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Research Paper

Modifying Kurchatov's method to find multiple roots of nonlinear equations

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ABSTRACT

We present a modification of Kurchatov's iterative method in order to solve a nonlinear equation with multiple roots, that is, for approximating solutions with multiplicity greater than one. One of its principal advantages is that you do not have to know a priori the multiplicity of the root, since it does not appear in the iterative expression. In order to examine the behaviour of the proposed method, we perform a dynamical analysis. Furthermore, we carry out some numerical experiments in order to confirm the theoretical results and compare the proposed method with other known methods for dealing with multiple roots.

1. Introduction

Many engineering and applied mathematics problems require the solution of nonlinear equations, $f(x)=0$.

They cannot always be solved exactly, so an approximation of the solution is sometimes obtained.

These approximations are usually obtained using an iterative method. A well-known method is Newton's method, which is expressed as follows:

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)},\ \text{for } k=0,1,\dots$$

It is required that the derivative of the function evaluated at the solution is non-zero to ensure that this method converges to a root of $f(x)=0$.

For this reason, there appear iterative methods that allow us to obtain solutions with a multiplicity greater than 1. In manuscripts [1–7] a variety of memoryless, iterative schemes with and without derivatives are created to approximate the multiple roots of a nonlinear equation $f(x)=0$.

Most of them make the assumption that the multiplicity is known and it appears in the iterative expression of the method.

It is known that Schröder scheme [8]

$$x_{k+1}=x_k-\frac{f(x_k)f'(x_k)}{f'(x_k)^2-f(x_k)f''(x_k)},\ \text{for } k=0,1,\dots$$

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has second-order of convergence for multiple roots of the $f(x)=0$. This method was designed from Newton's scheme applied to $g(x)=\dfrac{f(x)}{f'(x)}$. Its main feature is that you do not need to know a priori the multiplicity of the root, which does not appear in the iterative expression.

In a similar way, in paper [9], the authors construct an iterative method with memory for approximating the multiple roots, that avoids the need to know a priori the multiplicity. In this manuscript, we apply several techniques to Kurchatov's scheme to obtain an iterative method without derivatives and with memory for finding multiple roots. We see that the modification of this method maintains the order and has good dynamical behaviour. Other recent texts analyze the stability of schemes with memory, such as [10,11].

Kurchatov's method is an iterative scheme of second-order convergence obtained from Newton's method by replacing the derivative by the divide difference of Kurchatov $f[2x_k-x_{k-1},x_{k-1}]$

$$x_{k+1}=x_k-\frac{f(x_k)}{f[2x_k-x_{k-1},x_{k-1}]}=x_k-\frac{2(x_k-x_{k-1})f(x_k)}{f(2x_k-x_{k-1})-f(x_{k-1})}, k=1,2,\dots$$

The design and convergence analysis of the suggested iterative method with memory to identify multiple roots without being aware of their multiplicity are discussed in Section 2 of this paper. In Section 3, a dynamical analysis of the rational function obtained by using the suggested method with low-degree polynomials is described. In Section 4, an analogous method to the one proposed in Section 2 is presented but without the use of derivatives. Finally, in Section 5, we carry out a number of numerical experiments using the Kurchatov method for multiple roots and compare the results obtained by this method with those of other well-known methods for multiple roots.

2. Analysis of convergence

In an open set $D\subset\mathbb{R}$ containing a solution α of $f(x)=0$, let f, defined from D to $\mathbb{R}$, be a sufficiently differentiable function. To prove the order of convergence, we use the expression of the divided difference operator

$$f[y,z](y-z)=f(y)-f(z). \tag{1}$$

We can find in [12] the Ortega-Rheinboldt theorem, which is used to demonstrate the order of convergence of an iterative scheme with memory:

Theorem 1. *If ϕ is an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α that converges to α. If there exist some positive numbers t_i, for $i\in\{0,\dots,m\}$ and a nonzero constant η, such that the inequality*

$$|e_{k+1}|\le\eta\prod_{i=0}^{m}|e_{k-i}|^{t_i},$$

is satisfied, in this case, ϕ has, at least, order of convergence p, where p is the positive root of

$$p^{m+1}-\sum_{i=0}^{m}t_ip^{m-i}=0.$$

To estimate the roots of $f(x)=0$, we define the following method, denoted by KM,

$$x_{k+1}=x_k-\frac{g(x_k)}{g[2x_k-x_{k-1},x_{k-1}]},\qquad k=0,1,2,\dots$$

where $g(x)=\dfrac{f(x)}{f'(x)}$.

