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RESEARCH ARTICLE

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Using decomposition of the nonlinear operator for solving non-differentiable problems

Eva G. Villalba[1] | **Miguel Hernandez**[2] | **Jose L. Hueso**[1] | **Eulalia Martínez**[1]

[1]Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia, Spain
[2]Department of Mathematics and Computation, University of La Rioja, Logroño, Spain

Correspondence
Eva G. Villalba, Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia, Spain.
Email: egarvil@posgrado.upv.es

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Starting from the decomposition method for operators, we consider Newton-like iterative processes for approximating solutions of nonlinear operators in Banach spaces. These iterative processes maintain the quadratic convergence of Newton's method. Since the operator decomposition method has its highest degree of application in non-differentiable situations, we construct Newton-type methods using symmetric divided differences, which allow us to improve the accessibility of the methods. Experimentally, by studying the basins of attraction of these methods, we observe an improvement in the accessibility of the derivative-free iterative processes that are normally used in these non-differentiable situations, such as the classic Steffensen's method. In addition, we study both the local and semilocal convergence of the considered Newton-type methods.

KEYWORDS
Kurchatov method, Newton–Kantorovich method, non-differentiable operator, semilocal convergence

MSC CLASSIFICATION
65H10, 65B99

1 | INTRODUCTION

In this paper, we study the problem of approximating a solution of a nonlinear equation

$$H(x) = 0, \tag{1}$$

where $H : \Omega \subseteq X \to X$ is an operator defined on a nonempty convex subset Ω of a Banach space X. If H is a Fréchet differentiable operator, Newton's method [1] is the most used point-to-point iterative process to solve (1), due to its computational efficiency, and it is given by

$$x_{n+1} = x_n - \left[H'(x_n)\right]^{-1} H(x_n), \ \ n \geq 0; \ \ x_0 \in \Omega \ \text{ is given.} \tag{2}$$

In addition, this method has good accessibility, so that the domain of starting points of the method is large, but this method needs the existence of $H'(x)$. For this reason, Newton's method cannot be applied when H is non-differentiable.

In this situation, point-to-point derivative-free iterative processes are usually considered. Our main goal in this work is to consider an iterative process that has the characteristics of Newton's method, for H differentiable, but it is also applicable in situations where the operator H is not differentiable, maintaining, in the non-differentiable situation, the important properties that the considered iterative process verifies in the differentiable case. To achieve this goal, the first step is to approximate the operator H' when the operator H is non-differentiable. It is common to approximate the derivatives by divided differences using a numerical derivation formula, and as a consequence, iterative processes that use divided differences instead of derivatives are obtained. We shall use, as in Ezquerro et al. [2], the following well-known definition for the divided differences of an operator. Let us denote by $\mathcal{L}(X,X)$ the space of bounded linear operators from X to$X$. An operator $[x,y;D] \in \mathcal{L}(X,X)$ is called a first-order divided difference for the operator $D : \Omega \subseteq X \to X$, on the points x and y $(x \neq y)$, if the following equality holds:

$$[x,y;D]\,(x-y) = D(x) - D(y). \tag{3}$$

Kung and Traub [3] presented a class of point-to-point derivative-free iterative processes which are as simple as Newton's method, having quadratic convergence and the same computational efficiency as Newton's method, but in contrast, this class of iterative processes does not use $H'(x)$. These iterative processes contain Steffensen's method as a special case, where the evaluation of $H'(x)$ in each step of Newton's method is approximated by the first-order divided difference $[x, x + H(x); H]$. Steffensen's method has been widely studied [4–6], and their algorithm is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_n + H(x_n); H]^{-1} H(x_n), \ n \geq 0. \end{cases} \tag{4}$$

This method has quadratic convergence and the same computational efficiency as Newton's method.

Other types of approximations to the derivative of the operator have been also considered and Newton-like methods are obtained:

$$x_{n+1} = x_n - [A(x_n)]^{-1} H(x_n), \ n \geq 0, \ x_0 \in \Omega,$$

where $A(x) \in \mathcal{L}(X,X)$ is an approximation of $H'(x)$ for each $x \in X$. The convergence analysis has been given by several authors [1, 7].

Methods using divided differences in their algorithm have a drawback. As we discuss in Section 2, the accessibility of these methods to the solution of the equation is poor, so that the domains of starting points are reduced. In contrast, this is one of the favorable features of Newton's method (2). So, in this work, we try to improve the accessibility of Steffensen's method. We use a new idea that is to perform a decomposition of the operator H that defines the equation $H(x) = 0$. So we consider

$$H(x) = F(x) + G(x),$$

where $F, G : \Omega \subseteq X \to X$ are nonlinear operators, F is differentiable, and G is continuous but non-differentiable.

Then, we consider

$$A(x_n) = F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G]\,, \text{ with } \ \varepsilon > 0,$$

and obtain the following Newton–Steffensen-type iterative processes:

$$\begin{cases} x_0 \in \Omega \text{ and } \varepsilon > 0 \text{ are given}, \\ x_{n+1} = x_n - \left(F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G]\right)^{-1} H(x_n), \ n \geq 0. \end{cases} \tag{5}$$

Notice that if we consider the case in which the operator H is continuous but non-differentiable, such that $H(x) = F(x) + G(x)$, where F is Fréchet differentiable and G is continuous but non-differentiable, there are two advantages of process (5): First, the differentiable part of the operator is considered in the optimal situation, namely, $F'(x_n)$; and second, for the non-differentiable part, the symmetric first-order divided difference $[x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G]$ is considered, which improves the results given by $[x_n, x_n + H(x_n); H]$. Thus, a more suitable situation for $A(x_n)$ is considered than the known ones until now. Notice that if H is Fréchet differentiable and $G = 0$, Newton's method (2) is obtained and if $F = 0$, then we obtain a Steffensen-type method with quadratic convergence [8].

In this paper, we start in Section 2 by seeing that the iterative processes considered improve the accessibility of the classical Steffensen's method, one of the most widely used derivative-free methods. After this motivation of our work, we dedicate Sections 3 and 4 to the study of local and semilocal convergence of methods (5), respectively. Finally, a numerical test is presented where the application of methods (5) is justified.

Throughout the paper, we suppose that there exists a divided difference operator $[x, y; G]$ for each pair of distinct points $x, y \in \Omega$.

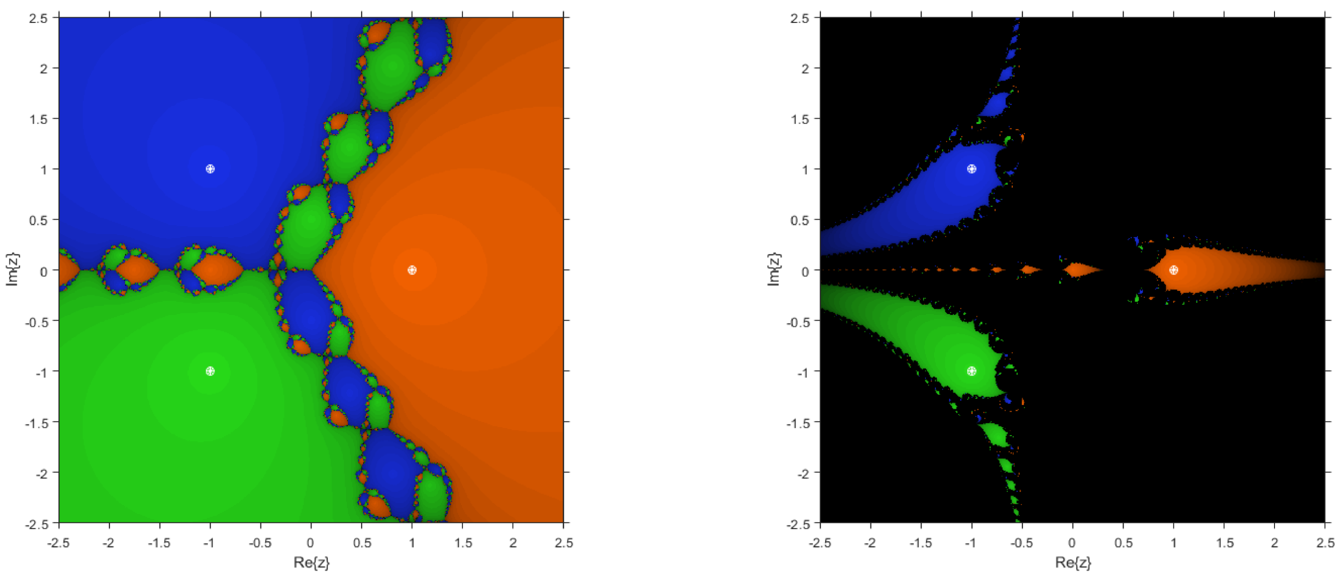
2 | MOTIVATION

As we have already mentioned, Steffensen's method is a version of Newton's method in which the derivative is replaced by a divided difference operator. This method is as simple as Newton's, it has the same efficiency, the quadratic order of convergence that characterizes Newton's method, and it is also applicable to non-differentiable operators. A priori it seems a more useful method than Newton's. However, it has a major flaw, which is that the region of accessibility of this method is considerably reduced with respect to Newton's. Notice that method (5) is the Newton's method when the operator H is differentiable and we consider $G = 0$.

In practice, we estimate the accessibility of a method by depicting its basins of attraction (the set of points in space such that the initial conditions chosen in the dynamic set evolve to a particular attractor [9, 10]) when it is applied to solve a complex equation $H(z) = 0$, where $H : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}$.

In this article, we are going to show the accessibility of Newton's and Steffensen's methods that are generated by solving the complex equation $H(z) = z^3 + z^2 - 2$. We know that the solutions of this equation are $z_1^* = 1$, $z_2^* = -1 + -i$ and $z_3^* = -1 + i$.

