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An Application of wt-Distances to Characterize Complete b-Metric Spaces

Salvador Romaguera

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain; sromague@mat.upv.es

Abstract: The notion of wt-distance, introduced by Hussain et al. provides a natural generalization to the b-metric framework of the well-known and fruitful concept of w-distance, initiated by Kada et al. Since then, several authors have obtained fixed point theorems for complete b-metric spaces with the help of wt-distances. In this note, we generalize the b-metric version of the celebrated Matkowski fixed point theorem, stated by Czerwik, by replacing the involved b-metric with any wt-distance on the corresponding complete b-metric space. From this result, we derive characterizations of complete b-metric spaces that constitute full generalizations of both a prominent characterization of metric completeness due to Suzuki and Takahashi, and the classical characterization of metric completeness obtained by Hu.

Keywords: b-metric space; complete; wt-distance; fixed point; Matkowski's theorem

MSC: 54H25; 54E50; 47H10

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1. Introduction

It is well known that important examples of non-normable locally bounded topological vector spaces such as $l^p(\mathbb{R})$ and $L^p([0,1])$, $0 < p < 1$, can be endowed with the structure of a quasi-normed space. In fact, this follows from the nice result that a topological vector space is quasi-normable if and only if it is locally bounded (see, e.g., [1] (Section 2), [2] (SubSection 3.2.5)). In this context, the recent deep discussion by Berinde and Păcurar on the origins of the concept of a quasi-normed space [3] (Section 2) is worthy of mention and appreciation.

The natural metric generalization of quasi-normed spaces are the so-called b-metric spaces, as defined by Czerwik [4,5]. Actually, b-metric spaces have been also considered and examined by several authors under other names and approaches (the references [3,6,7] provide numerous and interesting details to this respect). As expected, the study of the topological properties of these spaces has been the subject of careful and extensive research (see, e.g., [6–10]). In parallel, a broad theory of fixed point for b-metric spaces has emerged due to the contributions furnished by many authors. In order not to make the bibliography on this subject too prolix, we will limit ourselves to recommending the reader recent contributions [3,11–15] and the references therein.

Among the various and diverse possible ways to conduct the research to establishing fixed point theorems in this framework, we will be settle in this note to deal with those related to the concept of wt-distance as defined in [16,17]. Indeed, the notion of a wt-distance constitutes a natural b-metric extension of the successful notion of w-distance introduced and analyzed by Kada et al. in [18]. Specifically, Darko et al. presented in their very recent paper [15] a valuable update on the study of wt-distances and their applicability to the fixed point theory for b-metric spaces.

On the other hand, it is well known the relevance of the famous Matkowski fixed point theorem [19] (Theorem 1.2). In particular, Czerwik stated in [4] (Theorem 1) a b-metric full generalization of Matkowski's theorem. Kajántó and Lukács observed in [20] (pp. 85-86) that Czerwik's proof had an inaccuracy. Then, they successfully corrected the original proof, validating Czerwik's theorem. In [21] (Theorem 3.1 and Corollary 3.1), Miculescu and Mihail also gave a correct proof of Czerwik's theorem.

In this note, we generalize the b-metric version of Matkowski's theorem by replacing the involved b-metric with any wt-distance on the corresponding complete b-metric space. With the help of this result, we deduce characterizations of complete b-metric spaces that yield full generalizations of both the featured characterization of metric completeness by Suzuki and Takahashi [22] (Theorem 4), and the first characterization of metric completeness obtained via fixed point results, due to Hu [23].

2. Preliminaries

In this section, we recollect some definitions and properties that will be useful in Sections 3 and 4. By $\mathbb{N}$ and $\mathbb{R}^+$, we appoint the set of positive integers and the set of nonnegative reals, respectively. Our main source for general topology is [24].

According to [4,5], a triple $(\mathcal{S}, \mathfrak{b}, K)$ is a b-metric space provided $\mathcal{S}$ is a set, K is a real constant such that $K \geq 1$, and $\mathfrak{b}$ is a function from $\mathcal{S} \times \mathcal{S}$ to $\mathbb{R}^+$ verifying the following conditions for any $x, y, z \in \mathcal{S}$:

(b1) $\mathfrak{b}(x, y) = 0$ if and only if $x = y$;
(b2) $\mathfrak{b}(x, y) = \mathfrak{b}(y, x)$;
(b3) $\mathfrak{b}(x, z) \leq K(\mathfrak{b}(x, y) + \mathfrak{b}(y, z))$.

In such a case, the function $\mathfrak{b}$ is called a b-metric (on $\mathcal{S}$). If $K = 1$, $\mathfrak{b}$ is a metric and $(\mathcal{S}, \mathfrak{b}, K)$ is a metric space.

To find relevant examples of b-metric spaces, the reader can consult [3,6,8,9,25], among others.

Given a b-metric space $(\mathcal{S}, \mathfrak{b}, K)$, we have (see e.g., [8] (pp. 4310), [7] (Section 2), [6] (Chapter 12)):

- The b-metric in a natural way $\mathfrak{b}$ induces a topology $\sigma(\mathfrak{b})$ on $\mathcal{S}$ which is constructed, as in the metric case, as follows:

$$\sigma(\mathfrak{b}) = \{\mathcal{A} \subseteq \mathcal{S} : \text{ for each } x \in \mathcal{A} \text{ there is } \varepsilon > 0 \text{ such that } B_{\mathfrak{b}}(x, \varepsilon) \subseteq \mathcal{A}\},$$

  where $B_{\mathfrak{b}}(x, \varepsilon) = \{y \in \mathcal{S} : \mathfrak{b}(x, y) < \varepsilon\}$ for all $x \in \mathcal{S}$ and $\varepsilon > 0$.
- The topology $\sigma(\mathfrak{b})$ is generated by the uniformity which has as a base the countable family $\{U_n : n \in \mathbb{N}\}$, where

$$U_n = \{(x, y) \in \mathcal{S} \times \mathcal{S} : \mathfrak{b}(x, y) < 2^{-n}\},$$

