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On Protected Quasi-Metrics

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Abstract: In this paper, we introduce and examine the notion of a protected quasi-metric. In particular, we give some of its properties and present several examples of distinguished topological spaces that admit a compatible protected quasi-metric, such as the Alexandroff spaces, the Sorgenfrey line, the Michael line, and the Khalimsky line, among others. Our motivation is due, in part, to the fact that a successful improvement of the classical Banach fixed-point theorem obtained by Suzuki does not admit a natural and full quasi-metric extension, as we have noted in a recent article. Thus, and with the help of this new structure, we obtained a fixed-point theorem in the framework of Smyth-complete quasi-metric spaces that generalizes Suzuki's theorem. Combining right completeness with partial ordering properties, we also obtained a variant of Suzuki's theorem, which was applied to discuss types of difference equations and recurrence equations.

Keywords: general topology; protected quasi-metric; partial order; Smyth-complete; right-complete; Suzuki-type contraction; fixed point; difference equation

MSC: 54E35; 54E50; 54H25; 54F05; 54C35; 39A05

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1. Introduction

In the realm of general topology, the terms quasi-metric and quasi-metric space were introduced by Wilson [1], as asymmetric generalizations of the notions of the metric and metric spaces, respectively (related asymmetric structures were discussed by Niemytzki [2] and Frink [3]). A systematized study of quasi-metric spaces and their relation to other concepts of general topology begins with Kelly's article [4] in the framework of bitopological spaces. Since then, numerous authors have contributed to the topological development of quasi-metric spaces and other related structures. In fact, relevant non-metrizable topological spaces, such as the Alexandroff spaces, the Sorgenfrey line, the Michael line, and the Khalimsky line, among others, are quasi-metrizable. The books of Fletcher and Lindren [5] and Cobzaş [6], as well as the survey article by Künzi [7] provide suitable sources to the study of these spaces.

Applications of quasi-metric spaces to theoretical computer science, the complexity of algorithms, and to the study of dissipation systems began to be formalized and became relevant in the last decade of the Twentieth Century (cf. [8–13]). In this period were also published some articles in which quasi-metric generalizations of several important fixed-point theorems in metric spaces were obtained (cf. [14–18]).

As expected, these attractive research lines have continued to make significant advances. On the one hand, in constructing mathematical models in some fields of computer science and in obtaining (potential) applications to asymmetric functional analysis, the calculus of variations, aggregations functions, dynamic systems, fractal theory, and machine learning, among others (cf. [19–25]), and, on the other hand, in developing extensive research on the fixed-point theory for quasi-metric spaces (due to the numerous articles published in the last ten years in this field and with the aim not to make the bibliography too extensive, we will limit ourselves to the references [26–31] and the most-recent ones [32–37] together with the references therein).

In a recent paper [38], we gave an example showing that the natural and full quasi-metric generalization of a nice, and already celebrated, fixed-point theorem obtained by Suzuki in [39] does not hold. Motivated, in part, by this fact, we here introduce the notion of a protected quasi-metric. We analyzed some of its properties and give several examples of noteworthy quasi-metric spaces whose quasi-metric is protected. For instance, the quasi-metrics naturally induced by the Alexandroff topology, the Khalimski line, the Sorgenfrey line, and the Michael line, among others, are protected. Furthermore, we obtained a fixed-point theorem that generalizes Suzuki's theorem to Smyth-complete quasi-metric spaces, under the assumption that the involved quasi-metric is protected. Combining right completeness with partial ordering properties, we also obtained a variant of Suzuki's theorem, which was applied to discuss types of difference equations and recurrence equations.

2. Background

In the rest of this paper, the sets of real numbers, rational numbers, non-negative real numbers, integer numbers, and natural (or positive integers) numbers will be denoted by $\mathbb{R}, \mathbb{Q}, \mathbb{R}^+, \mathbb{Z}$, and $\mathbb{N}$, respectively.

Our main references for the general topology are [40,41].

With the aim of helping non-specialist readers, we next give some basic concepts and properties that will be used later on.

A quasi-metric on a set X is a function d from $X \times X$ to $\mathbb{R}^+$ fulfilling the following two conditions for every $u, v, w \in X$:

(qm1) $d(u,v) = d(v,u) = 0$, if and only if $u = v$;

(qm2) $d(u,v) \leq d(u,w) + d(w,v)$.

We say that the quasi-metric d is a T_1 quasi-metric on X if it fulfills the following condition stronger than (qm1):

$d(u,v) = 0$, if and only if $u = v$.

By a (T_1) quasi-metric space, we mean a pair (X,d), where X is a set and d is a (T_1) quasi-metric on X.

Given a (T_1) quasi-metric d on a set X, the function d^* defined on $X \times X$ as $d^*(u,v) = d(v,u)$ is also a (T_1) quasi-metric on X, called the conjugate (or the reverse) quasi-metric of d, while the function d^s defined on $X \times X$ as $d^s(u,v) = \max\{d(u,v), d(v,u)\}$ is a metric on X.

Each quasi-metric d on a set X induces a T_0 topology τ_d on X that has as a base the family of τ_d-open sets:

$$\{B_d(u,\varepsilon) : u \in X, \varepsilon > 0\},$$

where $B_d(u,\varepsilon) = \{v \in X : d(u,v) < \varepsilon\}$ for all $u \in X$ and all $\varepsilon > 0$.

We say that a sequence $(u_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X,d) is τ_d-convergent if there is $u \in X$ such that $(u_n)_{n\in\mathbb{N}}$ converges to u in the topological space (X,τ_d). Therefore, a sequence $(u_n)_{n\in\mathbb{N}}$ in (X,d) is τ_d-convergent to $u \in X$, if and only if $d(u,u_n) \to 0$ as $n \to \infty$. In the sequel, we simply write $d(u,u_n) \to 0$ if no confusion arises.

Clearly, d is T_1, if and only if τ_d is a T_1 topology.

We will say that d is a Hausdorff quasi-metric if τ_d is a Hausdorff (or T_2) topology. If both τ_d and τ_{d^*} are Hausdorff topologies, we refer to d as a doubly Hausdorff quasi-metric.

Let (X,τ) be a topological space. If there is a quasi-metric d on X such that $\tau = \tau_d$, we will say that d is compatible with τ. Then, a topological space (X,τ) is called quasi-metrizable if there is a quasi-metric d on X compatible with τ.

The absence of symmetry yields the existence of several different notions of the Cauchy sequence and quasi-metric completeness in the literature (see, e.g., [6]). For our goals here, we will consider the following ones.

A sequence $(u_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X,d) is called left Cauchy if, for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $d(u_n,u_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$, and it is called right Cauchy in (X,d) if it is left Cauchy in (X,d^*). Note that, if (X,d) is a metric space, these notions coincide with the classical notion of a Cauchy sequence for metric spaces.

A quasi-metric space (X, d) is called:

- Smyth-complete if every left Cauchy sequence is τ_{d^s}-convergent.
- Left-complete if every left Cauchy sequence is τ_d-convergent.
- Right-complete if every right Cauchy sequence is τ_d-convergent.

It is clear that Smyth completeness implies left completeness, but the converse does not hold, in general (see, e.g., Example 7 below).

It is also well known that the notions of left completeness and right completeness are independent of each other: for instance, the quasi-metric space of Example 4 below is right-complete, but not left-complete, whereas the quasi-metric space of [38] (Example 2) is Smyth-complete (hence, left-complete), but not right-complete.

We finish this section by recalling the following well-known notion.

A relation $\preceq$ on a set X is said to be a partial order on X if it satisfies the next conditions for every $u, v, w \in X$:

(i) $u \preceq u$ (reflexivity);
(ii) $u \preceq v$ and $v \preceq u$, implying $u = v$ (antisymmetry);
(iii) $u \preceq v$ and $v \preceq w$, implying $u \preceq w$ (transitivity).

