm{Sp}(F) = \mathrm{Sp}(\mathbf{L}(F))$.

Before stating the next result, we introduce some fundamental definitions. If $X \subseteq \mathbb{R}^n$, then we denote by $\mathrm{Conv}(X)$ the convex hull of X in $\mathbb{R}^n$.

Let us fix coordinates $(x_1, \dots, x_n)$ in $\mathbb{K}^n$. If $F = (F_1, \dots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, then we denote by $\mathbf{S}(F)$ the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $x^k \in \mathrm{Sp}(F)$. We remark that the set $\mathbf{S}(F)$ depends on the given coordinate system. We will refer to $\mathbf{S}(F)$ as the *monomial zone* of $\mathrm{Sp}(F)$. Obviously, if $\mathbf{S}(F)$ contains some point different from 0, then there exists an $M > 0$ such that

$$F^{-1}(0) \cap \{x \in \mathbb{K}^n : \|x\| \geqslant M\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}. \tag{12}$$

We denote by $\mathbf{S}'(F)$ the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $x^k \in \mathrm{Sp}(F, 1)$. That is, $\mathbf{S}'(F) = \mathbf{S}(F, 1)$. Since $1 \in \mathrm{Sp}(F, 1)$, it follows that 0 always belongs to $\mathbf{S}'(F)$. Obviously $\mathbf{S}(F) \subseteq \mathbf{S}'(F)$ and equality holds if and only if $0 \in \mathbf{S}(F)$.

In the next result we summarize some of the main properties of $\mathrm{Sp}(F)$.

Theorem 2.15 *Let* $F = (F_1, \dots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ *be a polynomial map.*

(a) $\mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)$*, for any* $h \in \mathrm{Sp}(F)$*. In particular* $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F)$*,* $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+(\mathrm{Sp}(F))$ *and* $\mathrm{Sp}(F)$ *is a finite-dimensional* $\mathbb{K}$*-vector subspace of* $\mathbb{K}[x_1, \dots, x_n]$.

(b) $\mathbf{S}(F) = \mathrm{Conv}(\mathbf{S}(F)) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) *If $G : \mathbb{K}^n \longrightarrow \mathbb{K}^q$ is another polynomial map, then the following conditions are equivalent:*

   (a) $\mathrm{Sp}(F) \subseteq \mathrm{Sp}(G)$.
   (b) $F_i \in \mathrm{Sp}(G)$, *for all* $i \in [p]$.
   (c) *there exists a constant* $C > 0$ *such that* $\|F(x)\| \leqslant C\|G(x)\|$, *for all* $\|x\| \gg 1$.

(d) *Given any* $h \in \mathbb{K}[x_1, \ldots, x_n]$, *we have* $h \in \mathrm{Sp}(F) \Longleftrightarrow \mathrm{Sp}(F) = \mathrm{Sp}(F, h)$.
(e) $\mathrm{Sp}(F) = \mathrm{Sp}\big(\mathrm{Sp}(F)\big)$.
(f) *If* $\mathbb{K} = \mathbb{C}$, $p = n$, $F^{-1}(0)$ *is finite and* h *is any non-zero polynomial of* $\mathbb{C}[x_1, \ldots, x_n]$, *then* $h \in \mathrm{Sp}(F)$ *if and only if* $\mu(F) = \mu(F + h\alpha)$, *for all* $\alpha \ll 1$.

Proof Items (a) and (b) are in [6, Lemma 3.4]. Items (c), (d) and (e) follow as direct applications of the definition of special closure. Item (f) is in [6, Theorem 3.8]. □

By Theorem 2.15 (a), we have $\dim_{\mathbb{K}} \mathrm{Sp}(F) \leqslant |\widetilde{\Gamma}_+ \cap \mathbb{Z}^n_{\geqslant 0}|$. When equality holds, that is, when $\mathbf{S}(F) = \widetilde{\Gamma}_+(F)$, then we will say that F *has maximal special closure*.

In the following result we relate the notions of special closure of a polynomial map and integral closure of an ideal. Let $g \in \mathbb{K}[x_1, \ldots, x_n]$ and let d be an integer with $d \geqslant \deg(g)$. Let x_{n+1} a variable independent from $x_1, \ldots, x_n$. We define the *homogenization of g of degree d* as the polynomial $\mathrm{H}_d(g) \in \mathbb{K}[x_1, \ldots, x_n, x_{n+1}]$ such that

$$\mathrm{H}_d(g)(x_1, \ldots, x_n, x_{n+1}) = x_{n+1}^d g\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$$

for any $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{K}^{n+1}$ with $x_{n+1} \neq 0$. Once the number d is fixed, we will also denote $\mathrm{H}_d(g)$ simply g^*. Let us remark that $\mathrm{H}_d(g)(x_1, \ldots, x_n, 1) = g(x_1, \ldots, x_n)$, for all $(x_1, \ldots, x_n) \in \mathbb{K}^n$.

Let $F = (F_1, \ldots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ be a polynomial map. We recall that $\deg(F)$ denotes the maximum of the degrees of the components of F. If d is an integer with $d \geqslant \deg(F)$, then $\mathrm{H}_d(F)$ is the map $\mathbb{K}^{n+1} \longrightarrow \mathbb{K}^p$ given by $\mathrm{H}_d(F) = (\mathrm{H}_d(F_1), \ldots, \mathrm{H}_d(F_p))$. Analogously, we will also denote the map $\mathrm{H}_d(F)$ by F^*.

Given any $d \in \mathbb{Z}_{\geqslant 1}$, let $\mathbb{K}_d[x_1, \ldots, x_n]$ denote the vector space of all homogeneous polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ of degree d. As usual, if I is an ideal of the ring of analytic germs $(\mathbb{K}, 0) \longrightarrow \mathbb{K}$, we denote by $\overline{I}$ the integral closure of I (see [6, Remark 3.2]).

Theorem 2.16 *Let $F = (F_1, \ldots, F_p) : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ be a polynomial map. Let $d = \deg(F)$ and let $f^* = \mathrm{H}_d(f)$, for any given polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ such that $d \geqslant \deg(f)$. Let I be the ideal of $\mathcal{O}_{n+1}$ generated by the polynomials $F_1^*, \ldots, F_p^*$. Given an $h \in \mathbb{K}[x_1, \ldots, x_n]$, if $h^* \in \overline{I}$, then $h \in \mathrm{Sp}(F)$. In particular we have the following inclusion*

$$\left\{g(x_1, \ldots, x_n, 1) : g \in \overline{\langle F_1^*, \ldots, F_p^* \rangle},\ g \in \mathbb{K}_d[x_1, \ldots, x_{n+1}]\right\} \subseteq \mathrm{Sp}(F). \tag{13}$$

Proof Let us consider the map $\rho : \mathbb{K}^n \longrightarrow \mathbb{K}^{n+1}$ given by

$$\rho(x_1, \ldots, x_n) = \left(e^{-\|x\|}x_1, \ldots, e^{-\|x\|}x_n, e^{-\|x\|}\right)$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. Let us observe that this map verifies

$$\lim_{\|x\| \to +\infty} \|\rho(x)\| = 0. \tag{14}$$

By definition, the condition $h^* \in \overline{I}$ says that there exists an open neighbourhood U of $0 \in \mathbb{K}^n$ for which there exists a constant $C > 0$ such that

$$|h^*(x_1, \dots, x_n, x_{n+1})| \leqslant C\|F^*(x_1, \dots, x_n, x_{n+1})\|$$

for all $(x_1, \dots, x_{n+1}) \in U$. By (14), let $M > 0$ such that $\rho(x) \in U$, for all $x \in \mathbb{K}^n$ with $\|x\| \geqslant M$. Hence

$$|h^*(\rho(x))| \leqslant C\|F^*(\rho(x))\| \tag{15}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Since h^* and the components of F^* are homogeneous polynomials of degree d, we conclude that

$$h^*(\rho(x)) = h^*(e^{-\|x\|}x_1, \dots, e^{-\|x\|}x_n, e^{-\|x\|}) = e^{-d\|x\|}h^*(x_1, \dots, x_n, 1) = e^{-d\|x\|}h(x_1, \dots, x_n)$$
$$F^*(\rho(x)) = e^{-d\|x\|}F^*(x_1, \dots, x_n, 1) = e^{-d\|x\|}F(x_1, \dots, x_n).$$

Substituting the above relations in (15) and cancelling the term $e^{-d\|x\|}$, we conclude that $|h(x)| \leqslant C\|F(x)\|$, for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$.

The inclusion (13) is a direct application of the implication $h^* \in \overline{I} \Longrightarrow h \in \mathrm{Sp}(F)$. □

As shown in the following example, the converse of Theorem 2.16 does not hold in general.

Example 2.17 Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the polynomial map with $\deg(F) = 4$ given by $F(x, y) = (x^3y, xy^3) = xy(x^2, y^2)$. Let $g^* = \mathrm{H}_4(g)$, for any $g \in \mathbb{C}[x, y]$ with $\deg(g) \leqslant 4$. We observe that the homogenized map $F^* : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ is given by $F^*(x, y, z) = (x^3y, xy^3)$. Let $h = xy \in \mathbb{C}[x, y]$. It is straightforward to see that $h \in \mathrm{Sp}(F)$. We observe that $h^* = xyz^2$, which does not belong to the Newton polyhedron of the ideal I of $\mathcal{O}_3$ generated by the components of F^*. In particular, $h^* \notin \overline{I}$.

It is easy to see that $\mathrm{Sp}(F)$ is equal to the vector subspace of $\mathbb{C}[x, y]$ generated by the monomials $xy, xy^2, xy^3, x^2y, x^2y^2, x^3y$ (see also [6, Lemma 4.11]). Therefore the inclusion (13) can be strict.

The following equivalence was proven in [6, Proposition 4.8].