  for all $n \in \mathbb{N}$.
  Therefore, there is a metric on $\mathcal{S}$ that induces the topology $\sigma(\mathfrak{b})$, i.e., $\sigma(\mathfrak{b})$ is a metrizable topology.
- Contrarily to the metric setting, the set $B_{\mathfrak{b}}(x, \varepsilon)$ is not necessarily open for $\sigma(\mathfrak{b})$ (see [8] (Example on pp. 4310–4311), [9] (Example 3.9)), and there exist b-metrics that are not continuous functions (see [9] (Examples 3.9 and 3.10)).
  However, from the fact that for each $x \in \mathcal{S}$ and $\varepsilon > 0$ it follows that $x \in \mathrm{Int}(B_{\mathfrak{b}}(x, \varepsilon))$ [24] (Corollary 8.1.3), we obtain the important property that a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{S}$ converges to $x \in \mathcal{S}$ for $\sigma(\mathfrak{b})$ if and only if $\mathfrak{b}(x, x_n) \to 0$ as $n \to +\infty$.
- Precisely as in the metric framework, a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{S}$ is a Cauchy sequence in $(\mathcal{S}, \mathfrak{b}, K)$ provided for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $\mathfrak{b}(x_n, x_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$.
  In addition, $(\mathcal{S}, \mathfrak{b}, K)$ is called complete if every Cauchy sequence is convergent for $\sigma(\mathfrak{b})$.

3. A b-Metric Generalization of Matkowski's Theorem That Involves wt-Distances

We begin this section by reminding the key concepts of a comparison function and of a wt-distance.

Following [26–29], by a comparison function we mean a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi^n(t) \to 0$ as $n \to +\infty$, for all $t \geq 0$.

The next are two useful properties of a comparison function φ:

(i) $\varphi(t) < t$ for all $t > 0$;

(ii) φ is continuous at $t = 0$ and $\varphi(0) = 0$.

According to [16,17], if $(\mathcal{S}, \mathfrak{b}, K)$ is a b-metric space, a function $f : \mathcal{S} \to \mathbb{R}^+$ is K-lower semicontinuous (K-lsc, in short) on $(\mathcal{S}, \mathfrak{b}, K)$ provided the following condition holds:

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}$ such that $x_n \to x \in \mathcal{S}$ for $\sigma(\mathfrak{b})$, then $f(x) \leq \liminf_n Kf(x_n)$.

Remark 1. *Let $(\mathcal{S}, \mathfrak{b}, K)$ be a b-metric space. A function $f : \mathcal{S} \to \mathbb{R}^+$ is K-lsc if and only if whenever a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ converges to $x \in \mathcal{S}$ for $\sigma(\mathfrak{b})$, then for each $\varepsilon > 0$, $f(x) - Kf(x_n) < \varepsilon$ eventually.*

Let $(\mathcal{S}, \mathfrak{b}, K)$ be a b-metric space. A function $\mathfrak{p} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ is said to be a wt-distance on $(\mathcal{S}, \mathfrak{b}, K)$ ([16,17]) if it fulfills the following conditions:

(wt1) $\mathfrak{p}(x, y) \leq K(\mathfrak{p}(x, z) + \mathfrak{p}(z, y))$, for all $x, y, z \in \mathcal{S}$.

(wt2) For each $x \in \mathcal{S}$, the function $\mathfrak{p}(x, \cdot) : \mathcal{S} \to \mathbb{R}^+$ is K-lsc on $(\mathcal{S}, \mathfrak{b}, K)$.

(wt3) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathfrak{p}(x, y) \leq \delta$ and $\mathfrak{p}(x, z) \leq \delta$ imply $\mathfrak{b}(y, z) \leq \varepsilon$.

If $K = 1$, the notion of wt-distance coincides with the notion of w-distance as defined in [18].

Several examples of wt-distances may be found in [15–17] (a novel instance is given in Example 3 below). In particular, if $(\mathcal{S}, \mathfrak{b}, K)$ is a b-metric space, the b-metric $\mathfrak{b}$ is a wt-distance on $(\mathcal{S}, \mathfrak{b}, K)$. The following representative example will be used in Example 2 below.

Example 1. *Consider the complete b-metric space $(\mathbb{R}, \mathfrak{b}, 2)$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathfrak{b}$ is given by $\mathfrak{b}(x, y) =\mid x - y \mid^2$ for all $x, y \in \mathbb{R}$ (see, e.g., [6] (Example 12.2)). Then, the function $\mathfrak{p} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined as $\mathfrak{p}(x, y) = y^2$ for all $x, y \in \mathbb{R}$, is a wt-distance on $(\mathbb{R}, \mathfrak{b}, 2)$.*

Definition 1. *Let $(\mathcal{S}, \mathfrak{b}, K)$ be a b-metric space. A self map $\mathcal{T}$ of $\mathcal{S}$ is said to be a wt-Matkowski contraction (on $(\mathcal{S}, \mathfrak{b}, K)$) if there exists a wt-distance $\mathfrak{p}$ on $(\mathcal{S}, \mathfrak{b}, K)$ and a comparison function φ such that $\mathfrak{p}(\mathcal{T}x, \mathcal{T}y) \leq \varphi(\mathfrak{p}(x, y))$ for all $x, y \in \mathcal{S}$.*

In this case, we say that $\mathfrak{p}$ is a wt-distance associated with $\mathcal{T}$.

If $\varphi(t) = ct$, where $c \in (0, 1)$ is a constant, we say that $\mathcal{T}$ is a wt-Banach contraction (on $(\mathcal{S}, \mathfrak{b}, K)$).

Next, we establish and show the main result of this section. Our proof consists of an adaptation and refinement of a method developed by Kajántó and Lukács in [20].