It is clear that, if $\preceq$ is a partial order on X, the relation $\preceq^*$ on X given by $u \preceq^* v$, if and only if $v \preceq u$, is also a partial order on X.

3. Protected Quasi-Metrics

We start this section by introducing the main concept of our paper.

Definition 1. *We say that a quasi-metric d on a set X is protected by d^* (protected, in short) if it satisfies the following condition:*

Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in X that τ_d-converges to some $u \in X$, there is a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $d(u, u_{j_n+1}) \le d^*(u, u_{j_n})$ for all $n \in \mathbb{N}$.

A quasi-metric d is doubly protected provided that d is protected by d^* and d^* is protected by d.

Remark 1. *In the rest of this paper, the following obvious fact will be used without quoting it explicitly: If $(u_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d(u, u_{n+1}) \le d^*(u, u_n)$ eventually for some $u \in X$ (i.e., if there is $n_0 \in \mathbb{N}$ such that the above inequality holds for all $n > n_0$), then there is a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $d(u, u_{j_n+1}) \le d^*(u, u_{j_n})$ for all $n \in \mathbb{N}$.*

Remark 2. *We have chosen the term "protected" because, roughly speaking, the inequality $d(u, u_{j_n+1}) \le d^*(u, u_{j_n})$ may be seen as that value $d^*(u, u_{j_n})$ acting as a "bodyguard" (protector) for value $d(u, u_{j_n+1})$.*

As desirable, every metric is a (doubly) protected quasi-metric. Indeed, let d be a metric on a set X, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X that τ_d-converges to some $u \in X$. Since $d(u, u_n) \to 0$, there is a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ such that $d(u, u_{j_n+1}) \le d(u, u_{j_n})$ for all $n \in \mathbb{N}$. In fact, if $d(u, u_n) < d(u, u_{n+1})$ eventually, we have a contradiction. As $d = d^*$, we have that d is doubly protected.

It seems natural and tempting to propose an alternative statement of Definition 1, in the next simpler and, apparently, more-manageable terms:

A quasi-metric d on a set X is protected by d^* provided that it satisfies the following condition:

Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in X that τ_d-converges to some $u \in X$, then $d(u, u_{n+1}) \le d^*(u, u_n)$ eventually.

Unfortunately, there exist metrics that do not meet this alternative proposal, as the next example shows.

Example 1. *Let $X = \mathbb{N} \cup \{\infty\}$, and let $d : X \times X \to \mathbb{R}^+$ be defined as:*

$d(u,v) = 0$ *if* $u = v$;
$d(\infty,v) = d(v,\infty) = 2^{-v}$ *if v is odd;*
$d(\infty,v) = d(v,\infty) = 2^{-(v+2)}$ *if v is even;*
$d(u,v) = 2^{-u} + 2^{-v}$ *if u and v are odd with $u \neq v$,*
$d(u,v) = 2^{-(u+2)} + 2^{-(v+2)}$ *if u and v are even with $u \neq v$,*
and
$d(u,v) = d(v,u) = 2^{-(u+2)} + 2^{-v}$ *if u is even and v is odd.*

It is routine to check that d is a metric on X. Let $u_n = n$ for all $n \in \mathbb{N}$. Then, we have $d(\infty, u_n) \to 0$. However, for n even, we obtain

$$d(\infty, u_{n+1)} = d(\infty, n+1) = 2^{-(n+1)} > 2^{-(n+2)} = d(\infty, u_n).$$

The following easy property of protected quasi-metrics will be fundamental in obtaining our fixed-point results.

Proposition 1. *Let d be a protected quasi-metric on a set X. If $(u_n)_{n\in\mathbb{N}}$ is a sequence in X that τ_d-converges to some $u \in X$, then there is a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that*

$$d(u_{j_n}, u_{j_n+1}) \leq 2d(u_{j_n}, u),$$

for all $n \in \mathbb{N}$.

Proof. Since d is protected, there exists a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $d(u, u_{j_n+1}) \leq d(u_{j_n}, u)$ for all $n \in \mathbb{N}$. Hence,

$$d(u_{j_n}, u_{j_n+1}) \leq d(u_{j_n}, u) + d(u, u_{j_n+1}) \leq 2d(u_{j_n}, u),$$

for all $n \in \mathbb{N}$. □

There are several interesting examples of protected quasi-metrics. In this direction, Propositions 2 and 3 below will be useful.

Let (X,d) be a quasi-metric space. We say that a partial order $\preceq$ on X is compatible with τ_d if the following condition is satisfied:

Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d(u, u_n) \to 0$ for some $u \in X$, then $u \preceq u_n$ eventually.

Proposition 2. *Let (X,d) be a quasi-metric space. If there are a partial order $\preceq$ on X that is compatible with τ_d and a constant $c > 0$ such that $d(u,v) \geq c$ whenever $u \npreceq v$, then d is protected.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X that τ_d-converges to some $u \in X$. Then, $u \preceq u_n$ eventually. Assume, without loss of generality, that $u \neq u_n$ eventually. Thus, $u_n \npreceq u$ eventually. Therefore, $d(u, u_{n+1}) < c \leq d(u_n, u)$ eventually. We conclude that d is protected. □

Proposition 3. *Let (X,d) be a quasi-metric space such that τ_d is the discrete topology on X. Then, d is protected.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X that τ_d-converges to some $u \in X$. Since τ_d is the discrete topology, $u_n = u$ eventually. So, $d(u, u_n) = 0$ eventually. We conclude that d is protected. □

It seems natural to ask whether Proposition 3 can be generalized to the case in which the topology τ_d is finer than τ_{d^*}. The following example shows that this question has a negative answer, even in the case that $\tau_d = \tau_{d^*}$.

Example 2. *Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be defined as*

$$d(u,v) = \begin{cases} v-u & \text{if } u \leq v, \\ 2(u-v) & \text{if } v < u. \end{cases}$$

It is well known (cf. [29] (Example 3.2)) that d is a doubly Hausdorff quasi-metric on $\mathbb{R}$. In fact, for each $u \in \mathbb{R}^+$ and each $\varepsilon > 0$, we obtain

$$B_d(u,\varepsilon) = (u - \varepsilon/2, u + \varepsilon) \quad \text{and} \quad B_{d^*}(u,\varepsilon) = (u - \varepsilon, u + \varepsilon/2),$$

which implies that $\tau_d = \tau_{d^*} = \tau_E$, where, by τ_E, we denote the Euclidean (usual) topology on $\mathbb{R}$.

However, neither d nor d^* are protected quasi-metrics. Indeed, pick $u \in \mathbb{R}^+$ and the sequence $(u_n)_{n\in\mathbb{N}}$, where $u_n = u - 1/n$ for all $n \in \mathbb{N}$. Then,

$$d(u, u_{n+1}) = 2/(n+1) > 1/n = d^*(u, u_n),$$

for all $n > 1$, which implies that d is not protected. Similarly (taking the sequence $(u + 1/n)_{n\in\mathbb{N}}$), we infer that d^* is not protected.

Example 3. *A T_0 topology τ on a set X is an Alexandroff topology provided that every intersection of open sets is an open set [42]. In that case, the relation $\preceq$ on X defined as $u \preceq v$, if and only if $u \in \mathrm{cl}\{v\}$, is a partial order on X ($\mathrm{cl}\{v\}$ denotes the closure of $\{v\}$ in (X,τ), and note that τ is not T_1 if $\mathrm{cl}\{v\} \neq \{v\}$ for some $v \in X$). Moreover, the function d_A: $X \times X \to \mathbb{R}^+$, defined as*

$$d_A(u,v) = \begin{cases} 0 & \text{if } u \preceq v, \\ 1 & \text{if } u \not\preceq v, \end{cases}$$

is a quasi-metric on X compatible with τ.

We proceed to show that d_A is doubly protected.