Theorem 2. *Assume $f:D\longrightarrow\mathbb{R}$ is a function sufficiently differentiable in an neighbourhood of α, denoted by $D\subset\mathbb{R}$, such that α is a multiple root of $f(x)=0$ with unknown multiplicity $m\in\mathbb{N}-\{1\}$. Based on an initial estimation x_0 close to α, method KM generates a sequence of iterations $\{x_k\}$ that converges to α with order 2, and the error equation is:*

$$e_{k+1}=\left(\frac{-1}{m}C_1e_k^2+\frac{(m+1)C_1^2-2mC_2}{m^2}\left(-5e_k^3+2e_k^2e_{k-1}-e_ke_{k-1}^2\right)\right)+O_4(e_k,e_{k-1}),$$

being $C_j=\dfrac{m!}{(m+j)!}\dfrac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j\in\{2,3,\dots\}$ and where O_4 denotes all terms for which the sum of the exponents of e_k and e_{k-1} is at least 4.

Proof. Applying, around α, the Taylor expansion of $f(x_k)$ where $e_k=x_k-\alpha$:

$$f(x_k)=\frac{f^{(m)}(\alpha)}{m!}\left(e_k^m+C_1e_k^{m+1}+C_2e_k^{m+2}+C_3e_k^{m+3}\right)+O(e_k^{m+4}).$$

Calculating the derivative of the above expression we obtain

$$f'(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(me_k^{m-1} + (m+1)C_1e_k^m + (m+2)C_2e_k^{m+1} + (m+3)C_3e_k^{m+2}\right) + O(e_k^{m+3}).$$

Then, from the above expressions, we calculate $g(x_k)$

$$g(x_k) = \frac{f(x_k)}{f'(x_k)} = \frac{1}{m}\left(e_k - \frac{1}{m}C_1e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}e_k^3\right) + O(e_k^4).$$

In an equivalent way we obtain the following expressions for $g(x_{k-1})$ and $g(2x_k - x_{k-1})$

$$\begin{aligned} g(x_{k_1}) &= \frac{f(x_{k-1})}{f'(x_{k-1})} = \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1e_{k-1}^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}e_{k-1}^3\right) + O(e_{k-1}^4), \\ g(2x_k - x_{k-1}) &= \frac{f(2x_k - x_{k-1})}{f'(2x_k - x_{k-1})} \\ &= \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}(2e_k - e_{k-1})^3\right) + O_4(e_k, e_{k-1}), \end{aligned}$$

with $e_{k-1} = x_{k-1} - \alpha$.

From the above relations, we obtain

$$\begin{aligned} g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g\left(2x_k - x_{k-1}\right) - g\left(x_{k-1}\right)}{2(x_k - x_{k-1})} \\ &= \frac{\left(2e_k - 2e_{k-1} - \frac{1}{m}C_1\left(\left(2e_k - e_{k-1}\right)^2 - e_{k-1}^2\right) + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(\left(2e_k - e_{k-1}\right)^3 - e_{k-1}^3\right)\right) + O(e_k^4)}{2m\left(e_k - e_{k-1}\right)} \\ &= \frac{1}{m}\left(1 - \frac{2}{m}C_1e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_ke_{k-1} + e_{k-1}^2\right)\right) + O_3(e_k, e_{k-1}). \end{aligned}$$

Thus, applying the above relationship, the error equation is:

$$\begin{aligned} x_{k+1} - \alpha &= x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]} \\ &= e_k - \frac{\left(e_k - \frac{1}{m}C_1e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}e_k^3\right) + O(e_k^4)}{\left(1 - \frac{2}{m}C_1e_k + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(4e_k^2 - 2e_ke_{k-1} + e_{k-1}^2\right)\right) + O_3(e_k, e_{k-1})} \\ &= \frac{-1}{m}C_1e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(-e_k^3 - e_k\left(4e_k^2 - 2e_ke_{k-1} + e_{k-1}^2\right)\right) + O_4(e_k, e_{k-1}) \\ &= \frac{-1}{m}C_1e_k^2 + \frac{(m+1)C_1^2 - 2mC_2}{m^2}\left(-5e_k^3 + 2e_k^2e_{k-1} - e_ke_{k-1}^2\right) + O_4(e_k, e_{k-1}). \end{aligned}$$

Different possibilities exist for the behaviour of e_{k+1} in relation to e_k and e_{k-1}.

Based on the previous expression, we are only going to the behaviour like e_k^2 or $e_ke_{k-1}^2$, since e_k^3 and $e_k^2e_{k-1}$ converge faster to 0 than e_k^2.

Then,

$$e_{k+1} \sim \frac{-1}{m}C_1e_k^2 - \frac{(m+1)C_1^2 - 2mC_2}{m^2}e_ke_{k-1}^2.$$

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- We assume that $e_{k+1} \sim e_ke_{k-1}^2$. Then, we assume that the method has R-order p, that means,

$$e_{k+1} \sim e_k^p.$$

In the same way $e_k \sim e_{k-1}^p$. From the above relations, we get that

$$e_{k+1} \sim e_{k-1}^{p^2}.$$

Then, the error equation is

$$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$

By equating the exponents of e_{k-1} of the above relations, we obtain the following polynomial $p^2 = p + 2$, whose positive root is $p = 2$, then, by Theorem 1, the order is 2. □

3. Dynamical analysis

Since we later perform a dynamical analysis of the proposed method for some family of functions, we review some theoretical concepts involved in the dynamical analysis of an iterative method with memory in this section. We assume that the method only needs two previous iterations to obtain the next one.