Theorem 1. *Every wt-Matkowski contraction $\mathcal{T}$ on a complete b-metric space $(\mathcal{S}, \mathfrak{b}, K)$ has a unique fixed point $u \in \mathcal{S}$. Furthermore, $\mathfrak{p}(u, u) = 0$ for any wt-distance $\mathfrak{p}$ associated with $\mathcal{T}$.*

Proof. Let $\mathcal{T}$ be a wt-Matkowski contraction on the complete b-metric space $(\mathcal{S}, \mathfrak{b}, K)$. Then, there exist a wt-distance $\mathfrak{q}$ on $(\mathcal{S}, \mathfrak{b}, K)$ and a comparison function φ such that

$$\mathfrak{q}(\mathcal{T}x, \mathcal{T}y) \leq \varphi(\mathfrak{q}(x, y)), \quad (1)$$

for all $x, y \in \mathcal{S}$.

Fix an $x_0 \in \mathcal{S}$ and let, as usual, $x_n := \mathcal{T}^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

If there is $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of $\mathcal{T}$.

If there is $n_0 \in \mathbb{N}$ such that $\mathfrak{q}(x_{n_0}, x_{n_0+1}) = 0$, we have $\mathfrak{q}(x_{n_0+1}, x_{n_0+2}) = 0$ by the contraction condition (1), and thus $\mathfrak{q}(x_{n_0}, x_{n_0+2}) = 0$ by **(wt1)**. Hence, by **(wt3)**, $x_{n_0+1} = x_{n_0+2}$, i.e., x_{n_0+1} is a fixed point of $\mathcal{T}$.

Therefore, in the sequel, we assume that $x_n \neq x_{n+1}$ and $\mathfrak{q}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

We shall show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}, \mathfrak{b}, K)$ whose limit is the unique fixed point of $\mathcal{T}$.

Given $\varepsilon > 0$ let $\delta{:=}\delta(\varepsilon) > 0$ satisfying **(wt3)** with respect to ε. We assume, without loss of generality, that $\delta < \varepsilon/4K$.

Put $\mu := \min\{1, \delta/2K^2\}$. Thus, $\mu < \delta$, and $2\mu K^2 < \varepsilon/4K$.

Now define a function $\mathcal{Q} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ as:

$$\mathcal{Q}(x, y) = \max\{\mathfrak{q}(x, y), \mathfrak{q}(y, x)\},$$

for all $x, y \in \mathcal{S}$.

The following two assertions, **(a1)** and **(a2)**, are easily checked.

(a1) $\mathcal{Q}$ satisfies condition **(wt1)**, i.e.,

$$\mathcal{Q}(x, y) \leq K(\mathcal{Q}(x, z) + \mathcal{Q}(z, y)),$$

for all $x, y, z \in \mathcal{S}$.

Furthermore $\mathcal{Q}$ is symmetric, i.e., $\mathcal{Q}(x, y) = \mathcal{Q}(y, x)$ for all $x, y \in \mathcal{S}$.

(a2) $\mathcal{Q}(\mathcal{T}x, \mathcal{T}y) \leq \varphi(\mathcal{Q}(x, y))$, for all $x, y \in \mathcal{S}$, because φ is nondecreasing.

Notice that, by **(a2)**, for any $m \in \mathbb{N}$ we obtain

$$\mathcal{Q}(x_m, \mathcal{T}x_m) \leq \varphi(\mathcal{Q}(x_{m-1}, x_m)) \leq \varphi^2(\mathcal{Q}(x_{m-2}, x_{m-1})) \leq \cdots \leq \varphi^m(\mathcal{Q}(x_0, x_1)),$$

$$\mathcal{Q}(x_{2m}, \mathcal{T}^2x_{2m}) \leq \varphi(\mathcal{Q}(x_{2m-1}, \mathcal{T}x_{2m})) \leq \varphi^2(\mathcal{Q}(x_{2m-2}, x_{2m})) \leq \cdots \leq \varphi^{2m}(\mathcal{Q}(x_0, x_2)),$$

and, in general,

$$\mathcal{Q}(x_{nm}, \mathcal{T}^n x_{nm}) \leq \varphi^{nm}(\mathcal{Q}(x_0, x_n)), \tag{2}$$

for all $n, m \in \mathbb{N}$.

Choose an $n_0 > 1$ such that $\varphi^{n_0}(\mu/2K) < \mu^2/4K^2$, and, then, an $m_0 \in \mathbb{N}$ such that $\varphi^{n_0m_0}(\mathcal{Q}(x_0, x_{n_0})) < \mu/4K^2$. By (2) we infer that

$$\mathcal{Q}(x_{n_0m}, \mathcal{T}^{n_0}x_{n_0m}) < \frac{\mu}{4K^2}, \tag{3}$$

for all $m \geq m_0$.

The rest of the proof consists of seven claims and some complementary observations that link these claims.

Claim 1. *For each $z \in \mathcal{S}$*

$$\mathcal{Q}(x_{n_0m_0}, z) < \frac{\mu}{2K} \implies \mathcal{Q}(x_{n_0m_0}, \mathcal{T}^{n_0}z) < \frac{\mu}{2K}. \tag{4}$$

*Indeed, let $z \in \mathcal{S}$ such that $\mathcal{Q}(x_{n_0m_0}, z) < \mu/2K$. From **(a2)**, it follows that*

$$\mathcal{Q}(\mathcal{T}^{n_0}x_{n_0m_0}, \mathcal{T}^{n_0}z) \leq \varphi^{n_0}(\mathcal{Q}(x_{n_0m_0}, z)).$$

Hence, by using (3) joint with the two preceding inequalities, and taking into account that $\mu \leq 1$, we obtain

$$\begin{aligned}
\mathcal{Q}(x_{n_0m_0}, \mathcal{T}^{n_0}z) &\leq K(\mathcal{Q}(x_{n_0m_0}, \mathcal{T}^{n_0}x_{n_0m_0}) + \mathcal{Q}(\mathcal{T}^{n_0}x_{n_0m_0}, \mathcal{T}^{n_0}z)) \\
&< K(\frac{\mu}{4K^2} + \varphi^{n_0}(\mathcal{Q}(x_{n_0m_0}, z))) \leq K(\frac{\mu}{4K^2} + \varphi^{n_0}(\frac{\mu}{2K})) \\
&< K(\frac{\mu}{4K^2} + \frac{\mu^2}{4K^2}) \leq \frac{\mu}{2K}.
\end{aligned}$$

Claim 2. *For each $m > m_0$, the following inequality holds:*

$$\mathcal{Q}(x_{n_0 m_0}, x_{n_0 m}) < \frac{\mu}{2K}. \tag{5}$$

We prove it by mathematical induction.