We are interested in showing the basins of attraction of the methods for comparison. We will choose the rectangle $D = [-2.5, 2.5] \times [-2.5, 2.5]$ that contains the three solutions. We will take a grid of 1000×1000 points in D that will be the starting points and we will begin to iterate in both methods. The initial points that are in the rectangle can converge to one of the solutions or diverge according to the iterative method we use. For both methods we choose the stopping criterion 10^{-3} and a maximum of 50 iterations. If the tolerance is not less than 10^{-3} in 50 iterations, we will say that the method does not converge starting from that initial point and this point will be colored black. In case it converges, we have assigned a different color depending on which solution converges to. Specifically, we have chosen the orange color for the basins of attraction of the real root $z_1^* = 1$ and the colors green and blue for the basin of attraction of the complex roots z_2^* and z_3^* , respectively. Note that for all the starting points $z_0 \in D$, the real part is located on the abscissa axis and the imaginary part on the ordinate axis.



(A) Newton's method. **(B)** Steffensen's method.

FIGURE 1 Basins of attraction for the polynomial $H(z) = z^3 + z^2 - 2$. [Colour figure can be viewed at wileyonlinelibrary.com]

We observe in the Figure 1 that Steffensen's accessibility is clearly worse than Newton's and therefore of method (5). We can see this in the black areas that are the points of the rectangle that, after 50 iterations, do not match any of the three solutions. However, it is sometimes necessary to use this method without derivatives despite its low accessibility due to the non-applicability of Newton when H is non-differentiable.

If we now apply the method (5) to solve the complex equation $H(z) = z^3 + z^2 - 2$ assuming $F(z) = 0$, $G(z) = z^3 + z^2 - 2$ and $\varepsilon = 1$, we see in Figure 1 that the accessibility improves with respect to Steffensen's method since we use centered divided differences. If we also choose $\varepsilon = 0.001$, the basins of attraction shown in Figure 2 are more similar to those of Newton's method than to those of Steffensen's method even though we consider $F(z) = 0$ and do not use derivatives.

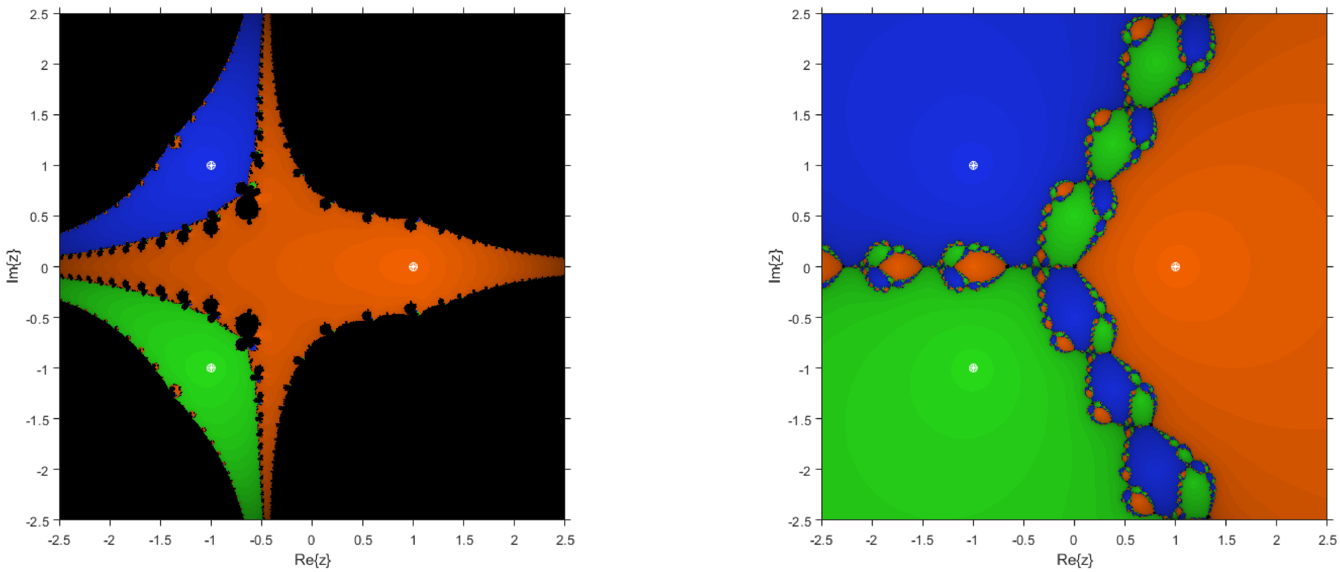
In addition to the dynamic planes, we would like to see numerically that the accessibility of the method (5) is quite close to that of Newton's method. In Table 1, we study the percentage of initial points that converge to some solution. Also in this table we can observe the average number of iterations required for each method to achieve convergence, taking into account that we work with a maximum of 50 iterations.

As we see both in the dynamical planes and in the table, the method (5) greatly improves the accessibility of Steffensen's method and, in turn, is applicable to non-differentiable operators. For this reason, our goal in this article is to decompose the nonlinear operator into the sum of a differentiable and a non-differentiable part, if possible, trying to preserve the good accessibility of Newton's method for the differentiable part of H.

Next, see Figure 3, we are going to see how the accessibility of Steffensen's method improves using Newton–Steffensen's method with advanced divided differences of the paper [11], whose scheme is given by

$$\begin{cases} x_0 \in \Omega \text{ and } \varepsilon > 0 \text{ are given,} \\ x_{n+1} = x_n - \left(F'(x_n) + [x_n x_n + H(x_n); G]\right)^{-1} H(x_n), \ n \geq 0. \end{cases} \tag{6}$$

and the method (5) for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$. We know that the solutions of this equation are $z_1^* = 1$ and $z_2^* = -1$. In the case of Newton–Steffensen-type methods, we decompose $H(z)$ into $F(z) = z^2 - 2$, the differentiable part of the equation, and $G(z) = |z|$, the non-differentiable part.



(A) $\varepsilon = 1$. (B) $\varepsilon = 0.001$.

FIGURE 2 $H(z) = z^3 + z^2 - 2$ for the iterative method (5). [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Comparison of the percentage and the mean of the methods.

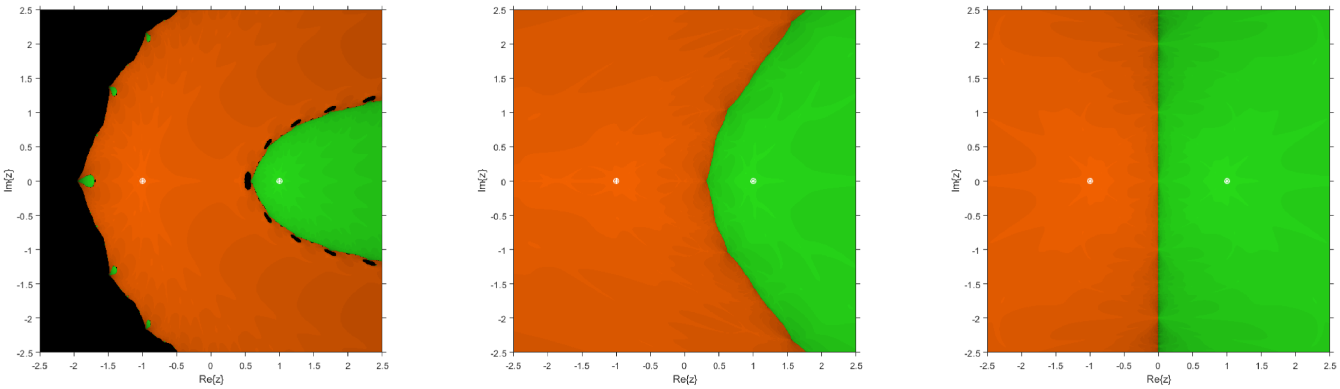
Method	ε	Percentage	Mean
Newton		99.9998%	5.7795
Steffensen		10.72645%	45.6607
Method (5)	1	30.2849%	36.7588
Method (5)	0.001	99.9378%	5.7988

We see how the both methods, Newton–Steffensen's method and (5), improve the accessibility of Steffensen's method and, in turn, are applicable to non-differentiable operators.

Another motivation for this work is to study how the centered divided difference operator used in (5) improves Newton–Steffensen's method, which uses leading divided differences. In the case of these two methods, which of the two has better accessibility is not so reflected, since no non-convergence zones are seen. The only thing that should be noted is that the basins of attraction of the solutions for the method (5) are symmetric while for Newton–Steffensen's method more initial points converge to the solution z_2^* than a z_1^*.

To better see the differences between three methods in the non-differentiable case, we study in Table 2 the percentage of initial points that converge to any of the solutions and the mean of iterations necessary to do so, taking into account that the maximum number iterations is 50.

Finally, see Figure 4, using the same non-differentiable operator, we are going to study the accessibility of method (5) for different values of the ε parameter. To better see the different dynamic planes generated by the different values of ϵ, we are going to study the accessibility of the previous equation for a maximum of 25 iterations. That is, all those points that have not satisfied the 10^{-3} tolerance in less than 25 iterations, will belong to the black zone in each of the dynamic planes.

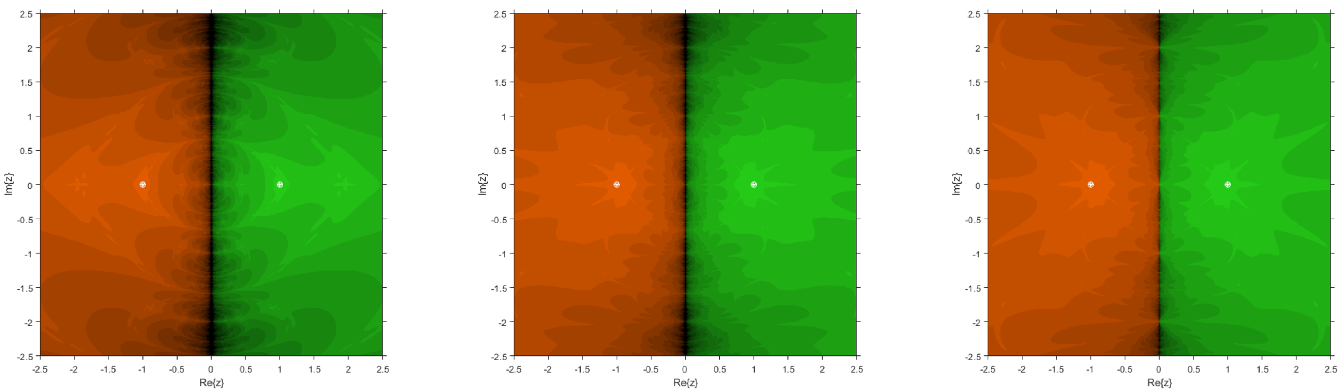


Steffensen's method. Newton-Steffensen's method. Method (5) for $\varepsilon = 0.001$.