If x_0 and x_1 are the initial estimations, then the standard form of an iterative method with memory has the form:

$$x_{k+1} = \phi(x_{k-1}, x_k), \quad \text{with } k \geq 1.$$

Since, the function is defined from $\mathbb{R}^2$ to $\mathbb{R}$, it is clear that this function cannot have fixed points.
To solve this problem, we define an auxiliary vectorial function O: $O(x_{k-1}, x_k) = (x_k, \phi(x_{k-1}, x_k)) = (x_k, x_{k+1})$, for $k = 1, 2, \ldots$
If ϕ is the operator of the iterative, then the discrete dynamical system $O : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$O(\bar{x}) = O(z, x) = (x, \phi(z, x)).$$

A fixed point (z, x) of O is a points that satisfies $z = x$ and $x = \phi(z, x)$. A strange fixed is a fixed point (z, x) of operator O that does not verify that $f(x) = 0$.

To study the stability of a fixed point we use the following result that can be found in [13].

Theorem 3. *Let O from $\mathbb{R}^2$ to $\mathbb{R}^2$ be a function sufficiently differentiable. Assuming we have a fixed point that is $\bar{x}$. Let denote the eigenvalues of the Jacobian matrix of O at $\bar{x}$ by λ_1 and λ_2. Therefore,*

- *The fixed point is an attractor, if all the eigenvalues satisfy $|\lambda_j| < 1$.*
- *The fixed points are unestable (repulsor or saddle), if one eigenvalue λ_i satisfy $|\lambda_i| > 1$.*
- *The fixed point is an repulsor, if all the eigenvalues satisfy $|\lambda_j| > 1$.*

Also, we called a fixed point superattractor if all the eigenvalues are zero.

Another relevant concept is the critical points. Those critical points are the ones whose eigenvalues of the Jacobian matrix are 0.

The set of pre-images of a fixed point is the basin of attraction, that can be defined as

$$\mathcal{A}(x^*) = \{y \in \mathbb{R}^n : O^r(y) \to x^*, \ r \to \infty\}.$$

Now, we analyse our proposed iterative method. We are going to perform this study for the following family of functions $p_m(x) = (x+1)(x-1)^m$, when m is a positive integer greater than 1. Now, we calculate the auxiliar vectorial operator where $z = x_{k-1}$ and $x = x_k$

$$Op(z, x) = \left(x, x - \frac{\left(x^2 - 1\right)(mz + m + z - 1)(2mx - mz + m + 2x - z - 1)}{(mx + m + x - 1)(m(z + 1)(2x - z + 1) + (z - 1)(2x - z - 1))} \right).$$

Theorem 4. *For the operator $Op(z, x)$, we obtain that the fixed points are the roots of the polynomial $p_m(x)$, that is, $(1, 1)$ and $(-1, -1)$, and a strange fixed point $\left(\frac{1-m}{1+m}, \frac{1-m}{1+m} \right)$. Also, the fixed points $(1, 1)$ and $(-1, -1)$ have superattractor character and the strange fixed point is an unestable point.*

Proof. The fixed points (z, x) are those that satisfies $z = x$ and $Op(z, x) = (x, x)$. First, we compute $Op(x, x)$

$$Op(x, x) = \left(x, \frac{m(x+1)^2 - (x-1)^2}{m(x+1)^2 + (x-1)^2} \right).$$

By equating $Op(x, x) = (x, x)$, we obtain that the fixed points satisfy:

$$\frac{m(x+1)^2 - (x-1)^2}{m(x+1)^2 + (x-1)^2} = x,$$
$$m(x+1)^2 - (x-1)^2 = xm(x+1)^2 + x(x-1)^2,$$

$$m(1-x)(x+1)^2 = (x+1)(x-1)^2.$$

If $x = 1$ or $x = -1$, then it is obvious that the above equation is satisfied.

Suppose that $x \neq 1$ and $x \neq -1$. Then, the above equation can be rewritten as:

$$\begin{aligned} -m(x-1)(x+1)^2 &= (x+1)(x-1)^2, \\ -m(x+1) &= x-1, \\ (-m-1)x &= -1+m, \\ x &= \frac{-1+m}{-m-1} = \frac{1-m}{1+m}. \end{aligned}$$

So, we obtain two fixed points from the roots of the equation, that is, $z = x = 1$ and $z = x = -1$, and one strange fixed point when $z = x = \frac{1-m}{1+m}$.