Indeed, if $m = m_0 + 1$, we have $x_{n_0 m} = x_{n_0 m_0 + n_0} = \mathcal{T}^{n_0} x_{n_0 m_0}$. Since, by (3), $\mathcal{Q}(x_{n_0 m_0}, \mathcal{T}^{n_0} x_{n_0 m_0}) < \mu/2K$, we obviously have the conclusion.

Then, suppose that (5) holds for $m_0 + k - 1$ with $k > 1$. By assumption we have

$$\mathcal{Q}(x_{n_0 m_0}, x_{n_0(m_0+k-1)}) < \frac{\mu}{2K}.$$

So, by (4),

$$\mathcal{Q}(x_{n_0 m_0}, \mathcal{T}^{n_0} x_{n_0(m_0+k-1)}) < \frac{\mu}{2K},$$

i.e.,

$$\mathcal{Q}(x_{n_0 m_0}, x_{n_0(m_0+k)}) < \frac{\mu}{2K},$$

which concludes the proof of this claim.

On the other hand, from the fact that φ is a comparison function we deduce the existence of an $m' > m_0$ such that

$$\varphi^{m'}(\mathcal{Q}(x_0, x_1)) < \frac{\mu}{2n_0 K^{n_0+1}}.$$

Since $\varphi^{m}(\mathcal{Q}(x_0, x_1)) \leq \varphi^{m'}(\mathcal{Q}(x_0, x_1)$ for all $m \geq m'$, and, by **(a2)**, $\mathcal{Q}(x_m, x_{m+1}) \leq \varphi^{m}(\mathcal{Q}(x_0, x_1))$ for all $m \in \mathbb{N}$, we deduce that

$$\mathcal{Q}(x_m, x_{m+1}) < \frac{\mu}{2n_0 K^{n_0+1}}, \tag{6}$$

for all $m \geq m'$.

Claim 3. *For every $m \geq m'$ and $p \in \{1, \ldots, n_0 - 1\}$ (recall that $n_0 > 1$), the following inequality holds:*

$$\mathcal{Q}(x_{n_0 m}, x_{n_0 m + p}) \leq \frac{\mu}{2K}. \tag{7}$$

*Indeed, if $m \geq m'$ we have $n_0 m + l > m'$ for all $l \in \mathbb{N} \cup \{0\}$, so we can apply (**a1**) and (6) to deduce that*

$$\begin{aligned} \mathcal{Q}(x_{n_0 m}, x_{n_0 m + p}) &\leq \sum_{l=0}^{p-1} K^{l+1} \mathcal{Q}(x_{n_0 m + l}, x_{n_0 m + l + 1}) \\ &< \sum_{l=0}^{n_0 - 1} K^{n_0} \frac{\mu}{2n_0 K^{n_0+1}} = \frac{\mu}{2K}. \end{aligned}$$

Claim 4. *For every $j \in \mathbb{N}$ such that $j > n_0 m'$, the following inequality holds:*

$$\mathcal{Q}(x_{n_0 m_0}, x_j) < \mu. \tag{8}$$

Indeed, since $j > n_0 m'$ there exists $m_j \geq m'$ and $p_j \in \{0, 1, \ldots, n_0 - 1\}$ such that $j = n_0 m_j + p_j$.

- *If $p_j = 0$, taking into account that $m_j \geq m' > m_0$, we obtain, by (5),*

$$\mathcal{Q}(x_{n_0 m_0}, x_j) = \mathcal{Q}(x_{n_0 m_0}, x_{n_0 m_j}) < \frac{\mu}{2K} < \mu.$$

- *If* $p_j \in \{1, \ldots, n_0 - 1\}$, *we obtain, by* (5) *and* (7),

$$\begin{aligned}\mathcal{Q}(x_{n_0m_0}, x_j) &\leq K(\mathcal{Q}(x_{n_0m_0}, x_{n_0m_j}) + \mathcal{Q}(x_{n_0m_j}, x_{n_0m_j+p_j})) \\ &< K(\frac{\mu}{2K} + \frac{\mu}{2K}) = \mu.\end{aligned}$$

Claim 5. *For every* $j, k \in \mathbb{N}$ *such that* $j, k > n_0m'$, *we have*

$$\mathfrak{b}(x_j, x_k) \leq \varepsilon. \tag{9}$$

Indeed, since $\mathfrak{q}(x_{n_0m_0}, x_j) \leq \mathcal{Q}(x_{n_0m_0}, x_j)$, $\mathfrak{q}(x_{n_0m_0}, x_k) \leq \mathcal{Q}(x_{n_0m_0}, x_k)$, *and* $\mu < \delta$, *we deduce from* (8) *that* $\mathfrak{q}(x_{n_0m_0}, x_j) < \delta$ *and* $\mathfrak{q}(x_{n_0m_0}, x_k) < \delta$, *so* $\mathfrak{b}(x_j, x_k) \leq \varepsilon$ *by* ***(wt3)***.

It follow from *Claim 5* that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}, \mathfrak{b}, K)$. Hence, there is a unique $u \in \mathcal{S}$ such that $\mathfrak{b}(u, x_n) \to 0$ as $n \to +\infty$.

We want to show that u is a fixed point of $\mathcal{T}$. To reach it, *Claim 6* below will be crucial (up to this point in the proof, we could have worked by simply replacing the function $\mathcal{Q}$ with the *wt*-distance $\mathfrak{q}$. However, the use of $\mathcal{Q}$ and its symmetry will be decisive in showing *Claim 6*).

Claim 6. $\mathfrak{q}(x_n, u) \to 0$ *as* $n \to +\infty$.