We first note that the partial order $\preceq$ on X is compatible with τ because, if $(u_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d_A(u,u_n) \to 0$ for some $u \in X$, we infer that $u \preceq u_n$ eventually. Moreover, we have $d_A(v,w) = 1$ whenever $v \not\preceq w$. Hence, d is protected by Proposition 2.

Now suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in X such that $(d_A)^*(u,u_n) \to 0$ for some $u \in X$. Then, $u_n \preceq u$ eventually, i.e., $u \preceq^* u_n$ eventually. Since $(d_A)^*(v,w) = 1$ whenever $v \not\preceq^* w$, Proposition 2 implies that $(d_A)^*$ is protected.

Example 4. *The celebrated Sorgenfrey line [43] is the topological space $(\mathbb{R},\tau_S)$ where the sets of the form $[u,v)$, with $u,v \in \mathbb{R}$ and $u < v$, constitute a base of the topology τ_S. Solving a question posed by Dieudonné [44], Sorgenfrey proved in [43] that $(\mathbb{R},\tau_S)$ is normal and paracompact, but the product space $(\mathbb{R}\times\mathbb{R}, \tau_S \times \tau_S)$ is neither normal nor paracompact. It is well known (see, e.g., [6,27]) that the function $d_S : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ given by*

$$d_S(u,v) = \begin{cases} v-u & \text{if } u \leq v, \\ 1 & \text{if } u > v, \end{cases}$$

is a doubly Hausdorff quasi-metric on $\mathbb{R}$ compatible with τ_S.

We shall show that d_S is doubly protected.

We first note that the usual order $\leq$ on $\mathbb{R}$ is compatible with τ_S, because if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that $d_S(u,u_n) \to 0$, we infer that $u \leq u_n$ eventually. Moreover, we have $d_S(v,w) = 1$ whenever $v > w$. Hence, d_S is protected by Proposition 2.

Similarly, we obtain that $(d_S)^*$ is protected.

Example 5. *The well-known Michael line on $\mathbb{R}$ is the topological space $(\mathbb{R},\tau_M)$ where the intervals of the form $(u-\varepsilon, u+\varepsilon)$, with $u \in \mathbb{Q}$ and $\varepsilon > 0$, and those ones of the form $\{u\}$, with $u \in \mathbb{R}\backslash\mathbb{Q}$, constitute a base of the topology τ_M. Observe that, in particular, each irrational is an isolated point* *in τ_M, whereas the basic neighborhoods of each rational are exactly its basic neighborhoods for the usual topology.*

In fact, the Michael line provides a nice and simple example of a normal Lindelöf hereditarily paracompact space whose product with a separable metric space need not be normal (see [45]). The function $d_M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ given by

$$d_M(u,v) = \begin{cases} 0 \quad if\ u = v, \\ \min\{1, |u-v|\} \quad if\ u \in \mathbb{Q}, \\ 1 \quad if\ u \in \mathbb{R}\backslash\mathbb{Q} \quad and\ u \neq v, \end{cases}$$

is a doubly Hausdorff quasi-metric on $\mathbb{R}$ compatible with τ_M.

We shall show that d_M is doubly protected.

We first check that d_M is protected. Let $(u_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{R}$ such that $d_M(u,u_n) \to 0$ for some $u \in \mathbb{R}$. Then, $u \in \mathbb{Q}$, and there exists a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $d_M(u,u_{j_n+1}) \leq d_M(u,u_{j_n}) < 1$ for all $n \in \mathbb{N}$:

- *If $u_{j_n} \in \mathbb{Q}$, we obtain $d_M(u,u_{j_n+1}) \leq d_M(u,u_{j_n}) = |u - u_{j_n}| = (d_M)^*(u,u_{j_n})$.*
- *If $u_{j_n} \in \mathbb{R}\backslash\mathbb{Q}$, we obtain $d_M(u,u_{j_n+1}) \leq d_M(u,u_{j_n}) = |u - u_{j_n}| < 1 = (d_M)^*(u,u_{j_n})$.*

Consequently, d_M is protected.

Now suppose that $(u_n)_{n\in\mathbb{N}}$ is a non-eventually constant sequence in $\mathbb{R}$ verifying that $(d_M)^*(u,u_n) \to 0$ for some $u \in \mathbb{R}$. Again, there exists a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $(d_M)^*(u,u_{j_n+1}) \leq (d_M)^*(u,u_{j_n}) < 1$ for all $n \in \mathbb{N}$. We can assume, without loss of generality, that $u \neq u_{j_n}$ for all $n \in \mathbb{N}$, and thus, $u_{j_n}, u_{j_n+1} \in \mathbb{Q}$ for all $n \in \mathbb{N}$:

- *If $u \in \mathbb{Q}$, we obtain $(d_M)^*(u,u_{j_n+1}) \leq (d_M)^*(u,u_{j_n}) = d_M(u,u_{j_n})$ for all $n \in \mathbb{N}$.*
- *If $u \in \mathbb{R}\backslash\mathbb{Q}$, we obtain $(d_M)^*(u,u_{j_n+1}) < 1 = d_M(u,u_{j_n})$ for all $n \in \mathbb{N}$.*

Consequently, $(d_M)^*$ is protected.

Example 6. *The famous Khalimsky line constitutes a well-established foundation for a digital topology (see [46]). It consists of the T_0 topological space $(\mathbb{Z}, \tau_K)$, where τ_K is the topology on $\mathbb{Z}$, which has as a base the family of open sets $\{\{2n+1\}, \{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$. Thus, each odd integer is an isolated point and each even integer n has an open base of neighborhoods consisting of a unique set, namely $\{2n-1, 2n, 2n+1\}$.*

It is clear that the quasi-metric d_K on $\mathbb{Z}$ given by

$$d_K(u,v) = \begin{cases} 0 \quad if\ u = v, \\ 0 \quad if\ u\ is\ even\ and\ v \in \{u-1, u+1\}, \\ 1 \quad otherwise, \end{cases}$$

is compatible with τ_K. Obviously, d_k is not a T_1 quasi-metric on $\mathbb{Z}$.

We show that d_K is doubly protected.

Let $(u_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{Z}$ such that $d_K(u,u_n) \to 0$ for some $u \in \mathbb{Z}$. Then, u is even and $d_K(u,u_n) = 0$ eventually. Therefore, d_K is protected.

Now, let $(u_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{Z}$ such that $(d_K)^*(u,u_n) \to 0$ for some $u \in \mathbb{Z}$. Then, u is odd and $(d_K)^*(u,u_n) = 0$ eventually. Therefore, $(d_K)^*$ is protected.

Example 7. *Denote by τ_{co} the co-finite topology on $\mathbb{N}$ (proper τ_{co}-closed subsets are the finite subsets of $\mathbb{N}$). It is well known that the quasi-metric d_{co} on $\mathbb{N}$ given by*

$$d_{co}(u,v) = \begin{cases} 0 \quad if\ u = v, \\ 1/v \quad otherwise, \end{cases}$$

is compatible with τ_{co}. Note that d_{co} is T_1, but not Hausdorff (in fact, the sequence $(n)_{n\in\mathbb{N}}$ τ_{co}-converges to any $u \in \mathbb{N}$). We show that d_{co} is doubly protected.

Let $(u_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{N}$ such that $d_{co}(u,u_n) \to 0$ for some $u \in \mathbb{N}$ (note that, in fact, we have $d_{co}(v,u_n) \to 0$ for all $v \in \mathbb{N}$). Then, there is $n_0 \in \mathbb{N}$ such that $d_{co}(u,u_{n+1}) < 1/u$ for all $n \geq n_0$. Since $(d_{co})^*(u,u_n) = 1/u$, we conclude that d_{co} is protected.

Finally, note that, for each $u \in \mathbb{N}$, $B_{(d_{co})^*}(u,1/u) = \{u\}$, so τ_{co} is the discrete topology on $\mathbb{N}$. By Proposition 3, $(d_{co})^*$ is protected.