We are going to see below that the fixed points coming from the roots are superattractors. First, we have to calculate the Jacobian matrix $Op'(z,x)$.

$$Op'(z,x) = \begin{pmatrix} 0 & 1 \\ dOp_z(z,x) & dOp_x(z,x) \end{pmatrix},$$

where

$$dOp_z(z,x) = -\frac{8m(m+1)(x-z)\left(x^2-1\right)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2},$$

$$\begin{aligned} dOp_x(z,x) = &-\frac{4m^3(z+1)\left(x^2(5z+1)+x\left(-4z^2+2z-2\right)+z^3-z^2-2\right)+8m^2\left(x^2\left(5z^2-3\right)-4xz^3+z^4-z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2} \\ &-\frac{4m(z-1)\left(x^2(5z-1)-2x\left(2z^2+z+1\right)+z^3+z^2+2\right)}{(mx+m+x-1)^2(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}. \end{aligned}$$

The eigenvalues of $Op'(x,x)$ are 0 and $-\frac{8m\left(x^2-1\right)}{\left(m(x+1)^2+(x-1)^2\right)^2}$.

Then, both eigenvalues are 0 when $x^2-1=0$, that is, $x=1$ or $x=-1$, so we find that the fixed points coming from the roots are superattractor fixed points.

In the case $x = \frac{1-m}{1+m}$, we obtain that the second eigenvalue is 2, so is a point with an unstable character (repulsor or saddle). □

Theorem 5. *The operator $Op(z,x)$ has only two critical points that are the superattractor fixed points, that is, the operator does not have free critical points.*

Proof. First, we calculate the determinant of $Op'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0.

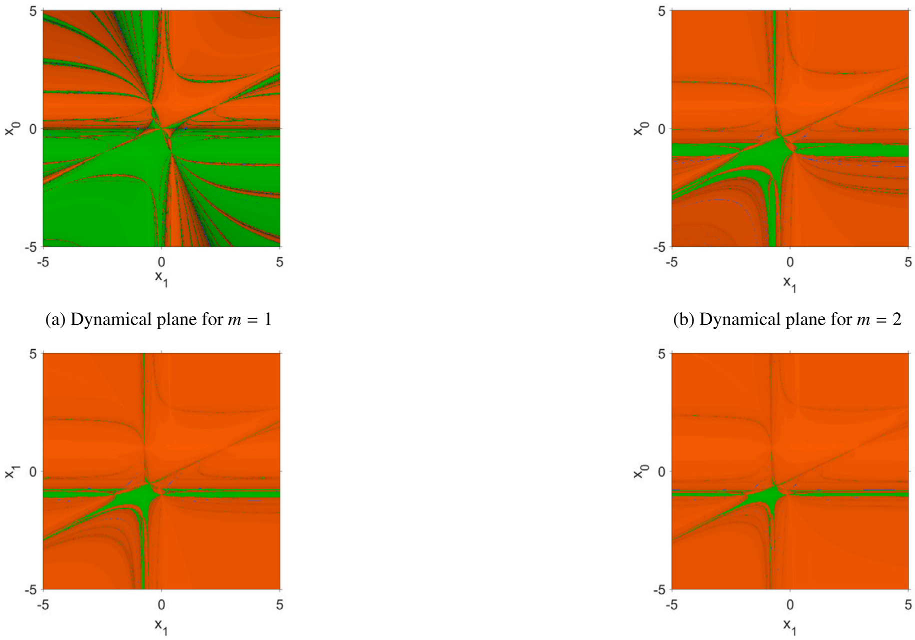
$$\det(Op'(z,x)) = \frac{8m(m+1)\left(x^2-1\right)(x-z)}{(mx+m+x-1)(m(z+1)(2x-z+1)+(z-1)(2x-z-1))^2}.$$

By equating that expression to 0, we obtain 3 types of possible critical points:

- The points (z,x) where $x=-1$. The eigenvalues of $Op'(z,-1)$ are 0 and $-\frac{m(1+m)(1+z)^2}{-3+2z+z^2+m(1+z)^2}$.
  Then, the second eigenvalue is 0 if $z=-1$. Then, there is only one critical point with this structure which is the fixed point $(-1,-1)$.
- The points (z,x) where $x=1$. The eigenvalues of $Op'(z,1)$ are 0 and $-\frac{(1+m)(-1+z)^2}{m((z-1)^2+m(z^2-2z-3))}$.
  Then, the second eigenvalue is 0 if $z=1$. Then, there is only one critical point with this structure which is the fixed point $(1,1)$.
- The points (z,x) where $z=x$. The eigenvalues of $Op'(z,z)$ are 0 and $-\frac{8m(-1+z^2)}{((-1+z)^2+m(1+z)^2)^2}$.
  The second eigenvalue is 0 if $z=\pm 1$. So, the critical points that verify this structure are the non strange fixed points, that is, $(1,1)$ and $(-1,-1)$.