FIGURE 3 Basins of attraction for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Comparison of the percentage and the mean of methods.

Method	Percentage	Mean
Steffensen	77.2919%	15.8885
Newton–Steffensen	100%	4.5630
Method (5)	100%	4.5091



$\varepsilon = 10$. $\varepsilon = 1$. $\varepsilon = 0.001$.

FIGURE 4 Basins of attraction for (5), for different values of ε, for the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$. [Colour figure can be viewed at wileyonlinelibrary.com]

Observing Figure 4, we can conclude that, for smaller values of ε, the convergence speed improves considerably since in the cases $\varepsilon = 1$ and $\epsilon = 10$ we can observe more extended black areas, that is, initial points that do not converge to any of the two solutions in 25 iterations.

3 | LOCAL ERROR AND LOCAL CONVERGENCE

In this section, firstly, we study the local order of convergence for the method given by (5), assuming that the nonlinear operator H is differentiable and we can obtain Taylor's expansion in a neighborhood of the solution x^*; thus, we can obtain an expression of the error. Secondly, we pay our attention to the analysis of the local convergence. The study of the local convergence of an iterative process is based on demanding conditions in the solution x^*, from certain conditions on the operator H, and provide the so-called ball of convergence of the iterative process, that shows the accessibility to the solution x^* from the initial approximations belonging to the ball.

3.1 | Local order of convergence

First of all, as usual [12–14], we consider the case in which the operator $H : \Omega \subseteq X \to X$ is sufficiently differentiable and any decomposition is performed $H(x) = F(x) + G(x)$. This will allow us to obtain the error equation for the decomposition method (5) and thus obtain its order of convergence.

From the Genochi–Hermite formula (see Kung and Traub [3]), for the divided difference operator

$$[x+h, x-h; G] = \frac{1}{2}\int_{-1}^{1} G'(x+th)dt, \tag{7}$$

and by the Taylor series expansion of $G'(x+th)$ at the point x and integrating, we obtain the following development:

$$[x+h, x-h; G] = G'(x) + \frac{1}{6}G'''(x)h^2 + O(h^3), \tag{8}$$

which we will use in the proof of the following theorem when we obtain the convergence order of the method.

Theorem 1. *Let $H : \Omega \subseteq X \to X$ such that $H(x) = F(x) + G(x)$, where F and G are sufficiently differentiable operators, and $x^* \in \Omega$ such that $H(x^*) = 0$. If the initial approximation x_0 is chosen sufficiently close to x^*, then the sequence $\{x_n\}$, given by (5), converges to x^* and the local order of convergence of the method (5) is at least 2. Moreover, the error equation is*

$$e_{n+1} = (\Gamma A_2 + \Upsilon B_2)(\Gamma + \Upsilon)^{-1}e_n^2 + O(e_n^3), \tag{9}$$

where $\Gamma = F'(x^*)$, $\Upsilon = G'(x^*)$ and $A_i = \frac{1}{i!}F'(x^*)^{-1}F^{(i)}(x^*)$ and $B_i = \frac{1}{i!}G'(x^*)^{-1}G^{(i)}(x^*)$ such that $A_i, B_i \in \mathcal{L}_i(X,X)$, $i = 1, 2, 3$, where $\mathcal{L}_i(X,X)$ is the space of bounded i-linear symmetric operators.

Proof. We calculate the Taylor series of $F(x_n)$ and $G(x_n)$ around x^*, which is the solution, and obtain

$$F(x_n) = \Gamma\left(A_0 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right),$$
$$G(x_n) = \Upsilon\left(B_0 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + O(e_n^4)\right),$$

where $e_n = x_n - x^*$, $\Gamma = F'(x^*)$, $\Upsilon = G'(x^*)$ and $A_i = \frac{1}{i!}F'(x^*)^{-1}F^{(i)}(x^*)$, $B_i = \frac{1}{i!}G'(x^*)^{-1}G^{(i)}(x^*) \in \mathcal{L}_i(X,X)$. By the way to define A_i and B_i, we have that $A_1 = B_1 = I$ in the expansions of $F(x_n)$ and $G(x_n)$. We calculate the expression of $H(x_n) = F(x_n) + G(x_n)$, using the previous developments. To do this, we are going to take into account that $\Gamma A_0 + \Upsilon B_0 = F(x^*) + G(x^*) = H(x^*) = 0$. We obtain

$$H(x_n) = (\Gamma + \Upsilon)e_n + (\Gamma A_2 + \Upsilon B_2)e_n^2 + (\Gamma A_3 + \Upsilon B_3)e_n^3 + O(e_n^4). \tag{10}$$

Now, deriving the Taylor series of $F(x_n)$ with respect to x_n, we obtain that $F'(x_n)$ is of the form:

$$F'(x_n) = \Gamma\left(I + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + O(e_n^4)\right).$$

Using the divided difference operator seen in (8), we have

$$[x_n + h, x_n - h; G] = G'(x_n) + \frac{1}{6}G'''(x_n)h^2 + O(h^3), \tag{11}$$

where $h = \varepsilon H(x_n)$.

To calculate the above equation, first, we will see what the expressions for $G'(x_n)$ and $G'''(x_n)$ look like. From the Taylor series of $G(x_n)$ around x^*, we obtain

$$G'(x_n) = \Upsilon\left(I + 2B_2e_n + 3B_3e_n^2 + 4B_4e_n^3 + O(e_n^4)\right),$$
$$G'''(x_n) = \Upsilon\left(6B_3 + 24B_4e_n + 60B_5e_n^2 + O(e_n^2)\right).$$

Substituting these developments in (11) and taking into account that $h = \varepsilon H(x_n)$ with $H(x_n)$ developed in (10), we calculate

$$[x_n - h, x_n + h; G] = \Upsilon(I + 2B_2e_n + (3I + \varepsilon^2(\Gamma + \Upsilon)^2)B_3e_n^2 + 2(\varepsilon^2(\Gamma + \Upsilon)(\Gamma A_2 + \Upsilon B_2)B_3 + 2B_4 + 2\varepsilon^2(\Gamma + \Upsilon)^2B_4)e_n^3 + O(e_n^4)).$$

We denote $A(x_n) = F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G]$. Using the Taylor series expansion of $F'(x_n)$ around x^* and the last expression obtained, $A(x_n)$ is expressed as follows:

$$A(x_n) = (\Gamma + \Upsilon) + (2\Gamma A_2 + 2\Upsilon B_2)e_n + (3\Gamma A_3 + \Upsilon(3I + \varepsilon^2(\Gamma + \Upsilon)^2)B_3)e_n^2 + O(e_n^3).$$

Finally, calculating $A(x_n)^{-1} = (F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G])^{-1}$ and substituting this development in the method (5) together with the expansion of $H(x_n)$ from (10), the following error equation is obtained:

$$e_{n+1} = x_{n+1} - x^* = x_n - x^* - [A(x_n)]^{-1}H(x_n) = (\Gamma A_2 + \Upsilon B_2)(\Gamma + \Upsilon)^{-1}e_n^2 + O(e_n^3).$$

□

3.2 | Local convergence and uniqueness of solutions

Second, we are going to obtain a local convergence result applicable to the operator H both when it is differentiable and non-differentiable.

We consider $H(x) = F(x) + G(x)$ where $F, G : \Omega \subseteq X \to X$, F is a differentiable Fréchet operator and G is continuous but not differentiable. Note that if $H(x)$ is differentiable, we consider $H(x) = F(x)$ and we would be studying local convergence for Newton's method. In the opposite case, that is, if $H(x)$ is a non-differentiable operator such that $H(x) = G(x)$, we would be studying the local convergence of Steffensen's method.

To ensure the local convergence of the method (5), we assume conditions on the operators F and G and also on the solution x^* of the equation $H(x) = 0$. Recall that local convergence provides what we call the convergence ball $B(x^*, r)$ where x^* is the center and r is its radius. Depending on how large the radius of convergence is, we can say that the method has more or less accessibility because the larger the ball, the more starting points within it will converge to the solution.

First, we consider the following conditions:

(A) H is μ_0-Lipschitz continuous operator on x^* such that

$$\|H(x) - H(x^*)\| \leq \mu_0(\|x - x^*\|), x \in \Omega, \tag{12}$$

where $\mu_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

(B) F' is μ_1-Lipschitz continuous operator such that

$$\|F'(x) - F'(y)\| \leq \mu_1(\|x - y\|), x, y \in \Omega, \tag{13}$$

where $\mu_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function. We assume that there exists a function $h : [0,1] \to \mathbb{R}^+$, which is continuous and non-decreasing, such that $\mu_1(tz) \leq h(t)\mu_1(z)$, with $t \in [0,1]$ and $z \in [0,+\infty)$. In addition, we define $M = \int_0^1 h(t)dt$.

(C) We suppose that there exists the divided difference operator of the form $[u,v;G]$ for each pair $u,v \in \Omega$, $u \neq v$. Therefore, $[-,-;G]$ is a μ_2-Lipschitz continuous operator such that

$$\|[x,y;G] - [u,v;G]\| \leq \mu_2(\|x-u\|, \|y-v\|), x,y,u,v \in \Omega, \tag{14}$$

where $\mu_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function respect to both arguments.
Note that since G is a non-differentiable operator, $\mu_2(0,0) > 0$. If we consider that $\mu_2(0,0) = 0$, this would imply that G is differentiable as proved in Hernández and Rubio [15] and, therefore, we would not be doing a study of local convergence of the method (5) for any operator $H(x)$, be it differentiable or non-differentiable.

Now, we are going to introduce a lemma to make sure that the sequence of iterations $\{x_n\}$ given by the method (5) is well defined.

