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# Reasons for stability in the construction of derivative-free multistep iterative methods

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In this paper, a deep dynamical analysis is made, by using tools from multidimensional real discrete dynamics, of some derivative-free iterative methods with memory. All of them have good qualitative properties, but one of them (due to Traub) shows to have the same behavior as Newton's method on quadratic polynomials. Then, the same techniques are employed to analyze the performance of several multipoint schemes with memory, whose first step is Traub's method, but their construction was made using different procedures. Therefore, their stability is analyzed, showing which is the best in terms of wideness of basins of convergence or the existence of free critical points that would yield to convergence toward different elements from the desired zeros of the nonlinear function. Therefore, the best stability properties are linked with the best estimations made in the iterative expressions, rather than in their simplicity. These results have been checked with numerical and graphical comparison with many other known methods with and without memory, with different order of convergence, with excellent performance.

**KEYWORDS**

derivative-free, multipoint root-solver with memory, nonlinear equations, stability analysis, Traub's method

**MSC CLASSIFICATION**

65H05, 68W25, 37D45

## 1 | INTRODUCTION

A wide variety of physical processes observed in real life are nonlinear, as are many systems underlying engineering problems. If, in order to simplify the problem, they are linearized, much of the complexity disappears, but the solution obtained is a worse approximation to the real solution. Iterative processes are very useful in this context, approximating the solution of the nonlinear equations, $f(x) = 0$, that model this type of problem.

Although the best-known fixed-point iterative method is Newton's method, it represents only a subclass of numerical procedures: Memoryless iterative processes, which use only the current iteration to compute the next one, building the sequence that will eventually converge to the solution. However, there are iterative schemes that use more than one known iterate to calculate the next: These are known as iterative procedures with memory, and the best known is the secant method, whose iterative expression is

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})},\ n = 1, 2, \ \dots\ ,$$

where $x_0$ and $x_1$ are the initial estimations. The simplicity of its expression makes it very useful but the quadratic order of convergence of Newton's scheme is lost, reaching superlinear convergence. To overload this inconvenient, Traub in [1] designed, among others, the derivative-free scheme (DF, for short) with memory,

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-2}] - f[x_{n-1}, x_{n-2}] + f[x_n, x_{n-1}]}, \tag{1}$$

denoted by $TM$, where $f[x, y] = \frac{f(x)-f(y)}{x-y}$, which increases the order of convergence from 1.618 (of secant scheme) up to 1.839. Traub [1] has shown that the method is of order 1.839. See also [2, 3]. It is lower than other DF methods as Steffensen's scheme (without memory), but it has good numerical properties. In fact, this scheme has been used as first step of several higher order multipoint methods, with good results in terms of robustness and applicability (see the works by Neta [3–5] and other authors [6–10]).

In recent years, different iterative schemes with memory have been designed (a good overview can be found in [11]), mostly derivative-free. These have been constructed with increasing order of convergence and, therefore, with increasing computational complexity. In terms of stability, some researchers compared the amplitude of the set of initial points converging to the same attractor, using complex discrete dynamics techniques. In [12], the authors observed that iterative schemes with seventh-order memory convergence showed better stability properties than many eighth-order optimal procedures without memory. This graphical comparison was subsequently used by different authors; observe, for instance, the work of Wang et al. in [13] and Cordero et al. [14] in 2016 or the investigations of Bakhtiari et al. [15] in 2016 and Howk et al. [16] in the following years.

The authors developed in [17, 18] a technique that, using multidimensional real discrete dynamics tools, is able to study the qualitative performance of iterative with memory schemes, not only in graphical terms but essentially in analytical terms. Using this technique, the stability of the fixed and critical points of secant, Steffensen's and Kurchatov's methods (among others) were studied in [17]. It was also used to analyze other procedures, such as those described in [18], that defined by Choubey et al. in [19], or those by Chicharro et al. in [20–22]. In this kind of analysis, the performance of the numerical procedure on the simplest nonlinear functions (i.e., quadratic polynomials) is studied. As it has been corroborated by many researchers in the area, this kind of study allows us to select those elements of a class of iterative schemes with better qualitative performance. Those schemes are shown to be the best also on nonpolynomial functions (see, e.g., [12, 19, 22]), among others. Also Behl et al. in [23] presents a similar study on derivative-free eighth-order iterative schemes without memory, both with polynomial and nonpolynomial functions.

The design of high-order multipoint iterative methods is based on the scheme used as first step: It defines the starting order of convergence, the use of derivatives or not, the employment of only one previous iterate, or the use of memory. Our aim is to analyze in depth the qualitative performance of some DF methods with memory so that we can select the one with the best stability properties. Then, that one is used as first step of different iterative methods, which were designed by means of diverse techniques. The qualitative behavior of those multipoint schemes is studied in order to deduce how the qualitative properties are inherited. The positive aspect of this study is that we have objective tools to select which technique is more suitable in the construction of iterative multipoint methods with memory. On the opposite sense, we are working with derivative-free iterative methods with memory, but schemes with memory using derivatives are out of this analysis and it will be a forthcoming issue.

In this context, we made in Section 2 a deep dynamical analysis of several DF iterative schemes with memory, defined by using three previous iterates. We find the most stable one and, therefore, compare in Section 3 the performance of several multistep methods based on the previous methods. All this analysis is made by using multidimensional discrete dynamics. By using these results, we select the most stable scheme, and in Section 4, we check numerically the performance of the methods on nonpolynomial functions, showing their basins of attraction. Therefore, the applicability of the schemes and the dynamical results are checked.

## 2 | QUALITATIVE STUDY OF ONE-STEP ITERATIVE WITH MEMORY SCHEMES

An iterative procedure that uses three previous iterate to calculate the next one is

$$x_{n+1} = \Psi(x_{n-2}, x_{n-1}, x_n),\ n \geq 2,$$

being the starting guesses $x_0, x_1$ and $x_2$. The authors described in [17, 18] a procedure that allows us to describe any iterative with memory scheme as a multidimensional real discrete dynamical system, so that its stability performance can be studied.

To get the fixed points of an iterative scheme defined by $\Psi$, we define a multidimensional fixed point function $H : \mathbb{R}^3 \to \mathbb{R}^3$, related to $\Psi$ as

$$H(x_{n-2}, x_{n-1}, x_n) = (x_{n-1}, x_n, \Psi(x_{n-2}, x_{n-1}, x_n)),$$

for $n = 1, 2, \ldots$, where $x_0$, $x_1$, and $x_2$ are the initial guesses. Then, any fixed point of $H$ must satisfy $x_{n+1} = x_n$, $x_{n-2} = x_n$ and $x_{n-1} = x_n$.

From function $H : \mathbb{R}^3 \to \mathbb{R}^3$, the associate discrete dynamical system in $\mathbb{R}^3$ is defined by $H(x_{n-2}, x_{n-1}, x_n) = (x_{n-1}, x_n, x_{n+1})$, where $\Psi$ is the operator of the iterative method with memory. Let us define the sequence of vectors $\bar{x}_n = (x_{n-1}, x_n, x_{n+1})$ by taking three consecutive iterates. The fixed points $\bar{x}$ of $H$ satisfy $\bar{x} = \Psi(\bar{x})$ and all three components are identical. This notation implies $x_{n-2} = x_{n-1} = x_n$. Now, let us introduce some definitions (see [24]).

Let us consider the vectorial rational function $H : \mathbb{R}^3 \to \mathbb{R}^3$, usually obtained by applying an iterative method on a scalar polynomial $q(x)$. Then, if a fixed point $\bar{x}$ of operator $H$ is different from $(r, r, r)$, being $r$ a zero of $q(x)$, it is called strange fixed point. Moreover, the orbit of a point $x^* \in \mathbb{R}^3$ is defined as the set of successive images from $x^*$ by the vector function, that is, $\operatorname{orbit}(x^*) = \{x^*, H(x^*), \ldots, H^n(x^*), \ldots\}$. Indeed, if a point $\bar{x}^* \in \mathbb{R}^3$ satisfy $H^k(\bar{x}^*) = \bar{x}^*$ and $H^p(\bar{x}^*) \neq \bar{x}^*$, $p = 1, 2, \ldots, k-1$ is called $k$-periodic point. Let us remark that a $k$-periodic point $x^*$ is a fixed point if $k = 1$.