Proposition 2.18 *Under the conditions of Theorem 2.16, given a polynomial $h \in \mathbb{K}[x_1, \dots, x_n]$, the following conditions are equivalent:*

(a) $h \in \mathrm{Sp}(F, 1)$
(b) $h^* \in \overline{J}$, *where J is the ideal of $\mathcal{O}_{n+1}$ generated by $F_1^*, \dots, F_p^*, x_{n+1}^d$.*

As a consequence

$$\mathrm{Sp}(F, 1) = \left\{ g(x_1, \dots, x_n, 1) : g \in \overline{\langle F_1^*, \dots, F_p^*, x_{n+1}^d \rangle},\ g \in \mathbb{K}_d[x_1, \dots, x_n] \right\}. \tag{16}$$

Remark 2.19 It is known that, in general, if $f \in \mathcal{O}_n$ and $f(0) = 0$, then $f \in \overline{I(f)}$, where $I(f)$ denotes the ideal of $\mathcal{O}_n$ generated by

$$x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n}.$$

The proof of this fact can be found in [17, p. 144] and [28, p. 290]. In particular, f is integral over $J(f)$. In view of this result and the existing analogies between the notions of special and integral closures, we would expect that any $f \in \mathbb{K}[x_1, \dots, x_n]$ such that $f(0) = 0$ is special with respect to ∇f, but this is not the case, as the following example shows. Let

$f(x, y) = x^3 + y^2 + x^4y^5 \in \mathbb{C}[x, y]$. We observe that $\widetilde{\Gamma}_+(\nabla f) = \widetilde{\Gamma}_+(x^2, x^3y^5, y, x^4y^4)$. Hence $\widetilde{\Gamma}_+(f)$ is not contained in $\widetilde{\Gamma}_+(\nabla f)$, which shows that $f \notin \mathrm{Sp}(\nabla f)$, by Theorem 2.15(a).

We also remark that if $f \in \mathbb{C}[x_1, \dots, x_n]$ then f is not special over $\mathrm{G}(f)$ in general, as the function $f(x, y) = xy + x^2y^2 \in \mathbb{C}[x, y]$ shows.

We end this subsection relating the notion of tame map with the special closure. We remark that tameness is an important condition on the gradient of a given polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$ leading to results about the bifurcation set B_f and the homotopy type of the generic fiber of f (see [8] and [14]).

Definition 2.20 If $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, $p \geqslant 1$, then we say that F is a *tame map* when there exists some $\delta > 0$ and a compact set U such that $\delta < \|F(x)\|$, for all $x \in \mathbb{K}^n \smallsetminus U$. In other words, F is tame if and only if there exists some $\delta > 0$ such that the set $\{x \in \mathbb{K}^n : \|F(x)\| \leqslant \delta\}$ is compact.

We remark that, given a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, when the gradient map $\nabla f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is tame then in [8, Definition 3.1] the function f is also called tame.

Corollary 2.21 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map. Then the following conditions are equivalent:*

(a) *F is a tame map.*
(b) $1 \in \mathrm{Sp}(F)$.
(c) $0 \in \mathbf{S}(F)$. *When $p = n$, then the above conditions are equivalent to the following:*
(d) *$\mu(F)$ is finite and $\mu(F) = \mu(F + (\alpha_1, \dots, \alpha_n))$, for all sufficiently small $\alpha \in \mathbb{C}^n$.*

Proof The equivalence between (a), (b) and (c) follows as a direct consequence of the corresponding definitions. The equivalence between (b) and (d) is a direct application of Theorem 2.15(f). □

Remark 2.22 Let $f \in \mathbb{C}[x_1, \dots, x_n]$. If we apply Corollary 2.21 to the gradient map of f we deduce the characterization of the tameness of ∇f already obtained in [8, Corollary 3.1]. That is: ∇f is tame function if and only if $\mu_\infty(f) < \infty$ and $\mu_\infty(f + \alpha_1 x_1 + \cdots + \alpha_n x_n) = \mu_\infty(f)$, for all sufficiently small $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.

3 Maps with maximal special closure

Let us recall that if $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ is a polynomial map, then $\mathbf{S}'(F)$ denotes the set of exponents $k \in \mathbb{Z}^n_{\geqslant 0}$ for which $x^k \in \mathrm{Sp}(F, 1)$. We also recall the following result, which is contained in [6, Theorem 4.9].

Theorem 3.1 *Let $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ be a polynomial map. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}'(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.

Remark 3.2 (a) Let us observe that, under the conditions of the above result, the equality $\mathbf{S}'(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$ is equivalent to the equality $\mathrm{Sp}(F, 1) = \big\{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\big\}$, by Theorem 2.15 (a).

(b) Moreover, if we assume that F is tame, which is equivalent to the condition $1 \in \mathrm{Sp}(F)$ (see Corollary 2.21), then we conclude that F is Newton non-degenerate at infinity if and only if $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$, which in turn is equivalent to $\mathrm{Sp}(F) = \big\{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\big\}$.

Corollary 3.3 *Let $F : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ be a convenient polynomial map. If F is Newton non-degenerate at infinity, then F is tame. In particular, if $\mathbb{K} = \mathbb{C}$, then $F^{-1}(0)$ is finite.*

Proof Since F is convenient, for all $i \in [n]$ there exists some $r_i \in \mathbb{Z}_{\geqslant 1}$ such that $x_i^{r_i} \in \mathrm{Sp}(F, 1)$, by Theorem 3.1. Let $r_0 = \min\{r_1, \dots, r_n\}$. Then there exists some constants $C, M > 0$ such that $\|x\|^{r_0} \leqslant C\|(F(x), 1)\|$, for all $x \in \mathbb{K}^n$ with $\|x\| \geqslant M$.

Let $N > \max\{M, (C\sqrt{n+1})^{\frac{1}{r_0}}, 1\}$. Let us suppose that there exists some $x \in \mathbb{K}^n$ for which $\|x\| \geqslant N$ and $\|F(x)\| < 1$. In particular $\|(F(x), 1)\|^2 = \sum_{i=1}^n |F_i(x)|^2 + 1 \leqslant n + 1$. Moreover

$$N^{r_0} \leqslant \|x\|^{r_0} \leqslant C\|(F(x), 1)\| \leqslant C\sqrt{n+1} < N^{r_0},$$

which is a contradiction. Therefore, for any $x \in \mathbb{K}^n$ such that $\|x\| \geqslant N$ we have $\|F(x)\| > 1$. That is, $1 \in \mathrm{Sp}(F)$, which means that F is tame, by Corollary 2.21. □

Corollary 3.4 *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a convenient polynomial map. If f is Newton non-degenerate at infinity, then the maps $\mathrm{G}(f)$ and ∇f are tame. In particular $\mathrm{G}(f)^{-1}(0)$ and $(\nabla f)^{-1}(0)$ are finite.*

Proof By Corollary 3.3 and Definition 2.10 (i) we have that $\mathrm{G}(f)$ is tame.

Since $\mathrm{G}(f)$ is convenient and $\mathrm{G}(f)$ is Newton non-degenerate at infinity, there exist $r_1, \dots, r_n \in \mathbb{Z}_{\geqslant 1}$ such that $x_i^{r_i} \in \mathrm{Sp}(\mathrm{G}(f))$, for all $i \in [n]$ (see Remark 3.2 (b)). Therefore

$$\begin{aligned}\|(x_1^{r_1}, \dots, x_n^{r_n})\|^2 &\leqslant C \left\| \left(x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \right) \right\|^2 = C \sum_{i=1}^n \left| x_i \frac{\partial f}{\partial x_i} \right|^2 \\ &\leqslant C \|(x_1^{r_1}, \dots, x_n^{r_n})\|^2 \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i} \right|^2\end{aligned}$$

for all $|x| \gg 1$. By cancelling $\|(x_1^{r_1}, \dots, x_n^{r_n})\|^2$ in the above chain of inequalities it follows that $1 \in \mathrm{Sp}(\nabla f)$. Hence the result follows. □

Let us remark that, under the conditions of Corollary 3.4, the tameness of ∇f is proven in [8, Proposition 3.4] by applying a different argument.

Let $A \subseteq \mathbb{R}^n_{\geqslant 0}$. If $A \cap \mathbb{Z}^n_{\geqslant 0} \neq \emptyset$, then we denote by $\mathbf{L}(A)$ the vector subspace of $\mathbb{C}[x_1, \dots, x_n]$ generated by all polynomials $h \in \mathbb{C}[x_1, \dots, x_n]$ for which $\mathrm{supp}(h) \subseteq A$. If $A \cap \mathbb{Z}^n_{\geqslant 0} = \emptyset$, then we set $\mathbf{L}(A) = 0$. In particular, if $\widetilde{\Gamma}_+$ denotes a global Newton polyhedron in $\mathbb{R}^n_{\geqslant 0}$, then $\mathbf{L}(\widetilde{\Gamma}_+)$ is a finite-dimensional subspace of $\mathbb{C}[x_1, \dots, x_n]$ and $\dim_{\mathbb{C}} \mathbf{L}(\widetilde{\Gamma}_+) = |\widetilde{\Gamma}_+ \cap \mathbb{Z}^n_{\geqslant 0}|$.

The notion of reduction of ideals (see for instance [17, §1] or [29, §1]) motivates us to introduce the following definition.

Definition 3.5 (a) Let V and V' be vector subspaces of finite dimension of $\mathbb{C}[x_1, \dots, x_n]$. We say that V is a *global reduction* of V' when $V \subseteq V'$ and $\mathrm{Sp}(V) = \mathrm{Sp}(V')$. We will write $V \subseteq_{\mathrm{red}} V'$ to indicate that V is a global reduction of V'. In particular, if $V \subseteq_{\mathrm{red}} V'$ then $\widetilde{\Gamma}_+(V) = \widetilde{\Gamma}_+(V')$, by Theorem 2.15. We say that V is a *minimal global reduction* of V' when $V \subseteq_{\mathrm{red}} V'$ and there is no global reduction of V' strictly contained in V.

