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Picturing the Growth Order of Solutions in Complex Linear Differential–Difference Equations with Coefficients of φ-Order

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Abstract: Given an unbounded non-decreasing positive function φ, we studied what the relations are between the growth order of any solution of a complex linear differential–difference equation whose coefficients are entire or meromorphic functions of finite φ-order. Our findings extend some earlier well-known results.

Keywords: Nevanlinna; complex solution; growth order; complex linear ODE

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1. Introduction

Our notation is standard and currently used when working with meromorphic functions and Nevanlinna's value distribution theory [1,2]. Nevertheless, for the sake of completeness and to facilitate the reading of this paper, we recall some of its fundamentals in Section 2.

The meromorphic functions, i.e., those that are analytic in the whole complex field, but in a set of isolated points that are poles of the function, have been widely studies in Complex Functions Theory. Researchers have gone further by adding insight into their growth order when they are solutions of linear complex differential, and difference, equations with entire or meromorphic coefficients, looking at how the possible growth order of the former is determined by the growth order of the latter ones [3–8].

Firstly, let us recall that an entire function f is said to have finite-order when its maximum modulus function, M_f, is dominated by the exponential of some real power $a > 0$, as displayed in the following inequality for r large enough:

$$M_f(r) \;:=\; \max_{|z|=r} |f(z)| \leq \exp\left(r^a\right). \tag{1}$$

If there is no a such that Equation (1) holds for r large enough, the growth order of f is said to be infinite. Otherwise, the infimum of all $a > 0$ that satisfy Equation (1) is called the order of growth of f. It is represented by $\sigma(f)$, and in general, it may be calculated by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2}$$

If we replace lim sup by lim inf in Equation (2), we obtain the so-called *lower order* of growth of f and represent it by $\underline{\sigma}(f)$ [9]. On the other hand, the notions of type-order (τ) and hyper-order (σ_2) [10] are defined, respectively, by

$$\tau(f) = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\sigma(f)}}, \quad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}. \tag{3}$$

lSimilarly, if lim sup is replaced by lim inf in Equation (3), we obtain the so-called *lower type* of f and represent it by $\underline{\tau}(f)$.

In this setting, Laine and Yang [4] obtained the following growth order property concerning the solutions of any complex linear difference equation with entire coefficients.

Theorem 1 ([4])**.** *Let $w_1, \ldots, w_n$ be distinct complex numbers, and assume that $A_0(z), \ldots, A_n(z)$ are entire functions of finite-order, which are the coefficients of the difference equation:*

$$A_n(z)f(z+w_n) + \cdots + A_1(z)f(z+w_1) + A_0(z)f(z) = 0. \tag{4}$$

If there is exactly one A_{k_0}, $0 \le k_0 \le n$, so that $\sigma := \sigma(A_{k_0}) = \max_{0\le k\le n} \sigma(A_k)$, and $f(\not\equiv 0)$ is a meromorphic solution of Equation (4), then $\sigma(f) \ge \sigma + 1$ holds.

The particular case that arises in Equation (4) when $w_k = k$ is considered, $0 \le k \le n$, has been subject to further study on the relationship between the growth order of its coefficients and its solution; cf. [6,11,12].

Higher-order complex linear differential equations:

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_0(z)f = 0 \tag{5}$$

have been studied, as well. During the last four decades, the growth order of the solutions of Equation (5) has been related to the growth order of the coefficients when these are entire functions and satisfying some given growth conditions; cf. [13–17].

Let us mention that, in fact, there is a significant amount of recent research on difference equations and their applications, as the ones developed in [18–25]. A number of authors (cf. [6,26–29]) have studied the growth rate of any meromorphic solution of linear differential–difference equations defined by

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i) = 0, \tag{6}$$

where all the $A_{ij}(z)$, $0 \le i \le n$, $1 \le j \le m$, are meromorphic or entire functions with finite growth order and the c_i, $1 \le i \le n$, are distinct complex constants.

Finally, let us recall that Chyzhykov et al. [30] considered the concept of the φ-order of a function f, meromorphic in the unit disc, where $\varphi : [0,\infty) \to (0,\infty)$ is an unbounded non-decreasing real function. Later on, this concept was revisited by Shen et al. [31] and Bouabdelli/Belaidi [32], who extended it and, additionally, explicated the corresponding φ-lower-order definitions that we recall in the following section.

Remark 1. *Throughout this paper, we assume that $\varphi : [0,\infty) \to (0,\infty)$ is an unbounded non-decreasing real function that satisfies the following two conditions:*

(i) $\lim_{r\to+\infty} \frac{\log\log r}{\log\varphi(r)} = 0.$

(ii) $\lim_{r\to+\infty} \frac{\log\varphi(\alpha r)}{\log\varphi(r)} = 1$ *for some* $\alpha > 1$.

In this context, the following question arises naturally, and it will be the focus of our attention in this paper.

Research question : Assuming that the coefficients of a homogeneous linear differential–difference equation defined by Equation (6) are functions of finite-φ-order, entire or meromorphic, can we infer somehow the growth rate of any of its solutions?

2. Notation and Background

Let us recall some notation concerning the measure and Nevanlinna theory concepts that will be used throughout this paper.

Given a subset $E \subset [0, \infty)$, its Lebesgue linear measure, $m(E)$, and its upper density, $\overline{\text{dens}}(E)$, are, respectively, defined by

$$m(E) = \int_E dt, \quad \overline{\text{dens}}(E) = \limsup_{r \to \infty} \frac{m(E \cap [0, r])}{r}.$$

Furthermore, if $E \subset [1, \infty)$, then we also consider its logarithmic measure, $m_l(E)$, and its upper logarithmic density, $\overline{\log \text{dens}}\,(E)$, which are, respectively, defined by

$$m_l(E) = \int_E \frac{dt}{t}, \quad \overline{\log \text{dens}}\,(E) = \limsup_{r \to \infty} \frac{m_l(E \cap [1, r])}{\log r}.$$

Remark 2. *Given a subset $H \subset [1, +\infty)$, the following implications hold:*

(i) $m_l(H) = \infty \implies m(H) = \infty$.
(ii) $\overline{\text{dens}}(H) > 0 \implies m(H) = \infty$.
(iii) $\overline{\log \text{dens}}(H) > 0 \implies m_l(H) = \infty$.