The qualitative performance of a point of $\mathbb{R}^3$ is classified depending on its asymptotic performance. So, in order to declare the stability of multidimensional fixed points, the following result from Robinson [25] is used.

**Theorem 1.** *Let $H$ be a function of class $C^2$, defined from $\mathbb{R}^m$ to $\mathbb{R}^m$. Let us also assume that $x^*$ is a $k$-periodic point. If we denote by $\lambda_1, \lambda_2, \ldots, \lambda_m$, the eigenvalues of $H'(x^*)$, then*

*a) $x^*$ is attracting if $|\lambda_j| < 1$, for all $j = 1, 2, \ldots, m$.*
*b) If $\exists j_0 \in \{1, 2, \ldots, m\}$ such that $|\lambda_{j_0}| > 1$, then $x^*$ is unstable (repelling or saddle).*
*c) $x^*$ is repelling if $|\lambda_j| > 1$, for all $j = 1, 2, \ldots, m$.*

Moreover, a fixed point $\bar{x} \in \mathbb{R}^3$ is said to be hyperbolic if $|\lambda_j| \neq 1$ for all $j = 1, 2, \ldots, m$. Specifically, if $\exists i, j \in \{1, 2, \ldots, m\}$ satisfying $|\lambda_i| < 1$ and $|\lambda_j| > 1$, then the fixed point is a saddle point.

Nevertheless, sometimes the Jacobian is not well defined at the fixed points. In these cases, we impose to the rational operator $H$ the condition that all components are identical, so that it is reduced to a real-valued function. Therefore, the stability of the fixed point can be inferred from the absolute value of its first derivative at the fixed point, as it is done in scalar complex dynamics.

By considering $\bar{x}$ an attracting fixed point of function $H$, we define its basin of attraction $\mathcal{B}(\bar{x})$ as the set

$$\mathcal{B}(\bar{x}) = \left\{ \bar{x} \in \mathbb{R}^3 : H^m(\bar{x}) \to \bar{x}, \text{for } m \to \infty \right\}.$$

A key element in the stability analysis of an iterative method is the set of critical points of its associated rational function $H$: If $H'(\bar{x})$ satisfies $\det(H'(\bar{x})) = 0$, $\bar{x}$ is said to be a critical point. This definition usually do not provide a finite set of points, but one or several curves in the domain of the rational function or even that all points are critical. Therefore, we calculate them by finding those points satisfying that $H'$ has zero eigenvalues on them; this is a more restrictive definition, but often necessary. Moreover, if the critical points are also fixed points, they are called superattracting points; if not, they are called free critical points (let us remark that components of critical points can be different). Indeed, Julia and Fatou [24] proved that there is at least one critical point associated with each basin of attraction. Therefore, by studying the orbit of the free critical points, all the attracting elements can be found.

## 2.1 | Preliminary analysis: How to select the first step

In this section, we analyze the performance on quadratic polynomials of three different schemes with memory due to Traub [1], (1), denoted by $TM$, that of Jarratt and Nudds [26],

$$x_{n+1} = x_n - f(x_n)\frac{(x_{n-1} - x_n)(x_{n-2} - x_n)(f(x_{n-2}) - f(x_{n-1}))}{(x_{n-1} - x_n)(f(x_{n-2}) - f(x_n))f(x_{n-1}) + (x_{n-2} - x_n)(f(x_n) - f(x_{n-1}))f(x_{n-2})}, \tag{2}$$

denoted by *JNM*, and the procedure presented by Popovski et al. in [27],

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-2})(f(x_{n-2}) - f(x_{n-1}))(x_n - x_{n-1})}{(f(x_{n-2}) - f(x_n))(x_{n-2} - x_{n-1})(f(x_{n-1}) - f(x_n))}, \tag{3}$$

denoted by *PM*.

All these schemes have similar iterative expressions, the same order of convergence ($p = 1.839$), and our first aim is to decide, under qualitative considerations, which is the most stable one in order to add two more steps, increasing its convergence order and showing the best performance in terms of wideness of the sets of initial estimations converging to the roots.

With the aim of extending the results to any polynomial of second degree, this study is constructed on $q(x) = x^2 - c$, so that the value of $c$ yields to a situation with real, complex, or multiple roots depending on $c > 0$, $c < 0$, or $c = 0$, respectively. This analysis can be summarized in the following results.

**Theorem 2.** *The multidimensional rational operator associated with Traub's scheme TM, when it is mapped on polynomial* $q(x) = x^2 - c$, $c \neq 0$, *is*

$$T(w, z, x) = \left(z, x, \frac{c + x^2}{2x}\right),$$

*and it is*

$$T(w, z, x) = \left(z, x, \frac{x}{2}\right),$$

*for* $c = 0$. *Moreover, TM satisfies:*

a) *The only fixed points are the roots of* $q(x)$.
b) *The only critical points are the roots of* $q(x)$.

*So, there is no other possible performance of TM scheme than convergence to the roots.*

*Proof.* Let us remark that the third component of operator $T(w, z, x)$ is equal to the rational function obtained when classical Newton's method is applied on polynomial $q(x)$. This is the reason why, when we force the three consecutive iterates to be equal ($x = z = w$) in order to get the fixed points, then the only fixed points are the roots $x = \pm\sqrt{c}$.

Regarding the critical points, the Jacobian matrix $T'$ is

$$T'(w, z, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} - \frac{c}{2x^2} \end{pmatrix},$$

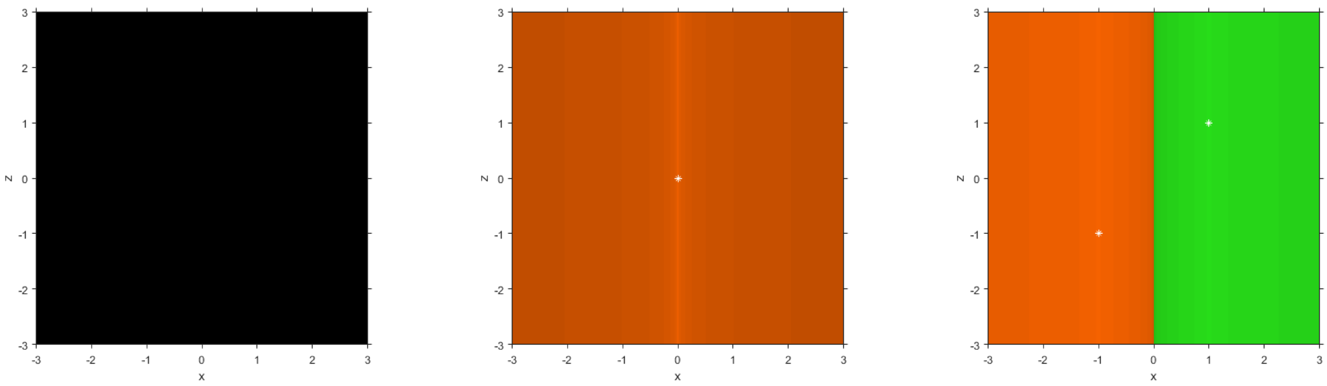
with eigenvalues $\left\{0, 0, \frac{1}{2} - \frac{c}{2x^2}\right\}$. So, there are no free critical points. □

A very useful tool to visualize the analytical results is the dynamical plane of the system, composed by the set of the different basins of attraction. They are drawn by means of the programs presented in [28] using Matlab R2021a, after some changes to adapt them to schemes with memory. The dynamical plane of a method is built by calculating the orbit of a mesh of $400 \times 400$ starting points $(z, x)$ ($w$ does not appear in the rational function $T$) and painting each of them in different colors (orange and green in this case) depending on the attractor they converge to (marked as a white star), with a tolerance of $10^{-3}$. Also they appear in black color if the orbit has not reached any attracting fixed point in a maximum of 80 iterations. In Figure 1, we show the dynamical planes of this method for selected values of $c$, in order to show its performance. Let us remark that, as by definition all the fixed points have equal components, they will always appear in the main diagonal of the dynamical plane. It can be observed that, when there are no real root ($c < 0$; Figure 1A), no other attracting element appear; when $c = 0$, the only root is multiple and the convergence is linear, so there are global convergence to $x = 0$ as can be seen in Figure 1B. In Figure 1C, the convergence to the roots is also observed to be global, being their basins of attraction two symmetrical half-planes.