Indeed, given $\varepsilon > 0$, *we shall apply the preceding parts of the proof as follows.*

Choose any $j > n_0m'$. *Since* $\mathfrak{b}(u, x_n) \to 0$ *as* $n \to +\infty$, *and* $\mathfrak{q}(x_j, \cdot) : \mathcal{S} \to \mathbb{R}^+$ *is K-lsc, we find* $k > j$ *such that*

$$\mathfrak{q}(x_j, u) < \varepsilon + K\mathfrak{q}(x_j, x_k).$$

Therefore,

$$\begin{aligned}\mathfrak{q}(x_j, u) &< \varepsilon + K\mathcal{Q}(x_j, x_k) \leq \varepsilon + K^2(\mathcal{Q}(x_j, x_{n_0m_0}) + \mathcal{Q}(x_{n_0m_0}, x_k)) \\ &= \varepsilon + K^2(\mathcal{Q}(x_{n_0m_0}, x_j) + \mathcal{Q}(x_{n_0m_0}, x_k)),\end{aligned}$$

so, by Claim 4, $\mathfrak{q}(x_j, u) < \varepsilon + 2\mu K^2$. *Hence* $\mathfrak{q}(x_j, u) < 2\varepsilon$ *for all* $j > n_0m'$.

We conclude that $\mathfrak{q}(x_n, u) \to 0$ *as* $n \to +\infty$.

Claim 7. *u is the unique fixed point of* $\mathcal{T}$. *Furthermore* $\mathfrak{q}(u, u) = 0$.

Indeed, since $\mathfrak{q}(x_{n+1}, \mathcal{T}u) \leq \varphi(\mathfrak{q}(x_n, u))$, *we deduce that* $\mathfrak{q}(x_n, \mathcal{T}u) \to 0$ *as* $n \to +\infty$, *by Claim 6. Then, condition* ***(wt3)*** *immediately implies that* $\mathfrak{b}(u, \mathcal{T}u) \leq \varepsilon$ *for all* $\varepsilon > 0$, *so* $u = \mathcal{T}u$.

Hence, from the contraction condition (1) *we infer that* $\mathfrak{q}(u, u) = 0$.

If v is another fixed point of $\mathcal{T}$, *we obtain*

$$\mathfrak{q}(u, v) = \mathfrak{q}(\mathcal{T}u, \mathcal{T}v) \leq \varphi(\mathfrak{q}(u, v)),$$

so $\mathfrak{q}(u, v) = 0$. *Since* $\mathfrak{q}(u, u) = 0$, *condition* ***(wt3)*** *implies that* $u = v$. *Thus, u is the unique fixed point of* $\mathcal{T}$.

Finally, if $\mathfrak{p}$ is any *wt*-distance associated with $\mathcal{T}$ we deduce that

$$\mathfrak{p}(u, u) = \mathfrak{p}(\mathcal{T}u, \mathcal{T}u) \leq \varphi(\mathfrak{p}(u, u)),$$

so $\mathfrak{p}(u, u) = 0$. This concludes the proof. □

Corollary 1. *Every wt-Banach contraction on a complete b-metric space has a unique fixed point.*

Corollary 2 (Czerwik's theorem)**.** *Let $\mathcal{T}$ be a self map of a complete b-metric space $(\mathcal{S}, \mathfrak{b}, K)$. If there is a comparison function φ such that*

$$\mathfrak{b}(\mathcal{T}x, \mathcal{T}y) \leq \varphi(\mathfrak{b}(x, y)),$$

for all $x, y \in \mathcal{S}$, then $\mathcal{T}$ has a unique fixed point.

Example 2. *Let $\mathfrak{p}$ be the wt-distance on the complete b-metric space $(\mathbb{R}, \mathfrak{b}, 2)$ of Example 1. Let $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ be defined as $\mathcal{T}x = x/(1+ \mid x \mid)$ for all $x \in \mathbb{R}$. We shall show that we can apply Theorem 1 but not Corollary 1 to this self map and the wt-distance $\mathfrak{p}$.*

Let φ be the comparison function given by $\varphi(t) = t/(1+t)$ for all $t \geq 0$. Then, for each $x, y \in \mathbb{R}$ we obtain

$$\mathfrak{p}(\mathcal{T}x, \mathcal{T}y) = \left(\frac{y}{1+ \mid y \mid}\right)^2 \leq \frac{y^2}{1+y^2} = \varphi(y^2) = \varphi(\mathfrak{p}(x, y)).$$

Consequently, $\mathcal{T}$ is a wt-Matkowski contraction on $(\mathbb{R}, \mathfrak{b}, 2)$, and thus all conditions of Theorem 1 are fulfilled. In fact, 0 is the unique fixed point of $\mathcal{T}$.

Finally, fix $c \in (0, 1)$. Choose $y \in (0, c^{-1/2} - 1)$. Then $(1+y)^2 < 1/c$, so, for any $x \in \mathbb{R}$ we have

$$\mathfrak{p}(\mathcal{T}x, \mathcal{T}y) = \left(\frac{y}{1+y}\right)^2 > cy^2 = c\mathfrak{p}(x, y).$$

We conclude this section with an example where we can apply Corollary 1 but not Corollary 2.

Example 3. *Let $\sum = \{0, 1\}$ and $\sum^F$ be the set of finite words (sequences) of elements of $\sum$, where we assume that the empty word $\varnothing$ is an element of $\sum^F$.*

Denote by $\ell(x)$ the length of each $x \in \sum^F$. Thus, we have $\ell(x) = n$ if $x = x_1 ... x_n$, with $x_j \in \sum$, $j \in \{1, ..., n\}$. In particular $\ell(\varnothing) = 0$.

For each $x, y \in \sum^F$ we denote by $x \sqcap y$ the common prefix of x and y.

Fix $k \in \mathbb{N} \backslash \{1\}$. Put $\sum^k = \{x \in \sum^F : \ell(x) \leq k\}$, and let $\mathfrak{d}$ be the metric on $\sum^k$ given by

$$\mathfrak{d}(x, x) = 0 \text{ for all } x \in \textstyle\sum^k, \text{ and } \mathfrak{d}(x, y) = 2^{-\ell(x \sqcap y)} \text{ if } x \neq y.$$

(See, e.g., [30] for details).

