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Spectra and dynamics of generalized Cesàro operators in (LF) and (PLB) sequence spaces

Angela A. Albanese[1] · Vicente Asensio[2,3]

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Abstract

In this paper, we introduce inductive limits of the Fréchet spaces $\ell(p+)$, $\mathrm{ces}(p+)$, and $d(p+)$ $(1 \le p < \infty)$ and projective limits of the (LB)-spaces $\ell(p-)$, $\mathrm{ces}(p-)$, and $d(p-)$ $(1 < p \le \infty)$. After having established some topological properties of such spaces as acyclicity and ultrabornologicity, we prove that the generalized Cesàro operators C_t $(0 \le t \le 1)$ act continuously in these sequence spaces, and we determine the spectra. Finally, we study the ergodic properties, that is, power boundedness, (uniform) mean ergodicity, and supercyclicity, of the operators C_t acting in the (LF)-spaces and in the (PLB)-spaces mentioned above.

Keywords Generalized Cesàro operator · Spectrum · Power boundedness · Uniform mean ergodicity · Supercyclicity · Sequence spaces · (LF)-space · (PLB)-space

Mathematics Subject Classification 46A45 · 47B37 · 46A13 · 47A10 · 47A35 · 47A16

1 Introduction

Let $\omega := \mathbb{C}^{\mathbb{N}_0}$ and $0 \le t \le 1$. The discrete generalized Cesàro operator $C_t : \omega \to \omega$ is given by

$$C_t(x) := \left(\frac{t^n x_0 + t^{n-1} x_1 + \cdots + x_n}{n+1}\right)_{n \in \mathbb{N}_0}, \qquad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \tag{1.1}$$

✉ Vicente Asensio
viaslo@upv.es; vasensio@edem.es

Angela A. Albanese
angela.albanese@unisalento.it

1 Dipartimento di Matematica e Fisica "E. di Giorgi", Università del Salento, C.P. 193, 73100 Lecce, Italy

2 Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Camino de Vera, s/n, 46022 Valencia, Spain

3 Centro Universitario EDEM, Plaça de L'aigua, s/n, 46024 Valencia, Spain

For $t = 1$, we recover the Cesàro operator C_1, given by

$$C_1(x) := \left(\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right)_{n\in\mathbb{N}_0}, \qquad x = (x_n)_n \in \omega. \tag{1.2}$$

The spectra of the Cesàro operator C_1 defined in (1.2) have been analyzed in several Banach sequence spaces. For instance, in ℓ_p, for $1 < p < \infty$, [17, 18, 24, 31], in the discrete Cesàro space ces(p) [19] and in $d(p)$ [14], for $1 < p < \infty$, which is isomorphic to the strong dual of ces(p'), with p' the conjugate exponent of p, [12]. We refer the reader to [8] for a vast list of references concerning the study of the Cesàro operator C_1 in other Banach sequence spaces. The research of the operators defined in (1.1) started later, in the 1980s. This study has been focused on the spectrum of the generalized Cesàro operator C_t $(0 \le t < 1)$ in ℓ_p, for $1 < p < \infty$, [38, 44], in ces(p) [19], and in $d(p)$ [14], for $1 < p < \infty$, among others, showing, surprisingly enough, a behaviour rather different from that of the Cesàro operator C_1. See also [20, 39, 45]. All the spaces considered satisfy that they are continuously included in the Fréchet space ω, when it is endowed with the coordinatewise convergence topology.

Since the inclusions $\ell_q \subseteq \ell_p$, ces$(q) \subseteq$ ces(p), and $d(q) \subseteq d(p)$ are continuous if $1 < q \le p < \infty$, for given $1 \le p < \infty$ and $\{p_n\}_{n\in\mathbb{N}_0}$ a sequence satisfying $p < p_{n+1} < p_n$, for $n \in \mathbb{N}_0$, with $p_n \downarrow p$, we can define the Fréchet spaces

$$\ell(p+) = \bigcap_{n\in\mathbb{N}_0} \ell_{p_n}, \quad \mathrm{ces}(p+) = \bigcap_{n\in\mathbb{N}_0} \mathrm{ces}(p_n), \quad d(p+) = \bigcap_{n\in\mathbb{N}_0} d(p_n). \tag{1.3}$$

Besides, for given $1 < p \le \infty$ and $\{p_n\}_{n\in\mathbb{N}_0}$ a sequence satisfying $p_n < p_{n+1} < p$, for $n \in \mathbb{N}_0$, with $p_n \uparrow p$, we can define the (LB)-spaces

$$\ell(p-) = \bigcup_{n\in\mathbb{N}_0} \ell_{p_n}, \quad \mathrm{ces}(p-) = \bigcup_{n\in\mathbb{N}_0} \mathrm{ces}(p_n), \quad d(p-) = \bigcup_{n\in\mathbb{N}_0} d(p_n). \tag{1.4}$$

Recently, the spectra of the generalized Cesàro operators C_t $(0 \le t < 1)$ acting in these Fréchet and (LB)-spaces have been studied by the first author, Bonet, and Ricker in [8]. On the other hand, the same authors described the spectra of the Cesàro operator C_1 acting in the spaces mentioned above [2, 4, 6–8, 15, 16], whose behaviour is still diverse from that of the generalized Cesàro operators C_t for $t \ne 1$.