Now, we analyze the performance of Jarratt–Nudds method with memory [26] on quadratic polynomials.

**Theorem 3.** *The multidimensional rational operator associated with method JNM, when it is applied on polynomial* $q(x) = x^2 - c$, $c \neq 0$, *is*

$$JN(w, z, x) = \left(z, x, \frac{c(x + z + w) + xzw}{c + x(z + w) + zw}\right),$$



**(A)** $c = -1$ **(B)** $c = 0$ **(C)** $c = 1$

**FIGURE 1** Dynamical planes of scheme *TM* on $q(x)$. [Colour figure can be viewed at wileyonlinelibrary.com]

*and it is*

$$JN(w, z, x) = \left( z, x, \frac{xzw}{x(z+w)+zw} \right),$$

*for* $c = 0$. *Moreover, JNM satisfies:*

a) *There are no attracting strange fixed points. If* $c \neq 0, x = 0$ *is a strange fixed point, that is, a saddle point. If* $c = 0, x = 0$ *is an attracting fixed point, as it is a multiple zero of* $q(x)$.
b) *There exists an infinite set of free critical points* $(w, z, x)$*, defined by the lines* $x = \pm\sqrt{c}$ *or* $z = \pm\sqrt{c}$*, being* $c > 0$ *and* $w$ *arbitrary, provided that* $c + x(z + w) + zw \neq 0$.

*Proof.* By applying Jarratt–Nudds' method on $q(x)$ and constructing the auxiliary multidimensional operator, $JN(w, z, x)$ is found. To get the fixed points of $JN$, we solve $JN(x, x, x) = (x, x, x)$ and find

$$\frac{2x\left(c - x^2\right)}{c + 3x^2} = 0,$$

so the fixed points are those whose three components coincide at $x = \pm\sqrt{c}$ and $x = 0$, provided that $c + 3x^2 \neq 0$. To study their qualitative behavior, we calculate

$$JN'(w, z, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{(c-x^2)(c-z^2)}{(c+x(z+w)+zw)^2} & \frac{(c-x^2)(c-w^2)}{(c+x(z+w)+zw)^2} & \frac{(c-z^2)(c-w^2)}{(c+x(z+w)+zw)^2} \end{pmatrix}$$

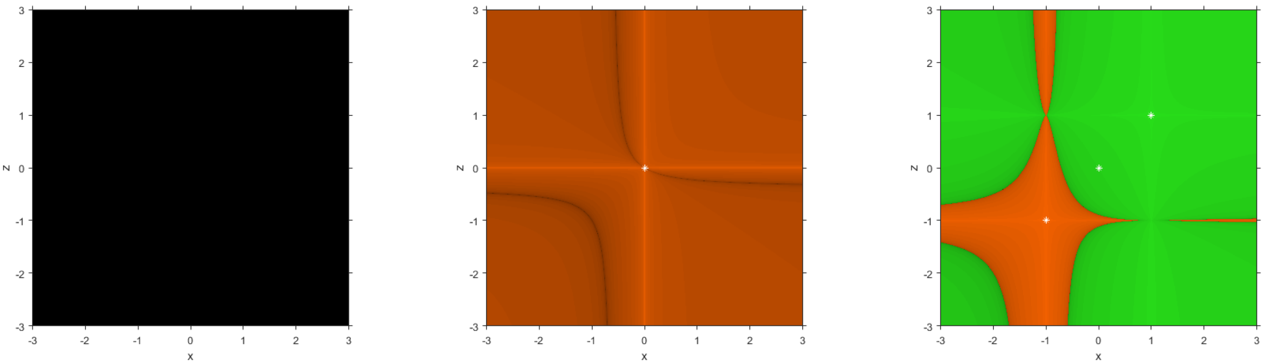
and its eigenvalues at the fixed points are $(0, 0, 0)$ in the case of $w = z = x = \pm\sqrt{c}$ and (approximately) $\{1.83929, -0.419643 + 0.606291i, -0.419643 - 0.606291i\}$ for $w = z = x = 0$. So, the roots of $q(x)$ are superattracting fixed points and $(0, 0, 0)$ is saddle, since $|\lambda_1| = 1.83929 > 1$ and $|\lambda_2| = |\lambda_2| \approx 0.737353 < 1$.

Regarding the critical points, it is not possible to gent an analytical expression of the eigenvalues of $JN'(w, z, x)$. Then, it can be checked that

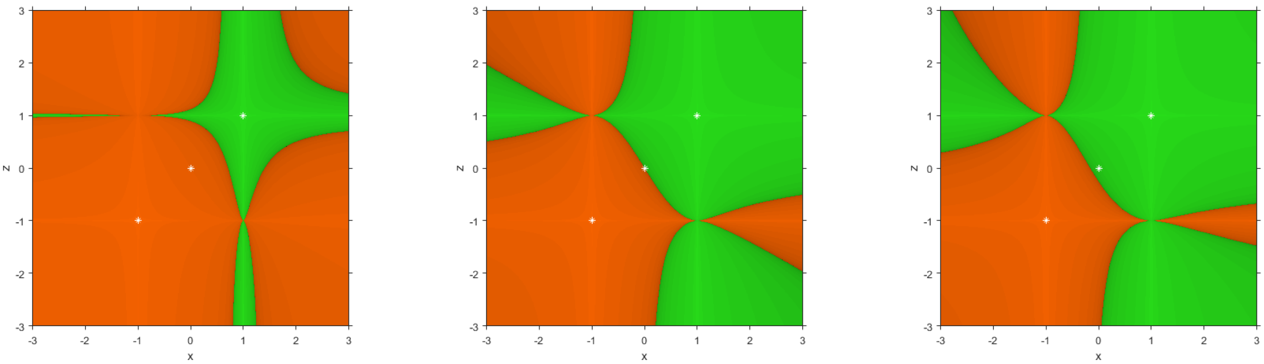
$$\det(JN'(w, z, x)) = \frac{\left(c - x^2\right)\left(c - z^2\right)}{(c + x(z + w) + zw)^2},$$

and, therefore, $x = \pm\sqrt{c}$ or $z = \pm\sqrt{c}$ are curves of critical points, provided that $c + x(z + w) + zw \neq 0$ and they are free as the third component $w$ is not fixed. □

In Figure 2, we show the dynamical planes of this method for selected values of $c$, in order to show its performance. For all the dynamical planes, different values of $w$ has been used, in order to observe the dependence of the wideness of the basins of attraction on it. It can be noticed that, for $c \geq 0$, global convergence to the roots is found, being slower in case of multiplicity (see Figure 2B). Moreover, a symmetry is observed for opposite values of $w$ in the wideness of the basins of attraction of both roots (see Figure 2C,D).




**(A)** $c = -1$, $w = 2$ **(B)** $c = 0$, $w = 2$ **(C)** $c = 1$, $w = 2$




**(D)** $c = 1$, $w = -2$ **(E)** $c = 1$, $w = 0$ **(F)** $c = 1$, $w = 10$

**FIGURE 2** Dynamical planes of scheme *JNM* method on $q(x)$, for different values of $w$ and $c$. [Colour figure can be viewed at wileyonlinelibrary.com]

Finally, by means of a similar analysis, we found the main result about the stability of Popovski's scheme [27]. The proof is omitted as it is similar to the previous ones.