Lemma 1. *Under conditions (A), (B), and (C), we assume that* $x_{n-1}, x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ *for* $\varepsilon > 0$, $n \geq 1$, *and the following conditions are verified:*

(D) Let x^* *be the solution of* $H(x) = 0$ *and considering* $\tilde{x} \in \Omega$, *with* $\|\tilde{x} - x^*\| \leq \delta$ *and* $\delta > 0$, *so that* $L^{-1} = (F'(x^*) + [x^*, \tilde{x}; G])^{-1}$ *exists and* $\|L^{-1}\| \leq \gamma$.

(E)

$$\alpha_{n-1} = \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \delta + \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|))) < 1.$$

Then, x_n *is well defined, and it holds that*

$$\|x_n - x^*\| \leq \Gamma_{n-1}\|x_{n-1} - x^*\| \text{ where } \Gamma_{n-1} = \frac{\tilde{\alpha}_{n-1}}{1 - \alpha_{n-1}} \text{ and}$$
$$\tilde{\alpha}_{n-1} = \gamma\left(M\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \varepsilon\mu_0(\|x_{n-1} - x^*\|))\right).$$

Proof. Note that $x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ for $\varepsilon > 0$ and we assume that $x_{n-1} - \varepsilon H(x_{n-1}) \neq x_{n-1} + \varepsilon H(x_{n-1})$. We assume the above because otherwise we would have to $2\varepsilon H(x_{n-1}) = 0$, and since $\varepsilon > 0$, we have that $H(x_{n-1}) = 0$ and so $x_m = x_{n-1} = x^*$, $m \geq n-1$ so that the result would be obtained in a simple way. So if $x_{n-1} - \varepsilon H(x_{n-1}) \neq x_{n-1} + \varepsilon H(x_{n-1})$, the operator exists $[x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]$.

To simplify the proof notation, we define $A(x) = F'(x) + [x - \varepsilon H(x), x + \varepsilon H(x); G]$. Now, taking into account conditions (A) and (B),

$$\begin{aligned}
\|I - L^{-1}A(x_{n-1})\| &= \|I - L^{-1}(F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])\| \\
&\leq \|L^{-1}\|\|L - F'(x_{n-1}) - [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]\| \\
&\leq \|L^{-1}\|(\|F'(x^*) - F'(x_{n-1})\| + \|[x^*, \tilde{x}; G] - [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G]\|) \\
&\leq \gamma\left(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x^* - x_{n-1} + \varepsilon H(x_{n-1})\|, \|\tilde{x} - x_{n-1} - \varepsilon H(x_{n-1})\|)\right) \\
&\leq \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\|H(x_{n-1}) - H(x^*)\|, \|\tilde{x} - x^*\| + \|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\|)) \\
&\leq \gamma(\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|), \delta + \|x_{n-1} - x^*\| + \varepsilon\mu_0(\|x_{n-1} - x^*\|))) \\
&\leq \alpha_{n-1} < 1.
\end{aligned}$$

By Banach's lemma for invertible operators (see Argyros and Ren [16]), there exists $A(x_{n-1})^{-1}$ and

$$\|A(x_{n-1})^{-1}\| \leq \frac{\|L^{-1}\|}{1 - \|L^{-1}A(x_{n-1})\|} \leq \frac{\gamma}{1 - \alpha_{n-1}}.$$

Since method (5) says that

$$x_n = x_{n-1} + (F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])^{-1} H(x_{n-1}),$$

and we have seen that $A(x_{n-1})^{-1}$ exists, so x_n is well defined.

From the method (5), we deduce

$$\begin{aligned} x_{n-1} - x^* &= x_{n-1} - A(x_{n-1})^{-1} H(x_{n-1}) - x^* = A(x_{n-1})^{-1}(A(x_{n-1})(x_{n-1} - x^*) - H(x_{n-1})) \\ &= A(x_{n-1})^{-1}((F'(x_{n-1}) + [x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G])(x_{n-1} - x^*) - F(x_{n-1}) - G(x_{n-1})). \end{aligned}$$

Since $\int_{x_{n-1}}^{x^*} (F'(z) - F'(x_{n-1}))dz = F(x^*) - F(x_{n-1}) - F'(x_{n-1})(x^* - x_{n-1})$, substituting in the previous equation:

$$\begin{aligned} x_{n-1} - x^* &= A(x_{n-1})^{-1} \int_{x_{n-1}}^{x^*} (F'(z) - F'(x_{n-1}))dz \\ &\quad + A(x_{n-1})^{-1}([x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G](x_{n-1} - x^*) + G(x^*) - G(x_{n-1})) \\ &= A(x_{n-1})^{-1} \left(\int_0^1 (F'(x_{n-1} + t(x^* - x_{n-1})) - F'(x_{n-1}))(x^* - x_{n-1}) dt \right) \\ &\quad + A(x_{n-1})^{-1} \left([x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}); G] - [x^*, x_{n-1}; G](x_{n-1} - x^*) \right). \end{aligned}$$

Taking norms in the previous expression and taking into account conditions (A)–(C), we obtain

$$\begin{aligned} \|x_n - x^*\| &\leq \|A(x_{n-1})^{-1}\| \left(\int_0^1 \mu_1(\|t(x^* - x_{n-1})\|)dt + \mu_2(\|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\|, \|\varepsilon H(x_{n-1})\|) \right) |x_{n-1} - x^*\| \\ &\leq \frac{\gamma}{1 - \alpha_{n-1}} (M\mu_1(\|x_{n-1} - x^*\|) + \mu_2(\|x_{n-1} - x^*\| + \varepsilon \mu_0(\|x_{n-1} - x^*\|), \varepsilon \mu_0(\|x_{n-1} - x^*\|))) \|x_{n-1} - x^*\| \\ &= \frac{\tilde{\alpha}_{n-1}}{1 - \alpha_{n-1}} \|x_{n-1} - x^*\| = \Gamma_{n-1} \|x_{n-1} - x^*\|. \end{aligned}$$

□

To prove that $\{x_n\}$ given by the method (5) converges to x^*, we are interested in the conditions under which $\{\|x_n - x^*\|\}_{n=0}^{\infty}$ is a strictly decreasing sequence of positive real numbers. By Lemma 1, $\|x_n - x^*\| < \|x_{n-1} - x^*\|$ if $\Gamma_{n-1} < 1$ and this is true if only if $\tilde{\alpha}_{n-1} + \alpha_{n-1} < 1$.

Thus, if there exists at least a positive real root for the real equation:

$$\gamma\left(\mu_1(t) + \mu_2(t + \varepsilon \mu_0(t), \delta + t + \varepsilon \mu_0(t))\right) + \gamma\left(M\mu_1(t) + \mu_2(t + \varepsilon \mu_0(t), \varepsilon \mu_0(t))\right) - 1 = 0,$$

we denote by r the smallest positive real root.

So for $x_{n-1} \in B(x^*, r)$ and $x_{n-1} - \varepsilon H(x_{n-1}), x_{n-1} + \varepsilon H(x_{n-1}) \in \Omega$ such that

$$\begin{aligned} \|x_{n-1} + \varepsilon H(x_{n-1}) - x^*\| &\leq \|x_{n-1} - x^*\| + \varepsilon \mu_0(\|x_{n-1} - x^*\|) < r + \varepsilon \mu_0(r), \\ \|x_{n-1} - \varepsilon H(x_{n-1}) - x^*\| &\leq \|x_{n-1} - x^*\| + \varepsilon \mu_0(\|x_{n-1} - x^*\|) < r + \varepsilon \mu_0(r), \end{aligned}$$

we get that

$$\alpha_{n-1} < m = \gamma(\mu_1(r) + \mu_2(r + \varepsilon \mu_0(r), \delta + r + \varepsilon \mu_0(r))) < 1,$$

since μ_1 and μ_2 are non-decreasing functions in $\mathbb{R}^+ \times \mathbb{R}^+$. Also,

$$\tilde{\alpha}_{n-1} < \tilde{m} = \gamma(M\mu_1(r) + \mu_2(r + \varepsilon \mu_0(r), \varepsilon \mu_0(r))).$$

Thus, since $m + \tilde{m} = 1$, we have

$$\Gamma_{n-1} < \frac{\tilde{m}}{1-m} = 1,$$

where $\{\Gamma_{n-1}\}$ is a strictly decreasing sequence.

Bearing in mind these new results, we modify condition (E) from the previous study and consider the following condition.

(E′) We assume that the equation

$$\gamma\left[(1+M)\mu_1(t) + \mu_2(t+\varepsilon\mu_0(t), \delta + t + \varepsilon\mu_0(t)) + \mu_2(t+\varepsilon\mu_0(t), \varepsilon\mu_0(t))\right] - 1 = 0 \tag{15}$$

has at least one positive real root and $B(x^*, r + \varepsilon\mu_0(r)) \subset \Omega$, where r is the smallest positive real root of (15).

The following result shows the local convergence results.

Theorem 2. *Under onditions (A)–(D) and (E′), if we choose an initial approximation $x_0 \in B(x^*, r)$, then the sequence $\{x_n\}$ given by the method (5) is well defined, belongs to the ball of convergence $B(x^*, r)$ and converges to the solution x^* from the equation $H(x) = 0$.*

Proof. We have $x_0 + \varepsilon H(x_0) \neq x_0 - \varepsilon H(x_0)$ since otherwise $H(x_0) = 0$ and therefore $x_0 = x^*$ and $x_n = x^*$, $\forall n \leq 1$ and the result would be proved.