Given a meromorphic function f, let us denote by $n(t, f)$ the number of its poles, counting multiplicities, that lie in $\overline{D}(0, t) := \{z \in \mathbb{C} : |z| \leq t\}$, $t \geq 0$. Then, the *Nevanlinna counting function* of poles, $N(r, f)$, is defined by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f) \log r, \;\; r > 0, \; 0 \leq t \leq r.$$

If $\log^+ : \mathbb{R} \to [0, +\infty)$ stands for the real function defined by $\log^+ x := \log x$ for $x \geq 1$ and $log^+ x := 0$ for $x \leq 1$, the *proximity function* of f, $m(r, f)$, is defined by

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

The *Nevanlinna characteristic* function, represented by T, is the sum of the counting and proximity functions:

$$T(r, f) = N(r, f) + m(r, f).$$

Definition 1 ([1])**.** *Given $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we call the deficiency of a with respect to a given meromorphic function f, and represent it as $\delta(a, f)$, to the value given by*

$$\begin{aligned} \delta(a, f) &= \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \; a \neq \infty, \\ \delta(\infty, f) &= \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, f)}{T(r, f)}. \end{aligned}$$

Definition 2 ([30,31])**.** *The $\varphi-$ (respectively, lower-) order σ (respectively, $\underline{\sigma}$) of a given meromorphic function f is represented as $\sigma(f, \varphi)$ (respectively, $\underline{\sigma}(f, \varphi)$) and corresponds to the value given by*

$$\sigma(f, \varphi) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log \varphi(r)}, \quad (resp.\; \underline{\sigma}(f, \varphi) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log \varphi(r)}).$$

When f is entire, then

$$\sigma(f, \varphi) = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \varphi(r)}, \quad (resp.\; \underline{\sigma}(f, \varphi) = \liminf_{r \to \infty} \frac{\log \log M_f(r)}{\log \varphi(r)}).$$

Definition 3 ([30,32])**.** *Assume that f is a meromorphic function such that $0 < \sigma(f,\varphi) = \sigma < \infty$, then the $\varphi-$type of f is represented as $\tau(f,\varphi)$ and corresponds to the value defined as*

$$\tau(f,\varphi) = \limsup_{r\to\infty} \frac{T(r,f)}{\varphi(r)^{\sigma}}.$$

If f is entire, then

$$\tau(f,\varphi) = \limsup_{r\to\infty} \frac{\log M_f(r)}{\varphi(r)^{\sigma}}.$$

Similarly, if $0 < \underline{\sigma}(f,\varphi) = \underline{\sigma} < +\infty$, the corresponding $\varphi-$ lower types are represented and defined by

$$\underline{\tau}(f,\varphi) = \liminf_{r\to\infty} \frac{T(r,f)}{\varphi(r)^{\underline{\sigma}}} \quad (respectively,\ \ \underline{\tau}(f,\varphi) = \liminf_{r\to\infty} \frac{\log M_f(r)}{\varphi(r)^{\underline{\sigma}}}).$$

Remark 3. *If we take $\varphi(r) = r$ in Definitions 2 and 3, then we generate the order, lower-order, type, and lower-type standard definitions, respectively.*

3. Main Results

In this section, we announce the main findings of this paper, the first of which deal with coefficients that are entire functions and the last two with meromorphic coefficients.

Theorem 2. *Let $A_{ij}(z), 0 \le i \le n, 0 \le j \le m,$ be a family of entire functions such that the $\varphi-$order of some A_{l0}, $0 \le l \le n$, is finite and dominates the $\varphi-$ order of the rest of them, i.e.,*

$$\max\{\sigma(A_{ij},\varphi) : (i,j) \neq (l,0)\} \le \sigma(A_{l0},\varphi) < \infty, \tag{7}$$

and that the $\varphi-$ type of A_{l0} also satisfies that

$$\max\{\tau(A_{ij},\varphi) : \sigma(A_{ij},\varphi) = \sigma(A_{l0},\varphi),\ (i,j) \neq (l,0)\} < \tau(A_{l0},\varphi), \tag{8}$$

Then, if f $(\not\equiv 0)$ is a transcendental meromorphic solution of Equation (6), $\sigma(f,\varphi) \ge +\sigma(A_{l0},\varphi) + 1$.

Theorem 3. *Let $A_{ij}(z), 0 \le i \le n, 0 \le j \le m$, be a family of entire functions such that the $\varphi-$lower-order of some A_{l0}, $0 \le l \le n$, is finite and dominates the $\varphi-$ order of the rest of them, i.e.,*

$$\max\{\sigma(A_{ij},\varphi) : (i,j) \neq (l,0)\} \le \underline{\sigma}(A_{l0},\varphi) < \infty, \tag{9}$$

and that the $\varphi-$ lower type of A_{l0} also satisfies that

$$\max\{\tau(A_{ij},\varphi) : \sigma(A_{ij},\varphi) = \underline{\sigma}(A_{l0},\varphi),\ (i,j) \neq (l,0)\} < \underline{\tau}(A_{l0},\varphi). \tag{10}$$

Then, if f $(\not\equiv 0)$ is a transcendental meromorphic solution of Equation (6), $\underline{\sigma}(f,\varphi) \ge \underline{\sigma}(A_{l0},\varphi) + 1$.