Note that the topology $\sigma(\mathfrak{d})$ agrees with discrete topology on $\sum^k$ and that $(\sum^k, \mathfrak{d})$ is a complete metric space because the Cauchy sequences are eventually constant (observe that $\ell(x \sqcap y) \leq k$ and, hence, $2^{-\ell(x \sqcap y)} \geq 2^{-k}$ whenever $x \neq y$).

Therefore, for each $a \geq 1$, the b-metric space $(\sum^k, \mathfrak{b}_{\mathfrak{d},a} 2^{a-1})$ is also complete and the topology $\sigma(\mathfrak{b}_{\mathfrak{d},a})$ agrees with the discrete topology on $\sum^k$, where the b-metric $\mathfrak{b}_{\mathfrak{d},a}$ is given by

$$\mathfrak{b}_{\mathfrak{d},a}(x, y) = (\mathfrak{d}(x, y))^a$$

for all $x, y \in \sum^k$ (see, e.g., [6] (Example 12.2)). Notice that for $a = 1$, we obtain the (complete) metric space $(\sum^k, \mathfrak{d})$.

Let $\mathcal{T} : \sum^k \to \sum^k$ be defined as $\mathcal{T}x = \varnothing$ if $\ell(x) \leq 1$, and for each $x = x_1 x_2 ... x_n$ with $2 \leq n \leq k$, $\mathcal{T}x = x_2 ... x_n$.

Observe that for each $x \in \sum^k \backslash \{\varnothing\}$ we have $\ell(\mathcal{T}x) = \ell(x) - 1$.

Then, for $x, y \in \sum^k$ with $\mathcal{T}x \neq \mathcal{T}y$ we obtain

$$\mathfrak{b}_{\mathfrak{d},a}(\mathcal{T}x, \mathcal{T}y) = (2^{-\ell(Tx \sqcap Ty)})^a = (2^{-(\ell(x \sqcap y) - 1)})^a = 2^a \mathfrak{b}_{\mathfrak{d},a}(x, y),$$

so, we cannot apply Czerwik's theorem for any b-metric space $(\sum^k, \mathfrak{b}_{\mathfrak{d},a}, 2^{a-1})$, $a \geq 1$.

Finally, we shall prove that it is possible to apply Corollary 1 to any b-metric space $(\sum^k, \mathfrak{b}_{\mathfrak{d},a}, 2^{a-1})$, $a \geq 1$.

To achieve it define $\mathfrak{p} : \sum^k \times \sum^k \to \mathbb{R}^+$ as $\mathfrak{p}(x, y) = \ell(y)$ for all $x, y \in \sum^k$.

We show that $\mathfrak{p}$ is a wt-distance on $(\sum^k, \mathfrak{b}_{\mathfrak{d},a}, 2^{a-1})$ for all $a \geq 1$.

*In fact, condition **(wt1)** is obviously satisfied and **(wt2)** is an immediate consequence of the fact that $\sigma(\mathfrak{b}_{\mathfrak{d},a})$ is the discrete topology on $\sum^k$. To verify condition **(wt3)** take an arbitrary $\varepsilon > 0$ and $\delta = \min\{\varepsilon, 1/2\}$; if $\mathfrak{p}(x,y) \leq \delta$ and $\mathfrak{p}(x,z) \leq \delta$, we deduce that $\ell(y) < 1$ and $\ell(z) < 1$, so $y = z = \varnothing$.*

Since for each $x, y \in \sum^k$ (it suffices to consider the case $y \neq \varnothing$), we have

$$\mathfrak{p}(\mathcal{T}x, \mathcal{T}y) = \ell(\mathcal{T}y) = \ell(y) - 1 \leq \frac{k-1}{k}\ell(y) = \frac{k-1}{k}\mathfrak{p}(x,y),$$

then, all conditions of Corollary 1 are satisfied. In fact, $\varnothing$ is the unique fixed point of $\mathcal{T}$.

4. Characterizing Complete *b*-Metric Spaces

In order to simplify the proof of our main result (Theorem 2), we shall use Proposition 1 below, which will be presented in the more general context of quasi-distance spaces.

Definition 2. *A quasi-distance space is a pair $(\mathcal{S}, q)$ where $\mathcal{S}$ is a set and $q : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ is a function that satisfies the following condition for every $x, y \in \mathcal{S}$:*

$q(x,y) = 0$ if and only if $x = y$.

Of course, every *b*-metric space is a quasi-distance space.

The notion of a Cauchy sequence in a quasi-distance space is defined exactly as the classical metric case, as well as the notion of a Banach contraction.

Proposition 1. *Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of distinct points in a quasi-distance space $(\mathcal{S}, q)$. Suppose that for each $n \in \mathbb{N}$, $\inf_{m\in\mathbb{N}\setminus\{n\}} q(x_n, x_m) > 0$ and $\inf_{m\in\mathbb{N}\setminus\{n\}} q(x_m, x_n) > 0$. Then, there exists a Banach contraction on the quasi-distance (sub)space $(\mathcal{X}, q|_{\mathcal{X}})$ free of fixed points, where $\mathcal{X} := \{x_n : n \in \mathbb{N}\}$.*

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, for $n = 1$ there is $i(1) > 1$ such that

$$q(x_j, x_k) < \tfrac{1}{2}\min\{\inf_{m\in\mathbb{N}\setminus\{1\}} q(x_1, x_m), \inf_{m\in\mathbb{N}\setminus\{1\}} q(x_m, x_1)\},$$

for all $j, k \geq i(1)$.

Analogously, for $n = 2$ there is $i(2) > \max\{i(1), 2\}$ such that

$$q(x_j, x_k) < \tfrac{1}{2}\min\{\inf_{m\in\mathbb{N}\setminus\{2\}} q(x_2, x_m), \inf_{m\in\mathbb{N}\setminus\{2\}} q(x_m, x_2)\},$$

for all $j, k \geq i(2)$.