From Lemma 1,

$$\begin{aligned}\alpha_0 &= \gamma(\mu_1(\|x_0 - x^*\|) + \mu_2(\|x_0 - x^*\| + \varepsilon\mu_0(\|x_0 - x^*\|), \delta + \|x_0 - x^*\| + \mu_0(x_0 - x^*)))\\ &< \gamma(\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), \delta + r + \varepsilon\mu_0(r))) = m < 1,\end{aligned}$$

we have that there exists $A(x_0)^{-1}$ and $\|A(x_0)^{-1}\| \leq \frac{\gamma}{1-\alpha_0}$. Hence, x_1 is well defined and

$$\|x_1 - x^*\| \leq \Gamma_0\|x_0 - x^*\| < \|x_0 - x^*\| < r.$$

Therefore $x_1 \in B(x^*, r)$. On the other hand,

$$\begin{aligned}\|x_1 + \varepsilon H(x_1) - x^*\| &\leq \|x_1 - x^*\| + \varepsilon\|H(x_1)\| \leq \|x_1 - x^*\| + \varepsilon\mu_0(\|x_1 - x^*\|)\\ &< \|x_0 - x^*\| + \varepsilon\mu_0(\|x_0 - x^*\|) < r + \varepsilon\mu_0(r),\\ \|x_1 - \varepsilon H(x_1) - x^*\| &\leq \|x_1 - x^*\| + \varepsilon\mu_0(\|x_1 - x^*\|) < r + \varepsilon\mu_0(r).\end{aligned}$$

So $x_1 + \varepsilon H(x_1), x_1 - \varepsilon H(x_1) \in \Omega$.

By induction on $n \geq 2$, we prove that, if $x_{n-1} \in B(x^*, r)$ and $x_{n-1} + \varepsilon H(x_{n-1}), x_{n-1} - \varepsilon H(x_{n-1}) \in \Omega$, with $x_{n-1} + \varepsilon H(x_{n-1}) \neq x_{n-1} - \varepsilon H(x_{n-1})$, then x_n is well defined,

$$\|x_n - x^*\| < \|x_{n-1} - x^*\|, \|x_n + \varepsilon H(x_n) - x^*\| < r + \varepsilon\mu_0(r) \text{and} \|x_n - \varepsilon H(x_n) - x^*\| < r + \varepsilon\mu_0(r).$$

Assuming that the hypotheses are true for $n = 2, \ldots, k$, let's see that it is true for $n = k+1$.

Since $x_k \in B(x^*, r)$ and $x_k + \varepsilon H(x_k), x_k - \varepsilon H(x_k) \in \Omega$, by Lemma 1, there exists $A(x_k)^{-1}$. This implies that x_{k+1} is well defined and so

$$\|x_{k+1} - x^*\| \leq \Gamma_k\|x_k - x^*\| < \|x_k - x^*\| < \ldots < \|x_0 - x^*\| < r.$$

Moreover,

$$\|x_{k+1} + \varepsilon H(x_{k+1}) - x^*\| \leq \|x_{k+1} - x^*\| + \varepsilon \mu_0(\|x_{k+1} - x^*\|) < r + \varepsilon \mu_0(r),$$
$$\|x_{k+1} - \varepsilon H(x_{k+1}) - x^*\| \leq \|x_{k+1} - x^*\| + \varepsilon \mu_0(\|x_{k+1} - x^*\|) < r + \varepsilon \mu_0(r).$$

Then, $\{x_n\} \subset B(x^*, r)$ and $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive real numbers. Since $\{Gamma_{n-1}\}$ is also strictly decreasing, we have that

$$\|x_n - x^*\| < \Gamma_{n-1}\|x_{n-1} - x^*\| < \ldots < \Gamma_0^n \|x_0 - x^*\|.$$

Therefore, when $n \to \infty$, $\{x_n - x^*\} \to 0$ is satisfied, and therefore, $\{x_n\}$ converges to x^*. □

Now, we are going to present a result that ensures uniqueness of solution under certain conditions and allows us to define the uniqueness ball.

Theorem 3. *Under conditions (A)–(D) and (E′), we assume that the equation*

$$E_1(t) = \gamma(M\mu_1(t) + \mu_2(0, t + \delta)) - 1 = 0 \tag{16}$$

has at least one positive real root, where R is the smallest positive real root of (16). Then, the solution x^* is the only solution of the equation $H(x) = 0$ in $\overline{B(x^*, R)} \cap \Omega$.

Proof. Let $y^* \in \overline{B(x^*, R)} \cap \Omega$ and $H(y^*) = 0$. We define the following operator:

$$P = \int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G].$$

Using (A) and (B), we get

$$\begin{aligned}
\|L^{-1}P - I\| &\leq \|L^{-1}\|\|P - L\| \\
&\leq \|L^{-1}\| \left[\int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x^*)\|dt + \|[x^*, y^*; G] - [x^*, \tilde{x}; G]\| \right] \\
&\leq \gamma \left[\int_0^1 \mu_1(\|t(y^* - x^*)\|)dt + \mu_2(0, \|y^* - \tilde{x}\|) \right] \\
&< \gamma(M\mu_1(R) + \mu_2(0, R + \delta)) = 1.
\end{aligned}$$

In the last inequality, we have used the hypothesis that there is at least one positive real root of $E_1(t)$ since this implies that at least one of the functions $\mu_1(t)$ or $\mu_2(t)$ is strictly increasing.

Therefore, there exists $P^{-1} \in \mathcal{L}(X, Y)$, and for the identity

$$\begin{aligned}
0 &= H(x^*) - H(y^*) = F(x^*) - F(y^*) + G(x^*) - G(y^*) \\
&= \left(\int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G] \right)(x^* - y^*) \\
&= P(x^* - y^*),
\end{aligned}$$

we deduce that $x^* = y^*$. □

4 | SEMILOCAL CONVERGENCE

In this section, we are going to analyze the semilocal convergence of iterative method (5), that is, we are going to impose the necessary conditions so that, given a starting point x_0, the iterative method (5) converges to a solution. Moreover, we determine the existence and uniqueness convergence domains.

For this, we consider $x_0 \in \Omega$, and we assume:

(I) Exists $A_0^{-1} = A(x_0)^{-1} = (F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0)])^{-1}$ such that $\|A_0^{-1}\| \leq \beta$. Furthermore, $\|H(x_0)\| \leq \eta_0$ such that $\|A_0^{-1} H(x_0)\| \leq \beta\eta_0 = \omega$.

(II) $\|H(x) - H(y)\| \leq \mu_0(\|x - y\|)$, $x, y \in \Omega$ where $\mu_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function.

(III) $\|F'(x) - F'(y)\| \leq \mu_1(\|x - y\|)$, $x, y \in \Omega$ where $\mu_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function. We assume that exists a continuous non-decreasing function $h : [0, 1] \to \mathbb{R}^+$ such that $\mu_1(tz) \leq h(t)\mu_1(z)$ with $z \in [0, 1]$ and $t \in [0, \infty)$. We denote $M = \int_0^1 h(t)dt$.
Note that h always exists, by taking $h(t) = 1$, as a consequence of μ_1 is a non-decreasing function.

(IV) We suppose that exists $[x, y; G]$ for each pair $x, y \in \Omega$, $x \neq y$ such that the divided differences operator satisfies:

$$\|[x, y; G] - [u, v; G]\| \leq \mu_2(\|x - u\|, \|y - v\|); \text{for all} x, y, u, v \in \Omega,$$

where $\mu_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function with respect to both arguments.

Note that since G is non-differentiable, $\mu_2(0, 0) > 0$. Then, we will give a generalized result of semilocal convergence. We will do it by fixing the radius r of the domain of existence, and we will try to calculate it so that the sequence $\{x_n\}$ is contained in the ball of convergence $B(x_0, r)$ whose center is the initial iteration and whose radius is our fixed r.

Theorem 4. *Under conditions (I)–(IV), we consider the following parameter* $s = \mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega)$, *and the functions*

$$m(t) = \beta(\mu_1(t) + \mu_2(t + \varepsilon\mu_0(t), t + \varepsilon\mu_0(t))) \text{and } \lambda(t) = \frac{\beta s}{1 - m(t)},$$

with $t \in \mathbb{R}^+$. *Let us assume that the following equation has at least one positive real root:*

$$\frac{1}{1 - \lambda(t)}\omega - t = 0, \tag{17}$$

by denoting r the smallest one and if it is verified that $s\beta < 1$, $m(r) < 1$ *and* $\lambda(r) < 1$. *Then, the sequence* $\{x_n\}$ *given by the iterative method (5) is well defined, remains in* $B(x_0, r)$ *and converges to the solution* x^* *of the equation* $H(x) = 0$.

Proof. To simplify the notation, we denote

$$A_n = A(x_n) = F'(x_n) + [x_n - \varepsilon H(x_n), x_n + \varepsilon H(x_n); G].$$

First, we prove by induction that the sequence given by the method (5) is well defined, that is, in each step $n \geq 1$, the operator A_n is invertible and the iteration x_{n+1} can be obtained.

We start from the fact that x_1 is well defined since by condition (I), we have

$$\|x_1 - x_0\| = \|A_0^{-1} H(x_0)\| \leq \omega < r.$$

Considering conditions (I)–(IV) and the auxiliary function implied, we have

$$\begin{aligned}
\|I - A_0^{-1}A_1\| &\leq \|A_0^{-1}\| \, \|A_0 - A_1\| \\
&\leq \|A_0^{-1}\| \, \|F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G] - F'(x_1) - [x_1 + \varepsilon H(x_1), x_1 - \varepsilon H(x_1)]\| \\
&\leq \beta\left(\mu_1(\|x_1 - x_0\|) + \mu_2(\|x_1 - x_0\| + \varepsilon\mu_0(\|x_1 - x_0\|), \|x_1 - x_0\| + \varepsilon\mu_0(\|x_1 - x_0\|))\right) \\
&\leq \beta\left(\mu_1(r) + \mu_2(r + \varepsilon\mu_0(r), r + \varepsilon\mu_0(r))\right) = m(r) < 1.
\end{aligned}$$

So by applying Banach's lemma, there exists A_1^{-1} and

$$\|A_1^{-1}\| \leq \frac{\beta}{1-m(r)}.$$

Since F is Fréchet differentiable, one has

$$F(x_1) = F(x_0) + \int_0^1 F'(x_0 + t(x_1 - x_0))(x_1 - x_0)dt$$
$$= F(x_0) + F'(x_0)(x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt.$$

On the other hand, for the operator G, we use the definition of divided difference:

$$G(x_1) = G(x_0) - [x_0, x_1; G](x_0 - x_1).$$

Therefore, since $H(x) = F(x) + G(x)$, we have

$$H(x_1) = H(x_0) + F'(x_0)(x_1 - x_0) + [x_0, x_1; G](x_1 - x_0) + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt.$$