(b) If F and G are complex polynomial maps defined in $\mathbb{C}^n$ and if $\widetilde{\Gamma}_+$ and $\widetilde{\Gamma}'_+$ are global Newton polyhedra in $\mathbb{R}^n_{\geqslant 0}$, then we denote the conditions $\mathbf{L}(F) \subseteq_{\mathrm{red}} \mathbf{L}(G)$, $\mathbf{L}(F) \subseteq_{\mathrm{red}} \mathbf{L}(\widetilde{\Gamma}_+)$ and $\mathbf{L}(\widetilde{\Gamma}_+) \subseteq_{\mathrm{red}} \mathbf{L}(\widetilde{\Gamma}'_+)$ simply by $F \subseteq_{\mathrm{red}} G$, $F \subseteq_{\mathrm{red}} \widetilde{\Gamma}_+$ and $\widetilde{\Gamma}_+ \subseteq_{\mathrm{red}} \widetilde{\Gamma}'_+$, respectively.

(c) If $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is a global Newton polyhedron, then we say that $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is *homogeneous* when there exist a polynomial map $F = (F_1, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $F \subseteq_{\mathrm{red}} \widetilde{\Gamma}_+$ and F_i is a homogeneous polynomial of positive degree, for all $i \in [n]$ (we remark that the domain and codomain of F are equal to $\mathbb{C}^n$).

Example 3.6 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(y^2, xy^5, x^2, x^5y)$. Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $F(x, y) = (x^2 + xy^5, y^2 + x^5y)$, for all $(x, y) \in \mathbb{C}^2$. We observe that F is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+$. In particular $\mathrm{Sp}(F)$ is maximal, that is, $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$. In particular $F \subseteq_{\mathrm{red}} \widetilde{\Gamma}_+$. We remark that $\widetilde{\Gamma}_+$ is not homogeneous (this can be checked directly).

Example 3.7 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^a, y^b, x^c y^d) \subseteq \mathbb{R}^2_{\geqslant 0}$, where $a, b, c, d \in \mathbb{Z}_{\geqslant 1}$ and $a, b \leqslant c + d$. We observe that $\widetilde{\Gamma}_+$ is homogeneous if and only if $a = b$. In this case, a homogeneous reduction is given by the map $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by $F(x, y) = (x^a + y^a, x^c y^d)$, which verifies $\mu(F) = a(c + d) = 2\mathrm{V}_2(\widetilde{\Gamma}_+)$.

Joining [6, Corollary 4.10] and [7, Theorem 3.2], we have the following characterization of the Newton non-degeneracy at infinity of polynomial maps.

Theorem 3.8 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a convenient polynomial map with finite zero set. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F) = \left\{h \in \mathbb{C}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\right\}$.
(d) $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$.

We remark that in [6, Corollary 4.10] we proved the equivalence of (a), (b) and (c) for any convenient polynomial map $\mathbb{K}^n \longrightarrow \mathbb{K}^p$ (without assuming that $F^{-1}(0)$ is finite).

Corollary 3.9 *If $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is a convenient polynomial map with finite zero set, $p \geqslant n$, then Theorem 3.8 remains true when replacing item* (d) *by the condition* $\mathrm{m}(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$.

Proof Let us suppose that F is Newton non-degenerate at infinity. Let $d = \deg(F)$. Let us consider the homogeneous map $(\mathrm{H}_d(F_1), \dots, \mathrm{H}_d(F_p), x_{n+1}^d) : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{p+1}, 0)$, which we will denote by $\widetilde{F}$ (as in [6, Theorem 4.9]).

We observe that the global boundary of $\widetilde{\Gamma}_+(\widetilde{F})$ is formed by a unique face $\Delta_0 \subseteq \mathbb{R}^{n+1}_{\geqslant 0}$ of dimension n, which is contained in the hyperplane of equation $x_1 + \dots + x_{n+1} = d$. Let $\mathscr{F}$ be the set of faces of $\widetilde{\Gamma}_+(\widetilde{F})$ not passing through the origin. Given any $r \in \{0, 1, \dots, n\}$, let $\mathscr{F}_r$ denote the subset of $\mathscr{F}$ formed by those Δ such that $\dim(\Delta) = r$. We denote by $\mathscr{F}^0_{n-1}$ the set of faces $\Delta \in \mathscr{F}_{n-1}$ for which $de_{n+1} \notin \Delta$ (that is, Δ does not contain the exponent of the pure monomial x_{n+1}^d). Given any $\Delta \in \mathscr{F}$, we denote by $C(\Delta)$ the cone $\{\lambda x : \lambda \geqslant 0,\ x \in \Delta\}$ and by $\mathscr{A}_\Delta$ be the local ring formed by those function germs $f \in \mathcal{O}_n$ such that all monomials x^k in the Taylor expansion of f around the origin verify that $k \in C(\Delta)$. We also denote by I_Δ the ideal of $\mathscr{A}_\Delta$ generated by the component functions of $\widetilde{F}_\Delta$.

We remark that Δ_0 is also equal to the union of the compact faces of the Newton polyhedron of $\widetilde{F}$, when considering $\widetilde{F}$ as an analytic map (see [6, Definition 4.4]). By [6, Theorem 4.9], the Newton non-degeneracy at infinity of F is equivalent to the Newton non-degeneracy (in the local sense, as exposed in [6, Definition 4.4]) of the map $\widetilde{F}$. In turn this is equivalent to the condition that each ideal I_Δ has finite colength in $\mathscr{A}_\Delta$, for any $\Delta \in \mathscr{F}$ (see [15, Théorème 6.2]). In particular, we can apply the theorem of existence of reductions (see [21, Theorem 14.14]) to each ideal I_Δ, where $\Delta \in \mathscr{F}^0_{n-1}$, so that we obtain a polynomial map $H : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ whose components are sufficiently general $\mathbb{C}$-linear combinations of $\{\mathrm{H}_d(F_1), \dots, \mathrm{H}_d(F_p)\}$ and such that the ideal J_Δ generated by the components of H_Δ is a reduction of I_Δ, for all $\Delta \in \mathscr{F}^0_{n-1}$ (which, in particular, implies that J_Δ is also an ideal of finite colength, since J_Δ and I_Δ will have the same integral closure). In particular, the map H is Newton non-degenerate (in the local sense, as exposed in [6, Definition 4.4]) and $\widetilde{\Gamma}_+(H, x^d_{n+1}) = \widetilde{\Gamma}_+(\widetilde{F})$.

Hence, defining $G : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by $G(x_1, \dots, x_n) = H(x_1, \dots, x_n, 1)$, for all $(x_1, \dots, x_n) \in \mathbb{C}^n$, we obtain that the components of G are $\mathbb{C}$-linear combinations of $\{F_1, \dots, F_p\}$, $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+(G)$ and G is Newton non-degenerate at infinity (again by [6, Theorem 4.9], since $d = \deg(G)$ and $\mathrm{H}_d(G) = (H, x^d_{n+1})$).

By [6, Corollary 4.10], we have $\mathbf{S}(G) = \widetilde{\Gamma}_+(G) \cap \mathbb{Z}^n_{\geqslant 0}$, which in particular implies that for each $i \in [n]$, there exists some integer $r_i > 0$ for which $x_i^{r_i} \in \mathrm{Sp}(G)$. Therefore $G^{-1}(0)$ is finite (see also Corollary 3.3). Moreover we have that $n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(G)) = \mu(G) \leqslant \mathrm{m}(F) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$. In particular $n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)) = \mathrm{m}(F)$.

Let us suppose now that $n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)) = \mathrm{m}(F)$. By the definition of $\mathrm{m}(F)$, let us consider a matrix $A \in M_{p\times n}(\mathbb{C})$ for which $\mathrm{m}(F) = \mu(F_A)$. Let $G = F_A$. We have $\widetilde{\Gamma}_+(G) \subseteq \widetilde{\Gamma}_+(F)$, since the components of G are $\mathbb{C}$-linear combinations of the components of F. Hence we conclude that

$$n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)) = \mathrm{m}(F) = \mu(G) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(G) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)).$$

Thus $\widetilde{\Gamma}_+(G) = \widetilde{\Gamma}_+(F)$ and $\mu(G) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(G))$, which in turn implies that G is Newton non-degenerate at infinity, by Theorem 3.8. If Δ denotes any face of $\widetilde{\Gamma}_+(F)$ not passing through the origin, then the inclusion $\mathbf{L}(G) \subseteq \mathbf{L}(F)$ implies that $\mathbf{L}(G_\Delta) \subseteq \mathbf{L}(F_\Delta)$. In particular $F_\Delta^{-1}(0) \subseteq G_\Delta^{-1}(0) \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$. Therefore F is Newton non-degenerate at infinity. □

4 Homogeneous maps and the Newton non-degeneracy at infinity

Let $\widetilde{\Gamma}_+$ denote a convenient global Newton polyhedron in $\mathbb{R}^n_{\geqslant 0}$. In this section we show an easily computable sharp lower bound for $n!\mathrm{V}_n(\widetilde{\Gamma}_+)$ and we characterize the corresponding equality by using the notion of homogeneity (see Definition 3.5 (c)).

Given any $k = (k_1, \dots, k_n) \in \mathbb{R}^n$, we will denote by $|k|$ the sum $k_1 + \cdots + k_n$.