Continuing with this process, we obtain a sequence $(i(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $i(1) > 1$, $i(n+1) > \max\{i(n), n+1\}$ for all $n \in \mathbb{N}\setminus\{1\}$, and

$$q(x_j, x_k) < \tfrac{1}{2}\min\{\inf_{m\in\mathbb{N}\setminus\{n\}} q(x_n, x_m), \inf_{m\in\mathbb{N}\setminus\{n\}} q(x_m, x_n)\},$$

for all $j, k \geq i(n)$, with $n \in \mathbb{N}$.

Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ as $\mathcal{T}x_n = x_{i(n)}$ for all $n \in \mathbb{N}$.

Since $i(n) > n$, $\mathcal{T}$ has no fixed points.

We show that, nevertheless, it is a Banach contraction on $(\mathcal{X}, q|_{\mathcal{X}})$.

Indeed, let $k, l \in \mathbb{N}$ with $k \neq l$. Suppose $k < l$. Then $i(k) < i(l)$, so

$$\begin{aligned} q(\mathcal{T}x_k, \mathcal{T}x_l) &= q(x_{i(k)}, x_{i(l)}) \\ &< \tfrac{1}{2}\min\{\inf_{m\in\mathbb{N}\setminus\{k\}} q(x_k, x_m), \inf_{m\in\mathbb{N}\setminus\{k\}} q(x_m, x_k)\} \\ &\leq \tfrac{1}{2}\min\{q(x_k, x_l), q(x_l, x_k)\} \leq \tfrac{1}{2}q(x_k, x_l). \end{aligned}$$

Finally, if $k > l$ we obtain

$$\begin{aligned} q(\mathcal{T}x_k, \mathcal{T}x_l) &= q(x_{i(k)}, x_{i(l)}) \\ &< \tfrac{1}{2}\min\{\inf_{m\in\mathbb{N}\setminus\{l\}} q(x_l, x_m), \inf_{m\in\mathbb{N}\setminus\{l\}} q(x_m, x_l)\} \\ &\leq \tfrac{1}{2}\min\{q(x_l, x_k), q(x_k, x_l)\} \leq \tfrac{1}{2}q(x_k, x_l). \end{aligned}$$

The proof is complete. □

Theorem 2. *For a b-metric space $(\mathcal{S}, \mathfrak{b}, K)$ the following are equivalent:*

(A) $(\mathcal{S}, \mathfrak{b}, K)$ is complete.

(B) Every wt-Matkowski contraction on $(\mathcal{S}, \mathfrak{b}, K)$ has a fixed point.

(C) Every wt-Banach contraction on $(\mathcal{S}, \mathfrak{b}, K)$ has a fixed point.

(D) Every Banach contraction on any closed and bounded subset of $(\mathcal{S}, \mathfrak{b}, K)$ has a fixed point.

Proof. $(A) \Rightarrow (B)$ It follows from Theorem 1.

$(B) \Rightarrow (C)$ It is obvious.

$(C) \Rightarrow (D)$ Let $\mathcal{X}$ be a bounded closed subset of $\mathcal{S}$ and $\mathcal{T}$ be a Banach contraction on the b-metric (sub)space $(\mathcal{X}, \mathfrak{b}_{|\mathcal{X}}, K)$. Then, there is a constant $r \in (0,1)$ such that $\mathfrak{b}(\mathcal{T}x, \mathcal{T}y) \leq rb(x,y)$ for all $x, y \in \mathcal{X}$.

We shall prove that $\mathcal{T}$ has a fixed point.

Since $\mathcal{X}$ is bounded, there is a constant $L > r$ such that $\mathfrak{b}(x,y) < L$ for all $x, y \in \mathcal{X}$.

Fix $u \in \mathcal{X}$ and let $\mathcal{F}$ be the self map of $\mathcal{S}$ defined as $\mathcal{F}x = \mathcal{T}x$ for all $x \in \mathcal{X}$, and $\mathcal{F}x = u$ for all $x \in \mathcal{S} \backslash \mathcal{X}$.

Now let $\mathfrak{p} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ be defined as $\mathfrak{p}(x,y) = \mathfrak{b}(x,y)$ if $x, y \in \mathcal{X}$, and $\mathfrak{p}(x,y) = L/r$ otherwise.

We are going to check that $\mathfrak{p}$ fulfills conditions **(wt1)**, **(wt2)** and **(wt3)**. Thus, it will be a *wt*-distance on $(\mathcal{S}, \mathfrak{b}, K)$.

(wt1) : Let $x, y, z \in \mathcal{S}$. If $x, y, z \in \mathcal{X}$ we obtain

$$\mathfrak{p}(x,y) = \mathfrak{b}(x,y) \leq K(\mathfrak{b}(x,z) + \mathfrak{b}(z,y)) = K(\mathfrak{p}(x,y) + \mathfrak{p}(y,z)).$$

Otherwise, if $\mathfrak{p}(x,y) = \mathfrak{b}(x,y)$ we deduce that $x, y \in \mathcal{X}$ and $z \in \mathcal{S} \backslash \mathcal{X}$, so

$$\mathfrak{p}(x,y) < L < L/r < 2L/r = \mathfrak{p}(x,z) + \mathfrak{p}(z,y) \leq K(\mathfrak{p}(x,z) + \mathfrak{p}(z,y)),$$

and if $\mathfrak{p}(x,y) = L/r$ we deduce that $\mathfrak{p}(x,z) = L/r$ or $\mathfrak{p}(z,y) = L/r$, so

$$\mathfrak{p}(x,y) = \max\{\mathfrak{p}(x,z), \mathfrak{p}(z,y)\} \leq \mathfrak{p}(x,z) + \mathfrak{p}(z,y) \leq K(\mathfrak{p}(x,z) + \mathfrak{p}(z,y)).$$

(wt2) : Fix $x \in \mathcal{S}$ and let $y_n \to y$ for $\sigma(\mathfrak{b})$. Given $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\mathfrak{b}(y, y_n) < \varepsilon/K$ for all $n \geq n_\varepsilon$.