Then, the operator does not have free critical points. □

To illustrate the way the method and the basins of attractions for the function p_m behave when m changed, some dynamical planes are shown below.



(a) Dynamical plane for $m = 1$

(b) Dynamical plane for $m = 2$

(c) Dynamical plane for $m = 3$

(d) Dynamical plane for $m = 4$

Fig. 1. Real dynamical planes. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

These planes have been generated by making a mesh of 400×400 points, where every point represents the first iterations of the iterative method. Iteration x_1 is on the abscissa axis and iteration x_0 is on the ordinate axis.

We say that the initial points converge to one of the roots of the function if the distance between iterations to one of the roots is less than 10^{-3}. Additionally, it must happen before 100 iterations.

We represent the initial point in different colours according to its convergence. Orange is used to represent points that converge to 1 and green is used to represent points that converge to -1.

We represent in black those points that do not converge, but in this case, that does not happen.

As we can see in Figs. 1a, 1b, 1c, 1d, if we increase the value of m, the zone of convergence to the root 1 increases, which is the root of multiplicity m.

A family of polynomials with one simple root and one multiple root is shown in this study. We can see in the dynamical planes that all the initial points from the mesh converge to one of the roots.

Our next step is to perform a dynamical analysis to find out what happens when we have two multiple roots with different multiplicities.

The polynomials are $f_{m,n}(x) = (x+1)^n(x-1)^m$ where $m > 1$ and $n > 1$.

Now, we calculate the auxiliar vectorial operator

$$Of(z,x) = \left(x, \frac{m^2(x+1)(z+1)(2x-z+1) + 2mn\left(2xz - z^2 - 1\right) - n^2(x-1)(z-1)(2x-z-1)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))} \right).$$

Theorem 6. *The operator $Of(z,x)$ has that the fixed points are the roots of the polynomial $f_{m,n}(x)$, that is, $(1,1)$ and $(-1,-1)$, both fixed points have superattractor character, and an unestable strange fixed point that is $\left(\frac{n-m}{n+m}, \frac{n-m}{n+m}\right)$.*

Proof. The fixed points (z,x) are those that satisfies $z = x$ and $Op(z,x) = (x,x)$. First, we compute $Of(x,x)$

$$Of(x,x) = \left(x, \frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2} \right).$$

By equating $Of(x,x) = (x,x)$, we obtain that the fixed points are those that are satisfied:

$$\begin{aligned} &\frac{m(x+1)^2 - n(x-1)^2}{m(x+1)^2 + n(x-1)^2} = x, \\ &m(x+1)^2 - n(x-1)^2 = xm(x+1)^2 + xn(x-1)^2, \\ &\quad m(1-x)(x+1)^2 = n(x+1)(x-1)^2. \end{aligned}$$

If $x = 1$ or $x = -1$, then it is obvious that the above equation is satisfied. Suppose that $x \neq 1$ and $x \neq -1$. Then, the above equation can be rewritten as:

$$-m(x-1)(x+1)^2 = n(x-1)^2,$$
$$-m(x+1) = n(x-1),$$
$$(-m-n)x = -n+m,$$
$$x = \frac{-n+m}{-m-n} = \frac{n-m}{n+m}.$$

So, we obtain two fixed points from the roots of the equation, that is, $z = x = 1$ and $z = x = -1$, and one strange fixed point when $z = x = \frac{n-m}{n+m}$.

We see below that the fixed points coming from the roots are superattractors. First, we calculate the eigenvalues of the Jacobian matrix $Of'(x,x)$, that are 0 and $-\frac{8mn\left(z^2-1\right)}{\left(m(z+1)^2+n(z-1)^2\right)^2}$.

Then, both eigenvalues are 0 when $x^2 - 1 = 0$, that is, $x = 1$ or $x = -1$, so we find that the fixed points coming from the roots are superattractor fixed points.

In the case that $x = \frac{n-m}{n+m}$, we obtain that the second eigenvalue is 2, so is a point with an unstable character (repulsor or saddle). $\square$

Theorem 7. *The operator $Of(z,x)$ has only two critical points that are the superattractor fixed points.*

Proof. First, we analyze the determinant of $Of'(z,x)$, because when the determinant is 0, it means that at least one of the eigenvalues is 0,

$$\det\left(Of'(z,x)\right) \frac{8mn\left(x^2-1\right)(m+n)(x-z)}{(m(x+1)+n(x-1))(m(z+1)(2x-z+1)+n(z-1)(2x-z-1))^2}.$$

By equating that expression to 0, we obtain 3 types of possible critical points:

- The points (z,x) where $x = -1$. The eigenvalues of $Of'(z,-1)$ are 0 and $-\frac{m(z+1)^2(m+n)}{n\left(m(z+1)^2+n\left(z^2+2z-3\right)\right)}$.