Theorem 4. *Let $A_{ij}(z), 0 \le i \le n, 0 \le j \le m$, be a family of entire functions such that their $\varphi-$orders are finite and smaller than a real number $\sigma \in [1,+\infty)$, i.e.,*

$$\max\{\sigma(A_{ij},\varphi) : 0 \le i \le n, 0 \le j \le m, \} \le \sigma.$$

Assume that there exists some $H \subset \mathbb{C}$ *with* $\overline{\log \operatorname{dens}}\{|z| : z \in H\} > 0$, *such that, for some integer* $0 \leq l \leq n$, *there exist some constants* $0 \leq \beta < \alpha$ *and some sufficiently small* $\varepsilon, 0 < \varepsilon < \sigma$, *so that as* $|z| = r \to \infty$ *for* $z \in H$,

$$|A_{l0}(z)| \geq \exp\left\{\alpha(\varphi(r))^{\sigma-\varepsilon}\right\}, \tag{11}$$

while for the rest of functions:

$$|A_{ij}(z)| \leq \exp\left\{\beta(\varphi(r))^{\sigma-\varepsilon}\right\}, \ (i,j) \neq (l,0). \tag{12}$$

Then, if f $(\not\equiv 0)$ *is a transcendental meromorphic solution of Equation (6),* $\sigma(f,\varphi) \geq \sigma(A_{l0},\varphi) + 1$.

Theorem 5. *Let* $A_{ij}(z), 0 \leq i \leq n, 0 \leq j \leq m$, *be a family of entire functions of finite* $\varphi-$*orders so that, for some of them,* A_{l0}, $0 \leq l \leq n$, *it holds*

$$\limsup_{r\to\infty} \frac{\sum\limits_{(i,j)\neq(l,0)} m(r, A_{ij})}{m(r, A_{l0})} < 1. \tag{13}$$

Then, every meromorphic solution f $(\not\equiv 0)$ *of Equation (6) satisfies* $\sigma(f,\varphi) \geq \sigma(A_{l0},\varphi) + 1$.

The following results provide some growth properties of the solutions of Equation (6) when the coefficients are meromorphic functions.

Theorem 6. *Let* $A_{ij}(z), 0 \leq i \leq n, 0 \leq j \leq m$, *be a family of meromorphic functions such that, for some of them,* A_{l0}, $0 \leq l \leq n$, *it holds*

$$\max\{\sigma(A_{ij},\varphi) : (i,j) \neq (l,0)\} < \sigma(A_{l0},\varphi) \text{ and } \delta(\infty, A_{l0}) > 0.$$

Then, every meromorphic solution f $(\not\equiv 0)$ *of Equation (6) satisfies* $\sigma(f,\varphi) \geq \sigma(A_{l0},\varphi) + 1$.

Theorem 7. *Let* $A_{ij}(z), 0 \leq i \leq n, 0 \leq j \leq m$, *be a family of meromorphic functions such that, for some of them,* A_{l0}, $0 \leq l \leq n$, *it holds*

$$\limsup_{r\to\infty} \frac{\sum\limits_{(i,j)\neq(l,0)} m(r, A_{ij})}{m(r, A_{l0})} < 1 \text{ and } \delta(\infty, A_{l0}) > 0.$$

Then, every meromorphic solution f $(\not\equiv 0)$ *of Equation (6) satisfies* $\sigma(f,\varphi) \geq \sigma(A_{l0},\varphi) + 1$.

4. Preliminary Lemmas

Let us go through some results that will pave the way for the sequel.

Lemma 1 ([33])**.** *Let* $\alpha > 1$ *be a real number and* (m,n) *a pair of integers with* $0 \leq m < n$. *If* f *is a complex transcendental meromorphic function, then there exist some* $E_1 \subset (1,+\infty)$ *with* $m_l(E_1) < \infty$ *and a real constant* $B > 0$ *depending on* α *and* (m,n), *so that, for* $|z| = r \notin [0,1] \cup E_1$,

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{n-m}.$$

Taking advantage of this lemma, we deduce the following one.

Lemma 2. *Let $\varepsilon > 0$, $\alpha > 1$ be real constants and (m, n) a pair of integers, $0 \le m < n$. If f is a complex transcendental meromorphic function with $1 \le \sigma(f, \varphi) = \sigma < +\infty$, then there exist some $E_2 \subset (1, +\infty)$ with $m_l(E_2) < \infty$, so that, for $|z| = r \notin [0, 1] \cup E_2$, it holds*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \le \left(\frac{(\varphi(r))^{\sigma+\alpha+\varepsilon}}{r} \right)^{n-m}.$$

Proof. By the hypothesis, f has finite $\varphi-$order σ, so given $\varepsilon, 0 < \varepsilon < 2$, for sufficiently large $r > R$, it holds that

$$T(r, f) < (\varphi(r))^{\sigma+\frac{\varepsilon}{2}}. \tag{14}$$

Having in mind Lemma 1, Equation (14) implies that there exist some $E_2 \subset (1, +\infty)$ with $m_l(E_2) < \infty$, and a real constant $B > 0$, so that, if $|z| = r \notin [0, 1] \cup E_2$, then

$$\begin{aligned} \left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| &\le B \left(\frac{(\varphi(\alpha r))^{\sigma+\frac{\varepsilon}{2}}}{r} (\log^{\alpha} r) \log(\varphi(\alpha r))^{\sigma+\frac{\varepsilon}{2}} \right)^{n-m} \\ &\le \left(\frac{(\varphi(r))^{\sigma+\alpha+\varepsilon}}{r} \right)^{n-m}. \end{aligned}$$

This proves the lemma. □

Remark 4. *Goldberg and Ostrovskii ([34], p. 66) showed that the following inequalities hold for any arbitrary complex number $c \neq 0$:*

$$(1 + o(1))T(r - |c|, f(z)) \le T(r, f(z + c)) \le (1 + o(1))T(r + |c|, f(z)),$$

as $r \to \infty$ for an arbitrary meromorphic function f. Hence, it follows that

$$\sigma(f(z + c), \varphi) = \sigma(f, \varphi),\ \underline{\sigma}(f(z + c), \varphi) = \underline{\sigma}(f, \varphi).$$