Let $f \in \mathbb{C}[x_1, \dots, x_n]$, let us suppose that f is written as $f = \sum_k a_k x^k$. Let us fix a subset $\mathtt{I} \subseteq [n]$. If $\mathrm{supp}(f) \cap \mathbb{R}^n_{\mathtt{I}} = \emptyset$, then we set $f_{\mathtt{I}} = 0$. If $\mathrm{supp}(f) \cap \mathbb{R}^n_{\mathtt{I}} \neq \emptyset$ and we write $\mathtt{I} = \{i_1, \dots, i_r\}$, where $1 \leqslant i_1 < \cdots < i_r \leqslant n$, then we denote by $f_{\mathtt{I}}$ the restriction of f to $\mathbb{R}^n_{\mathtt{I}}$; that is, $f_{\mathtt{I}}$ will denote the polynomial of $\mathbb{C}[x_{i_1}, \dots, x_{i_r}]$ obtained as the sum of all terms $a_k x^k$ with $k \in \mathbb{R}^n_{\mathtt{I}}$. If $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is a polynomial map, then we denote by $F_{\mathtt{I}}$ the polynomial map $\mathbb{C}^r \longrightarrow \mathbb{C}^p$ obtained from F by restricting to $\mathbb{R}^n_{\mathtt{I}}$ componentwise. If J denotes an ideal of $\mathbb{C}[x_1, \dots, x_n]$, then we denote indistinctly by $J^{\mathtt{I}}$ or by $J_{\mathtt{I}}$ the ideal of $\mathbb{C}[x_{i_1}, \dots, x_{i_r}]$ generated by all elements $f_{\mathtt{I}}$ where f varies in J.

Definition 4.1 Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a convenient global Newton polyhedron at infinity.

(a) We define the *degree of* $\widetilde{\Gamma}_+$ as $\deg(\widetilde{\Gamma}_+) = \max\{|k| : k \in \widetilde{\Gamma}_+\}$.
(b) For any $i \in [n]$, let us define

$$d_i(\widetilde{\Gamma}_+) = \min\left\{\deg(\widetilde{\Gamma}_+^{\mathtt{I}}) : \mathtt{I} \subseteq [n],\ |\mathtt{I}| = n - i + 1\right\}$$

and $\mathbf{d}(\widetilde{\Gamma}_+) = \big(d_1(\widetilde{\Gamma}_+), \dots, d_n(\widetilde{\Gamma}_+)\big)$.

Let us fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. It is immediate to see from the definition of the numbers $d_i(\widetilde{\Gamma}_+)$ that

$$\deg(\widetilde{\Gamma}_+) = d_1(\widetilde{\Gamma}_+) \geqslant \cdots \geqslant d_n(\widetilde{\Gamma}_+).$$

For each $i \in [n]$, let $r_i e_i$ be the point of intersection of the Newton boundary of $\widetilde{\Gamma}_+$ with the x_i-axis, where $r_i \in \mathbb{Z}_{\geqslant 1}$ and we recall that $\{e_1, \dots, e_n\}$ denotes the canonical basis in $\mathbb{R}^n$. Then $\deg(\widetilde{\Gamma}_+^{\{i\}}) = r_i$, for all $i \in [n]$ and this implies that $d_n(\widetilde{\Gamma}_+) = \min\{r_1, \dots, r_n\}$.

The main result of this section is the following (this is inspired by the results of [3]).

Theorem 4.2 *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a convenient global Newton polyhedron. Then*

$$d_1(\widetilde{\Gamma}_+) \cdots d_n(\widetilde{\Gamma}_+) \leqslant n! \mathrm{V}_n(\widetilde{\Gamma}_+). \tag{17}$$

Moreover, the following conditions are equivalent:

(a) *equality holds in* (17).
(b) *there exists an integer $s \geqslant 1$ such that $s\widetilde{\Gamma}_+$ is homogeneous.*
(c) *there exists an integer $s \geqslant 1$ and a homogeneous polynomial map $G = (G_1, \dots, G_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with $\deg(G_i) = s d_i(\widetilde{\Gamma}_+)$, for all $i \in [n]$, such that $G \subseteq_{\mathrm{red}} \widetilde{\Gamma}_+$.*

In order to show the proof of the previous result we need to recall some preliminary concepts and result concerning the mixed multiplicity of ideals in local rings.

In [2] we studied the notion of Rees' mixed multiplicity attached to a set of n ideals in a local ring of dimension n, which generalizes the usual notion of mixed multiplicity of n-tuples of ideals not having finite colength in general. Given positive integers $d_1, \dots d_n$, we will apply this notion specially to the case where we consider n ideals $J_1, \dots, J_n$ of $\mathcal{O}_n$ such that each ideal J_i is generated by monomials of degree d_i, for all $i \in [n]$, as will be seen in Corollary 4.4.

Let $(R, \mathbf{m})$ be a local ring and let $J_1, \dots, J_n$ be ideals of R of finite colength of R. We denote by $e(J_1, \dots, J_n)$ the mixed multiplicity of $J_1, \dots, J_n$. We refer to [17, §17.4], [23] or [27] for fundamental results concerning mixed multiplicities of ideals. If $J_1, \dots, J_n$ are all equal to a given ideal J of finite colength of R, then we recall that $e(J_1, \dots, J_n) = e(J)$, where $e(J)$ denotes the Samuel multiplicity of J.

If $J_1, \dots, J_n$ denote arbitrary ideals of R, we defined in [2] the Rees' mixed multiplicity of $J_1, \dots, J_n$, denoted by $\sigma(J_1, \dots, J_n)$, as

$$\sigma(J_1, \dots, J_n) = \max_{r \geqslant 1} e(J_1 + \mathbf{m}^r, \dots, J_n + \mathbf{m}^r).$$

We recall in Proposition 4.3 a characterization of the finiteness of $\sigma(J_1, \dots, J_n)$ and how this number can be computed. Obviously, if J_i has finite colength, for all $i \in [n]$, then $\sigma(J_1, \dots, J_n) = e(J_1, \dots, J_n)$.

Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite and let us consider a generating system $\{a_{i1}, \dots, a_{is_i}\}$ of J_i, for any $i \in [n]$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $J_1 \oplus \cdots \oplus J_n$ if there exists a non-empty Zariski-open set

U in k^s such that all elements $(g_1, \ldots, g_n) \in J_1 \oplus \cdots \oplus J_n$ satisfy the said property provided that

$$g_i = u_{i1}a_{i1} + \cdots + u_{is_i}a_{is_i}$$

with $(u_{11}, \ldots, u_{1s_1}, \ldots, u_{n1}, \ldots, u_{ns_n}) \in U$, for all $i \in [n]$. It is immediate to check that this notion does not depend on the chosen generating sets for $J_1, \ldots, J_n$.

Proposition 4.3 [2, 2.9] *Let $(R, \mathbf{m})$ be a Noetherian local ring such that the residue field $k = R/\mathbf{m}$ is infinite. Let $J_1, \ldots, J_n$ be ideals of R. Then $\sigma(J_1, \ldots, J_n) < \infty$ if and only if there exist elements $g_i \in J_i$, for $i \in [n]$, such that $\langle g_1, \ldots, g_n\rangle$ has finite colength. In this case, we have that $\sigma(J_1, \ldots, J_n) = e(g_1, \ldots, g_n)$ for sufficiently general elements $(g_1, \ldots, g_n) \in J_1 \oplus \cdots \oplus J_n$.*

In [3, Theorem 3.2], we showed that if $J_1, \ldots J_n$ are monomial ideals of $\mathcal{O}_n$, then $\sigma(J_1, \ldots, J_n)$ is finite if and only if for any $\mathtt{I} \subseteq [n]$ we have $|\{i : J_i^{\mathtt{I}} \neq \emptyset\}| \geqslant |\mathtt{I}|$. The following result is a particular case of this result.

Corollary 4.4 *Let $d_1, \ldots, d_n$ be positive integers. Let $\Delta(d_i) = \{k \in \mathbb{R}^n_{\geqslant 0} : |k| = d_i\}$, for all $i \in [n]$. Let E_i be the convex hull of a finite subset of $\Delta(d_i) \cap \mathbb{Z}^n_{\geqslant 0}$ and let J_i be the ideal of $\mathcal{O}_n$ generated by the monomials x^k such that $k \in E_i$, for all $i \in [n]$. Then the following conditions are equivalent:*

(a) *$\sigma(J_1, \ldots, J_n)$ is finite.*
(b) *For each non-empty $\mathtt{I} \subseteq [n]$, we have that $|\{i : E_i^{\mathtt{I}} \neq \emptyset\}| \geqslant |\mathtt{I}|$.*

We remark that, under the conditions of the previous result, if $\sigma(J_1, \ldots, J_n) < \infty$, then $\sigma(J_1, \ldots, J_n) = d_1 \cdots d_n$.

Proof of Theorem 4.2. Let $d_i = d_i(\widetilde{\Gamma}_+)$, for all $i \in [n]$. Let us define $\Lambda_i = \left\{k \in \widetilde{\Gamma}_+ : |k| = d_i\right\}$, for all $i \in [n]$. By the definition of $d_i(\widetilde{\Gamma}_+)$, we have $\Lambda_i \cap \mathbb{Z}^n_{\geqslant 0} \neq \emptyset$, for all $i \in [n]$.

Let us fix any subset $\mathtt{I} \subseteq [n]$. Let $r = |\mathtt{I}|$. We us observe that

$$\deg(\widetilde{\Gamma}^{\mathtt{I}}_+) \geqslant d_{n-r+1} \geqslant d_{n-r+2} \geqslant \cdots \geqslant d_n.$$

In general, given any integer $j \in \{n-r+1, n-r+2, \ldots, n\}$, the intersection $\Lambda_j^{\mathtt{I}} \cap \mathbb{Z}^n_{\geqslant 0}$ can be empty. However, since $\mathbf{v}(\widetilde{\Gamma}_+) \subseteq \mathbb{Z}^n_{\geqslant 0}$, it is possible to find an integer $s \geqslant 1$ such that

$$\left\{k \in s\widetilde{\Gamma}^{\mathtt{I}}_+ : |k| = sd_i\right\} \cap \mathbb{Z}^n \neq \emptyset. \tag{18}$$

We remark that $d_i(s\widetilde{\Gamma}_+) = sd_i(\widetilde{\Gamma}_+)$, for all $i \in [n]$ and $(s\widetilde{\Gamma}_+)^{\mathtt{I}} = s\widetilde{\Gamma}^{\mathtt{I}}_+$, for all $\mathtt{I} \subseteq [n]$. Let s be a positive integer verifying condition (18), for all $\mathtt{I} \subseteq [n]$. Now, let us define

$$E_i = \left\{k \in s\widetilde{\Gamma}_+ : |k| = sd_i\right\} \cap \mathbb{Z}^n, \tag{19}$$

for all $i \in [n]$. Let us apply Corollary 4.4 to the family $\{E_1, \ldots, E_n\}$.