**Theorem 4.** *The multidimensional rational operator associated with method PM, when it is applied on polynomial* $q(x) = x^2 - c$, $c \neq 0$, *is*

$$P(w, z, x) = \left( z, x, \frac{c(z + w) + x^3 + xzw}{(x + z)(x + y)} \right),$$

*and it is*

$$P(w, z, x) = \left( z, x, \frac{x^3 + xzw}{(x + z)(x + w)} \right),$$

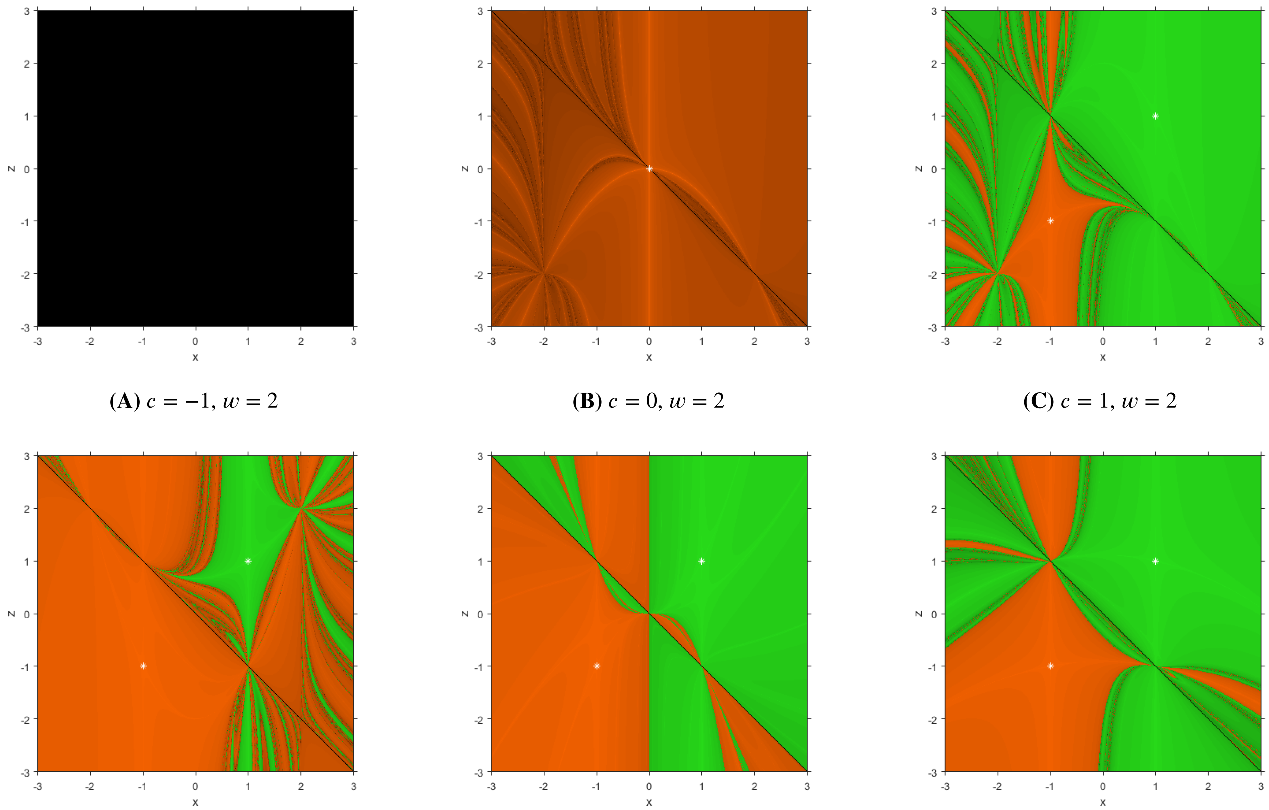
*for* $c = 0$. *Moreover, PM satisfies:*

*a) The only fixed points are the roots of* $q(x)$.
*b) There exists an infinite set of free critical points* $(w, z, x)$, *defined by the lines* $z = x$, *provided that* $x \neq z$ *and* $x \neq w$.

In Figure 3, we show the performance of this method with memory for several values of $c$. For all the dynamical planes, different values of $w$ has been used, in order to observe the dependence of the wideness of the basins of attraction on it. We observe that, depending of the value of $w$, the basins of both roots are symmetric ($w = 0$) or one of them is wider and this situation is reversed for the opposite value of $w$ (see, e.g., Figure 3C,D).

We notice that similar performance as in case of *JNM* is observe in terms of symmetry and convergence to the roots. However, their basins of attraction have more connected components and the Julia set (the boundary among the basins of attraction) is much more complicated. Also slow convergence to the multiple root in case of $c = 0$ is observed.

So, it can be concluded that the stability of Traub's scheme with memory is much better than the other methods with similar shape and order of convergence under analysis. Therefore, we study, in the following section, the qualitative behavior of two iterative schemes with three steps based on Traub's procedure as first step.



(A) $c = -1$, $w = 2$ (B) $c = 0$, $w = 2$ (C) $c = 1$, $w = 2$

(D) $c = 1$, $w = -2$ (E) $c = 1$, $w = 0$ (F) $c = 1$, $w = 10$

**FIGURE 3** Dynamical planes of scheme *PM* on $q(x)$, for different values of $w$ and $c$. [Colour figure can be viewed at wileyonlinelibrary.com]

## 3 | QUALITATIVE PERFORMANCE OF MULTIPOINT METHODS WITH THE SAME FIRST STEP

As it has been previously stated, we analyze the qualitative properties of two iterative with memory schemes based on Traub's scheme. We denote by method M1 that scheme with iterative expression

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_{n-2},x_n] + f[x_{n-1},x_n] - f[x_{n-2},x_{n-1}]}, \\ z_n &= y_n - \frac{f(y_n)}{f[y_n,x_n] + f[y_n,x_n,x_{n-1}](y_n - x_n) + f[y_n,x_n,x_{n-1},x_{n-2}](y_n - x_n)(y_n - x_{n-1})}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n,y_n] + f[z_n,y_n,x_n](z_n - y_n) + f[z_n,y_n,x_n,x_{n-1}](z_n - y_n)(z_n - x_n)}, \end{aligned} \tag{4}$$

presented in [4], with order of convergence 7.356. Also in [3], the scheme with memory that we denote by M2 was constructed,

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_{n-2},x_n] - f[x_{n-2},x_{n-1}] + f[x_{n-1},x_n]}, \\ z_n &= y_n - \frac{f(y_n)}{\alpha_1 f(x_n) + \alpha_2 f(x_{n-1}) + \alpha_3 f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{\beta_1 f(x_n) + \beta_2 f(y_n) + \beta_3 f(z_n)}, \end{aligned} \tag{5}$$

showing the order of convergence is 6.219.

Although both schemes are based on the same first step, they reach different order of convergence with also great divergence between their computational complexity: As M1 uses in the denominator of second and third steps high-order estimations of the derivatives $f'(y_n)$ and $f'(z_n)$, respectively, its expression is more complicated but it reaches higher order of convergence than its partner M2, with a simpler iterative expression but lower order of convergence. Moreover, usually the highest is the order of convergence, the closer to the root the initial estimate need to be in order to ensure convergence; so, it would be possible to get better stability properties for lower order methods. In any case, our aim is not to classify them by means of their order of convergence, but of their stability. In what follows, we construct the multidimensional discrete dynamical system associated to both schemes and analyze the existence of strange attracting fixed points or free critical points that might yield to undesirable numerical performances.

## 3.1 | Qualitative study of $M1$

We analyze now the performance of the rational operator related to $M1$ on quadratic polynomials. As in the previous section, this analysis is made on $q(x) = x^2 - c$. The results are condensed in the following theorem. It can be observed that the third component of the vectorial rational function does not depend on the two previous iterations, $w$ and $z$ as it happened in Traub's method.