Lemma 3 ([29])**.** *Let η_1, η_2 be two arbitrary complex numbers, $\eta_1 \neq \eta_2$. If f is a finite $\varphi-$order meromorphic function with $\varphi-$order σ, then for each $\varepsilon > 0$, it holds that*

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O\left((\varphi(r))^{\sigma-1+\varepsilon}\right).$$

Lemma 4 ([5])**.** *Let η be a non-zero complex number and $\gamma > 1$, $\varepsilon > 0$ be two real constants. If f is a meromorphic function, then there exist some subset $E_3 \subset (1, +\infty)$ with $m_l(E_3) < \infty$, and a constant A depending on γ and η, so that, for $|z| = r \notin E_3 \cup [0, 1]$, it holds that*

$$\left| \log \left| \frac{f(z + \eta)}{f(z)} \right| \right| \le A \left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r} \log^{\gamma} r \log^{+} n(\gamma r) \right),$$

where $n(t) = n(t, f) + n\left(t, \frac{1}{f}\right)$.

Lemma 5 ([33])**.** *Let j be a non-negative integer, a be a value in the extended complex plane, and $\alpha > 1$ be a real constant. If f is a transcendental meromorphic function, then there exists a constant $R > 0$, so that, for $r \ge R$, the number $n\left(r, f^{(j)}, a\right)$ of zeros of $f^{(j)}$ in $\overline{D_a(r)} = \{z \in \mathbb{C} : |z - a| \le r\}$ satisfies that*

$$n\left(r, f^{(j)}, a\right) \le \frac{2j + 6}{\log \alpha} T(\alpha r, f).$$

Now, we write down the following result, which comes from fixing $p = q = 1$ in Lemma 2.4 of [32].

Lemma 6 ([32]). *If f is a meromorphic function with $1 \leq \underline{\sigma}(f,\varphi) < +\infty$, then there exists some $E_4 \subset (1,+\infty)$ with $m_l(E_4) = +\infty$, so that, for $|z| = r \in E_4$,*

$$T(r,f) < (\varphi(r))^{\underline{\sigma}(f,\varphi)+\varepsilon}.$$

Lemma 7. *Let η be a non-zero complex number, and let $\beta > 1$ and $\varepsilon > 0$ be given real constants. If f is a meromorphic function that has finite $\varphi-$ order σ, then there exists some $E_5 \subset (1,+\infty)$ with $m_l(E_5) < \infty$, so that, for $|z| = r \notin E_5 \cup [0,1]$, it happens that*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\}.$$

Proof. From Lemma 4, it follows that there exist some $E_5 \subset (1,+\infty)$ with $m_l(E_5) < \infty$, and a constant A, depending on γ and η, so that, for $|z| = r \notin E_5 \cup [0,1]$ and denoting $n(t) = n(t,f) + n\left(t,\frac{1}{f}\right)$, it holds that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r,f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right). \tag{15}$$

Now, Lemma 5 and Equation (15) imply that

$$\begin{aligned}\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| &\leq A\left(\frac{T(\gamma r,f)}{r} + \frac{12}{\log\alpha}\frac{T(\alpha\gamma r,f)}{r}\log^{\gamma} r \log^{+}\left(\frac{12}{\log\alpha}T(\alpha\gamma r,f)\right)\right) \\ &\leq B\left(T(\beta r,f)\frac{\log^{\beta} r}{r}\log T(\beta r,f)\right),\end{aligned} \tag{16}$$

where $B > 0$ is a positive constant and $\beta = \alpha\gamma > 1$.

Since f has finite $\varphi-$order σ, given any ε, $0 < \varepsilon < 2$, for sufficiently large r, it holds

$$T(r,f) < (\varphi(r))^{\sigma+\frac{\varepsilon}{2}}. \tag{17}$$

Taking into account the inequality established by Equations (17) and (16), we deduce that

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq B(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}}\frac{\log^{\beta} r}{r}\log(\varphi(\beta r))^{\sigma+\frac{\varepsilon}{2}} \leq \frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}. \tag{18}$$

Finally, from Equation (18), it follows that

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\}.$$

This proves the lemma. □

Lemma 8. *Let η_1, η_2 be two arbitrary complex numbers $\eta_1 \neq \eta_2$ and $\beta > 1$, $\varepsilon > 0$ be two real numbers. If f is a meromorphic function of finite $\varphi-$order σ, then there exists some $E_6 \subset (1,+\infty)$ with $m_l(E_6) < \infty$, so that for $|z| = r \notin E_6$, it holds that*

$$\exp\left\{-\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\}.$$

Proof. Firstly, we write down the identity:

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right|,\ \eta_1 \neq \eta_2.$$

By Lemma 7 with the given ε, β, there exists some $E_5 \subset (1, +\infty)$ with $m_l(E_5) < \infty$, , so that, for $|z+\eta_2| = R \notin E_5 \cup [0,1]$, we obtain

$$\begin{aligned}
\exp\left\{-\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\} &\leq \exp\left\{-\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\beta+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\} \\
&= \exp\left\{-\frac{(\varphi(R))^{\sigma+\beta+\frac{\varepsilon}{2}}}{R}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \\
&= \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(R))^{\sigma+\beta+\frac{\varepsilon}{2}}}{R}\right\} \\
&\leq \exp\left\{\frac{(\varphi(|z|+|\eta_2|))^{\sigma+\beta+\varepsilon}}{|z+\eta_2|}\right\} \leq \exp\left\{\frac{(\varphi(r))^{\sigma+\beta+\varepsilon}}{r}\right\},
\end{aligned}$$

where $|z| = r \notin E_6$. □

By using Lemmas 4–6, we extend Lemmas 2 and 8 under the $\varphi-$lower-order setting in the following two results.