  Then, the second eigenvalue is 0 if $z = -1$. Then, there is only one critical point with this structure which is the fixed point $(-1,-1)$.
- The points (z,x) where $x = 1$. The eigenvalues of $Of'(z,1)$ are 0 and $-\frac{n(z-1)^2(m+n)}{m\left(m\left(z^2-2z-3\right)+n(z-1)^2\right)}$.

  Then, the second eigenvalue is 0 if $z = 1$. Then, there is only one critical point with this structure which is the fixed point $(1,1)$.
- The points (z,x) where $z = x$. The eigenvalues of $Of'(z,z)$ are 0 and $-\frac{8mn\left(z^2-1\right)}{\left(m(z+1)^2+n(z-1)^2\right)^2}$.

  The second eigenvalue is 0 if $z = \pm 1$. So, the critical points that verify this structure are the non strange fixed points, that is, $(1,1)$ and $(-1,-1)$.

Then, there are not free critical points for this rational operator. $\square$

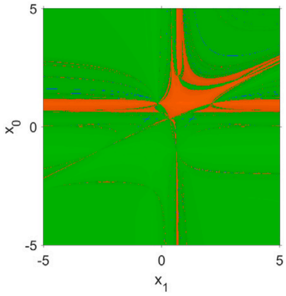
Below we show some real dynamical planes to see the behaviour of the method and the basins of attraction for the function $f_{m,n}$ varying the value of m and the value of n.

As with the previous dynamical planes, these planes were generated by making a mesh of 400×400 points, where each point represents the initial iterations of the method, in the abscissa axis we have the iteration x_1 and in the ordinate axis the iteration x_0.

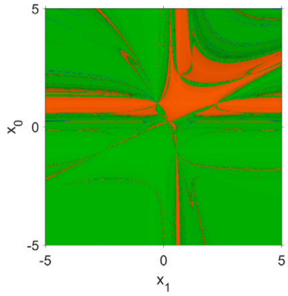
The convergence criteria are the same as in the previous dynamical planes. Let remember, that in orange are represented the points that tends to 1 and in green are represented the points that tends to -1.

As we can see in Figs. 2 and 3, if we the value of n is greater than the value of m, the zone of convergence to the root -1 is greater than the zone of convergence to the root 1. If both values are equal, then the convergence zones do not change if we increase the multiplicity value.

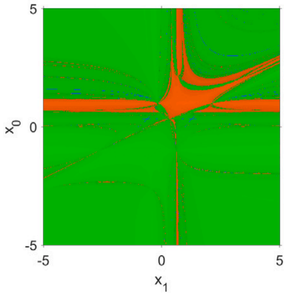
In the dynamical planes, all the initial points coming from the mesh converge to one of the roots. With this study we show that the method is stable for that family of polynomials that have two multiple roots.



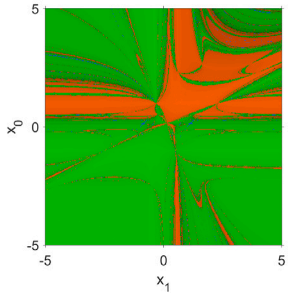
(a) Dynamical plane for $m = 1$ and $n = 2$



(b) Dynamical plane for $m = 2$ and $n = 3$

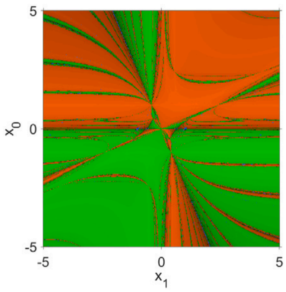


(c) Dynamical plane for $m = 2$ and $n = 4$

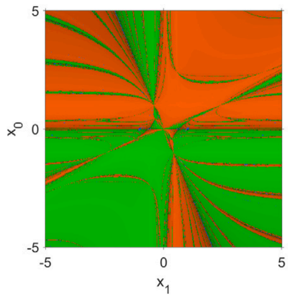


(d) Dynamical plane for $m = 3$ and $n = 4$

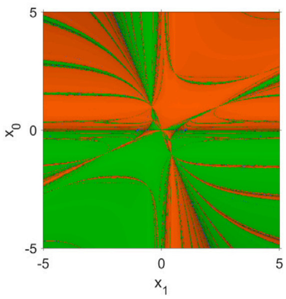
Fig. 2. Real dynamical planes.



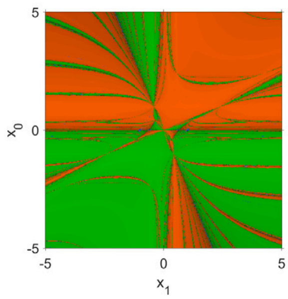
(a) Dynamical plane for $m = 1$ and $n = 1$



(b) Dynamical plane for $m = 2$ and $n = 2$



(c) Dynamical plane for $m = 3$ and $n = 3$



(d) Dynamical plane for $m = 4$ and $n = 4$

Fig. 3. Real dynamical planes.