Example 8. *Let $a \in \mathbb{R}^+$ be a constant and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+\backslash\{0\}$ be a bounded function. Put*

$$\mathcal{F}_{a,\varphi} = \{f : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that } \sup_{u\in\mathbb{R}^+} (\varphi(u)f(u)) < \infty \text{ and } f(0) = a\}.$$

For each $f \in \mathcal{F}_{a,\varphi}$, define $s(\varphi f) = \sup_{u\in\mathbb{R}^+}(\varphi(u)f(u))$.

Denote by $\preceq$ the usual (pointwise) partial order on $\mathcal{F}_{a,\varphi}$:

$$f \preceq g \; if \text{ and only } if \; f(u) \leq g(u)$$

,
for all $u \in \mathbb{R}^+$.

For each $f,g \in \mathcal{F}_{a,\varphi}$, put

$$d_{a,\varphi}(f,g) = \begin{cases} \sup_{u\in\mathbb{R}^+}(\varphi(u)(g(u) - f(u))) & if \; f \preceq g, \\ 1 + s(\varphi g) \quad if \; f \npreceq g. \end{cases}$$

We first observe that $d_{a,\varphi}$ defines a function from $\mathcal{F}_{a,\varphi} \times \mathcal{F}_{a,\varphi}$ to $\mathbb{R}^+$. Indeed, it suffices to consider the case that $f \preceq g$. Then, we obtain

$$\varphi(u)(g(u) - f(u)) \leq \varphi(u)g(u) \leq s(\varphi g),$$

for all $u \in \mathbb{R}^+$. Therefore, $d_{a,\varphi}(f,g) \leq s(\varphi g) < \infty$. It is routine to check that $d_{a,\varphi}$ is a quasi-metric on $\mathcal{F}_{a,\varphi}$.

Furthermore, it is a doubly Hausdorff quasi-metric. Indeed, suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}_{a,\varphi}$ such that $d_{a,\varphi}(f,f_n) \to 0$ and $d_{a,\varphi}(g,f_n) \to 0$. Then, we simultaneously have that $f \preceq f_n$ and $g \preceq f_n$ eventually, and for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that

$$\varphi(u)(f_n(u) - f(u)) < \varepsilon \quad and \quad \varphi(u)(f_n(u) - g(u)) < \varepsilon,$$

for all $n \geq n_\varepsilon$. From the preceding inequalities and the fact that $f \preceq f_n$ and $g \preceq f_n$ eventually, we deduce that

$$|\varphi(u)(f(u) - g(u))| < 2\varepsilon,$$

for all $u \in \mathbb{R}^+$. Since ε is arbitrary, we obtain $\varphi(u)(f(u) - g(u)) = 0$ for all $u \in \mathbb{R}^+$, so $f = g$, because $\varphi(u) > 0$ for all $u \in \mathbb{R}^+$. Hence, $d_{a,\varphi}$ is a Hausdorff quasi-metric on $\mathcal{F}_{a,\varphi}$. Similarly, we show that the quasi-metric $(d_{a,\varphi})^*$ is Hausdorff.

Finally, we shall prove that $d_{a,\varphi}$ is doubly protected.

Let $(f_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathcal{F}_{a,\varphi}$ such that $d_{a,\varphi}(f,f_n) \to 0$ for some $f \in \mathcal{F}_a$. Then, $d_{a,\varphi}(f,f_n) < 1$ eventually, which implies that $f \preceq f_n$ eventually. Hence, $\preceq$ is compatible with $\tau_{d_{a,\varphi}}$. Since $d_{a,\varphi}(f,g) \geq 1$ whenever $f \npreceq g$, we conclude, by Proposition 2, that $d_{a,\varphi}$ is protected.

An analogous argument shows that $(d_{a,\varphi})^*$ is also protected.

We conclude this section with two examples of protected quasi-metrics that are not doubly protected.

Example 9. *Given a (non-empty) set X, denote by X^F the set consisting of all finite sequences (finite words, in computer science) of elements of X and, by X^∞, the set of all infinite sequences (infinite words in computer science). Put $X^\omega = X^F \cup X^\infty$.*

Given $x \in X^\omega$, we design by $l(x)$ its length. Thus, $l(x) = j \in \mathbb{N}$ if $x \in X^F$ with $x := x_1...x_j$ and $l(x) = \infty$ if $x \in X^\infty$.

Now, define $[\infty] = \{x \in X^\omega : l(x) = \infty\}$ (i.e., $[\infty] = X^\infty$), $[n] = \{x \in X^F : l(x) = n\}$ for $n \in \mathbb{N}$, and $\mathbb{X} = \{[n] : n \in \mathbb{N}\} \cup \{[\infty]\}$.

Denoting by u and v the elements of $\mathbb{X}$ involved, we define a function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ as

$$d(u,v) = \begin{cases} 0 \quad if\ u = v, \\ 0 \quad if\ u = [\infty],\ and\ v = [n],\ n \in \mathbb{N}, \\ 2^{-n} \quad if\ u = [n],\ and\ v = [\infty],\ n \in \mathbb{N}, \\ 0 \quad if\ u = [n],\ and\ v = [m],\ with\ n > m, \\ 2^{-n} - 2^{-m} \quad if\ u = [n],\ and\ v = [m],\ with\ n < m. \end{cases}$$

Then, d is a quasi-metric on $\mathbb{X}$. We show that it is protected. Indeed, let $(u_k)_{k\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{X}$ such that $d(u, u_k) \to 0$ for some $u \in \mathbb{X}$. Then, $u = [\infty]$, so $d(u, u_k) = 0$ eventually, which implies that d is protected.

Finally, note that d^* is not protected because $d^*([\infty], [n+1]) \to 0$, but $d^*([\infty], [n]) > d([\infty], [n])$ for all $n \in \mathbb{N}$.

The quasi-metric of the preceding example is not T_1. We end this section with an example where the involved quasi-metric is doubly Hausdorff and protected, but its conjugate quasi-metric is not protected.

Example 10. *Let us recall that the Alexandroff (or the one-point) compactification of $\mathbb{N}$ consists of the set $\mathbb{N} \cup \{\infty\}$ endowed with the topology τ_0, where each natural is an isolated point and the neighborhoods of ∞ are of the form $X \backslash C$, where C is a finite subset of $\mathbb{N}$. It is well known that τ_0 is a compact and metrizable topology. We are going to construct a protected quasi-metric on X compatible with τ_0 and such that its conjugate quasi-metric is not protected.*

Let $d_0 : X \times X \to \mathbb{R}^+$ be defined as

$$d_0(u,v) = \begin{cases} 0 \quad if\ u = v, \\ 1/2v \quad if\ u = \infty \text{ and } v \in \mathbb{N}, \\ 1/u \quad if\ u \in \mathbb{N} \text{ and } v = \infty, \\ 1/u + 1/2v \quad if\ u, v \in \mathbb{N} \text{ and } u \neq v, \end{cases}$$

It is easy to check that d_0 is a quasi-metric on X. Furthermore, the topology τ_{d_0} is compact because every non-eventually constant sequence τ_{d_0}-converges to ∞. Note also that each natural n is an isolated point because $B_{d_0}(n, 1/n) = \{n\}$. Therefore, d_0 is compatible with τ_0, and thus, it is a Hausdorff quasi-metric. In fact, we clearly have $\tau_{d_0} = \tau_{(d_0)^*}$, so d_0 is doubly Hausdorff.

Next, we show that d_0 is protected. To achieve this, let $(u_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in X such that $d_0(u, u_n) \to 0$ for some $u \in X$. Then, $u = \infty$ and $(u_n)_{n\in\mathbb{N}}$ has a strictly increasing subsequence $(u_{j_n})_{n\in\mathbb{N}}$. Thus,

$$d_0(\infty, u_{j_n+1}) = 1/2u_{j_n+1} < 1/u_{j_n} = (d_0)^*(\infty, u_{j_n}),$$

for all $n \in \mathbb{N}$. Hence, d_0 is protected.