Lemma 9. *Let $\varepsilon > 0$, $\alpha > 1$ be two real numbers. If f is a transcendental meromorphic function with $1 \leq \underline{\sigma}(f,\varphi) = \underline{\sigma} < +\infty$, then there exist some $E_7 \subset (1, +\infty)$ with $m_l(E_7) = +\infty$, and a pair (m, n) of integers, $0 \leq m < n$, so that, for $|z| = r \in E_7$, it holds that*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq \left(\frac{(\varphi(r))^{\underline{\sigma}+\alpha+\varepsilon}}{r}\right)^{n-m}.$$

Lemma 10. *Let η_1, η_2 be two arbitrary complex numbers, $\eta_1 \neq \eta_2$, and $\varepsilon > 0$, $\beta > 1$ be two real numbers. If f is a meromorphic function of finite $\varphi-$lower-order $\underline{\sigma}$, then there exists some $E_8 \subset (1, +\infty)$ with $m_l(E_8) = +\infty$, such that, for $|z| = r \in E_8$, it holds that*

$$\exp\left\{-\frac{(\varphi(r))^{\underline{\sigma}+\beta+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\underline{\sigma}+\beta+\varepsilon}}{r}\right\}.$$

Lemma 11. *If f is a meromorphic function with $\sigma(f,\varphi) = \sigma \geq 1$, then there exists some $E_9 \subset (1, +\infty)$ with $m_l(E_9) = +\infty$, so that*

$$\lim_{\substack{r\to+\infty \\ r\in E_9}} \frac{\log T(r,f)}{\log \varphi(r)} = \sigma.$$

Proof. Taking into account the definition of $\sigma(f,\varphi)$, we may pick up some sequence $\{r_n\}$ diverging to $+\infty$, satisfying $\left(1+\frac{1}{n}\right)r_n < r_{n+1}$, and

$$\sigma(f,\varphi) = \lim_{r_n\to\infty} \frac{\log T(r_n,f)}{\log \varphi(r_n)}.$$

Hence, there exists some integer n_1, so that, for $n \geq n_1$, $r \in \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$,

$$\frac{\log T(r_n, f)}{\log \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} \leq \frac{\log T(r, f)}{\log \varphi(r)} \leq \frac{\log T\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log \varphi(r_n)}.$$

Set $E_9 = \bigcup\limits_{n=n_1}^{\infty}\left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$. Then, for $r \in E_9$, we deduce

$$\lim_{\substack{r\to+\infty \\ r\in E_9}} \frac{\log T(r, f)}{\log \varphi(r)} = \lim_{r_n\to+\infty} \frac{\log T(r_n, f)}{\log \varphi(r_n)}\sigma(f, \varphi) = \sigma,$$

and $m_l(E_9) = \sum\limits_{n=n_1}^{\infty} \int\limits_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty.$ □

The next lemma comes just from fixing $p = q = 1$ in Lemma 2.5 of [32].

Lemma 12 ([32])**.** *If f_1 and f_2 are two meromorphic functions satisfying $\sigma(f_1, \varphi) > \sigma(f_2, \varphi)$, then there exists some $E_{10} \subset (1, +\infty)$ with $m_l(E_{10}) = +\infty$, so that, for $r \in E_{10}$, it holds*

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Lemma 13. *It f is an entire function with $1 \leq \underline{\sigma}(f, \varphi) = \underline{\sigma} < +\infty$, then there exists some $E_{11} \subset (1, +\infty)$ with $m_l(E_{11}) = +\infty$, so that, for $r \in E_{11}$, it holds*

$$\underline{\tau}(f, \varphi) = \lim_{\substack{r\to+\infty \\ r\in E_{11}}} \frac{\log M_f(r)}{\varphi(r)^{\underline{\sigma}}}.$$

Proof. By the definition of $\underline{\sigma}(f, \varphi)$, we may pick up some sequence $\{r_n\}$ diverging to $+\infty$, with $\left(1+\frac{1}{n}\right)r_n < r_{n+1}$ and

$$\underline{\tau}(f, \varphi) = \lim_{r_n\to+\infty} \frac{\log M_f(r_n)}{\varphi(r_n)^{\underline{\sigma}}}.$$

Hence, there exists some integer n_1, so that, for $n \geq n_1$, $r \in \left[\frac{n}{n+1}r_n, r_n\right]$, it holds

$$\frac{\log M_f\left(\frac{n}{n+1}r_n\right)}{\varphi(r_n)^{\underline{\sigma}}} \leq \frac{\log M_f(r)}{\varphi(r)^{\underline{\sigma}}} \leq \frac{\log M_f(r_n)}{\varphi\left(\frac{n}{n+1}r_n\right)^{\underline{\sigma}}}.$$

Therefore,

$$\left(\frac{\varphi\left(\frac{n}{n+1}r_n\right)}{\varphi(r_n)}\right)^{\underline{\sigma}} \frac{\log M_f\left(\frac{n}{n+1}r_n\right)}{\varphi\left(\frac{n}{n+1}r_n\right)^{\underline{\sigma}}} \leq \frac{\log M_f(r)}{\varphi(r)^{\underline{\sigma}}} \leq \frac{\log M_f(r_n)}{\varphi(r_n)^{\underline{\sigma}}} \left(\frac{\varphi(r_n)}{\varphi\left(\frac{n}{n+1}r_n\right)}\right)^{\underline{\sigma}}.$$

If we fix $E_{11} = \bigcup\limits_{n=n_1}^{\infty}\left[\frac{n}{n+1}r_n, r_n\right]$, then, for $r \in E_{11}$, we obtain that

$$\lim_{\substack{r\to+\infty \\ r\in E_{11}}} \frac{\log M_f(r)}{\varphi(r)^{\underline{\sigma}}} = \lim_{r_n\to+\infty} \frac{\log M_f(r_n)}{\varphi(r_n)^{\underline{\sigma}}} = \underline{\tau}(f, \varphi),$$

and $m_l(E_{11}) = \sum\limits_{n=n_1}^{\infty} \int\limits_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty.$ □