4. Without derivatives

To estimate the roots of $f(x) = 0$ with the KM method we calculate the derivative of $f(x)$. In the following iterative method, which we denote by KMD, we modify the KM method, so that we do not use derivatives in the iterative expression:

$$x_{k+1} = x_k - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]},$$

where $g(x) = \dfrac{f(x)}{f[x+f(x),x]}$.

Theorem 8. *Assume $f : D \longrightarrow \mathbb{R}$ is a function sufficiently differentiable in an neighbourhood of α, denoted by $D \subset \mathbb{R}$, such that α is a multiple root of $f(x) = 0$ with unknown multiplicity $m \in \mathbb{N} - \{1\}$. Based on an initial estimation x_0 close to α, method KMD generates a sequence of iterations $\{x_k\}$ that converges to α with order 2.*

Proof. We first obtain, around α, the Taylor expansion of $f(x_k)$ where $e_k = x_k - \alpha$:

$$f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(e_k^m + C_1 e_k^{m+1}\right) + O(e_k^{m+2}),$$

being $C_j = \dfrac{m!}{(m+j)!}\dfrac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j = 2, 3, \ldots$

In the same way,

$$f(x_k + f(x_k)) = \frac{f^{(m)}(\alpha)}{m!}\left(\left(e_k + f(x_k)\right)^m + C_1\left(e_k + f(x_k)\right)^{m+1}\right) + O(e_k^{m+2}).$$

Then,

$$f(x_k + f(x_k)) - f(x_k) = \frac{f^{(m)}(\alpha)}{m!}\left(\left(e_k + f(x_k)\right)^m - e_k^m + C_1\left(\left(e_k + f(x_k)\right)^{m+1} - e_k^{m+1}\right)\right) + O(e_k^{m+2}).$$

Using Newton's binomial and the Taylor expansion of $f(x_k)$ around α we obtain

$$\frac{f(x_k + f(x_k)) - f(x_k)}{x_k + f(x_k) - x_k} = \frac{f^{(m)}(\alpha)}{m!}\left(m e_k^{m-1} + (m+1)C_1 e_k^m\right) + O(e_k^{m+1}).$$

We then calculate $g(x_k)$ from the above expressions:

$$\begin{aligned} g(x_k) &= \frac{f(x_k)}{f[x_k + f(x_k), x_k]} = \frac{e_k^m + C_1 e_k^{m+1} + O(e_k^{m+2})}{m e_k^{m-1} + (m+1)C_1 e_k^m + O(e_k^{m+1})} \\ &= \frac{1}{m}\left(e_k - \frac{1}{m}C_1 e_k^2\right) + O(e_k^3). \end{aligned}$$

In an equivalent way we obtain the following expressions for $g(x_{k-1})$ and $g(2x_k - x_{k-1})$

$$g(x_{k-1}) = \frac{1}{m}\left(e_{k-1} - \frac{1}{m}C_1 e_{k-1}^2\right) + O(e_{k-1}^3),$$

$$g(2x_k - x_{k-1}) = \frac{1}{m}\left(2e_k - e_{k-1} - \frac{1}{m}C_1(2e_k - e_{k-1})^2\right) + O_3(e_k, e_{k-1}),$$

with $e_{k-1} = x_{k-1} - \alpha$.

Then, applying the above relations, we obtain

$$\begin{aligned} g[2x_k - x_{k-1}, x_{k-1}] &= \frac{g(2x_k - x_{k-1}) - g(x_{k-1})}{2(x_k - x_{k-1})} \\ &= \frac{\left(2e_k - 2e_{k-1} - \frac{1}{m}C_1((2e_k - e_{k-1})^2 - e_{k-1}^2)\right) + O_3(e_k, e_{k-1})}{2m(e_k - e_{k-1})} \\ &= \frac{1}{m}\left(1 - \frac{2}{m}C_1 e_k\right) + O_2(e_k, e_{k-1}). \end{aligned}$$

Thus, the error equation obtained is

$$\begin{aligned} x_{k+1} - \alpha &= x_k - \alpha - \frac{g(x_k)}{g[2x_k - x_{k-1}, x_{k-1}]} \\ &= e_k - \frac{\left(e_k - \frac{1}{m}C_1 e_k^2\right) + O(e_k^3)}{\left(1 - \frac{2}{m}C_1 e_k\right) + O_2(e_k, e_{k-1})} \\ &= e_k - \frac{2}{m}C_1 e_k^2 + e_k O_2(e_k, e_{k-1}) - e_k + \frac{1}{m}C_1 e_k^2 + O(e_k^3) \\ &= -\frac{1}{m}C_1 e_k^2 + e_k O_2(e_k, e_{k-1}) + O(e_k^3). \end{aligned}$$

We have some different possibilities for the behaviour of e_{k+1} respect to e_k and e_{k-1}.