However, $(d_0)^*$ is not protected because $(d_0)^*(\infty, v) = 1/v > 1/2v = d_0(\infty, v)$ for all $v \in \mathbb{N}$.

4. Fixed-Point Theorems and an Application

According to [38], a self-map T of a quasi-metric space (X, d) is a basic contraction of Suzuki-type (an S-contraction, in short) provided that there is a constant $\lambda \in (0, 1)$ such that the following contraction condition holds for any $u, v \in X$:

$$d(u, Tu) \leq 2d(u, v) \Rightarrow d(Tu, Tv) \leq \lambda d(u, v). \tag{1}$$

Suzuki obtained in [39] an important generalization of Banach's contraction principle that, adapted to our context, we state as follows: Every S-contraction on a complete metric space has a unique fixed point.

In [38] was given an example of an S-contraction on a Smyth-complete quasi-metric space, which has no fixed points. Thus, the next quasi-metric generalization of Suzuki's theorem reveals a nice usefulness of protected quasi-metrics.

Theorem 1. *Let (X, d) be a Smyth-complete quasi-metric space. If d is protected, then each S-contraction on (X, d) has a unique fixed point.*

Proof. Let T be an S-contraction on (X, d). Then, there exists a constant $\lambda \in (0, 1)$ for which the contraction condition (1) holds.

Fix $u_0 \in X$. Since $d(u_0, Tu_0) \leq 2d(u_0, Tu_0)$, it follows that $d(Tu_0, T^2u_0) \leq \lambda d(u_0, Tu_0)$, and continuing this process, we deduce that $d(T^n u_0, T^{n+1}u_0) \leq \lambda^n d(u_0, Tu_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Then, by the triangle inequality (qm2), we obtain

$$d(T^n u_0, T^m u_0) \leq \frac{\lambda^n}{1-\lambda} d(u_0, Tu_0),$$

for all $n, m \in \mathbb{N}$ with $n \leq m$, which implies that $(T^n u_0)_{n \in \mathbb{N}}$ is a left Cauchy sequence in (X, d). Hence, there exists $u \in X$ such that $d^s(u, T^n u_0) \to 0$, so $d(u, T^n u_0) \to 0$ and $d^*(u, T^n u_0) \to 0$.

Since d is protected, it follows from Proposition 1 that there exists a subsequence $(T^{j_n} u_0)_{n \in \mathbb{N}}$ of $(T^n u_0)_{n \in \mathbb{N}}$ such that

$$d(T^{j_n} u_0, T^{j_n+1} u_0) \leq 2d(T^{j_n} u_0, u),$$

for all $n \in \mathbb{N}$. Therefore, by condition (1),

$$d(T^{j_n+1} u_0, Tu) \leq \lambda d(T^{j_n} u_0, v),$$

for all $n \in \mathbb{N}$. Thus, $d(T^{j_n+1} u_0, Tu) \to 0$ because $d(T^n u_0, u) \to 0$.

Since $d(u, T^{j_n+1} u_0) \to 0$, the triangle inequality implies that $d(u, Tu) = 0$. Hence, $d(u, Tu) \leq 2d(u, T^n u_0)$ for all $n \in \mathbb{N}$, so by condition (1), $d(Tu, T^{n+1} u_0) \leq \lambda d(u, T^n u_0)$ for all $n \in \mathbb{N}$. Consequently, $d(Tu, T^{n+1} u_0) \to 0$. Since

$$d(Tu, u) \leq d(Tu, T^{n+1} u_0) + d(T^{n+1} u_0, u),$$

for all $n \in \mathbb{N}$, we obtain $d(Tu, u) = 0$. Therefore, $d(u, Tu) = d(Tu, u) = 0$, so $u = Tu$.

Finally, let $v \in X$ be such that $v = Tv$. Then, $d(u, Tu) \leq 2d(u, v)$, so

$$d(u, v) = d(Tu, Tv) \leq \lambda d(u, v),$$

and thus, $d(u, v) = 0$. Analogously, $d(v, u) = 0$. Hence, $u = v$. We conclude that u is the unique fixed point of T in X. □

Example 11. *Let $(\mathbb{X}, d)$ be the quasi-metric space of Example 9. Recall that d is protected.*

We prove that $(\mathbb{X}, d)$ is Smyth-complete.

Let $(u_k)_{k \in \mathbb{N}}$ be a non-eventually constant left Cauchy sequence in $(\mathbb{X}, d)$. For given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $d(u_k, u_j) < \varepsilon$ whenever $k_0 \leq k \leq j$.

Since $(u_k)_{k \in \mathbb{N}}$ is non-eventually constant, we can assume, without loss of generality, that $u_k \neq u_j$ for all $k, j \in \mathbb{N}$ with $k \neq j$ and $u_k \neq [\infty]$ for all $k \in \mathbb{N}$. Thus, $u_k = [n_k] = \{x \in X^F : l(x) = n_k\}$ for all $k \in \mathbb{N}$, and there is a subsequence $(u_{i_k})_{k \in \mathbb{N}}$ of $(u_k)_{k \in \mathbb{N}}$ such that $n_{i_{k+1}} > n_{i_k}$ for all $k \in \mathbb{N}$.

Let $k \geq k_0$. Since $(n_{i_k})_{k\in\mathbb{N}}$ is a strictly increasing sequence, we can find an $m \in \mathbb{N}$ such that $n_{i_m} > k$ and $2^{-n_{i_m}} < \varepsilon$. Therefore,

$$d(u_k, [\infty]) \leq d(u_k, u_{n_{i_m}}) + d(u_{n_{i_m}}, [\infty]) < 2\varepsilon.$$

Since ε is arbitrary, we deduce that $d(u_k, [\infty]) \to 0$. On the other hand, we have $d([\infty], u_k) = 0$ for all $k \in \mathbb{N}$, so $d^s([\infty], u_k) \to 0$. Consequently, $(\mathbb{X}, d)$ is Smyth-complete.

Now, let $k \in \mathbb{N}$ be fixed, and let T be the self-map of $\mathbb{X}$ defined as $T[\infty] = [\infty]$; $T[n] = [\infty]$ if n is odd, and $T[n] = [n + 2k]$ if n is even.

We shall proceed to check that T is an S-contraction on the quasi-metric space $(\mathbb{X}, d)$. Since $(\mathbb{X}, d)$ is Smyth-complete and d is protected, all conditions of Theorem 1 will be satisfied.

Let $u, v \in \mathbb{X}$:

- *If $u = [\infty]$, we obtain $d(Tu, Tv) = d([\infty], Tv) = 0$ for all $v \in \mathbb{X}$.*
- *If $u = [n]$, with n odd, we obtain $d(Tu, Tv) = d([\infty], Tv) = 0$ for all $v \in \mathbb{X}$.*
- *If $u = [n]$ with n even and $v = [\infty]$, we obtain*

$$d(Tu, Tv) = d([n + 2k], [\infty]) = 2^{-(n+2k)} = 2^{-2k} d(u, v).$$

- *If $u = [n]$ and $v = [m]$, with n, m even and $n \geq m$, we obtain*

$$d(Tu, Tv) = d([n + 2k], [m + 2k]) = 0.$$

- *If $u = [n]$ and $v = [m]$, with n, m, even and $n < m$, we obtain*

$$d(Tu, Tv) = d([n + 2k], [m + 2k]) = 2^{-(n+2k)} - 2^{-(m+2k)} = 2^{-2k} d(u, v).$$

- *If $u = [n]$ with n even, $v = [m]$ with m odd, and $n < m$, we obtain*

$$d(Tu, Tv) = d([n + 2k], [\infty]) = 2^{-(n+2k)} \leq 2^{-(n+2)} \leq d(u, v)/2.$$

- *If $u = [n]$ with n even, $v = [m]$ with m odd, and $n > m$, we obtain*

$$d(u, Tu) = d([n], [n + 2k]) > 0 = 2d(u, v).$$

Therefore, T is an S-contraction with contraction constant $1/2$, and all conditions of Theorem 1 are fulfilled.