**Theorem 5.** *The multidimensional rational operator associated with method $M1$, when it is applied on $q(x) = x^2-c$, $c \neq 0$ is*

$$M1(w,z,x) = \left(z, x, \frac{c^4 + 28c^3x^2 + 70c^2x^4 + 28cx^6 + x^8}{8c^3x + 56c^2x^3 + 56cx^5 + 8x^7}\right),$$

*and it is*

$$M1(w,z,x) = \left(z, x, \frac{x}{8}\right),$$

*for $c = 0$. Moreover, $M1$ satisfies:*

*a) There are no strange attracting fixed points. If $c < 0$, there exist six real strange fixed points that are saddle points. If $c = 0$, $x = 0$ is the unique fixed point, that is, only attracting; finally, for $c > 0$, the only fixed points are the roots of $q(x)$.*
*b) There exists no critical points different from the roots of $q(x)$.*

*So, method $M1$ has global convergence.*

*Proof.* We calculate the fixed points of operator $M1$ by solving $M1(w,z,x) = (w,z,x)$, that must satisfy $w = x = z$. Specifically,

$$M1(w,z,x) = \left(z, x, \frac{c^4 + 28c^3x^2 + 70c^2x^4 + 28cx^6 + x^8}{8c^3x + 56c^2x^3 + 56cx^5 + 8x^7}\right) = (w,z,x),$$

if and only if $w = z = x$ and

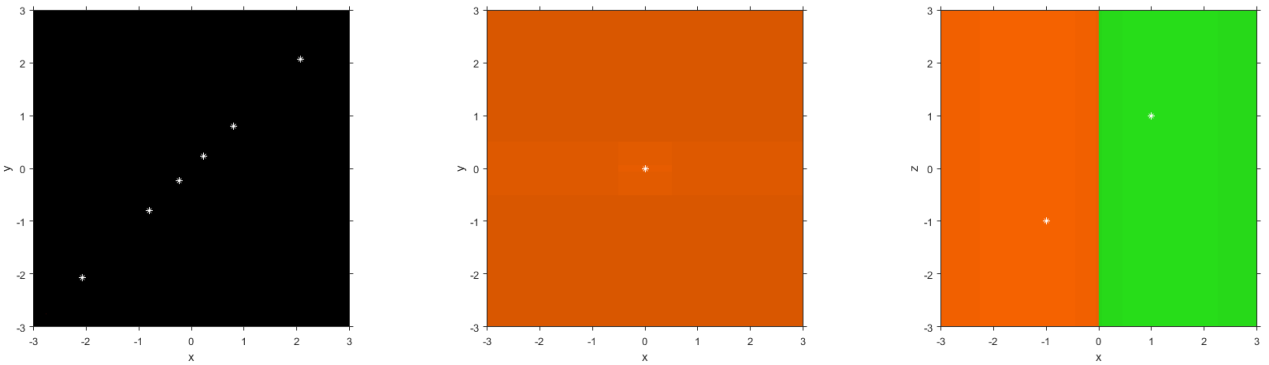
$$-\frac{\left(x^2 - c\right)\left(c^3 + 21c^2x^2 + 35cx^4 + 7x^6\right)}{8x\left(c + x^2\right)\left(c^2 + 6cx^2 + x^4\right)} = 0.$$

So, the fixed points of $M1(w,z,x)$ are the roots of $q(x)$ and also the zeros of the sixth-degree polynomial $c^3 + 21c^2x^2 + 35cx^4 + 7x^6$ (that are real if $c < 0$), meanwhile $c^2 + 6cx^2 + x^4 \neq 0$. Let us remark that in case $c > 0$, there are not strange fixed points and when $c = 0$, the rational function is reduced and the only fixed point is $x = 0$, that is attracting but not superattracting. The Jacobian matrix $M1'(w,z,x)$ is defined as

$$M1'(w,z,x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\left(x^2-c\right)^7}{8x^2\left(c+x^2\right)^2\left(c^2+6cx^2+x^4\right)^2} \end{pmatrix}.$$

It can be checked that the first two eigenvalues of $M1'$ evaluated at each one of these strange fixed points are null. Then, their character would be attracting or saddle depending on the absolute value of the third eigenvalue, depending on $c$ and $x$. In all cases, $|\lambda_3| = 8$, so they are saddle.

By calculating the eigenvalues of $M1'(w,z,x)$, we get $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -\frac{\left(c-x^2\right)^7}{8x^2\left(c+x^2\right)^2\left(c^2+6cx^2+x^4\right)^2}$. So, we conclude that the only critical points are the roots of $q(x)$, proving the global convergence for quadratic polynomials. □



**(A)** $c = -1$ **(B)** $c = 0$ **(C)** $c = 1$

**FIGURE 4** Dynamical planes of scheme $M1$ on $q(x)$. [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 4 shows the behavior stated at Theorem 5. Let us notice that only the convergence to the roots is reached (it is clear from the absence of free critical points), showing the best possible behavior in terms of stability. Fixed points are represented as white stars. This performance is qualitatively the same that those obtained by Newton's scheme.

## 3.2 | Qualitative study of $M2$

A similar study is made for $M2$, the rational function involved depends on the two previous iterations (in this case, $w = x_{n-2}$ has not any role). The proof is omitted as it can be developed in a similar way as in Theorem 5.

**Theorem 6.** *The multidimensional rational operator associated with method $M2$, when it is applied on $q(x) = x^2 - c$, $c \neq 0$, is*

$$M2(w,z,x) = \left( z, x, \frac{-c^5(3x+z)^3 + c^4x^2q_1(z,x) + c^3q_2(z,x) - 2c^2x^6q_3(z,x) + cx^8q_4(z,x) + x^{10}q_5(z,x)}{8x\left(c(x+z) + x^2(x-3z)\right)\left(-c^3(3x+z)^2 + c^2x^2r_1(z,x) + cx^4r_2(z,x) + x^6r_3(z,x)\right)} \right),$$

*where*

$$\begin{aligned}
q_1(z,x) &= -141x^3 - 77x^2z + 17xz^2 + 9z^3,\\
q_2(z,x) &= -222x^7 + 610x^6z + 438x^5z^2 + 70x^4z^3,\\
q_3(z,x) &= 69x^3 - 411x^2z + 471xz^2 + 319z^3,\\
q_4(z,x) &= -23x^3 + 425x^2z - 1373xz^2 + 1163z^3\\
q_5(z,x) &= 39x^3 - 217x^2z + 333xz^2 - 91z^3,\\
r_1(z,x) &= -17x^2 + 18xz + 15z^2,\\
r_2(z,x) &= -11x^2 + 62xz - 35z^2,\\
r_3(z,x) &= 5x^2 - 10xz - 11z^2,
\end{aligned}$$

*and it is*

$$M2(w,z,x) = \left( z, x, \frac{x\left(39x^3 - 217x^2z + 333xz^2 - 91z^3\right)}{8(x-3z)\left(5x^2 - 10xz - 11z^2\right)} \right),$$

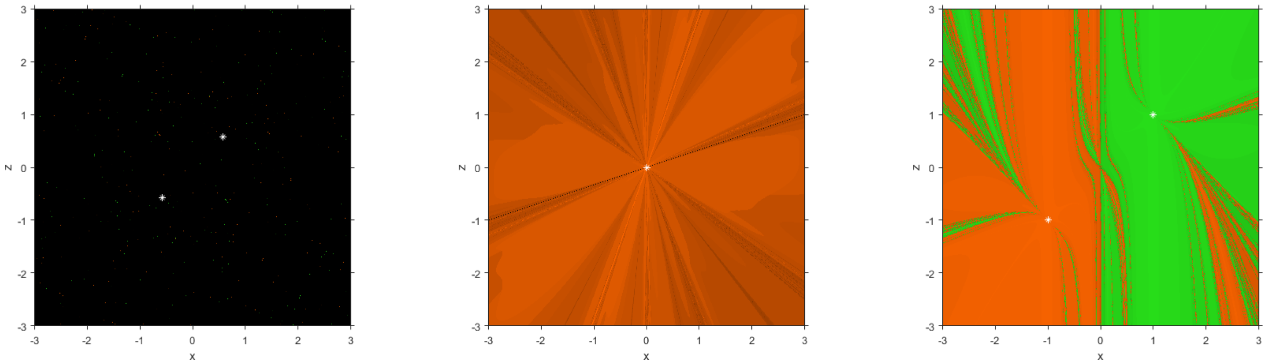
*for $c = 0$. Indeed, $M2$ satisfies:*

*a) There are no strange attracting fixed points. If $c < 0$, there are two real strange fixed points $\left(-\frac{\sqrt{-c}}{\sqrt{3}}, -\frac{\sqrt{-c}}{\sqrt{3}}\right)$ and $\left(\frac{\sqrt{-c}}{\sqrt{3}}, \frac{\sqrt{-c}}{\sqrt{3}}\right)$ that are saddle points. If $c = 0$, the unique fixed point is $x = 0$ that is attracting but not superattracting; finally, for $c > 0$, the only fixed points are the roots of $q(x)$.*