Again, given any subset $\mathtt{I} \subseteq [n]$, if $r = |I|$, we have

$$\deg((s\widetilde{\Gamma}_+)^{\mathtt{I}}) = \deg(s\widetilde{\Gamma}^{\mathtt{I}}_+) = s\deg(\widetilde{\Gamma}^{\mathtt{I}}_+) \geqslant sd_{n-r+1} \geqslant sd_{n-r+2} \geqslant \cdots \geqslant sd_n.$$

In particular, relation (18) says that

$$\{n-r+1, n-r+2, \ldots, n\} \subseteq \{i : E_i^{\mathtt{I}} \neq \emptyset\}.$$

Therefore: $|\{i : E_i^{\mathtt{I}} \neq \emptyset\}| \geqslant |\mathtt{I}|$. By Corollary 4.4, we deduce that $\sigma(J_1, \dots, J_n) < \infty$ and $\sigma(J_1, \dots, J_n) = s^n d_1 \cdots d_n$, where J_i denotes the ideal of $\mathcal{O}_n$ generated by the monomials x^k for which $k \in E_i$, for all $i \in [n]$.

Then, for sufficiently general elements $(F_1, \dots, F_n) \in J_1 \oplus \cdots \oplus J_n$, it follows that

$$s^n d_1 \cdots d_n = \sigma(J_1, \dots, J_n) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle F_1, \dots, F_n \rangle} = \mu(F),$$

where $F = (F_1, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and the last equality comes from Remark 2.7.

Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be any of such homogeneous maps. Since $E_i \subseteq s\widetilde{\Gamma}_+$, for all $i \in [n]$, it follows that $\widetilde{\Gamma}_+(F) \subseteq s\widetilde{\Gamma}_+$. Hence by Theorem 2.13 we have

$$s^n d_1 \cdots d_n = \mu(F) \leqslant n! \mathrm{V}_n(\widetilde{\Gamma}_+(F)) \leqslant n! \mathrm{V}_n(s\widetilde{\Gamma}_+(F)) \leqslant s^n n! \mathrm{V}_n(\widetilde{\Gamma}_+). \tag{20}$$

This shows relation (17).

Let us prove (a) $\Longrightarrow$ (b). Let us suppose that equality holds in (17). Then all inequalities of (20) become equalities. In particular $\mu(F) = s^n n! \mathrm{V}_n(\widetilde{\Gamma}_+)$ and $\mathrm{V}_n(\widetilde{\Gamma}_+(F)) = s^n \mathrm{V}_n(\widetilde{\Gamma}_+) = \mathrm{V}_n(s\widetilde{\Gamma}_+)$. Therefore $\widetilde{\Gamma}_+(F) = s\widetilde{\Gamma}_+$ and F is Newton non-degenerate at infinity, by Theorem 3.8. This means that $s\widetilde{\Gamma}_+$ is homogeneous, by definition.

In order to prove (b) $\Longrightarrow$ (c), let us suppose that there exists a map $G : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that G is homogeneous and $G \subseteq_{\mathrm{red}} s\widetilde{\Gamma}_+$, for some integer $s \geqslant 1$. Let $c_i = \deg(G_i)$, for all $i \in [n]$. Reordering the components of G, if necessary, we can suppose that $c_1 \geqslant \cdots \geqslant c_n$. Let us see that $c_i = sd_i$, for all $i \in [n]$.

Since $\widetilde{\Gamma}_+(G) = s\widetilde{\Gamma}_+$ and $\mathbf{S}(G) = s\widetilde{\Gamma}_+ \cap \mathbb{Z}^n_{\geqslant 0}$, the zero set of G is finite, by Corollary 3.3. We deduce that

$$c_1 \cdots c_n = \mu(G) = n! \mathrm{V}_n(\widetilde{\Gamma}_+(G)) = n! s^n \mathrm{V}_n(\widetilde{\Gamma}_+) \geqslant s^n d_1 \cdots d_n. \tag{21}$$

Let us fix an index $i \in [n]$ and a subset $\mathtt{I} \subseteq [n]$ with $|\mathtt{I}| = n - i + 1$. The fact that G is homogeneous with finite zero set implies that the zero set of the map $G^{\mathtt{I}} : \mathbb{C}^n_{\mathtt{I}} \longrightarrow \mathbb{C}^n$ is also finite. In particular, the number of non-zero functions in the set $\{G_1^{\mathtt{I}}, \dots, G_n^{\mathtt{I}}\}$ is at least equal to $n - i + 1$. Since $c_1 \geqslant \cdots \geqslant c_n$, this implies that $\deg(G^{\mathtt{I}}) \geqslant c_i$. Moreover, the equality $\widetilde{\Gamma}_+(G) = \widetilde{\Gamma}_+$ implies that $\widetilde{\Gamma}_+^{\mathtt{I}} = \deg(G^{\mathtt{I}})$. Hence we obtain the following:

$$sd_i = \min \left\{ \deg(s\widetilde{\Gamma}_+^{\mathtt{I}}) : |\mathtt{I}| = n - i + 1 \right\} = \min \left\{ \deg(G^{\mathtt{I}}) : |\mathtt{I}| = n - i + 1 \right\} \geqslant c_i.$$

That is, $sd_i \geqslant c_i$, for all $i \in [n]$. Joining this fact with (21) we conclude that

$$c_1 \cdots c_n = \mu(G) = n! \mathrm{V}_n(\widetilde{\Gamma}_+(G)) = n! s^n \mathrm{V}_n(\widetilde{\Gamma}_+) \geqslant s^n d_1 \cdots d_n \geqslant c_1 \cdots c_n,$$

which shows in particular that $sd_i = c_i$ for all $i \in [n]$.

The implication (c) $\Longrightarrow$ (a) follows easily by observing that any map G satisfying (c) must be Newton non-degenerate at infinity with finite zero set. □

Remark 4.5 In the proof of (b) $\Longrightarrow$ (c) of the above result, we actually have shown that if $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a homogeneous reduction of $\widetilde{\Gamma}_+$, then the set formed by the degrees of the components of F is $\{d_1(\widetilde{\Gamma}_+), \dots, d_n(\widetilde{\Gamma}_+)\}$.

Example 4.6 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^a, y^b, z^c, x^r y^r z^r) \subseteq \mathbb{R}^3$, where $a, b, c, r \in \mathbb{Z}_{\geqslant 1}$, $a \leqslant b \leqslant c$, and $r(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geqslant 1$, that is, the point (r, r, r) is on or above the plane determined by $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Then, an easy computation reveals that

$$3! \mathrm{V}_3(\widetilde{\Gamma}_+) = r(ac + bc + ab)$$
$$d_1(\widetilde{\Gamma}_+) d_2(\widetilde{\Gamma}_+) d_3(\widetilde{\Gamma}_+) = 3rba.$$

Hence $\widetilde{\Gamma}_+$ is homogeneous if and only if $r(ac+bc+ab) = 3rba$, which is to say that $a = b = c$. In this case a homogeneous reduction of $\widetilde{\Gamma}_+$ is given by the map $F : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ defined by $F(x, y, z) = (x^a + y^a + z^a, x^a + 2y^a + 3z^a, x^r y^r z^r)$.

5 Newton non-degeneracy at infinity of gradient maps

Mainly inspired by [4], in this section we compare the conditions of Newton non-degeneracy at infinity of the maps ∇f and $\mathrm{G}(f)$ (see (8)), for any given $f \in \mathbb{C}[x_1, \dots, x_n]$.

Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a Newton polyhedron at infinity, for some $n \geqslant 1$. As a natural attempt of establishing a canonical polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$ such that h is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(h) = \widetilde{\Gamma}_+$, we define $\mathbf{h}(\widetilde{\Gamma}_+)$ as the sum of all monomials x^k such that $k \in \mathbf{v}(\widetilde{\Gamma}_+)$.

Let Δ be a face of $\widetilde{\Gamma}_+$. If Δ is a d-dimensional face of $\widetilde{\Gamma}_+$, where $d \in \{0, 1, \dots, n-1\}$, we denote by P_Δ the d-dimensional affine subspace of $\mathbb{R}^n$ containing Δ (which in turn is the minimal affine subspace of $\mathbb{R}^n$ containing Δ). We say that Δ is a *simplex*, or a *simplicial face of* $\widetilde{\Gamma}_+$, when Δ equals the convex hull of $d+1$ vertices of $\widetilde{\Gamma}_+$. We say that $\widetilde{\Gamma}_+$ is *simplicial* when each face of $\widetilde{\Gamma}_+$ not passing through the origin is a simplex. Obviously, if $n = 1$ or 2, then $\widetilde{\Gamma}_+$ is always simplicial.

Lemma 5.1 *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global Newton polyhedron. Let $\Delta \subseteq \mathbb{R}^n$ be a simplicial face of $\widetilde{\Gamma}_+$ of dimension d such that $0 \notin \Delta$, where $d \in \{0, 1, \dots, n-1\}$. Let $v_0, v_1, \dots, v_d$ be the vertices of Δ. Then $\{v_0, v_1, \dots, v_d\}$ is linearly independent.*

Proof Let us first observe that $\Delta = P_\Delta \cap \widetilde{\Gamma}_+$, since Δ is a face of $\widetilde{\Gamma}_+$. Moreover, $v_i \neq 0$, for all $i \in \{0, 1, \dots, n\}$, since $0 \notin \Delta$.