We distinguish three cases.

Case 1. *$x, y \in \mathcal{X}$. Then $\mathfrak{p}(x,y) = \mathfrak{b}(x,y)$.*

Let $n \geq n_\varepsilon$. If $y_n \in \mathcal{X}$ we obtain $\mathfrak{p}(x, y_n) = \mathfrak{b}(x, y_n)$ and $\mathfrak{p}(y_n, y) = \mathfrak{b}(y_n, y)$, so

$$\mathfrak{p}(x,y) \leq K(\mathfrak{p}(x,y_n) + \mathfrak{p}(y_n,y)) < K\mathfrak{p}(x,y_n) + \varepsilon.$$

If $y_n \in \mathcal{S} \backslash \mathcal{X}$ we obtain $\mathfrak{p}(x, y_n) = \mathfrak{p}(y_n, y) = L/r$, so

$$\mathfrak{p}(x,y) = \mathfrak{b}(x,y) < L < L/r = \mathfrak{p}(x,y_n) < K\mathfrak{p}(x,y_n) + \varepsilon.$$

Case 2. *$x \in \mathcal{X}, y \in \mathcal{S} \backslash \mathcal{X}$. Then $\mathfrak{p}(x,y) = L/r$.*

From the fact that $\mathcal{X}$ is closed it follows the existence of an $n_0 \geq n_\varepsilon$ such that $y_n \in \mathcal{S} \backslash \mathcal{X}$ for all $n \geq n_0$.

Thus, $\mathfrak{p}(x, y_n) = L/r$ for all $n \geq n_0$, so

$$\mathfrak{p}(x,y) = p(x,y_n) < K\mathfrak{p}(x,y_n) + \varepsilon$$

for all $n \geq n_0$.

Case 3. *$x \in \mathcal{S} \backslash \mathcal{X}, y \in \mathcal{X}$, or $x, y \in \mathcal{S} \backslash \mathcal{X}$.*

We have $\mathfrak{p}(x,y) = L/r$ and $\mathfrak{p}(x, y_n) = L/r$ for all $n \geq n_\varepsilon$.

***(wt3)** : Given $\varepsilon > 0$, choose $\delta = \min\{\varepsilon/2K, 1\}$.*

Let $\mathfrak{p}(x,y) \leq \delta$ and $\mathfrak{p}(x,z) \leq \delta$. Since $\delta \leq 1$ and $L/r > 1$, we deduce that $\mathfrak{p}(x,y) = \mathfrak{b}(x,y)$ and $\mathfrak{p}(x,z) = \mathfrak{b}(x,z)$. Hence

$$\mathfrak{b}(y,z) \le K(\mathfrak{b}(y,x) + \mathfrak{b}(x,z)) \le 2\delta K \le \varepsilon.$$

Next, we show that $\mathcal{F}$ is a wt-Banach contraction on $(\mathcal{S}, \mathfrak{b}, K)$. Indeed, let $x, y \in \mathcal{S}$. If $x, y \in \mathcal{X}$ we have

$$\mathfrak{p}(\mathcal{F}x, \mathcal{F}y) = \mathfrak{b}(\mathcal{T}x, \mathcal{T}y) \le r\mathfrak{b}(x,y) = r\mathfrak{p}(x,y).$$

If $x \in \mathcal{S}\backslash\mathcal{X}$ and $y \in \mathcal{X}$ (the case $x \in \mathcal{X}$ and $y \in \mathcal{S}\backslash\mathcal{X}$ is analogous by the symmetry of $\mathfrak{p}$), we have

$$\mathfrak{p}(\mathcal{F}x, \mathcal{F}y) = \mathfrak{p}(u, \mathcal{T}y) = \mathfrak{b}(u, \mathcal{T}y) < L = r\mathfrak{p}(x,y).$$

If $x, y \in \mathcal{S}\backslash\mathcal{X}$, we have

$$\mathfrak{p}(\mathcal{F}x, \mathcal{F}y) = \mathfrak{p}(u,u) = \mathfrak{b}(u,u) = 0.$$

Hence, by our assumption, $\mathcal{F}$ has a fixed point which obviously belongs to $\mathcal{X}$ because $\mathcal{F}x = u \in \mathcal{X}$ for all $x \in \mathcal{S}\backslash\mathcal{X}$.

$(D) \Rightarrow (A)$ Suppose that $(\mathcal{S}, \mathfrak{b}, K)$ is not complete. Then, there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of $\mathcal{S}$ that does not converge for $\sigma(\mathfrak{b})$. Hence, for each $n \in \mathbb{N}$ we obtain $\inf_{m\in\mathbb{N}\backslash\{n\}} \mathfrak{b}(x_n, x_m) > 0$. By Proposition 1, there is a Banach contraction without fixed points in the bounded and closed subspace $(\mathcal{X}, \mathfrak{b}|_{\mathcal{X}}, K)$ of $(\mathcal{S}, \mathfrak{b}, K)$. We have reached a contradiction, that concludes the proof. □

Remark 2. *Hu [23], and Suzuki and Takahashi [22] proved, respectively,* $(A) \Leftrightarrow (D)$ *and* $(A) \Leftrightarrow (C)$ *when* $K = 1$*, while Czerwik's theorem is a special case of* $(A) \Rightarrow (B)$ *when the wt-distance is the b-metric* $\mathfrak{b}$.

5. Conclusions

Involving wt-distances, we have proved a generalization of the b-metric version of the celebrated Matkowski fixed point theorem, stated by Czerwik in 1993. With the help of this result, we derive characterizations of complete b-metric spaces that yield full generalizations of both a relevant characterization of metric completeness due to Suzuki and Takahashi, and the first characterization of metric completeness obtained via fixed point results, due to Hu. Appropriate adaptations of the technique used in the proof of Theorem 1 could be useful to reach wt-distance extensions of other important fixed point theorems obtained in the framework of complete metric spaces.

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