Lemma 14 ([13]). *Let $g, h : [0, \infty) \to \mathbb{R}$ be two monotone non-decreasing functions with $g(r) \le h(r)$ for $r \notin E_{12} \cup [0,1]$, where $E_{12} \subset (1, +\infty)$ satisfies that $m_l(E_{12}) < \infty$, and let $\gamma > 1$ be a real number. Then, there exists some $r_0 = r_0(\gamma) > 0$, so that $g(r) \le h(\gamma r)$ for $r > r_0$.*

5. Proof of Main Results

Proof of Theorem 7. Assume that $f(\not\equiv 0)$ is a transcendental meromorphic solution of Equation (6) such that $\sigma(f,\varphi) < \sigma(A_{l0},\varphi) + 1 < \infty$. Dividing both terms of Equation (6) by $f(z+c_l)$, we obtain

$$-A_{l0} = \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} A_{ij}(z)\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)}. \tag{19}$$

Let us write down $\sigma = \max\{\sigma(A_{ij},\varphi) : (i,j) \neq (l,0)\} \le \sigma(A_{l0},\varphi)$ and analogously, $\tau = \max\{\tau(A_{ij},\varphi) : \sigma(A_{ij},\varphi) = \sigma(A_{l0},\varphi), (i,j) \neq (l,0)\} < \tau(A_{l0},\varphi)$. Then, for a sufficiently large r, we have that, if $\sigma(A_{ij},\varphi) < \sigma(A_{l0},\varphi), (i,j) \neq (l,0)$, then

$$|A_{ij}(z)| \le \exp\left\{(\varphi(r))^{\sigma+\varepsilon}\right\}, \tag{20}$$

and, if $\sigma(A_{ij},\varphi) = \sigma(A_{l0},\varphi)$, $(i,j) \neq (l,0)$, then

$$|A_{ij}(z)| \le \exp\left\{(\tau+\varepsilon)(\varphi(r))^{\sigma(A_{l0},\varphi)}\right\}, \tag{21}$$

Lemma 2 and Remark 4 imply that, given $\varepsilon > 0$, $\alpha > 1$, there exists some $E_2 \subset (1, +\infty)$ with $m_l(E_2) < \infty$, for $|z| = r \notin [0,1] \cup E_2$ and $0 \le i \le n, 0 \le j \le m$; it holds

$$\left|\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right| \le \left(\frac{(\varphi(r))^{\sigma(f(z+c_i),\varphi)+\alpha+\varepsilon}}{r}\right)^j = \left(\frac{(\varphi(r))^{\sigma(f,\varphi)+\alpha+\varepsilon}}{r}\right)^j. \tag{22}$$

By Lemma 8, there exists some $E_6 \subset (1, +\infty)$ with $m_l(E_6) < \infty$, such that, for $|z| = r \notin E_6$, $\varepsilon > 0$ and $\beta > 1$, it holds that

$$\left|\frac{f(z+c_i)}{f(z+c_l)}\right| \le \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\beta+\varepsilon}}{r}\right\}, 0 \le i \le n,\ i \neq l. \tag{23}$$

We chose some $\varepsilon > 0$ small enough to satisfy

$$\tau + 2\varepsilon < \tau(A_{l0},\varphi), \quad \max\{\sigma, \sigma(f,\varphi) - 1\} + 2\varepsilon < \sigma(A_{l0},\varphi). \tag{24}$$

Carrying (20), (21), (22) and (23) into (19), for $|z| = r \notin [0,1] \cup E_2 \cup E_6$, we obtain that

$$M_{A_{l0}}(r) \le \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\beta+\varepsilon}}{r}\right\} O\left(\exp\left\{(\tau+\varepsilon)(\varphi(r))^{\sigma(A_{l0},\varphi)}\right\} + \exp\left\{(\varphi(r))^{\sigma+\varepsilon}\right\}\right)$$
$$\cdot\left(\frac{(\varphi(r))^{\sigma(f,\varphi)+\alpha+\varepsilon}}{r}\right)^m, \tag{25}$$

where $|A_{l0}(z)| = M_{A_{l0}}(r)$. By (24), (25) and Lemma 14, we obtain that

$$\tau(A_{l0},\varphi) = \limsup_{r\to\infty} \frac{\log M_{A_{l0}}(r)}{\varphi(r)^{\sigma(A_{l0},\varphi)}} \le \tau + \varepsilon < \tau(A_{l0},\varphi) - \varepsilon,$$

which is a contradiction. Hence, $\sigma(f,\varphi) \ge 1 + \sigma(A_{l0},\varphi)$. □

Proof of Theorem 8. Our reasoning will be similar to the one made for Theorem 2. Assume that $f(\not\equiv 0)$ is a transcendental meromorphic solution of Equation (6), satisfying $\underline{\sigma}(f,\varphi) < \underline{\sigma}(A_{l0},\varphi) + 1 < \infty$.

Let us set $\sigma_1 = \max\{\sigma(A_{ij},\varphi) : (i,j) \neq (l,0)\} \leq \underline{\sigma}(A_{l0},\varphi)$ and on the other hand, $\tau_1 = \max\{\tau(A_{ij},\varphi) : \sigma(A_{ij},\varphi) = \underline{\sigma}(A_{l0},\varphi),\ (i,j) \neq (l,0)\} < \underline{\tau}(A_{l0},\varphi)$. Then, for r large enough, we have that, if $\sigma(A_{ij},\varphi) < \underline{\sigma}(A_{l0},\varphi),\ \ (i,j) \neq (l,0)$,

$$|A_{ij}(z)| \leq \exp\Big\{(\varphi(r))^{\sigma_1+\varepsilon}\Big\}, \tag{26}$$

and if $\sigma(A_{ij},\varphi) = \underline{\sigma}(A_{l0},\varphi)$, $(i,j) \neq (l,0)$, then

$$|A_{ij}(z)| \leq \exp\Big\{(\tau_1+\varepsilon)(\varphi(r))^{\underline{\sigma}(A_{l0},\varphi)}\Big\}. \tag{27}$$