The origin belongs to $\widetilde{\Gamma}_+$, so the condition $0 \notin \Delta$ implies $0 \notin P_\Delta$. Let $\alpha_0, \alpha_1, \dots, \alpha_d \in \mathbb{R}$ such that $\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_d v_d = 0$. We observe that

$$\alpha_1(v_1 - v_0) + \cdots + \alpha_d(v_d - v_0) = (-\alpha_0 - \alpha_1 - \cdots - \alpha_d)v_0. \tag{22}$$

If $\alpha_0 + \alpha_1 + \cdots + \alpha_d \neq 0$, and we denote this number by β, then (22) implies that

$$0 = v_0 + \frac{\alpha_1}{\beta}(v_1 - v_0) + \cdots + \frac{\alpha_d}{\beta}(v_d - v_0).$$

This means that $0 \in P_\Delta$, which is a contradiction. Hence $\alpha_0 + \alpha_1 + \cdots + \alpha_d = 0$, that is, the member of the right hand side of (22) is zero. The condition $\dim \Delta = d$ is equivalent to saying that $\{v_1 - v_0, \dots, v_d - v_0\}$ is linearly independent. Then (22) implies that $\alpha_1 = \cdots = \alpha_d = 0$ and thus we also have $\alpha_0 = 0$. Therefore the result follows. □

Proposition 5.2 *Let $\widetilde{\Gamma}_+$ be a simplicial global Newton polyhedron in $\mathbb{R}^n_{\geqslant 0}$. Let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. Then h is Newton non-degenerate at infinity. If, in addition, $\widetilde{\Gamma}_+$ is convenient, then $\mathrm{G}(h)$ has finite zero set and $\mu(\mathrm{G}(h)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+)$.*

Proof Let Δ be any face of $\widetilde{\Gamma}_+$ not passing through the origin. Since $\widetilde{\Gamma}_+$ is simplicial, Δ is a simplex. Let $\{k^{(1)}, \dots, k^{(r)}\}$ be the vertices of Δ. We have that $h_\Delta = x^{k^{(1)}} + \cdots + x^{k^{(r)}}$. We need to check that the set of solutions of the system

$$\left(x_1 \frac{\partial h}{\partial x_1}\right)_\Delta (x) = \cdots = \left(x_n \frac{\partial h}{\partial x_n}\right)_\Delta (x) = 0 \tag{23}$$

is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

Let us write $k^{(j)} = (k_1^{(j)}, \dots, k_n^{(j)})$, for any $j \in [r]$. The system (23) can be rewritten as

$$\left.\begin{aligned} k_1^{(1)} x^{k^{(1)}} + \cdots + k_1^{(r)} x^{k^{(r)}} &= 0 \\ &\vdots \\ k_n^{(1)} x^{k^{(1)}} + \cdots + k_n^{(r)} x^{k^{(r)}} &= 0 \end{aligned}\right\} \tag{24}$$

which is a linear system in the unknowns $x^{k^{(1)}}, \dots, x^{k^{(r)}}$. By Lemma 5.1, the set of vertices $\{k^{(1)}, \dots, k^{(r)}\}$ is linearly independent. In particular we conclude that $r \leqslant n$ and therefore the system (24) has only the trivial solution. That is, $x^{k^{(1)}} = \cdots = x^{k^{(r)}} = 0$. In particular $x_1 \cdots x_n = 0$. Since we have fixed any face Δ of $\widetilde{\Gamma}_+$ not passing through the origin, it follows that h is Newton non-degenerate at infinity.

If we add the condition that h is convenient, then the finiteness of the zero set $\mathrm{G}(h)$ follows from Corollary 3.4. Moreover, the equality $\mu(\mathrm{G}(h)) = n!\mathrm{V}_n(\mathrm{G}(h))$ is a direct application of Theorem 3.8. □

Example 5.3 Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^3_{\geqslant 0}$ be the global Newton polyhedron given by the unit cube. That is, $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x, y, z, xy, xz, yz, xyz)$. So $\widetilde{\Gamma}_+$ is not simplicial. Let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. We observe that the map $\mathrm{G}(h)$ is given by

$$\mathrm{G}(h) = \big(x + xy + xz + xyz,\ y + xy + yz + xyz,\ z + xz + yz + xyz\big).$$

The set $\mathrm{G}(h)^{-1}(0)$ is not finite, since $\mathrm{G}(h)^{-1}(0)$ contains the points of the form $(0, 0, 0)$, $(\alpha, -1, -1)$, $(-1, \alpha, -1)$ and $(-1, -1, \alpha)$, where $\alpha \in \mathbb{C}$. Hence, we conclude that $\mathrm{G}(h)$ is not Newton non degenerate at infinity, since $\widetilde{\Gamma}_+$ is convenient (see Corollary 3.3).

Corollary 5.4 *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ be a polynomial map such that $F(0) = 0$ and $F^{-1}(0)$ is finite. Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F)$ and $h = \mathbf{h}(\widetilde{\Gamma}_+)$. Let us suppose that $\widetilde{\Gamma}_+$ is simplicial. Then F is Newton non-degenerate at infinity if and only if*

$$\mathrm{m}(F) = \mu(\mathrm{G}(h)).$$

Proof The polyhedron $\widetilde{\Gamma}_+$ is convenient, since $F^{-1}(0)$ is finite and $F(0) = 0$ (see [5, Lemma 2.7]). We suppose that $\widetilde{\Gamma}_+$ is simplicial, hence the polynomial h is Newton non degenerate at infinity and $\mu(\mathrm{G}(h)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+)$, by Proposition 5.2. Hence the result is a direct consequence of Corollary 3.9. □

Remark 5.5 Let $f \in \mathbb{C}[x_1, \dots, n]$ such that $\widetilde{\Gamma}_+(f)$ is simplicial, f is convenient and $\mathrm{G}(f)(0) = 0$. Let $h = \mathbf{h}(\widetilde{\Gamma}_+(f))$. As a consequence of Corollary 5.4, f is Newton non-degenerate at infinity (see Definition 2.10 (i)) if and only if the zero set of $\mathrm{G}(f)$ is finite and $\mu(\mathrm{G}(f)) = \mu(\mathrm{G}(h))$.

Let $f \in \mathbb{C}[x_1, \dots, n]$. Let $\mathtt{I} \subseteq [n]$ and let $r = |\mathtt{I}|$. If the polynomial $f_{\mathtt{I}} : \mathbb{C}^r \longrightarrow \mathbb{C}$ has a finite number of singularities, then we can speak about the global Milnor number $\mu_\infty(f_{\mathtt{I}})$. Of course, if $(\nabla f)^{-1}(0)$ is finite, then $(\nabla f_{\mathtt{I}})^{-1}(0)$ is not finite in general. But $((\nabla f)_{\mathtt{I}})^{-1}(0)$ is. In the remaining section, the functions for which the maps $\nabla f_{\mathtt{I}}$ and $(\nabla f)_{\mathtt{I}}$ are identical, for any $\mathtt{I} \subseteq [n]$, will deserve special consideration.

Lemma 5.6 *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map. Then*

$$\frac{\partial f_{\mathtt{I}}}{\partial x_i} = \left(\frac{\partial f}{\partial x_i}\right)_{\mathtt{I}}, \tag{25}$$

for all $i \in [n]$ and all $\mathtt{I} \subseteq [n]$ such that $i \in \mathtt{I}$.

Proof Let $\{e_1, \dots, e_n\}$ denote the canonical basis in $\mathbb{R}^n$ and let us suppose that f is written as $f = \sum_k a_k x^k$. Let us fix any index $i \in [n]$. For an arbitrary non-empty $\mathtt{I} \subseteq [n]$ we have that

$$\frac{\partial f_{\mathtt{I}}}{\partial x_i} = \sum_{\substack{k \in \mathrm{supp}(f_{\mathtt{I}}) \\ k_i > 0}} a_k k_i x^{k-e_i} \qquad \text{and} \qquad \left(\frac{\partial f}{\partial x_i}\right)_{\mathtt{I}} = \sum_{\substack{k \in \mathrm{supp}(f) \\ k_i > 0,\, k-e_i \in \mathbb{R}^n_{\mathtt{I}}}} a_k k_i x^{k-e_i}. \tag{26}$$

Therefore, if $i \in \mathtt{I}$, the polynomials of (26) are identical. That is, we obtain equality (25). □

Remark 5.7 If $f \in \mathbb{C}[x_1, \dots, x_n]$ and $\mathtt{I} \subseteq [n]$ then obviously $\frac{\partial f_{\mathtt{I}}}{\partial x_i} = 0$, for all $i \notin \mathtt{I}$. This fact and Lemma 5.6 show that

$$\mathrm{supp}\left(\frac{\partial f_{\mathtt{I}}}{\partial x_i}\right) \subseteq \mathrm{supp}\left(\left(\frac{\partial f}{\partial x_i}\right)_{\mathtt{I}}\right), \qquad \text{for all } i \in [n] \text{ and all } \mathtt{I} \subseteq [n]. \tag{27}$$

In particular $\widetilde{\Gamma}_+\big(\nabla(f_{\mathtt{I}})\big) \subseteq \widetilde{\Gamma}_+\big((\nabla f)_{\mathtt{I}}\big)$, for all $\mathtt{I} \subseteq [n]$, and hence we have the inequality

$$\mathrm{V}_{|\mathtt{I}|}\left(\widetilde{\Gamma}_+(\nabla(f_{\mathtt{I}}))\right) \leqslant \mathrm{V}_{|\mathtt{I}|}\left(\widetilde{\Gamma}_+(\nabla f)_{\mathtt{I}}\right), \qquad \text{for all } \mathtt{I} \subseteq [n].$$

As a direct consequence of Lemma 5.6 we also deduce that the following conditions are equivalent:

(a) $\dfrac{\partial f_{\mathtt{I}}}{\partial x_i} = \left(\dfrac{\partial f}{\partial x_i}\right)_{\mathtt{I}}$, for all $i \in [n]$ and all $\mathtt{I} \subseteq [n]$.

(b) $\left(\dfrac{\partial f}{\partial x_i}\right)_{\mathtt{I}} = 0$, for all $i \in [n]$ and all $\mathtt{I} \subseteq [n]$ such that $i \notin \mathtt{I}$.