So by substituting $H(x_0)$ by its obtained expression with the function iteration of (5), we obtain

$$\begin{aligned} H(x_1) = & -[F'(x_0) + [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G]](x_1 - x_0) \\ & + F'(x_0)(x_1 - x_0) + [x_0, x_1; G](x_1 - x_0) \\ & + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt \\ = & ([x_0, x_1; G] - [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G])(x_1 - x_0) \\ & + \int_0^1 (F'(x_0 + t(x_1 - x_0)) - F'(x_0))(x_1 - x_0)dt. \end{aligned}$$

Applying norms to the above expression, one gets

$$\begin{aligned} \|H(x_1)\| & \leq (\mu_2(\varepsilon\eta_0, \|x_1 - x_0\| + \varepsilon\eta_0) + \mu_1(\|x_1 - x_0\|) \int_0^1 h(t)dt)\|x_1 - x_0\| \\ & \leq (\mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega))\|x_1 - x_0\| \leq s\|x_1 - x_0\| < s\beta\eta_0 < \eta_0. \end{aligned}$$

Therefore, we can say that x_2 is well defined thanks to the existence of A_1^{-1} and verifies

$$\begin{aligned} \|x_2 - x_1\| & \leq \|A_1^{-1} H(x_1)\| \\ & \leq \frac{\beta}{1-m} s\|x_1 - x_0\| \\ & = \frac{\beta s}{1-m(r)}\|x_1 - x_0\| = \lambda(r)\|x_1 - x_0\| < \omega. \end{aligned}$$

So

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (\lambda(r)+1)\|x_1 - x_0\| \leq (\lambda(r)+1)\omega < r,$$

and $x_2 \in B(x_0, r)$.

By mathematical induction, we assume that for $k = 2, \dots, n-1$, the following are verified:

1. There exists $A_{k-1}^{-1} = \left(F'(x_k) + [x_k + \epsilon H(x_k), x_k - \epsilon H(x_k); G]\right)^{-1}$ such that $\|A_{k-1}^{-1}\| \leq \frac{\beta}{1-m(r)}$.
2. $\|x_k - x_{k-1}\| \leq \lambda(r)\|x_{k-1} - x_{k-2}\| \leq \lambda(r)^{k-1}\|x_1 - x_0\| < \omega$.
3. $\|H(x_k)\| \leq s\|x_k - x_{k-1}\| < \eta_0$.
4. $\|x_k - x_0\| \leq \frac{1-\lambda(r)^k}{1-\lambda(r)}\omega < r$ and $x_k \in B(x_0, r)$.

So we are going to see that (1)–(4) are true for $k = n$.

First, let us see that there exists A_{n-1}^{-1}.

$$\begin{aligned}\|I - A_0^{-1}A_{n-1}\| &\leq \beta\|F'(x_0) + [x_0 + \epsilon H(x_0), x_0 - \epsilon H(x_0)] - F'(x_{n-1}) - [x_{n-1} + \epsilon H(x_{n-1}), x_{n-1} - \epsilon H(x_{n-1}); G]\| \\ &\leq \beta\left(\mu_1(r) + \mu_2(r + \epsilon\mu_0(r), r + \epsilon\mu_0(r))\right) = m(r) < 1.\end{aligned}$$

So by Banach's lemma, there exists A_{n-1}^{-1} and

$$\|A_{n-1}^{-1}\| \leq \frac{\beta}{1-m(r)}.$$

Then, we have that x_n is well defined, and furthermore, (3) holds

$$\|x_n - x_{n-1}\| \leq \|A_{n-1}^{-1}H(x_{n-1})\| \leq \frac{\beta}{1-m(r)}s\|x_{n-1} - x_{n-2}\| \leq \lambda(r)^{n-1}\omega \leq \omega < r.$$

Also, thanks to this bound, we can ensure that $x_n \in B(x_0, r)$ since

$$\begin{aligned}\|x_n - x_0\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_0\| \\ &\leq \lambda(r)^{n-1}\omega + \frac{1-\lambda(r)^{n-1}}{1-\lambda(r)}\omega \leq \frac{1-\lambda(r)^n}{1-\lambda(r)}\omega < \frac{1}{1-\lambda(r)}\omega = r.\end{aligned}$$

On the other hand,

$$\begin{aligned}H(x_n) =& -[F'(x_{n-1}) + [x_{n-1} + \epsilon H(x_{n-1}), x_{n-1} - \epsilon H(x_{n-1}); G]](x_n - x_{n-1}) \\ &+ F'(x_{n-1})(x_n - x_{n-1}) + [x_{n-1}, x_n; G](x_n - x_{n-1}) \\ &+ \int_0^1 (F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1}))(x_n - x_{n-1})dt \\ =& ([x_{n-1}, x_n; G] - [x_{n-1} + \epsilon H(x_{n-1}), x_{n-1} - \epsilon H(x_{n-1}); G])(x_n - x_{n-1}) \\ &+ \int_0^1 (F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1}))(x_n - x_{n-1})dt.\end{aligned}$$

Applying norms to the above expression, one gets

$$\begin{aligned}\|H(x_n)\| &\leq (\mu_2(\epsilon\eta_0, \|x_n - x_{n-1}\| + \epsilon\eta_0) + \mu_1(\|x_n - x_{n-1}\|)\int_0^1 h(t)dt)\|x_n - x_{n-1}\| \\ &\leq (\mu_2(\epsilon\eta_0, \omega + \epsilon\eta_0) + M\mu_1(\omega))\|x_n - x_{n-1}\| \leq s\|x_n - x_{n-1}\| < s\lambda(r)^{n-1}\|x_1 - x_0\| \\ &\leq s\lambda(r)^{n-1}\beta\eta_0 \leq \eta_0.\end{aligned}$$

Therefore, since (1)–(4) holds for $k = n$, we have shown that these hypotheses are true $\forall n \in \mathbb{N}$.

Now, using these hypotheses, we are going to prove that the sequence of iterations $\{x_n\}$ is a Cauchy sequence. For $k \geq 1$,

$$\begin{aligned}\|x_{n+k} - x_{n-1}\| &\leq \|x_{n+k} - x_{n+k-1}\| + \ldots + \|x_n - x_{n-1}\| \\ &\leq \left(\lambda(r)^k + \lambda(r)^{k-1} + \ldots + 1\right) \|x_n - x_{n-1}\| \\ &\leq \frac{1-\lambda(r)^{k+1}}{1-\lambda(r)}\|x_n - x_{n-1}\| \leq \frac{1-\lambda(r)^k}{1-\lambda(r)}\lambda(r)^{n-1}\|x_1 - x_0\| \\ &< \frac{1}{1-\lambda(r)}\lambda(r)^{n-1}\|x_1 - x_0\|.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in B(x_0, r)$ where

$$r = \frac{1}{1-\lambda(r)}\omega.$$

Finally, we see that x^* is a zero of $H(x)$. Since

$$\|H(x_n)\| \leq (\mu_2(\varepsilon\eta_0, \omega + \varepsilon\eta_0) + M\mu_1(\omega))\|x_n - x_{n-1}\|.$$

Moreover, by taking limits when $n \to \infty$, as $\|x_n - x_{n-1}\| \to 0$, thus, by the continuity of the operator H, $H(x^*) = 0$. □

Theorem 5. *Under conditions (I)–(IV), we suppose that the equation*

$$E_2(t) = \beta(\mu_1(r+t) + \mu_2(r + \varepsilon\mu_0(t), r + \varepsilon\mu_0(t))) - 1 = 0 \tag{18}$$

has at least one positive real root, we denote R the smallest one. Then, the solution x^* is the unique solution of the equation $H(x) = 0$ in $\overline{B(x_0, R)} \cap \Omega$.

Proof. To see uniqueness, we assume that there is another solution $y^* \in \overline{B(x_0, R)} \cap \Omega$ and consider the operator

$$P = \int_0^1 F'(x^* + t(y^* - x^*))dt + [x^*, y^*; G].$$

Since $P(y^* - x^*) = H(y^*) - H(x^*) = 0$, if P is an invertible operator then $x^* = y^*$. So in order to apply Banach's lemma, we have

$$\begin{aligned}\|A_0^{-1}P - I\| &\leq \beta\left(\int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|dt + \|[x^*, y^*; G] - [x_0 + \varepsilon H(x_0), x_0 - \varepsilon H(x_0); G]\|\right) \\ &\leq \beta\left(\int_0^1 (\|x^* - x_0 + ty^* - tx^*)\|)dt + \mu_2(\|x^* - x_0\| + \varepsilon\|H(x_0)\|, \|y^* - x_0\| + \varepsilon\|H(x0)\|)\right) \\ &\leq \beta\left(\int_0^1 \mu_1(\|(1-t)(x^* - x_0) + t(y^* - x_0)\|)dt + \mu_2(\|x^* - x_0\| + \varepsilon\mu_0(\|x^* - x_0\|), \|y^* - x_0\| + \varepsilon\mu_0(\|y^* - x_0\|))\right) \\ &< \beta(\mu_1(r+R) + \mu_2(r + \varepsilon\mu_0(R), r + \varepsilon\mu_0(R))) = 1.\end{aligned}$$

In the last inequality, we have used the hypothesis that there is at least one positive real root of $E_2(t)$ since this implies that at least one of the functions $\mu_1(t)$ or $\mu_2(t)$ is strictly increasing.

Therefore, P^{-1} exists, and the result holds. □

5 | NUMERICAL EXPERIENCE

In this section, we deal with the application of the previously obtained local and semilocal convergence results. Moreover, we perform a numerical comparison with other existing iterative methods.