By Remark 4 and Lemmas 9 and 10, given $\varepsilon > 0$ and $\alpha,\beta > 1$, there exists some $E_8 \subset (1,+\infty)$ with $m_l(E_8) = +\infty$, so that, for $|z| = r \in E_8$ and $0 \leq i \leq n$, $0 \leq j \leq m$,

$$\left|\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right| \leq \left(\frac{(\varphi(r))^{\underline{\sigma}(f(z+c_i),\varphi)+\alpha+\varepsilon}}{r}\right)^j = \left(\frac{(\varphi(r))^{\underline{\sigma}(f,\varphi)+\alpha+\varepsilon}}{r}\right)^j \tag{28}$$

and

$$\left|\frac{f(z+c_i)}{f(z+c_l)}\right| \leq \exp\left\{\frac{(\varphi(r))^{\underline{\sigma}(f,\varphi)+\beta+\varepsilon}}{r}\right\}, (i = 0,1,\ldots,n,\ i \neq l) \tag{29}$$

hold. Let us pick some $\varepsilon > 0$ sufficiently small to satisfy

$$\tau_1 + 2\varepsilon < \underline{\tau}(A_{l0},\varphi), \quad \max\{\sigma_1, \underline{\sigma}(f,\varphi) - 1\} + 2\varepsilon < \underline{\sigma}(A_{l0},\varphi). \tag{30}$$

Carrying (26), (27), (28) and (29) into (19), for $|z| = r \in E_8$, we obtain

$$M_{A_{l0}}(r) \leq \exp\left\{\frac{(\varphi(r))^{\underline{\sigma}(f,\varphi)+\beta+\varepsilon}}{r}\right\} O\left(\exp\Big\{(\tau_1+\varepsilon)(\varphi(r))^{\underline{\sigma}(A_{l0},\varphi)}\Big\} + \exp\Big\{(\varphi(r))^{\sigma_1+\varepsilon}\Big\}\right)$$
$$\cdot\left(\frac{(\varphi(r))^{\underline{\sigma}(f,\varphi)+\alpha+\varepsilon}}{r}\right)^m, \tag{31}$$

where $|A_{l0}(z)| = M_{A_{l0}}(r)$. By (30), (31) and Lemma 13, for $|z| = r \in E_8$, we deduce

$$\underline{\tau}(A_{l0},\varphi) = \varliminf_{r\to\infty} \frac{\log M_{A_{l0}}(r)}{\varphi(r)^{\underline{\sigma}(A_{l0},\varphi)}} \leq \tau_1 + \varepsilon < \underline{\tau}(A_{l0},\varphi) - \varepsilon,$$

a contradiction. Hence, $\underline{\sigma}(f,\varphi) \geq \underline{\sigma}(A_{l0},\varphi) + 1$. □

Proof of Theorem 9. Assume that $f(\not\equiv 0)$ is a transcendental meromorphic solution of Equation (6) satisfying $\sigma(f,\varphi) < \sigma(A_{l0},\varphi) + 1 < \infty$. By hypothesis, there is some $H \subset \mathbb{C}$ with $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$, so that, if $z \in H$, Equations (11) and (12) hold as $r \to \infty$.

Hence, if we set $H_1 = \{|z| = r : z \in H\}$, Remark 2 yields that $\int_{H_1} \frac{dr}{r} = \infty$, it being immediate that Equations (22) and (23) are true for $|z| = r \notin [0,1] \cup E_2 \cup E_6$.

Carrying (11), (12), (22) and (23) into (19), for $|z| = r \in H_1 \setminus [0,1] \cup E_2 \cup E_6$, and considering any $\varepsilon \in \left(0, \frac{\sigma-\sigma(f,\varphi)+1}{2}\right)$, it follows that

$$\exp\Big\{\alpha(\varphi(r))^{\sigma-\varepsilon}\Big\} \leq n\exp\Big\{\beta(\varphi(r))^{\sigma-\varepsilon}\Big\} \cdot \exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\beta+\varepsilon}}{r}\right\} \cdot \left(\frac{(\varphi(r))^{\sigma(f,\varphi)+\alpha+\varepsilon}}{r}\right)^m.$$

Consequently,

$$\exp\left\{(\alpha-\beta)(\varphi(r))^{\sigma-\varepsilon}\right\} \le n\exp\left\{\frac{(\varphi(r))^{\sigma(f,\varphi)+\beta+\varepsilon}}{r}\right\}\left(\frac{(\varphi(r))^{\sigma(f,\varphi)+\alpha+\varepsilon}}{r}\right)^{m}. \quad (32)$$

Equation (32) and $\varepsilon \in \left(0, \frac{\sigma-\sigma(f,\varphi)+1}{2}\right)$ are contradictory. Hence, $\sigma(f,\varphi) \ge \sigma(A_{l0},\varphi) + 1$. □

Proof of Theorem 10. Assume that $f(\not\equiv 0)$ is a meromorphic solution of Equation (6). The result is trivial if $\sigma(f,\varphi) = \infty$; thus, we will suppose that $\sigma(f,\varphi) < +\infty$.

From Equation (19), it follows that

$$m(r, A_{l0}) \le \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} m(r, A_{ij}) + \sum_{j=1}^{m} m\left(r, A_{lj}\right) + \sum_{i=0}^{n}\sum_{j=1}^{m} m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)$$
$$+\sum_{\substack{i=0\\ i\neq l}}^{n} m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) + O(1). \quad (33)$$

Let us assume that

$$\frac{\sum_{(i,j)\neq(l,0)} m(r, A_{ij})}{m(r, A_{l0})} = \underline{\sigma} < \lambda < 1. \quad (34)$$

Consequently, for r large enough, it holds that

$$\sum_{(i,j)\neq(l,0)} m(r, A_{ij}) < \lambda m(r, A_{l0}). \quad (35)$$

By Lemma 3, for r large enough and any $\varepsilon > 0$, we obtain that

$$m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) = O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right), \quad i \neq l. \quad (36)$$