The dynamics of the generalized Cesàro operators $(0 \le t < 1)$ acting in the Banach sequence spaces ℓ_p, ces(p), and $d(p)$ $(1 < p < \infty)$, in the Fréchet spaces defined in (1.3), in the (LB)-spaces defined in (1.4), and in ω, are discussed in [8], showing that such operators C_t are power bounded and uniformly mean ergodic, but not supercyclic. This behaviour holds true for the Cesàro operator C_1 in ω [2, 7]. However, this contrasts with the fact that C_1 is not power bounded nor mean ergodic nor supercyclic in the aforementioned Banach spaces [4, 5, 11, 14], Fréchet spaces [2, 7, 16] and (LB)-spaces [6, 8, 16].

The aim of this paper is twofold: to introduce and analyze the inductive limits of the Fréchet spaces $\ell(p+)$, ces$(p+)$, and $d(p+)$ $(1 \le p < \infty)$, and the projective limits of the (LB)-spaces $\ell(p-)$, ces$(p-)$, and $d(p-)$ $(1 < p \le \infty)$, and to study the spectra and the dynamics of the generalized Cesàro operators C_t (1.1), $0 \leq t < 1$, acting in these new sequence spaces, comparing them to those of the generalized Cesàro operator C_t acting in the Fréchet and (LB)-spaces, and making the appropriate rearrangements for the case of the Cesàro operator C_1 (1.2).

The paper is divided as follows: in Sect. 2, we introduce some notation and present preliminary results on general (LF) and (PLB)-spaces. We also recall some properties regarding the Fréchet and (LB)-spaces mentioned above (see (1.3) and (1.4)).

In Sect. 3 we define the (LF)-spaces $L(p-)$, $C(p-)$, $D(p-)$, with $1 < p \leq \infty$ (the (PLB)-spaces $L(p+)$, $C(p+)$, $D(p+)$, with $1 \leq p < \infty$) as inductive limits of the Fréchet spaces given in (1.3) (as projective limits of the (LB)-spaces given in (1.4)), and study their topological properties such as acyclicity, completeness, reflexitivity or Montel (such as bornologicity, barrelledness, reflexitivity or Montel), see Proposition 2, Corollary 3 and Proposition 4. In Sect. 3 we also establish that $L(p-) \subseteq L(q-)$, $C(p-) \subseteq C(q-)$, $D(p-) \subseteq D(q-)$ ($L(p+) \subseteq L(q+)$, $C(p+) \subseteq C(q+)$, $D(p+) \subseteq D(q+)$), with continuous inclusions, for $1 < p \leq q$ (for $1 \leq p \leq q$), as in the (LB)-space case (as in the Fréchet-space case), see Propositions 7 and 9.

In Sect. 4 we determine the spectra of generalized Cesàro operators C_t ($0 \leq t \leq 1$) acting in the (LF) and (PLB)-spaces introduced in Sect. 3 (see Theorems 15, 20, and 22). The proofs of the results are based on Lemmas 10 and 12. In particular, we show that the spectra of C_t ($0 \leq t < 1$) acting in these (LF) and (PLB)-spaces coincide with those in the corresponding Fréchet and (LB)-spaces (Theorem 15). We obtain similar results also for the spectra of C_1 acting in these (LF) and (PLB)-spaces (Theorems 20 and 22).

Finally, in Sect. 5 we study the ergodic properties, i.e., the power boundedness, the mean ergodicity, and the uniform mean ergodicity, of continuous linear operators acting in (LF)-spaces (in (PLB)-spaces). We compare such properties with the same of the continuous linear operators acting in the steps of their inductive spectrum (projective spectrum), see Theorems 24 and 26. We then apply Theorems 24 and 26 to establish that the generalized Cesàro operators C_t ($0 \leq t < 1$) acting in the (LF) and (PLB)-spaces introduced in Sect. 3 are power bounded, mean ergodic, and uniformly mean ergodic (Corollaries 25 and 27). We conclude showing that the Cesàro operator C_1 acting in these sequence spaces is, in contrast, not power bounded nor mean ergodic, and that C_t is never supercyclic for every $0 \leq t \leq 1$.

2 Notation and preliminary results

For two locally convex Hausdorff spaces (lcHs, for short) X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators $T : X \to Y$. We write $\mathcal{L}(X) = \mathcal{L}(X, X)$ and we denote by $\Gamma = \Gamma_X$ a collection of continuous seminorms that determine the topology of X. The identity operator on $\mathcal{L}(X)$ is denoted by I and the set of bounded sets of X is denoted by $\mathcal{B}(X)$. When the space $\mathcal{L}(X)$ is endowed with the strong operator topology (the topology of uniform convergence on $\mathcal{B}(X)$), it is denoted by $\mathcal{L}_s(X)$ (by $\mathcal{L}_b(X)$). The topological dual of X is denoted by $X' := \mathcal{L}(X, \mathbb{C})$. We write X'_σ (we write X'_β) for the space X' endowed with the weak* topology $\sigma(X', X)$

(the strong topology $\beta(X', X)$). Some standard references of functional analysis are [27, 30, 34, 42].