5.1 | Local and semilocal convergence balls

We consider the nonlinear elliptic boundary value problem solved in Ezquerro and Hernández-Verón [17] that describes properties in the gas dynamic theory. But now, we add a non-differentiable part in order to illustrate the theory developed in our study.

$$\begin{aligned} &u_{ss} + u_{tt} = u^3 + |u| \text{ with } S = \{(s,t) \in \mathbb{R}^2 : s,t \in [0,1]\},\\ &u(s,0) = 2s^2 - s + 1, u(s,1) = 2, 0 \le s \le 1,\\ &u(0,t) = 2t^2 - t + 1, u(1,t) = 2, 0 \le t \le 1. \end{aligned}$$

We create a mesh for discretizing the problem being: $h = \frac{1}{n+1}$ and $\tau = \frac{1}{m+1}$, we have the mesh points (s_i, t_j) with $s_i = ih$, $i = 0, \dots, n+1$ and $t_j = j\tau$, $j = 0, \dots, m+1$, such that:

$$u_{ss}(s_i,t_j) + u_{tt}(s_i,t_j) = u(s_i,t_j)^3 + |u((s_i,t_j)|.$$

So by approximating the partial derivatives by central divided differences, we have the following equation:

$$\frac{u(s_{i+1},t_j) - 2u(s_i,t_j) + u(s_{i-1},t_j)}{h^2} + \frac{u(s_i,t_{j-1}) - 2u(s_i,t_j) + u(s_i,t_{j+1})}{\tau^2} = u(s_i,t_j)^3 + |u(s_i,t_j)|,$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$.

The boundary conditions are

$$\begin{aligned} &u(s_i,t_0) = 2s_i^2 - s_i + 1, u(s_i,t_{q+1}) = 2, i = 1, \dots, n,\\ &u(s_0,t_j) = 2t_j^2 - t_j + 1, u(s_{p+1},t_j) = 2, j = 0, \dots, m+1. \end{aligned}$$

Now, we denote $u(s_i,t_j) = u_{i,j}$ simplifying the notation we obtain:

$$2\left(1 + \left(\frac{h}{\tau}\right)^2\right)u_{i,j} - (u_{i-1} + u_{i+1,j}) - \left(\frac{h}{\tau}\right)^2(u_{i,j-1} + u_{i,j+1}) = -h^2(u_{ij} + |u_{ij}|), \tag{19}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$ with

$$\begin{aligned} &u_{i,0} = 2s_i^2 - s_i + 1, u_{i,q+1} = 2, i = 1, \dots, n,\\ &u_{0,j} = 2t_j^2 - t_j + 1, u_{p+1,j} = 2, j = 0, \dots, m+1. \end{aligned}$$

Equation (19) together with the boundary conditions form a nonlinear system of size $(nm) \times (nm)$ given by

$$Tu + h^2\nu(u) = \omega,$$

where

$$T = \begin{pmatrix} A & B & 0 & \dots & 0\\ B & A & B & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & B\\ 0 & \dots & 0 & B & A \end{pmatrix}, A = \begin{pmatrix} 2(1+(\frac{h}{\tau})^2) & -1 & 0 & \dots & 0\\ -1 & 2(1+(\frac{h}{\tau})^2) & -1 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \dots & 0 & -1 & 2(1+(\frac{h}{\tau})^2) \end{pmatrix}.$$

$B = -(\frac{h}{\tau})^2 I_{n\times n}$, $u = (u_1, \dots, u_d)^T$, $v(u) = (u_1^3, \dots, u_d^3)^T + (|u_1|, \dots, |u_d|)^T$ being $d = nm$, and W is a vector formed by the boundary conditions. Then, the nonlinear system can be formulated as follows:

$$H(u) = Tu + h^2 v(u) - W = 0, \tag{20}$$

being its differentiable part $F(u) = T(u_1, \dots, u_d) + h^2(u_1^3, \dots, u_d^3)^T - W$, and the non-differentiable part $G(u) = h^2(|u_1|, \dots, |u_d|)^T$, such that $F(u) + G(u) = 0$.

We calculate the linear operator $F'(u) = T + 3h^2 \text{diag}[u_1^2, \dots, u_d^2]$. Moreover, we characterize the divided differences considering $[u, v; G] = ([u, v; G]_{ij})_{i,j=1}^d \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ where

$$[u, v; G]_{ij} = \frac{1}{u_j - v_j}(G_i(u_1, \dots, u_j, v_{j+1}, \dots, v_d) - G_i(u_1, \dots, u_{j-1}, v_j, \dots, v_d)),$$

and $u = (u_1, \dots, u_d)^T$, $v = (v_1, \dots, v_d)^T$, then

$$[u, v; G] = h^2 \cdot diag\left(\frac{|u_1| - |v_1|}{u_1 - v_1}, \dots, \frac{|u_d| - |v_d|}{u_d - v_d}\right).$$

To solve the problem (20), we approximate the exact solution u^* with starting point $u^{(0)} = (1, 1, \dots, 1)$, denoting the iteration ith by the $u^{(i)} = (u_1^{(i)}, \dots, u_d^{(i)}$ when running the iterative scheme (5) in Matlab2020 by using variable precision arithmetic with 100 digits, using as stopping criteria $\|u^{(n+1)} - u^{(n)}\| < 10^{-30}$ and with $n = m = 4$ and $\varepsilon = 1$. The solution obtained after six iterations verifies $\|u^{(6)} - u^{(5)}\| < 5.2930\text{e-}37$ and the norm of nonlinear operator H at this point verifies $\|H(u^{(6)})\| \leq 3.3391\text{e-}75$, reshaping the approximated solution in $\mathbb{R}^{16}$ to the initial size 4×4 for $i, j = 1, \dots, 4$ is

$$u^{(6)}(s_i, t_j) = \begin{pmatrix} 0.92416 & 0.98713 & 1.11328 & 1.39532 \\ 1.01807 & 1.10905 & 1.25039 & 1.51249 \\ 1.20200 & 1.27954 & 1.39495 & 1.60315 \\ 1.50792 & 1.54714 & 1.61111 & 1.73408 \end{pmatrix}.$$

Finally, we change the starting point to $u^{(0)} = (1.5, 1.5, \dots, 1.5)$ noting that the iterative method converges to the same solution; now, we have $\|u^{(6)} - u^{(5)}\| < 1.4351\text{e-}41$, and the norm of nonlinear operator H at this point verifies $\|H(u^{(6)})\| \leq 2.1086\text{e-}84$.

So we use this approximated solution to establish that if there exists the exact solution u^* then $u^* \in B(\bar{1}, l)$, and so we take $\Omega = B(\bar{1}, l)$ with $l \geq 1$ and $\bar{1} = (1, 1, \dots, 1)$. Therefore, we can obtain the bounds for the conditions assumed in the semilocal convergence study and establishing the auxiliary functions μ_0, μ_1 and μ_2 involved in conditions $(I) - (IV)$, we observe that

$$\begin{aligned} \|H(u) - H(v)\| &= \|F(u) + G(u) - F(v) - G(v)\| \\ &\leq \|F(u) - F(v)\| + \|G(u) - G(v)\|. \end{aligned}$$

But taking into account that F is differentiable, we have, by applying the mean value theorem, that

$$\|F(u) - F(v)\| \leq F'(\tilde{u})\|u - v\| \text{ with } \tilde{u} \in B(\bar{1}, l),$$

so

$$\begin{aligned} \|F(u) - F(v)\| &\leq (\|T\| + 3h^2\| \text{ diag } (\tilde{u}_1^2, \dots, \tilde{u}_d^2))\| \|u - v\| \\ &\leq (\|T\| + 3h^2(1 + l)^2)\|u - v\|. \end{aligned}$$

On the other hand, by using the definition of the divided differences, we have

$$\|G(u) - G(v)\| \leq \|[u, v, G]\| \|u - v\|.$$

Therefore, we define $\mu_0(t) = (\|T\| + 3h^2(1 + l)^2 + h^2)t$. From the expression of $F'(u)$ deduced before we get $\mu_1(t) = 6h^2(1+l)t$, and using the characterization of divided differences, we define $\mu_2(s, t) = 2h^2$.

By using these auxiliary functions, we focus on applying the semilocal convergence results obtained previously, by taking the initial approximation

$$u^{(0)} = (1, 1, 1.2, 1.6, 1.1, 1.2, 1.4, 1.6, 1.2, 1.4, 1.5, 1.7, 1.4, 1.6, 1.6, 1.8)^T,$$

we obtain

$$\beta = 0.96767, \eta_0 = 0.34784, \omega = 0.33659,$$

and solving Equations 17 and (18), we obtain the existence and uniqueness radius, r and R, involved in our semilocal convergence study. We consider different values for l, while $\varepsilon = 0.1$ in scheme (5). In Table 3, we can see the radii obtained r and R while in the last two columns we check the conditions $m(r) < 1$ and $\lambda(r) < 1$. We obtain better results when l is smaller because in this case, the bounds used are more accurate.

Finally, once the existence of an exact solution has been proved, u^*, we focus on the local convergence study also in $\Omega = B(\bar{1}, l)$ with $l \geq 1$, so we set in condition (D) that $\|\tilde{u} - u^*\| \leq \delta < l$ obtaining parameter γ, the convergence and uniqueness radius, r and R, of Section 3.2. We take different values for l and $u_0 = \tilde{u}$ while we consider u^* as the approximate solution given by the scheme (5) and $\varepsilon = 0.1$. In Table 4, we can observe the results, it can be checked that the accessibility improves when we work in a smaller domain, that is due to the fact that we can find more precise bounds for our local convergence restrictions.

5.2 | Comparing numerical results

Now, we consider the nonlinear integral equation of Hammerstein type that appears in different applied problems, see Bruns and Bailey [18] and Wazwaz [19], given by

$$[\mathcal{H}(x)](s) = x(s) - \frac{1}{4} - \frac{1}{4}\int_0^1 G(s,t)(x(t)^2 + |x(t)|),\, dt,\;\; s \in [0,1], \tag{21}$$

where G is the Green's function and x is the solution to be obtained.