From the logarithmic derivative lemma and Remark 4, we obtain that

$$m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) = O\left((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\right), \ j = 0, 1, \ldots, m. \quad (37)$$

Taking (35), (36) and (37) into (33), for r large enough and any $\varepsilon > 0$,

$$m(r, A_{l0}) \le \lambda m(r, A_{l0}) + O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right) + O\left((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\right). \quad (38)$$

From (38), it follows that

$$(1-\lambda)m(r, A_{l0}) \le O\left((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\right) + O\left((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\right). \quad (39)$$

By (39), we deduce $\sigma(f,\varphi) \ge \sigma(A_{l0},\varphi) + 1$. □

Proof of Theorem 11. Assume that $f(\not\equiv 0)$ is a meromorphic solution of Equation (6). The result is trivial if $\sigma(f,\varphi) = \infty$; thus, we will suppose that $\sigma(f,\varphi) < +\infty$ and set

$$\delta(\infty, A_{l0}) = \liminf_{r\to\infty} \frac{m(r, A_{l0})}{T(r, A_{l0})} = \delta > 0. \quad (40)$$

From Equation (40), for r large enough, it follows that

$$m(r, A_{l0}) > \frac{1}{2}\delta T(r, A_{l0}). \tag{41}$$

Taking (36), (37) and (41) into (33), for r large enough and any $\varepsilon > 0$, we obtain

$$\begin{aligned} \frac{1}{2}\delta T(r, A_{l0}) &< m(r, A_{l0}) \leq \sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m} m(r, A_{ij}) + \sum_{j=1}^{m} m\Big(r, A_{lj}\Big) \\ &+ \sum_{i=0}^{n}\sum_{j=1}^{m} m\left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) + \sum_{\substack{i=0\\i\neq l}}^{n} m\left(r, \frac{f(z+c_i)}{f(z+c_l)}\right) + O(1) \\ &\leq \sum_{\substack{i=0\\i\neq l}}^{n}\sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T\Big(r, A_{lj}\Big) \\ &+ O\Big((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\Big) + O\Big((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\Big). \end{aligned} \tag{42}$$

Since $\max\{\sigma(A_{ij}, \varphi) : (i,j) \neq (l,0)\} < \sigma(A_{l0}, \varphi)$, Lemma 12 provides some $E_{10} \subset (1, +\infty)$ with $m_l(E_{10}) = +\infty$, so that, for $r \in E_{10}$ and $r \to +\infty$, it holds

$$\max\left\{\frac{T(r, A_{ij})}{T(r, A_{l0})} : (i,j) \neq (l,0)\right\} \to 0. \tag{43}$$

From (42) and (43), for all $r \in E_{10}$ and $r \to +\infty$, we have

$$\left(\frac{\delta}{2} - o(1)\right)T(r, A_{l0}) \leq O\Big((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\Big) + O\Big((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\Big). \tag{44}$$

It follows from (44) and Lemma 11 that $\sigma(f, \varphi) \geq \sigma(A_{l0}, \varphi) + 1$. □

Proof of Theorem 12. Assume that $f(\not\equiv 0)$ is a meromorphic solution of Equation (6). The result is trivial for $\sigma(f, \varphi) = \infty$. Thus, we will suppose that $\sigma(f, \varphi) < +\infty$. As in the proof of Theorem 5, by taking (35), (36) and (37) into (33), for r large enough and any $\varepsilon > 0$, it holds

$$(1-\lambda)m(r, A_{l0}) \leq O\Big((\varphi(r))^{\sigma(f,\varphi)-1+\varepsilon}\Big) + O\Big((\log(\varphi(r)))^{\sigma(f,\varphi)-1+\varepsilon}\Big). \tag{45}$$

From Lemma 11, it follows that there is some $E_9 \subset (1, +\infty)$ with $m_l(E_9) = +\infty$, so that

$$\lim_{\substack{r\to+\infty\\r\in E_9}} \frac{\log T(r, A_{l0})}{\log \varphi(r)} = \sigma(A_{l0}, \varphi), \tag{46}$$

Since $\delta(\infty, A_{l0}) = \liminf_{r\to\infty} \frac{m(r,A_{l0})}{T(r,A_{l0})} = \delta > 0$, we obtain that

$$\lim_{\substack{r\to+\infty\\r\in E_9}} \frac{\log m(r, A_{l0})}{\log \varphi(r)} = \sigma(A_{l0}, \varphi). \tag{47}$$

Finally, from Equations (45) and (47), it follows that $\sigma(f, \varphi) \geq \sigma(A_{l0}, \varphi) + 1$. □

The results obtained in this paper are true whenever the corresponding hypothesis requested in each of them holds. The next example shows the way in which their validity may be checked.

Example 1. *By considering the homogeneous differential–difference equation with entire coefficients:*

$$A_{11}(z)f'(z-1) + A_{20}(z)f(z+3) + A_{00}(z)f(z) = 0, \quad (48)$$

where

$$A_{00}(z) = 1, A_{11}(z), A_{20}(z) = (4\pi i(1-z) - \exp(4\pi iz))\exp(-16\pi iz)$$

depict the order of growth of the exponential function $f(z) = \exp(2\pi iz^2)$.

Proof. Considering the increasing function $\varphi(r) = r$, the conditions of Theorem 2 and Theorem 3 are enjoyed. Since the entire function $f(z) = \exp 2\pi iz^2$ is indeed a solution of the differential–difference Equation (48), f satisfies that $\sigma(f) \geq \sigma(A_{20}) + 1 = 2$. In fact, in this case, $\sigma(f) = 2$ [35]. □

6. Future Research

Keeping in mind the results already established, it looks interesting to find out what happens when the coefficients A_{ij} of the differential–difference equation are bi-complex-valued functions with a finite logarithmic order of growth in the unit disc. Furthermore, it is worthwhile for interested researchers in this field to study the case that arises when the above setting is restricted to a sector of the unit disc.

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