*b) If $c > 0$, there are two infinite sets of free critical points, $(w, \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, -\frac{\sqrt{c}}{\sqrt{85}})$ and $(w, -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, \frac{\sqrt{c}}{\sqrt{85}})$, for any real value of $w$.*

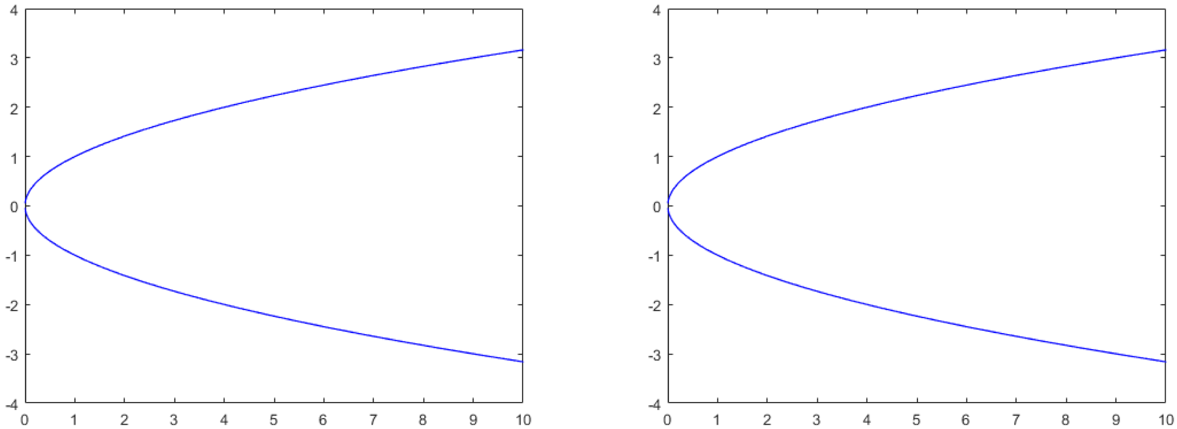
The existence of free critical points led us to infer the possibility of convergence to attracting elements (points, orbits, etc.) different from the roots. As a first step to check if other performances are possible, some dynamical planes can be seen in Figure 5.

In Figure 5, $M2$ scheme is found to have a very stable performance. That is, in case there exist strange fixed points, they are repelling or neutral. Global convergence is observed, despite the existence of free critical points that lie inside




**(A)** $c = -1$ **(B)** $c = 0$ **(C)** $c = 1$

**FIGURE 5** Dynamical planes of $M2$ method on $q(x)$for different values of $c$. [Colour figure can be viewed at wileyonlinelibrary.com]



**(A)** $z = \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c},\ x = -\frac{\sqrt{c}}{\sqrt{85}}$ **(B)** $z = -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c},\ x = \frac{\sqrt{c}}{\sqrt{85}}$

**FIGURE 6** Bifurcation diagrams of $M2$ for real critical points. [Colour figure can be viewed at wileyonlinelibrary.com]

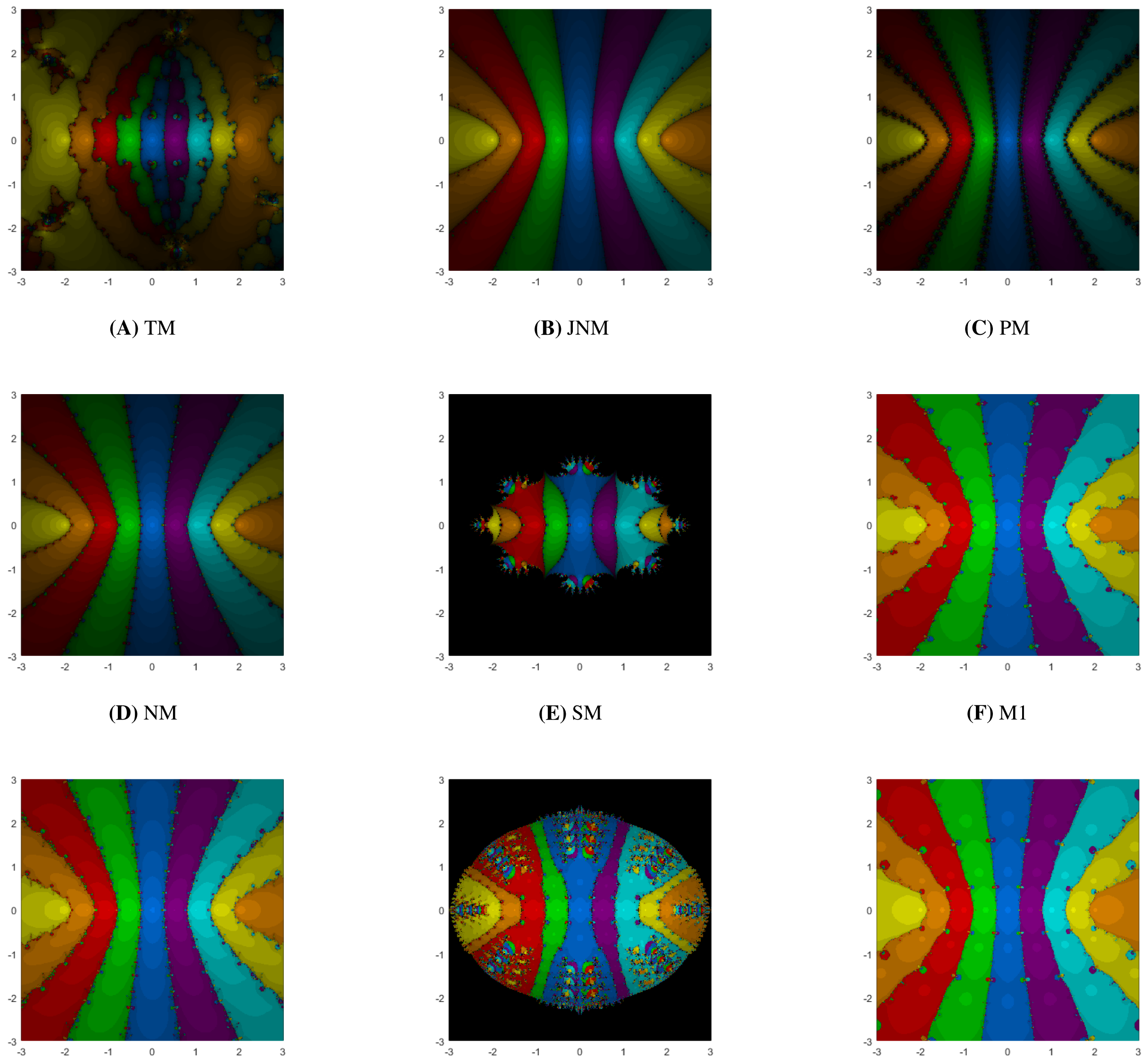
the basins of attraction of the roots in Figure 5C, where $c > 0$. The observed performance is similar to that of $M1$, but the basins of attraction are divided in infinite connected components. However, this existence of free critical points does not allow to assure that there exist another values of $c$ with convergence to attracting periodic orbits, or even with chaotical performance. So, it should be possible that for any value of $c > 0$, those free critical points where not in the basins of attraction of the roots, but inside the basin of any other attractor, maybe a periodic orbit or an strange attractor. In order to detect this performance, we use Feigenbaum's diagram.

### 3.2.1 | Feigenbaum's diagrams

We use bifurcation diagrams of $M2$, depending on the value of $c$, by means of the use of each real critical point $s_1(c) = (w, \frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, -\frac{\sqrt{c}}{\sqrt{85}})$ and $s_2(c) = (w, -\frac{9}{11}\sqrt{\frac{5}{17}}\sqrt{c}, \frac{\sqrt{c}}{\sqrt{85}})$ as a starting point, $w$ arbitrary, (described in Theorem 6) and observing the range $[0, 10]$ of the parameter $c$, where free critical points are real.

Both Feigenbaum's diagrams can be observed in Figure 6A,B, with the same performance. We use blue color for plotting the last 100 from a total amount of 500 iterations, for each $c \in [0, 10]$ (if a wider interval is used, the results are the same). We notice that the same curve appears in both. It corresponds with the real roots of $q(x)$ in this interval. This behavior is in accordance with the dynamical planes shown in Figure 5.