In order to solve the equation $\mathcal{H}(x) = 0$, where $\mathcal{H} : \Omega \subset C[0,1] \to C[0,1]$, we transform the problem into a nonlinear system in $\mathbb{R}^n$ and we consider the max-norm. So we approximate the integral by Gauss–Legendre quadrature formula with the corresponding weights q_j and nodes t_j, $j = 1, 2, \ldots, n$; with this discretization of the problem, we have the following nonlinear system:

$$x_j = \frac{1}{4} + \frac{1}{4}\sum_{i=1}^{n} p_{ji}(x_i^2 + |x_i|)^T \;\; j = 1, 2, \ldots, n, \tag{22}$$

where

$$p_{ij} = q_i G(t_j, t_i) = \begin{cases} q_i(1-t_j)t_i, & i \leq j, \\ q_i(1-t_i)t_j, & i > j, \end{cases}$$

TABLE 3 Numerical results with different values of l.

l	r	R	$m(r)$	$\lambda(r)$
1.5	0.45918	1.12985	0.34401	0.26696
2.0	0.51367	0.81052	0.43529	0.34473
2.25	0.58617	0.63616	0.51984	0.42577

TABLE 4 Numerical results with different values of u_0 and l.

$u^{(0)} = \tilde{u}$	l	$\delta = \|u^* - \tilde{u}\|$	γ	r	R
$(1, 1, 1, \ldots, 1)^T$	1.5	0.73408	1.27893	0.69101	2.33968
$(1, 1, 1, \ldots, 1)^T$	2	0.73408	1.27893	0.575838	1.94974
$(1, 1, 1, \ldots, 1)^T$	3	0.73408	1.27893	0.43188	1.46231
$(-0.25, \ldots, -0.25)^T$	1.5	1.98408	1.29837	0.67799	2.30065
$(-0.25, \ldots, -0.25)^T$	2	1.98408	1.29837	0.56499	1.91721
$(-0.25, \ldots, -0.25)^T$	3	1.98408	1.29837	0.42375	1.43791
$(-1, -1, \ldots, -1)^T$	2	2.73408	1.33098	0.54753	1.86479
$(-1, -1, \ldots, -1)^T$	3	2.73408	1.33098	0.41064	1.39859

TABLE 5 Numerical results with different iterative methods.

Method	Steffensen (4)	Newton–Steffensen-type (5) $\varepsilon = 0.5$	Kurchatov-type (23) $\mu = 0.5$	Secant-like (24) $\lambda = 0.5$
k	8	7	7	8
$\|\mathbf{x_{k+1}} - \mathbf{x_k}\|$	4.0655e-28	1.0053e-30	4.3004e-29	3.2834e-27
$\|H(\mathbf{x_{k+1}})\|$	3.3937e-28	6.9341e-59	6.7530e-59	1.1283e-44
Time	59.1824	53.5696	54.6592	63.7366

with $x_j = x(t_j)$ and with $j = 1, \ldots, n$.

We take $n = 8$ and solve the nonlinear system (22) with different iterative methods for non-differentiable operators, with the aim of comparing the results. First, we apply Newton–Steffensen's method given by (6). Next, we consider Kurchatov-type methods defined in Hernández–Verón et al. [20] by the iteration function:

$$\begin{cases} z_0, z_{-1} \text{ given in } \Omega, \ \mu \in [0,1], \\ x_n = (1-\mu)z_n + \mu z_{n-1}, \\ y_n = (1+\mu)z_n - \mu z_{n-1}, \\ z_{n+1} = z_n - [x_n, y_n; H]^{-1} H(z_n), \ n \geq 0. \end{cases} \tag{23}$$

Finally, we compare the results with the ones obtained with the following uniparametric family of secant-type iterative methods for solving $H(\mathbf{x}) = 0$, defined in Hernández and Rubio [21] as an improvement of [22] given by

$$\begin{cases} z_{-1}, z_0 \text{ given in } \Omega, \ \lambda \in [0,1], \\ x_n = \lambda z_n + (1-\lambda) z_{n-1}, \ n \geq 0, \\ x_{n+1} = x_n - [x_n, z_n; H]^{-1} H(x_n), \end{cases} \tag{24}$$

which depends on the parameter $\lambda \in [0,1]$. The family (24) reduces to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$ and H is differentiable.

We use Matlab20 with variable precision arithmetic with 50 digits, using as stopping criteria $\|\mathbf{x_{n+1}} - \mathbf{x_n}\| < 10^{-25}$ and with the starting point $\mathbf{z_0} = (0.2, 0.2, \ldots, 0.2)^T$ and $\mathbf{z_{-1}} = (0.5, 0.5, \ldots, 0.5)^T$. Then, for different values of parameters ε, μ, and λ, we obtain the approximated solution to the problem with six digits:

$$\mathbf{x}^* = (0.295568, 0.292161, 0.287938, 0.285097, 0.285097, 0.287938, 0.292161, 0.295568)^T.$$

In Table 5, we can compare the numerical results, k, for the iterations number needed to reach the tolerance required, $\|\mathbf{x_{k+1}} - \mathbf{x_k}\|$ the residual error, $\|H(\mathbf{x_{k+1}})\|$ the vale of the nonlinear operator at the approximated solution, and the execution time in seconds. We can conclude that new Newton–Steffesen-type methods, (5), are competitive reaching seven iterations and less execution time for reaching the given tolerance.

6 | CONCLUSIONS

In this work, we use a decomposition technique that consists on considering the non-differentiable and nonlinear operator $H(x)$ as $H(x) = F(x) + G(x)$, where F is Fréchet differentiable and G is continuous but non-differentiable. So we define a new iterative method to solve $H(x) = 0$ that allows us to maintain the Newton's method quadratic order of convergence due to Newton's method for non-differentiable operators. Besides, by means of a dynamical study, we check that the corresponding iterative process considered has as good accessibility as Newton's method. We focus on the local and semilocal convergence study by using weaker conditions in previous studies. The numerical examples show that our method favorably compares with some other existing ones from the points of view of accuracy, convergence rate, and computational cost.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

ORCID

Eva G. Villalba https://orcid.org/0000-0003-1357-8410
Eulalia Martínez https://orcid.org/0000-0003-2869-4334

REFERENCES

1. I. K. Argyros, *A convergence theorem for Newton-like methods under generalized Chen-Yamamoto-type assumptions*, Appl. Math. Comput. **61** (1994), 25–37.
2. J. A. Ezquerro, M. A. Hernández, N. Romero, and A. I. Velasco, *On Steffensen's method on Banach spaces*, J. Comput. Appl. Math. **249** (2013), 9–23.
3. H. T. Kung and J. F. Traub, *Optimal order of one-point and multipoint iteration*, Comput. Sci. Dep. Paper **1747** (1973).
4. V. Alarcón, S. Amat, S. Busquier, and D. J. López, *A Steffensen's type method in Banach spaces with applications on boundary-value problems*, J. Comput. Appl. Math. **216** (2008), 243–250.
5. I. K. Argyros, *A new convergence theorem for Steffensen's method on Banach spaces and applications*, Southwest J. Pure Appl. Math. **1** (1997), 23–29.
6. I. K. Argyros, M. A. Hernández-Verón, and M. J. Rubio, *Convergence of Steffensen's method for non-differentiable operators*, Numer. Algor. **75** (2017), 229–244.
7. J. E. Dennis, Toward a unified convergence theory for Newton-like methods, *Nonlinear functional analysis and application*, L. B. Rall, (ed.), Academic Press, New York, 1971, pp. 425–472.
8. S. Amat, J. A. Ezquerro, and M. A. Hernández-Verón, *On a Steffensen-like method for solving nonlinear equations*, Calcolo **53** (2016), 171–188.
9. K. Kneisl, *Julia sets for the super-Newton method, Cauchy's method, and Halley's method*, Chaos **11** (2001), 359–370.
10. J. L. Varona, *Graphic and numerical comparison between iterative methods*, Math. Intelligencer **24** (2002), 37–46.
11. M. A. Hernández-Verón and E. Martínez, *Improving the accessibility of Steffensen's method by de composition of operators*, J. Comput. Appl. Math. **330** (2018), 536–552.
12. M. Grau-Sánchez and M. Noguera, On convergence and efficiency in the resolution of systems of nonlinear equations from a local analysis, *Advances in iterative methods for nonlinear equations*, S. Amat and S. Busquier, (eds.), SEMA SIMAI Springer Series, Vol. **10**, Springer International Publishing, Switzerland, 2016.
13. M. Grau-Sánchez, A. Grau, and M. Noguera, *Frozen divided differences scheme for solving systems of nonlinear equations*, J. Comput. Appl. Math. **235** (2011), 1739–1743.
14. M. Grau-Sánchez, M. Noguera, and S. Amat, *On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods*, J. Comput. Appl. Math. **237** (2013), 363–372.
15. M. A. Hernández and M. J. Rubio, *A uniparametric family of iterative processes for solving nondifferentiable equations*, J. Math. Anal. Appl. **275** (2002), 821–834.
16. I. K. Argyros and H. Ren, *On an improved local convergence analysis for the secant method*, Numer. Algor. **52** (2009), 257–271.
17. J. A. Ezquerro and M. A. Hernández-Verón, *A new concept of convergence for iterative methods: restricted global convergence*, J. Comput. Appl. Math. **405** (2022), 11305.
18. D. D. Bruns and J. E. Bailey, *Nonlinear feedback control for operating a nonisothermal CSTR near an unstable steady state*, Chem. Eng. Sci. **32** (1977), 257–264.
19. A. M. Wazwaz, *Applications of integral equations; linear and nonlinear integral equations*, Springer, Berlin/Heidelberg, 2011.
20. M. A. Hernández-Verón, N. Yadav, A. Magreñán, E. Martínez, and S. Singh, *An improvement of the Kurchatov method by means of aparametric modification*, Math. Meth. Appl. Sci. **45** (2022), 6844–6860.
21. M. A. Hernández and M. J. Rubio, *An uniparametric family of iterative processes for solving nondifferentiable equations*, J. Math. Anal. Appl. **275** (2002), 821–834.
22. M. A. Hernández and M. J. Rubio, *The secant method for nondifferentiable operators*, Appl. Math. Lett. **4** (2002), 395–399.
23. S. M. Shakhno, *On the secant method under generalized Lipschitz conditions for the divided difference operator*, PAMM-Proc. Appl. Math. Mech. **7** (2007), 2060083–2060084.
24. J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970.

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