We will say that f is *adjusted to the coordinate subspaces* when any the above equivalent conditions hold.

It is immediate to see that if f is adjusted to the coordinate subspaces, then

$$\widetilde{\Gamma}_+\big(\nabla(f_{\mathtt{I}})\big) = \widetilde{\Gamma}_+\big((\nabla f)_{\mathtt{I}}\big) \tag{28}$$

for all $\mathtt{I} \subseteq [n]$, $\mathtt{I} \neq \emptyset$. As remarked in (27), in general we only have the inclusion $\subseteq$ in (28). This inclusion can be strict. For instance, let $f = x^2 + x^6y^2 + y^3 + x^4y \in \mathbb{C}[x, y]$, then $\widetilde{\Gamma}_+(\nabla f_{\{1\}})) = \widetilde{\Gamma}_+(x)$ and $\widetilde{\Gamma}_+\big((\nabla f)_{\{1\}}\big) = \widetilde{\Gamma}_+(x^4)$.

Theorem 5.8 *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial function such that f is adjusted to the coordinate axis and* $\mathrm{G}(f)$ *has finite zero set. If ∇f is Newton non-degenerate at infinity and $\nabla f(0) = 0$, then* $\mathrm{G}(f)$ *is also Newton non-degenerate at infinity.*

Proof Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(f)$. Let $g \in \mathbb{C}[x_1, \dots, x_n]$ such that g is Newton non-degenerate at infinity and $\mathrm{supp}(g) = \mathbf{v}(\widetilde{\Gamma}_+)$ (such a polynomial always exists, by [15, §6]). We have $\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$. This implies that

$$\widetilde{\Gamma}_+(\nabla g_{\mathtt{I}}) \subseteq \widetilde{\Gamma}_+(\nabla f_{\mathtt{I}}), \qquad \text{for all non-empty } \mathtt{I} \subseteq [n]. \tag{29}$$

Moreover, the zero set of the map $\mathrm{G}(g)$ is finite, by Corollary 3.3. Hence $(\nabla g)^{-1}(0)$ is also finite. Let us also remark that, since $(\nabla f)^{-1}(0)$ is finite and $\nabla f(0) = 0$, then $\widetilde{\Gamma}_+(\nabla f)$ is convenient.

We deduce the following chain of inequalities

$$n!\mathrm{V}_n(\widetilde{\Gamma}_+) \geqslant \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\left\langle x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \right\rangle} = \sum_{r=1}^{n} \sum_{\substack{\mathtt{I} \subseteq [n] \\ |\mathtt{I}| = r}} \mu_\infty(f_{\mathtt{I}}) \tag{30}$$

$$= \sum_{r=1}^{n} \sum_{\substack{\mathtt{I}\subseteq[n] \\ |\mathtt{I}|=r}} \mu((\nabla f)_{\mathtt{I}}) = \sum_{r=1}^{n} \sum_{\substack{\mathtt{I}\subseteq[n] \\ |\mathtt{I}|=r}} r!\mathrm{V}_r(\widetilde{\Gamma}_+(\nabla f_{\mathtt{I}})) \tag{31}$$

$$\geqslant \sum_{r=1}^{n} \sum_{\substack{\mathtt{I}\subseteq[n] \\ |\mathtt{I}|=r}} r!\mathrm{V}_r(\widetilde{\Gamma}_+(\nabla g_{\mathtt{I}})) \geqslant \sum_{r=1}^{n} \sum_{\substack{\mathtt{I}\subseteq[n] \\ |\mathtt{I}|=r}} \mu_\infty(\nabla g_{\mathtt{I}}) \tag{32}$$

$$= \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1,\ldots,x_n]}{\left\langle x_1\frac{\partial g}{\partial x_1},\ldots,x_n\frac{\partial g}{\partial x_n}\right\rangle} = n!\mathrm{V}_n(\widetilde{\Gamma}_+(g)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+). \tag{33}$$

The inequality $\geqslant$ of (30) is a direct application of Theorem 2.13. The equality of (30) is an application of (2) and the additivity of the intersection index (see for instance [15, 3.2]).

Similarly to (28), since f is adjusted to the coordinate axis, we deduce that $\mathbf{I}(\nabla(f_{\mathtt{I}})) = \mathbf{I}((\nabla f)_{\mathtt{I}})$, for any $\mathtt{I} \subseteq [n]$. This proves the first equality of (31). The condition of Newton non-degeneracy at infinity of ∇f implies that, if we fix any $\mathtt{I} \subseteq [n]$, then the map $(\nabla f)_{\mathtt{I}}$ is also Newton non-degenerate at infinity, as a map $\mathbb{C}^r \longrightarrow \mathbb{C}^n$, where $r = |\mathtt{I}|$. In particular, $\mu((\nabla f)_{\mathtt{I}}) = r!\mathrm{V}_r(\widetilde{\Gamma}_+((\nabla f)_{\mathtt{I}})) = r!\mathrm{V}_r(\widetilde{\Gamma}_+(\nabla f_{\mathtt{I}}))$, by Theorem 3.8 and relation (28), respectively. This proves the second equality of (31).

The first inequality ($\geqslant$) of (32) comes from (29). The second inequality ($\geqslant$) of (32) is another application of Theorem 2.13. The first equality of (33) is analogous to the equality of (30). Since $\mathrm{G}(g)$ is Newton non-degenerate at infinity, we have that $\mu(\mathrm{G}(g)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(g))$, by Theorem 3.8. The last equality of (33) is obvious, since $\mathrm{supp}(g) = \mathbf{v}(\widetilde{\Gamma}_+)$. This completes the proof of (30)-(33).

As a consequence of the above discussion, the inequality ($\geqslant$) of (30) becomes an equality and, again by Theorem 3.8, we conclude that the map $\mathrm{G}(f)$ is Newton non-degenerate at infinity. □

Given a global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$, let us recall that $\mathbf{P}(\widetilde{\Gamma}_+)$ denotes the set of those $f \in \mathbb{C}[x_1,\ldots,x_n]$ for which $\widetilde{\Gamma}_+(f) = \widetilde{\Gamma}_+$. Let us define the following subsets of $\mathbf{P}(\widetilde{\Gamma}_+)$:

$$\mathbf{P}_0(\widetilde{\Gamma}_+) = \left\{f \in \mathbf{P}(\widetilde{\Gamma}_+) : (\nabla f)^{-1}(0) \text{ is finite}\right\}$$
$$\mathscr{B}(\widetilde{\Gamma}_+) = \left\{f \in \mathbf{P}(\widetilde{\Gamma}_+) : \nabla f \text{ is Newton non-degenerate at infinity}\right\}$$
$$\mathscr{K}(\widetilde{\Gamma}_+) = \left\{f \in \mathbf{P}(\widetilde{\Gamma}_+) : G(f) \text{ is Newton non-degenerate at infinity}\right\}.$$

We will discuss the inclusion $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$ in Proposition 5.12. In general, $\mathscr{K}(\widetilde{\Gamma}_+)$ and $\mathscr{B}(\widetilde{\Gamma}_+)$ constitute two different classes of polynomials, as we see in the following examples.

Example 5.9 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^2, y^2, x^2y^3, x^4y^2) \subseteq \mathbb{R}^2_{\geqslant 0}$ and let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. This polynomial verifies $h \in \mathscr{K}(\widetilde{\Gamma}_+)$, by Proposition 5.2, but $h \notin \mathscr{B}(\widetilde{\Gamma}_+)$, as is easy to check.

Example 5.10 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^3, y^2, z^4, x^2y^4, y^3z) \subseteq \mathbb{R}^3_{\geqslant 0}$. Let us consider the polynomial $g \in \mathbf{P}_0(\widetilde{\Gamma}_+)$ given by $g(x,y,z) = x^3 + y^2 + z^4 + x^2y^4 - 2xy^3 + zy^3$. This polynomial is not Newton non-degenerate at infinity, since

$$\mu(\mathrm{G}(g)) = 71, \qquad 3!\mathrm{V}_3\big(\widetilde{\Gamma}_+(g)\big) = 76.$$

However, we have

$$\mu_\infty(g) = \mu(\nabla g) = 42, \qquad 3!\mathrm{V}_3\big(\widetilde{\Gamma}_+(\nabla g)\big) = 42.$$

So ∇g is Newton non-degenerate at infinity, that is, $f \in \mathscr{B}(\widetilde{\Gamma}_+) \setminus \mathscr{K}(\widetilde{\Gamma}_+)$.

Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global Newton polyhedron. We define the numbers

$$\lambda(\widetilde{\Gamma}_+) = \max\left\{n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big) : f \in \mathbf{P}(\widetilde{\Gamma}_+)\right\}$$
$$\lambda_0(\widetilde{\Gamma}_+) = n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla h)\big),$$

where $h = \mathbf{h}(\widetilde{\Gamma}_+)$. Obviously we have $\lambda_0(\widetilde{\Gamma}_+) \leqslant \lambda(\widetilde{\Gamma}_+)$.

If η denotes the function obtained as the sum of all monomials x^k such that $k \in \widetilde{\Gamma}_+ \cap \mathbb{Z}^n_{\geqslant 0}$, then the maximum $\lambda(\widetilde{\Gamma}_+)$ is attained when computing $n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla\eta))$. In other words, let us define $A_i = \{k - e_i : k \in \widetilde{\Gamma}_+ \cap \mathbb{Z}^n_{\geqslant 0},\ k_i > 0\}$ and $\partial_i\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(A_i)$, for any $i \in [n]$. Moreover, let $\mathscr{J}(\widetilde{\Gamma}_+) = \mathrm{Conv}(\partial_1\widetilde{\Gamma}_+ \cup \cdots \cup \partial_n\widetilde{\Gamma}_+)$. Then $\lambda(\widetilde{\Gamma}_+) = n!\mathrm{V}_n(\mathscr{J}(\widetilde{\Gamma}_+))$.