Note that T is not a Banach contraction on $(\mathbb{X}, d)$ because, for $u = [n]$ with n even, $v = [m]$ with m odd, and $n > m$, we obtain

$$d(Tu, Tv) = d([n + 2k], [\infty]) = 2^{-(n+2k)} > 0 = d(u, v).$$

The next example shows that Theorem 1 cannot be fully generalized to left-complete, nor to right-complete quasi-metric spaces, nor even for T_1 quasi-metric spaces whose quasi-metric is doubly protected.

Example 12. *Let $(\mathbb{N}, d_{co})$ be the T_1 quasi-metric space of Example 7. We have noted that d_{co} is doubly protected. Furthermore, $(\mathbb{N}, \tau_{co})$ is compact, because every non-eventually constant sequence in $\mathbb{N}$ is τ_{co}-convergent to any $n \in \mathbb{N}$. Consequently, $(\mathbb{N}, d_{co})$ is left- and right-complete.*

Let T be the self-map of $\mathbb{N}$ defined as $Tn = 2n$ for all $n \in \mathbb{N}$. Then,

$$d_{co}(Tn, Tm) = d_{co}(2n, 2m) = 1/2m = d_{co}(n, m)/2,$$

for all $n, m \in \mathbb{N}$ with $n \neq m$. Thus, T is a Banach contraction and, hence, an S-contraction, on $(\mathbb{N}, d_{co})$ without fixed points.

Our next result provides a quasi-metric variant of Suzuki's theorem that involves the properties of partial orders. It will be a fundamental piece later on.

Let (X, d) be a quasi-metric space, and let $\preceq$ be a partial order on X. We say that (X, d) is $\preceq$-co-right-complete if every $\preceq$-non-decreasing left Cauchy sequence is τ_{d^*}-convergent.

As usual, a self-map T of X is $\preceq$-non-decreasing if $Tu \preceq Tv$ whenever $u \preceq v$.

Theorem 2. *Let (X, d) be a quasi-metric space such that d^* is Hausdorff and protected. Suppose that there is a partial order $\preceq$ on X for which (X, d) is $\preceq$-co-right-complete. If T is a $\preceq$-non-decreasing self-map of X satisfying the following two conditions (a) and (b), then T has a fixed point:*

(a) There is $u_0 \in X$ such that $u_0 \preceq Tu_0$.

(b) There is a constant $\lambda \in (0, 1)$ such that the following contraction condition holds for any $u, v \in X$ with $u \preceq v$:

$$d(Tu, u) \leq 2d(v, u) \implies d(Tu, Tv) \leq \lambda d(u, v).$$

Proof. Since T is $\preceq$-non-decreasing, it follows from condition (a) that $T^n u_0 \preceq T^{n+1} u_0$ for all $n \in \mathbb{N} \cup \{0\}$. So, condition (b) implies that $d(T^n u_0, T^{n+1} u_0) \leq \lambda d(T^{n-1} u_0, T^n u_0)$, and consequently, $d(T^n u_0, T^{n+1} u_0) \leq \lambda^n d(u_0, Tu_0)$ for all $n \in \mathbb{N}$. Therefore, $(T^n u_0)_{n \in \mathbb{N}}$ is a $\preceq$-non-decreasing left Cauchy sequence in (X, d). Hence, there is $u \in X$ such that $d^*(u, T^n u_0) \to 0$ and $T^n u_0 \preceq u$ for all $n \in \mathbb{N}$.

Since d^* is protected, it follows from Proposition 1 that there exists a subsequence $(T^{j_n} u_0)_{n \in \mathbb{N}}$ of $(T^n u_0)_{n \in \mathbb{N}}$ such that

$$d^*(T^{j_n} u_0, T^{j_n+1} u_0) \leq 2d^*(T^{j_n} u_0, u),$$

for all $n \in \mathbb{N}$. By condition (b) and the fact that $T^n u_0 \preceq u$ for all $n \in \mathbb{N}$, we deduce that $d(T^{j_n+1} u_0, Tu) \leq \lambda d(T^{j_n} u_0, u)$ for all $n \in \mathbb{N}$. So, $d(T^{j_n+1} u_0, Tu) \to 0$. Hence, $u = Tu$ because d^* is Hausdorff. □

Example 13. *Let $(\mathbb{R}, d_S)$ be the quasi-metric space of Example 4. Recall that d_S is doubly Hausdorff and doubly protected.*

We shall prove that it is $\leq$-co-right-complete where, by $\leq$, we denote the usual order on $\mathbb{R}$.

Indeed, let $(u_n)_{n \in \mathbb{N}}$ be a $\leq$-non-decreasing left Cauchy sequence in $(\mathbb{R}, d_S)$. Then, $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$, and there is $n_1 \in \mathbb{N}$ such that $u_n < 1 + u_{n_1}$ for all $n > n_1$. Thus, the set $\{u_n : n \in \mathbb{N}\}$ is upper bounded, so there is $w \in \mathbb{R}$ such that $w = \sup_{n \in \mathbb{N}} u_n$. Therefore, $u_n \leq w$ for all $n \in \mathbb{N}$, and $d_S(u_n, w) \to 0$.

Define a self-map T of $\mathbb{R}$ as $Tu = (u + 1)/2$ if $u \geq 0$, and $Tu = u - c$ if $u < 0$, with $c > 2$ a constant.

It is clear that T is non-decreasing. We also observe that $0 < T0$.

Now, let $u, v \in \mathbb{R}$ be such that $u \leq v$. Then:

- *If $u < 0$, we obtain $d_S(Tu, u) = c > 2 = 2d_S(v, u)$.*
- *If $u \geq 0$, we obtain $d_S(Tu, Tv) = (v - u)/2 = d_S(u, v)/2$.*

Consequently, T is an $\leq$-S-contraction of (X, d_S). Thus, all conditions of Theorem 2 are satisfied. Hence, T has a fixed point, and in this case, $w = 1$ is its unique fixed point.

The self-map of the preceding example has a unique fixed point. However, it is easy to yield simple instances that satisfy the conditions of Theorem 2 and where the involved self-map has more than one fixed point, as we see now.

Example 14. *Let $X = \mathbb{N} \cup \{0\}$, and let d be the discrete metric on X. Then, $d = d^*$, so d^* is Hausdorff and protected.*

Define a relation $\preceq$ on X as

$$u \preceq v \text{ if and only if } v \leq u, \text{ with } u, v \in \mathbb{N}, \text{ or } u = v = 0.$$

It is obvious that $\preceq$ is a partial order on X and that (X, d) is $\preceq$-co-right-complete because the right Cauchy sequences are only those that are eventually constant.

Now, let T be the self-map of X given by $T0 = 0$ and $Tu = 1$ for all $u \in \mathbb{N}$.

Note that, for instance, $2 \preceq T2$. Moreover, T is clearly $\preceq$-non-decreasing.

Finally, given $u, v \in X$ with $u \preceq v$, we have $u = v = 0$, or $v \leq u$, with $u, v \in \mathbb{N}$. In all cases, we obtain $d(Tu, Tv) = 0$.

Hence, all conditions of Theorem 2 are fulfilled, and we have that 0 and 1 are the fixed points of T.

The last part of the paper is devoted to present a method for constructing suitable self-operators on the function space given in Example 8 and deducing the existence and uniqueness of the solution for the difference equations induced by such operators. This approach will be applied to directly deduce the existence and uniqueness of the solution for the recurrence equations associated with several distinguished algorithms. It is appropriate to point out that the idea of proving the existence and uniqueness of the solution for recursive algorithms using iteration techniques and fixed-point theorems in the realm of quasi-metric spaces is not new. However, while such a study has been usually performed in the context of certain sequence spaces (see, e.g., [10,27,47,48]), our procedure allows us to derive, in a unified and direct fashion, the study of such recurrence equations as a consequence of a more-general framework.