So, both schemes have shown good stability properties on quadratic polynomials. The performance of M1, in spite of the higher complexity of its iterative expression, has shown to be globally convergent due to the absence of free critical points, although the final performance of M2 has been similar. It seems that the best estimation of the derivatives has a




**(A)** TM **(B)** JNM **(C)** PM

**(D)** NM **(E)** SM **(F)** M1

**(G)** M2 **(H)** ZOM **(I)** SAM

**FIGURE 7** Dynamical planes of analyzed methods for the roots of the function $f_1(x)$. [Colour figure can be viewed at wileyonlinelibrary.com]

key role in the qualitative properties. In what follows, these schemes are numerically checked on some other nonlinear functions in order to test the applicability of these qualitative results.

## 4 | NUMERICAL EXPERIMENTS

In this section, we compare nine methods of various orders, some of which are derivative free (DF, for short) and other are optimal eighth-order schemes without memory. The methods and their order of convergence are as follows:

1. TM, Traub's DF method (1) of order 1.839 [1] (Method 7a on page 234)
2. JNM, Jarratt–Nudds' DF method (2) of order 1.839 [26]
3. PM, Popovski's DF method (3) of order 1.839 [27]
4. NM, Newton's second order method
5. SM, Steffensen's DF second order method [29]
6. M1, Neta's DF method of order 7.356 [4]
7. M2, Neta's DF method of order 6.219 [3]

**TABLE 1** Average number of function evaluations per point for each example and each of the methods.

| Method | Ex1 | Ex2 | Ex3 | Average |
|---|---|---|---|---|
| TM | 16.11 | 14.31 | 14.94 | 15.12 |
| JNM | 11.62 | 9.48 | 9.76 | 10.29 |
| PM | 14.92 | 12.81 | 13.18 | 13.64 |
| NM | 23.25 | 18.63 | 19.71 | 20.53 |
| SM | 63.90 | 39.36 | 49.17 | 50.81 |
| M1 | 13.67 | 11.84 | 12.20 | 12.57 |
| M2 | 16.72 | 14.38 | 14.98 | 15.36 |
| ZOM | 84.23 | 53.03 | 72.46 | 69.91 |
| SAM | 14.57 | 13.41 | 13.66 | 13.88 |

**TABLE 2** CPU time (ms) for each example and each of the methods.

| Method | Ex1 | Ex2 | Ex3 | Average |
|---|---|---|---|---|
| TM | 1050.964 | 842.572 | 774.008 | 889.181 |
| JNM | 568.898 | 574.472 | 551.134 | 564.835 |
| PM | 717.363 | 656.711 | 665.299 | 679.791 |
| NM | 926.910 | 530.669 | 522.581 | 660.054 |
| SM | 1498.093 | 920.992 | 975.164 | 1131.416 |
| M1 | 582.13 | 532.055 | 582.332 | 565.506 |
| M2 | 726.71 | 354.122 | 685.238 | 655.357 |
| ZOM | 1404.49 | 1110.924 | 1148.6 | 1221.338 |
| SAM | 805.488 | 512.445 | 492.982 | 603.638 |

**TABLE 3** Number of black points for each example and each of the methods and average across examples.

| Method | Ex1 | Ex2 | Ex3 | Average |
|---|---|---|---|---|
| TM | 278 | 11515 | 9022 | 6938 |
| JNM | 0 | 174 | 152 | 109 |
| PM | 2446 | 10625 | 11515 | 8195 |
| NM | 20 | 1742 | 1806 | 1189 |
| SM | 273404 | 140192 | 192616 | 202071 |
| M1 | 0 | 1730 | 1679 | 1136 |
| M2 | 0 | 1600 | 1900 | 1167 |
| ZOM | 166138 | 94779 | 142562 | 134439 |
| SAM | 0 | 1894 | 1827 | 1240 |

8. ZOM, Zhanlav-Otgondroj's DF method of optimal order 8 [30]
9. SAM, Sharma-Arora's method of optimal order 8 [31]

We ran these methods on three examples on a $6 \times 6$ square with center at $(0, 0)$. The functions are as follows:

1. Wilkinson-type polynomial

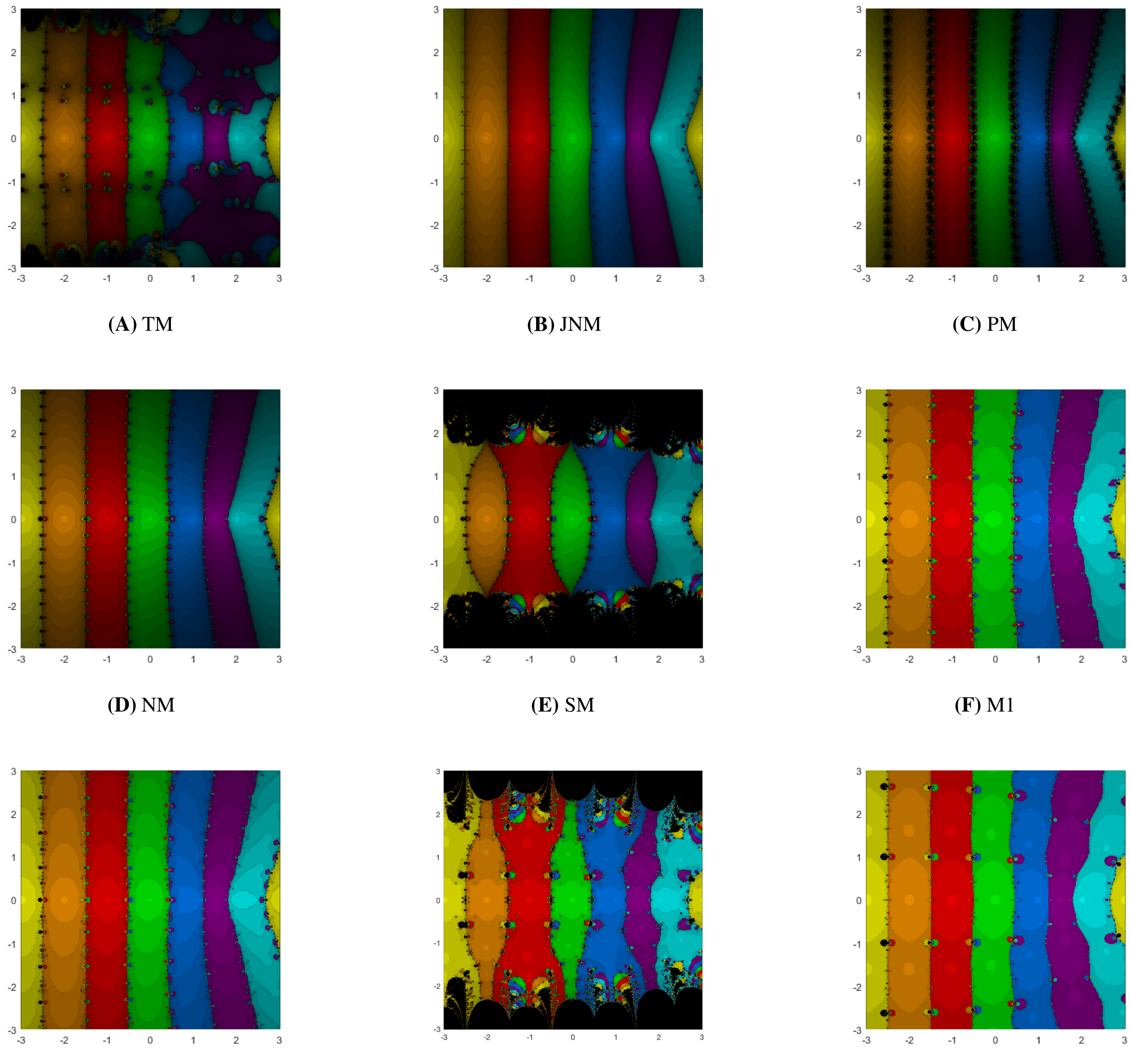
$$f_1(x) = x(x^2 - 1/4)(x^2 - 1)(x^2 - 9/4)(x^2 - 4). \tag{6}$$