By Theorem 2.12, the Newton number $\nu(\widetilde{\Gamma}_+)$ verifies

$$\nu(\widetilde{\Gamma}_+) = \max\left\{\mu_\infty(f) : f \in \mathbf{P}_0(\widetilde{\Gamma}_+)\right\}.$$

Theorem 2.13 shows that $\mu_\infty(f) \leqslant n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big)$, for any $f \in \mathbb{C}[x_1, \ldots, x_n]$ with a finite singular set. Therefore, the inequality $\nu(\widetilde{\Gamma}_+) \leqslant \lambda(\widetilde{\Gamma}_+)$ holds. Let us remark that these numbers are different in general, as the following example shows.

Example 5.11 Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^2_{\geqslant 0}$ be the global Newton polyhedron described in Example 5.9. We have $\nu(\widetilde{\Gamma}_+) = 13$. Moreover

$$\partial_1\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x, x^3y^2, xy^3, y^2)$$
$$\partial_2\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^3, x^4y, x^2y^2, y)$$
$$\mathscr{J}(\widetilde{\Gamma}_+) = \widetilde{\Gamma}_+(x^3, x^4y, x^3y^2, xy^3, y^2).$$

Therefore $\lambda(\widetilde{\Gamma}_+) = 2\mathrm{V}_2(\mathscr{J}(\widetilde{\Gamma}_+)) = 17$.

Proposition 5.12 *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_+$ be a convenient global Newton polyhedron. Then the following conditions are equivalent:*

(a) $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$
(b) $\nu(\widetilde{\Gamma}_+) = \lambda(\widetilde{\Gamma}_+)$.

Moreover, if $\widetilde{\Gamma}_+$ is simplicial, then condition (b) *can be replaced by*

(c) $\nu(\widetilde{\Gamma}_+) = \lambda(\widetilde{\Gamma}_+) = \lambda_0(\widetilde{\Gamma}_+)$.

Proof Let us prove (a)$\Longrightarrow$(b). Let us suppose that $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$. By the genericity of the Newton non-degeneracy at infinity (see [15, 6.1]), we can find a polynomial $f \in \mathscr{K}(\widetilde{\Gamma}_+)$ such that $\lambda(\widetilde{\Gamma}_+) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f))$. Since $\widetilde{\Gamma}_+$ is convenient, the zero set of $\mathrm{G}(f)$ is finite, by Corollary 3.3. So $(\nabla f)^{-1}(0)$ is finite, by (9), and we have $\mu_\infty(f) = \nu(\widetilde{\Gamma}_+)$. On the other hand, the inclusion $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$ implies that ∇f is also Newton non-degenerate at infinity. In particular $\mu_\infty(f) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f))$. Joining these equalities we deduce that $\nu(\widetilde{\Gamma}_+) = \lambda(\widetilde{\Gamma}_+)$.

Let us prove (b)$\Longrightarrow$(a). Let us suppose that $\nu(\widetilde{\Gamma}_+) = \lambda(\widetilde{\Gamma}_+)$. Let $f \in \mathscr{K}(\widetilde{\Gamma}_+)$. Since $\widetilde{\Gamma}_+$ is convenient, the zero set of $\mathrm{G}(f)$ is finite, by Corollary 3.3. This implies that $(\nabla f)^{-1}(0)$ is also finite. Then $\mu_\infty(f) = \nu(\widetilde{\Gamma}_+)$. Thus we deduce the following:

$$\lambda(\widetilde{\Gamma}_+) = \nu(\widetilde{\Gamma}_+) = \mu_\infty(f) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f)) \leqslant \lambda(\widetilde{\Gamma}_+).$$

Therefore $\mu_\infty(f) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f)$, which implies that ∇f is Newton non-degenerate at infinity. That is, $f \in \mathscr{B}(\widetilde{\Gamma}_+)$. Hence the inclusion $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$ follows.

Let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. If $\widetilde{\Gamma}_+$ is simplicial, then h is Newton non-degenerate at infinity, by Proposition 5.2. If (a) holds, then $h \in \mathscr{B}(\widetilde{\Gamma}_+)$, which leads to the following equalities: $\nu(\widetilde{\Gamma}_+) = \mu_\infty(h) = \lambda_0(\widetilde{\Gamma}_+) \leqslant \lambda(\widetilde{\Gamma}_+) = \nu(\widetilde{\Gamma}_+)$. Hence (c) follows. □

Remark 5.13 (a) Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_+$ be a convenient global Newton polyhedron and let $f \in \mathscr{K}(\widetilde{\Gamma}_+)$. Then we observe that $\nu(\widetilde{\Gamma}_+) = \mu_\infty(f) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f)) \leqslant \lambda(\widetilde{\Gamma}_+)$. Then the condition $\nu(\widetilde{\Gamma}_+) = \lambda(\widetilde{\Gamma}_+)$ forces that $n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla f))$ attains the maximum $\lambda(\widetilde{\Gamma}_+)$.

(b) By Proposition 5.12, for any polyhedron in $\mathbb{R}^2$, the condition $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$ implies $\lambda(\widetilde{\Gamma}_+) = \lambda_0(\widetilde{\Gamma}_+)$, which is a relatively simple condition to check in this case.

(c) By Theorem 3.8, if $f \in \mathbb{C}[x_1, \dots, x_n]$ verifies that ∇f is convenient and Newton non-degenerate at infinity, then the Łojasiewicz exponent at infinity of ∇f, denoted by $\mathcal{L}_\infty(\nabla f)$ (see for instance [16]) is equal to $\min\{r_1, \dots, r_n\}$, where $r_i e_i$ is the intersection of the global boundary of $\widetilde{\Gamma}_+(\nabla f)$ (see Definition 2.10(c)) with the x_i-axis, for all $i = 1, \dots, n$. In particular, $\mathcal{L}_\infty(\nabla f)$ is a positive integer in this case.

Example 5.14 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^5, xy^5, y^3) \subseteq \mathbb{R}^2_{\geqslant 0}$ and let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. We have that $\mathscr{J}(\widetilde{\Gamma}_+) = \widetilde{\Gamma}_+(y^5, xy^4, x^4) = \widetilde{\Gamma}_+(\nabla h)$. Therefore $\lambda(\widetilde{\Gamma}_+) = \lambda_0(\widetilde{\Gamma}_+) = 21$. Since h is Newton non-degenerate at infinity, we have $\mu_\infty(h) = \nu(\widetilde{\Gamma}_+) = 21$. Hence $\mathscr{K}(\widetilde{\Gamma}_+) \subseteq \mathscr{B}(\widetilde{\Gamma}_+)$, by Proposition 5.12.

Similarly, if we consider the global Newton polyhedron $\widetilde{\Gamma}'_+ = \widetilde{\Gamma}_+(x^5, y^5, x^3y^3) \subseteq \mathbb{R}^2_{\geqslant 0}$, then Proposition 5.12 also shows that $\mathscr{K}(\widetilde{\Gamma}'_+) \subseteq \mathscr{B}(\widetilde{\Gamma}'_+)$.

By [15, 6.1], it is clear that $\mathscr{K}(\widetilde{\Gamma}_+) \neq \emptyset$ for any global Newton polyhedron at infinity $\widetilde{\Gamma}_+$. However, the family $\mathscr{B}(\widetilde{\Gamma}_+)$ is not always non-empty, as the following proposition shows.

Proposition 5.15 *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a simplicial and convenient Newton polyhedron. Let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. Then $\mathscr{B}(\widetilde{\Gamma}_+) \neq \emptyset$ if and only if the polynomial map ∇h is Newton non-degenerate at infinity.*

Proof The if part is obvious. Let $f \in \mathscr{B}(\widetilde{\Gamma}_+)$. The condition $\widetilde{\Gamma}_+(f) = \widetilde{\Gamma}_+$ implies that $\mathrm{supp}(h) = \mathbf{v}(\widetilde{\Gamma}_+) \subseteq \mathrm{supp}(f)$. In particular the inclusion $\widetilde{\Gamma}_+(\nabla h) \subseteq \widetilde{\Gamma}_+(\nabla f)$ holds, which in turn implies that

$$\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla h)\big) \leqslant \mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big). \tag{34}$$

The map $\mathrm{G}(h)$ is Newton non degenerate at infinity, by Proposition 5.2. By hypothesis ∇f is Newton non-degenerate at infinity. Then, using (34) and Theorems 2.12 and 2.13, we deduce that

$$\nu(\widetilde{\Gamma}_+) = \mu_\infty(h) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla h)) \leqslant n!\mathrm{V}_n\big(\widetilde{\Gamma}_+(\nabla f)\big) = \mu_\infty(f) \leqslant \nu(\widetilde{\Gamma}_+). \tag{35}$$

Therefore, all inequalities in (35) become equalities. In particular, we obtain that $\mu_\infty(h) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(\nabla h))$, which implies that ∇h is Newton non-degenerate at infinity, by Theorem 3.8. □

Example 5.16 Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x^2, y^2, x^2y^3, x^4y^2) \subseteq \mathbb{R}^2$ and let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. The function h is Newton non-degenerate at infinity, by Proposition 5.2. We observe that $\widetilde{\Gamma}_+(\nabla h) = \widetilde{\Gamma}_+(x, x^4y, x^3y^2, xy^3, y)$ and ∇h is not Newton non-degenerate at infinity. Therefore, there is not any function $f \in \mathbf{P}_0(\widetilde{\Gamma}_+)$ such that ∇f is Newton non-degenerate at infinity, by Proposition 5.15.

Example 5.17 Let us consider the Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^3_{\geqslant 0}$ and the function $g \in \mathbf{P}_0(\widetilde{\Gamma}_+)$ of Example 5.10. Let $h = \mathbf{h}(\widetilde{\Gamma}_+)$. Since $\widetilde{\Gamma}_+$ is simplicial and ∇g is Newton non-degenerate at infinity, we conclude that ∇h is also Newton non-degenerate at infinity, by Proposition 5.15.

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