With the aim of being able to apply Theorem 2 to Example 8, we first made the following observation.

Remark 3. *The quasi-metric space of Example 8 is $\preceq$-co-right complete. Indeed, let $(f_n)_{n\in\mathbb{N}}$ be a $\preceq$-non-decreasing left Cauchy sequence in $(\mathcal{F}_{a,\varphi}, d_{a,\varphi})$. Then, $f_n \preceq f_m$ for $n \leq m$, and there is $n_1 \in \mathbb{N}$ such that $d(f_n, f_m) < 1$ for $n_1 \leq n \leq m$. So, $\sup_{u\in\mathbb{R}^+}(\varphi(u)(f_n(u) - f_{n_1}(u))) < 1$ for $n \geq n_1$, which implies that $\sup_{n\geq n_1} f_n(u) \leq (1 + s(\varphi f_{n_1}))/\varphi(u) < \infty$ for all $u \in \mathbb{R}^+$. Thus, we may define a function $F : \mathbb{R}^+ \to \mathbb{R}^+$ as $F(u) = \sup_{n\geq n_1} f_n(u)$ for all $u \in \mathbb{R}^+$. Observe that, actually, $F \in \mathcal{F}_{a,\varphi}$ because $F(0) = a$, and from the fact that for each $u \in \mathbb{R}^+$, there is $n_u \geq n_1$ such that $F(u) < 1 + f_{n_u}(u)$, it follows that $\sup_{u\in\mathbb{R}^+}(\varphi(u)F(u)) \leq M + s(\varphi f_{n_1}) + 1$, where $M > 0$ is an upper bound of φ.*

It remains to check that $d^*(F, f_n) \to 0$. To achieve this, choose an arbitrary $\varepsilon \in (0, 1)$. Then, there is $n_\varepsilon \geq n_1$ such that $d(f_n, f_m) < \varepsilon$ for $n_\varepsilon \leq n \leq m$.

Fix $n \geq n_\varepsilon$, and let $u \in \mathbb{R}^+$. By the definition of F, we find $n_u \geq n_1$ such that $F(u) < f_{n_u}(u) + \varepsilon$. If $n_u \leq n$, we obtain $f_{n_u} \preceq f_n$, and thus, $\varphi(u)F(u) < \varphi(u)f_n(u) + M\varepsilon$. If $n < n_u$, we obtain $d(f_n, f_{n_u}) < \varepsilon$, so $\varphi(u)f_{n_u}(u) - \varphi(u)f_n(u) < \varepsilon$, and thus, $\varphi(u)F(u) < \varphi(u)f_n(u) + (M+1)\varepsilon$. Therefore, for each $u \in \mathbb{R}^+$ and $n \geq n_\varepsilon$, $\varphi(u)(F(u) - f_n(u)) < (M+1)\varepsilon$. Hence, $d^*(F, f_n) \leq (M+1)\varepsilon$ for all $n \geq n_\varepsilon$. So, $(\mathcal{F}_{a,\varphi}, d_{a,\varphi})$ is $\preceq$-co-right complete.

Proposition 4. *Let $a \in \mathbb{R}^+$ be a constant, $p : \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded function on $\mathbb{R}^+$, and $q : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that φq is bounded on $\mathbb{R}^+$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+\backslash\{0\}$ is defined as $\varphi(u) = e^{-u}/M$ for all $u \in \mathbb{R}^+$, where $M > 0$ is an upper bound of p on $\mathbb{R}^+$.*

For each $f \in \mathcal{F}_{a,\varphi}$, put

$$\Phi f(u) = \begin{cases} a \quad if\ 0 \leq u \leq 1, \\ p(u)f(u-1) + q(u) \quad if\ u > 1. \end{cases}$$

Then, the correspondence Φ defines a self-map of $\mathcal{F}_{a,\varphi}$ that has a unique fixed point $f_{a,\varphi}$ in $\mathcal{F}_{a,\varphi}$.

Proof. Let $L > 0$ be such that $\varphi(u)q(u) \leq L$ for all $u \in \mathbb{R}^+$.

Next, we check that, given $f \in \mathcal{F}_{a,\varphi}$, we have that $\Phi f \in \mathcal{F}_{a,\varphi}$.

First, note that $\Phi f(0) = a$. Moreover, for each $u \in [0,1]$, we have $\varphi(u)\Phi f(u) = ae^{-u}/M \leq a/M$, and for each $u > 1$,

$$\begin{aligned}
\varphi(u)\Phi f(u) &= \frac{e^{-u}}{M}(p(u)(f(u-1)+q(u)) \\
&\leq \frac{e^{-u}}{M}(Mf(u-1)+q(u)) = e^{-1}(e^{-(u-1)}f(u-1)) + L \\
&\leq e^{-1}\sup_{u\in\mathbb{R}^+}(\varphi(u)f(u)) + L.
\end{aligned}$$

Since $\sup_{u\in\mathbb{R}^+}(\varphi(u)f(u)) < \infty$, we infer that $\sup_{u\in\mathbb{R}^+}(\varphi(u)\Phi f(u)) < \infty$. Consequently, $\Phi f \in \mathcal{F}_{a,\varphi}$, which implies that Φ is a self-map of $\mathcal{F}_{a,\varphi}$.

Furthermore, $f_0 \preceq \Phi f_0$, where $f_0 \in \mathcal{F}_{a,\varphi}$ is defined as $f_0(0) = a$ and $f_0(u) = 0$ if $u > 0$.

It is clear that Φ is non-decreasing. Indeed, given $f, g \in \mathcal{F}_{a,\varphi}$ with $f \preceq g$, we obtain $\Phi f(u) = \Phi g(u)$ if $0 \leq u \leq 1$, and $\Phi f(u) \leq \Phi g(u)$ if $u > 1$.

Now, let $f, g \in \mathcal{F}_{a,\varphi}$ be such that $f \preceq g$. Then,

$$\begin{aligned}
d_{a,\varphi}(\Phi f, \Phi g) &= \sup_{u\in\mathbb{R}^+}(\varphi(u)(\Phi g(u) - \Phi f(u)))) \\
&= \sup_{u>1}(\varphi(u)(p(u)(g(u-1) - f(u-1)))) \\
&\leq M\sup_{u\in\mathbb{R}^+}(\varphi(u+1)(g(u) - f(u))) = M\sup_{u\in\mathbb{R}^+}(\frac{e^{-(u+1)}}{M}(g(u) - f(u))) \\
&= e^{-1}\sup_{u\in\mathbb{R}^+}(\varphi(u)(g(u) - f(u))) = e^{-1}d_{a,\varphi}(f,g).
\end{aligned}$$

Taking into account Remark 3, we have that all conditions of Theorem 2 are fulfilled. So, there is $f_{a,\varphi} \in \mathcal{F}_{a,\varphi}$ satisfying that $f_{a,\varphi} = \Phi f_{a,\varphi}$.

Finally, we show that $f_{a,\varphi}$ is the unique fixed point of Φ in $\mathcal{F}_{a,\varphi}$. To achieve this, let $h \in \mathcal{F}_{a,\varphi}$ be such that $h = \Phi h$. By the construction of Φ, we have $h(u) = f_{a,\varphi}(u) = a$ for all $u \in [0,1]$. Suppose that there is $u_0 > 1$ such that $f(u_0) > h(u_0)$, i.e., $\Phi f(u_0) > \Phi h(u_0)$ Thus, $f(u_0 - 1) > h(u_0 - 1)$. Repeating this process, we will find an $m \in \mathbb{N}$ such that $u_0 - m \leq 1$ and $f(u_0 - m) > h(u_0 - m)$, a contradiction. Hence, $f_{a,\varphi} \preceq h$. Similarly, we deduce that $h \preceq f_{a,\varphi}$. This finishes the proof. □

Remark 4. *The following particular cases for which Proposition 4 applies will be useful later on:*

(A) $a > 0$, $p(u) = 2$ for all $u \in \mathbb{R}^+$, and $q(u) = c > 0$ for all $u \in \mathbb{R}^+$.