2. A function vanishing at $\pm 3, \pm 2, \pm 1, 0, 3/2$ on $[-3, 3]$

$$f_2(x) = \sin(\pi x)\left(e^{x-1.5} - 1\right). \tag{7}$$

3. A function vanishing at $\pm 2.5, \pm 1.5, -1, \pm 1/2$ on $[-3,3]$

$$f_3(x) = \cos(\pi x)\left(e^{x+1} - 1\right). \tag{8}$$



(A) TM (B) JNM (C) PM

(D) NM (E) SM (F) M1

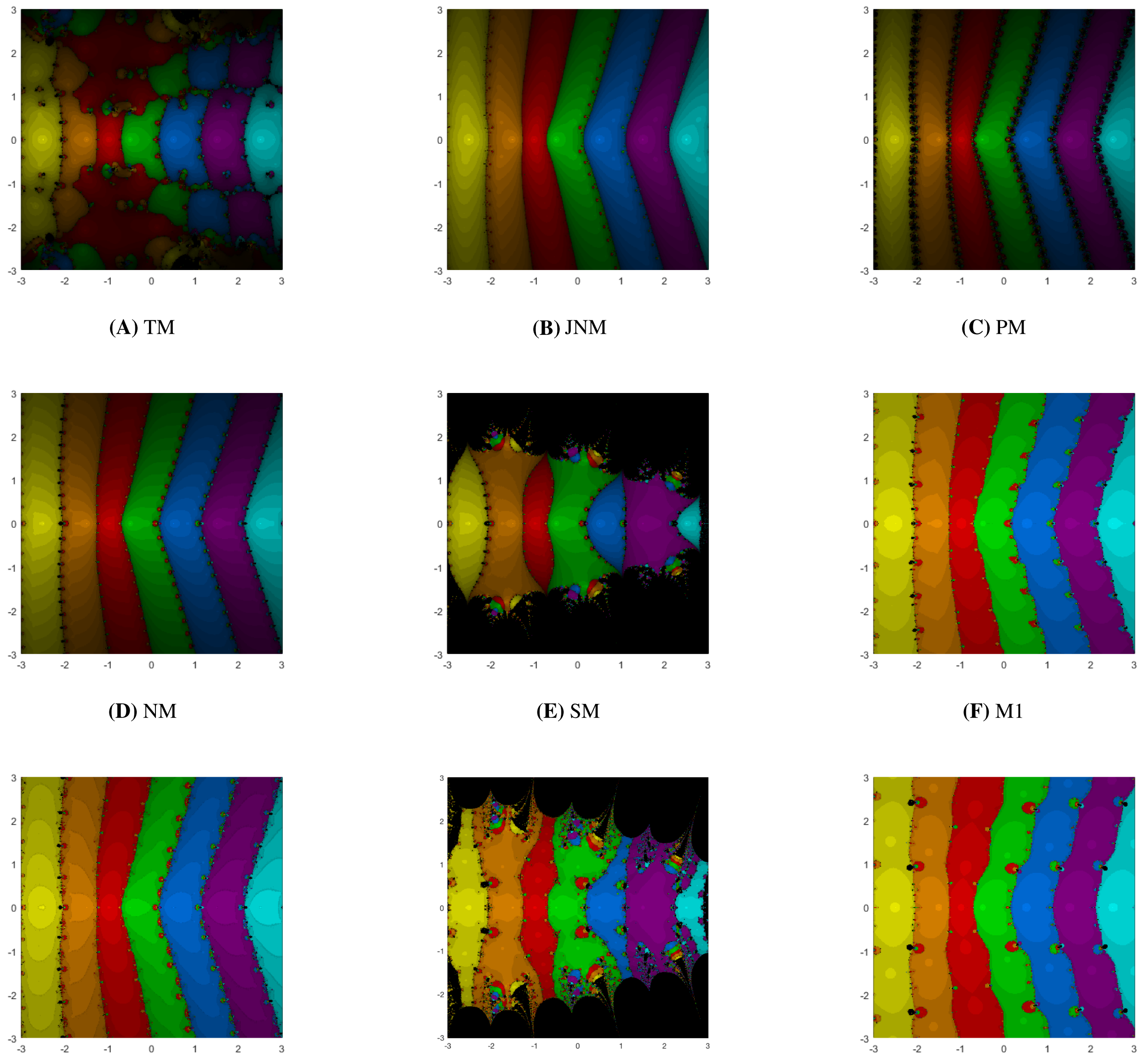
(G) M2 (H) ZOM (I) SAM

**FIGURE 8** Dynamical planes of analyzed methods for the roots of the function $f_2(x)$. [Colour figure can be viewed at wileyonlinelibrary.com]

The square is divided in a mesh of initial points of the complex plane in order to apply on them the iterative procedures. For those methods requiring additional starting values, we have taken $x_{-1} = x_0 + 0.01$ and $x_{-2} = x_0 + 0.02$. Also the number of function evaluations to converge within a tolerance of $10^{-7}$ is collected and the root the sequence has converged to. If the iterates have not converged in 40 iterations, we denote it as a divergent point. Each point is colored by the color corresponding to the root. Note that we have used six different colors; therefore, some roots will have the same color but they are far apart. Moreover, the color is brighter for lower number of iterations needed to converge to the root. A divergent point is colored black. We also have annotated the CPU time needed to run the code on all initial guesses of the mesh using MacBook Pro computer.

In Figure 7, we have depicted the basins of attraction for the nine methods of the first function. It is clear that SM and ZOM (having SM as first step) have too many divergent points. Also, the basins of M1, M2, and SAM are brighter than the rest, showing the fastest convergence. Also, in these cases, the basins of attraction of the roots are similar in terms of width to those of Newton's method.

We have also collected in Tables 1–3 the average number of function-evaluation per point for each scheme, the CPU run time in seconds and the number of divergent points. The methods SM and ZOM use the highest number of function-evaluations per point. Clearly, the CPU runtime for these schemes is the highest since they have the most



(A) TM (B) JNM (C) PM

(D) NM (E) SM (F) M1

(G) M2 (H) ZOM (I) SAM

**FIGURE 9** Dynamical planes of analyzed methods for the roots of the function $f_3(x)$. [Colour figure can be viewed at wileyonlinelibrary.com]

divergent points. The methods M1, M2, JNM, and SAM have no divergent point. TM and NM have very few divergent points.

The basins of attraction for the methods in the second example are given in Figure 8. Again SM and ZOM are inferior. The methods M2, SAM, and M1 are the fastest. Similar results can be observed in Figure 9.

We averaged the numerical results over the three examples, and we can conclude that JNM is the top scheme in all three categories followed by M1. In previous comparison of TM, JNM, and PM using four polynomials of degrees 2–5 and one nonpolynomial function [3], we found that TM was best.

## 5 | CONCLUSIONS

In this manuscript, we have delved deeper in the reasons of the better stability of derivative-free iterative methods with memory based on DF Traub's scheme relative to other methods of the same kind but based on other schemes as first step. The absence of critical points different from the roots in case of Traub's method yields to global convergence on quadratic polynomials, exactly the same performance as Newton's scheme. Other procedures also under analysis show stable behavior, but the complexity of the basins of attraction is much higher. Once Traub's method is selected as the most stable, some schemes constructed with this method as first step are also analyzed with the same dynamical technique, finding only convergence to the roots but global convergence in case of M1. This scheme was designed by using high-order estimations of the derivatives in the iterative expression, versus a simpler construction of the denominators in the design of M2. Therefore, the dynamical analysis has shown that the computational complexity is not a key fact in the stability, even if it is a sufficient element to assure a good performance, when it comes from fine estimations of the derivatives. Numerically, these schemes show to hold this good performance, compared with other schemes with memory, of different orders of convergence. In fact, it shows to be better than optimal iterative procedures without memory, of higher order of convergence. Even in these cases, the performance of M1 shows lower computational time and better efficiency. For further work, we will intend to do a similar study for methods with memory using derivatives.

## AUTHOR CONTRIBUTION

**Alicia Cordero**: Formal analysis; resources; software; writing—review editing. **Beny Neta**: Investigation; software; supervision; writing—original draft. **Juan Ramon Torregrosa**: Formal analysis; investigation; supervision writing—review editing.

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## CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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