Table 1
Results for the first equation, $f_1(x)=0$.

	x_0	x_{-1}	x_{-2}	$\|x_{k+1}-x_k\|$	$\|g(x_{k+1})\|$	Iter	ACOC
KM	0.5	0.1		1.5776e-13	0	8	1.9994
KMD	0.5	0.1		6.1173e-14	0	6	1.8434
gTM	0.5	0.1	-0.1	1.7764e-15	0	42	1.5850

By the expression, we only are going to take into account if the behaviour is like e_k^2 or $e_k e_{k-1}^2$, because e_k^3 and $e_k^2 e_{k-1}$ converge faster to 0 than e_k^2.

Then

- If $e_{k+1} \sim e_k^2$, then the order of convergence is 2.
- If we assume that $e_{k+1} \sim e_k e_{k-1}^2$. Then, we assume that the method has R-order p, that means,

$$e_{k+1} \sim D_{k,p} e_k^p.$$

At the same time, $e_k \sim e_{k-1}^p$, then we obtain that

$$e_{k+1} \sim e_{k-1}^{p^2}.$$

From the last relation and the expression of the error equation, we have

$$e_{k+1} \sim e_k e_{k-1}^2 \sim e_{k-1}^{p+2}.$$

By equating the exponents of e_{k-1} of the last two equations, we obtain the following polynomial $p^2 = p+2$, whose positive root is $p=2$, then the order is 2. □

5. Numerical experiments

Using Matlab R2020b with accuracy in arithmetic of 500 digits, we compute calculations. Our stopping criterion is

$$|f(x_{k+1})| < 10^{-25}.$$

Also, is used as a stopping criterion a maximum number of iterations that can be done, in this case is 100. We compare the proposed methods with the method coming from [9], which we denote by gTM.

The numerical results we are going to compare in the different examples are:

- last approximation given, x_{k+1},
- absolute value of $f(x_{k+1})$,
- distance between x_k and x_{k+1},
- iterations performed to verify the stopping criterion,
- the computational time
- and the ACOC (approximate computational convergence order), defined in [14], which has the expression

$$p \approx ACOC = \frac{\ln\left(\dfrac{|x_{k+1}-x_k|}{|x_k - x_{k-1}|}\right)}{\ln\left(\dfrac{|x_k - x_{k-1}|}{|x_{k-1}-x_{k-2}|}\right)}.$$

We are going to solve two nonlinear equations:

- The equation $f_1(x) = (x^3-1)^4 = 0$, which has three roots with multiplicity four.
- In [15], they considered the isothermal CSTR problem, with the following equation for the transfer function of the reactor: $KC2.98(x+2.25)/((x+1.45)(x+2.85)^2(x+4.35)) = -1$, where KC is the gain of the proportional controller. If we choose KC = 0, the nonlinear equation to solve is the following one:

$$f_2(x) = x^4 + 11.50x^3 + 47.49x^2 + 86.0325x + 51.23266875 = 0.$$

There is one multiple root with multiplicity 2.

As we can see in Table 1, all the methods obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one. What is interesting from the table is that, for the initial points chosen, we see

Table 2
Results for the second equation, $f_2(x) = 0$.

	x_0	x_{-1}	x_{-2}	$\|x_{k+1} - x_k\|$	$\|g(x_{k+1})\|$	Iter	ACOC
KM	-3	-3.25		1.9884e-09	1.6566e-30	4	2.2725
KMD	-3	-3.25		2.4269e-08	2.0293e-29	4	2.0649
gTM	-3	-3.25	-3.5	2.5116e-11	1.0354e-29	5	1.7914

that the KMD method performs less iterations to verify the stopping criterion than KM, but both perform far less iterations than the gTM method.

As we can see in Table 2, all the methods obtain good results for the chosen initial points. The approximate computational convergence order coincides with the theoretical one and the number of iterations to verify the stopping criterion is almost the same for all methods.

6. Conclusions

In this work, we have modified Kurchatov's method to make it applicable to obtaining multiple roots while maintaining the quadratic order of convergence of Kurchatov's method.

We have modified the method so that it does not use the multiplicity of the solution in its expression, so that it is not necessary to know this value before applying the iterative method.

We have performed the dynamical analysis of the iterative method for two families of functions, one of the polynomials with one simple root and one multiple root, and another with two multiple roots, showing that the method is stable in both cases.

We also modify the method we propose to obtain the KMD method, which is a method with free memory of derivatives, with the same characteristics as the KM method, that is, it can be applied to obtain solutions with multiplicity greater than one, and does not involve the value of this multiplicity in its iterative expression.

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CRediT authorship contribution statement

All the authors have contributed in the same way in this manuscript.

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