(B) $a > 0$, $p(u) = 1$ for all $u \in \mathbb{R}^+$; $q(u) = 0$ if $u \in [0,1]$; and $q(u) = (2u-1)/u$ if $u > 1$.

(C) $a > 0$, $p(u) = 1$ for all $u \in \mathbb{R}^+$, and $q(u) = cu$, $c > 0$, for all $u \in \mathbb{R}^+$.

(D) $a = 0$, $p(u) = 0$ if $u \in [0,1]$; $p(u) = (u+1)/u$ if $u > 1$; $q(u) = 0$ if $u \in [0,1]$; $q(u) = 2(u-1)/u$ if $u > 1$.

(E) $a = 0$, $p(u) = 0$ if $u \in [0,2)$; $p(u) = 2/u(u-1)$ if $u \geq 2$; $q(u) = 0$ if $u \in [0,1]$; $q(u) = u - 1$ if $u > 1$.

Denote by F the restriction of the function $f_{a,\varphi}$ on $\mathbb{N}$, where $f_{a,\varphi}$ is the fixed point for the self-map Φ of $\mathcal{F}_{a,\varphi}$ that was obtained in Proposition 4.

Then, we obtain $F(1) = \Phi F(1) = a$, and

$$F(n) = \Phi F(n) = p(n)F(n-1) + q(n),$$

for all $n > 1$. Hence, F is the (unique) solution of the recurrence equation $R : \mathbb{N} \to \mathbb{R}^+$ given by

$$R(n) = \begin{cases} a & if\ n = 1, \\ p_{\mathbb{N}}(n)R(n-1) + q_{\mathbb{N}}(n) & if\ n > 1, \end{cases} \tag{2}$$

where, by $p_{\mathbb{N}}$ and $q_{\mathbb{N}}$, we design the restrictions on $\mathbb{N}$ of the functions p and q, respectively.

Next, we specify some relevant particular cases of the recurrence Equation (2) (we remind that, in all these cases, the existence and uniqueness of the solution is guaranteed by virtue of the preceding discussion):

- The restrictions on $\mathbb{N}$ of the functions p and q of Remark 4 (A) are given by $p_{\mathbb{N}}(n) = 2$ and $q_{\mathbb{N}}(n) = c > 0$ for all $n > 1$. Thus, the recurrence Equation (2), with $R(1) > 0$), corresponds to the running time of the computing of the well-known problem of the Towers of Hanoi (cf. [49]).
- The restrictions on $\mathbb{N}$ of the functions p and q of Remark 4 (B) are given by $p_{\mathbb{N}}(n) = 1$ and $q_{\mathbb{N}}(n) = (2n-1)/n$ for all $n > 1$. Thus, the recurrence Equation (2), with $R(1) > 0$), corresponds to the running time of the computing of the well-known Largetwo algorithm (cf. [50]).
- The restrictions on $\mathbb{N}$ of the functions p and q of Remark 4 (C) are given by $p_{\mathbb{N}}(n) = 1$ and $q_{\mathbb{N}}(n) = cn > 0$, $c > 0$, for all $n > 1$. Thus, the recurrence Equation (2), with $R(1) > 0$), corresponds to the running time of the computing of the well-known Quicksort algorithm, being the worst case (cf. [51]).
- The restrictions on $\mathbb{N}$ of the functions p and q of Remark 4 (D) are given by $p_{\mathbb{N}}(n) = (n+1)/n$ and $q_{\mathbb{N}}(n) = 2(n-1)/n$ for all $n > 1$. Thus, the recurrence Equation (2), with $R(1) = 0$), corresponds to the running time of the computing of the well-known Quicksort algorithm, being the average case (cf. [51,52]).
- The restrictions on $\mathbb{N}$ of the functions p and q of Remark 4 (E) are given by $p_{\mathbb{N}}(n) = 2/n(n-1)$ and $q_{\mathbb{N}}(n) = n-1$ for all $n > 1$. Thus, the recurrence Equation (2), with $R(1) = 0$), corresponds to the running time of the computing of the well-known Quicksort algorithm, being the median of the three cases (cf. [51]).

The method developed above can be adapted to other cases. For instance, denote by R_F the recurrence equation defined as

$$R_F(n) = \begin{cases} 0 \quad if\ n = 0, \\ 1 \quad if\ n = 1, \\ bR_F(n-1) + cR_F(n) \quad \text{if } n > 1, \end{cases} \tag{3}$$

with $b, c > 0$ constants.

Note that, for $b = c = 1$, R_F is the recurrence equation associated with the celebrated Fibonacci sequence.

Now, let $a = 0$ and $p \in \mathbb{N}$ be such that $e^{-p}(b + ce^{-p}) < 1$. Define a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \backslash \{0\}$ as $\varphi(u) = e^{-pu}$ for all $u \in \mathbb{R}^+$.

For each $f \in \mathcal{F}_{0,\varphi}$, put

$$\Psi f(u) = \begin{cases} 0 \quad if\ u = 0, \\ 1 \quad if\ 1 \leq u \leq 2, \\ bf(u-1) + cf(u-2) \quad if\ u > 1. \end{cases}$$

A slight modification of the proof of Proposition 4 allows us to deduce that Ψ defines a self-map of $\mathcal{F}_{0,\varphi}$.

We also have that $f_0 \preceq \Psi f_0$, where f_0 is the zero function on $\mathbb{R}^+$ and Ψ is non-decreasing on $\mathcal{F}_{0,\varphi}$.

Now, let $f, g \in \mathcal{F}_{0,\varphi}$ be such that $f \preceq g$. Then,

$$\begin{aligned} d_{0,\varphi}(\Psi f, \Psi g) &= \sup_{u \in \mathbb{R}^+}(\varphi(u)(\Psi g(u) - \Psi f(u))) \\ &= \sup_{u > 2}(\varphi(u)(b(g(u-1) - f(u-1)) + c(g(u-2) - f(u-2)))) \\ &= \sup_{u \in \mathbb{R}^+}(\varphi(u+2)(b(g(u+1) - f(u+1)) + c(g(u) - f(u)))) \\ &\leq \sup_{u \in \mathbb{R}^+}(be^{-p}\varphi(u+1)(g(u+1) - f(u+1))) + \sup_{u \in \mathbb{R}^+}(ce^{-2p}\varphi(u)(g(u) - f(u))) \end{aligned}$$

$$\leq be^{-p}d_{0,\varphi}(f,g) + ce^{-2p}d_{0,\varphi}(f,g) = \lambda d_{0,\varphi}(f,g).$$

Since $0 < \lambda < 1$, all conditions of Theorem 2 are satisfied. Hence, the self-map Ψ has a fixed point $f_{0,\varphi} \in \mathcal{F}_{0,\varphi}$, which is unique by a similar argument to the one given in the proof of Proposition 4.

It immediately follows that the restriction to $\mathbb{N} \cup \{0\}$ of $f_{0,\varphi}$ constitutes the unique solution of the recurrence Equation (3).

5. Conclusions

Motivated by the difficulties of obtaining a full quasi-metric generalization of an outstanding generalization of Banach's contraction principle due to Suzuki, we have introduced and examined the notion of a protected quasi-metric. With the help of this new structure, we have obtained a fixed-point theorem in the framework of Smyth-complete quasi-metric spaces that generalizes Suzuki's theorem. Combining right completeness with partial ordering properties, we have also obtained a variant of Suzuki's theorem, which was applied to discuss a kind of difference equations and recurrence equations. We emphasize that several classical non-metrizable topological spaces as the Alexandroff spaces, the Sorgenfrey line, the Michael line, and the Khalimsky line, among others, can be endowed with the structure of a protected quasi-metric.

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