Let X be a lcHs. We will say that X is an (LF)-space if there exists an increasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of Fréchet spaces such that the inclusion $X_n \subseteq X_{n+1}$, for $n \in \mathbb{N}$, is continuous and the topology in $X = \bigcup_n X_n$ coincides with the finest locally convex topology for which each inclusion $X_n \subseteq X$ is continuous. We will denote it by $X = \text{ind}_n X_n$; the sequence $\{X_n\}_{n\in\mathbb{N}}$ is called a *defining inductive spectrum* of X. We will say that X is an (LB)-space if each X_n is a Banach space. We remark that every (LF)-space is ultrabornological [34, Remark 24.36], hence bornological and barrelled.

An (LF)-space $X = \text{ind}_n X_n$ is said to be *regular* if every $B \in \mathcal{B}(X)$ is contained and bounded in X_n, for some n. Every *complete* (LF)-space is regular. We will say that X satisfies condition (M) (condition (M_0)) of Retakh [37] if there exists an increasing sequence $\{U_n\}_{n\in\mathbb{N}} \subseteq X$ such that U_n is an absolutely convex zero-neighbourhood of (X_n, τ_n), $n \in \mathbb{N}$, for which

(M) for every n there exists $m \geq n$ such that for every $l \geq m$, the topologies τ_l and τ_m induce the same topology on U_n.
((M_0) for every n there exists $m \geq n$ such that for every $l \geq m$, the topologies $\sigma(X_l, X_l')$ and $\sigma(X_m, X_m')$ induce the same topology on U_n.)

Recall that Vogt [41, Theorem 2.10] established that an (LF)-space X satisfies condition (M) (condition (M_0)) if, and only if, it is *acyclic* (*weakly acyclic*) in the sense of Palamodov [36]. Acyclic (LF)-spaces are complete [36, Corollary 7.1] (cf. [41, Theorem 3.2]). Moreover, a standard duality proof shows that (M) implies (M_0).

If $\{\|\cdot\|_{n,\ell}\}_{\ell\in\mathbb{N}}$ denotes a fundamental system of seminorms of the Fréchet space X_n, for $n \in \mathbb{N}$, we then say that the (LF)-space $X = \text{ind}_n X_n$ satisfies the condition (Q) (the condition (wQ)) if

(Q) $\forall n\ \exists m > n,\ N \in \mathbb{N}\ \forall k > m,\ M \in \mathbb{N},\ \varepsilon > 0,\ \exists K \in \mathbb{N},\ S > 0\ \forall x \in X_n,$

$$\|x\|_{m,M} \leq S\,\|x\|_{k,K} + \varepsilon\,\|x\|_{n,N}\,.$$

((wQ) $\forall n\ \exists m > n,\ N \in \mathbb{N}\ \forall k > m,\ M \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0\ \forall x \in X_n,$

$$\|x\|_{m,M} \leq S(\|x\|_{k,K} + \|x\|_{n,N}).)$$

The condition (wQ) is clearly weaker than the condition (Q). The conditions (Q) and (wQ) were introduced and studied in [41]. In particular, in [41] it was shown that such conditions are necessary for the acyclicity and the weak acyclicity and, under further assumptions, also sufficient. Thereafter, Wengenroth [43, Theorems 2.7 and 3.3] proved that the conditions (M) and (Q) ((wQ) under suitable assumptions on the spaces X_n) are equivalent. More precisely,

Theorem 1 *Let $X = \text{ind}_n X_n$ be an (LF)-space. The space X satisfies condition (M) (i.e., it is acyclic) if, and only if, X satisfies condition (Q). Furthermore, if X is an inductive limit of Fréchet-Montel spaces, then X satisfies condition (M) (i.e., it is acyclic) if, and only if, it is complete if, and only if, it satisfies (wQ).*

We will say that a lcHs X is a (PLB)-space if there exists a decreasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of (LB)-spaces such that the inclusion $X_{n+1} \subseteq X_n$ is continuous, for $n \in \mathbb{N}$, and the topology in $X = \bigcap_n X_n$ coincides with the coarsest locally convex topology for which each inclusion $X \subseteq X_n$ is continuous. We will denote it by $X = \mathrm{proj}_n X_n$; the sequence $\{X_n\}_{n\in\mathbb{N}}$ is called a *defining projective spectrum* of X. It follows that a (PLB)-space X is complete provided X_n is a complete (LB)-space for infinitely many n. The (PLB)-space X is called *reduced* if the inclusion $X \subseteq X_n$ has dense range for every $n \in \mathbb{N}$. The (PLB)-space X is a Fréchet space if each X_n is a Banach space.

The main problem concerning (PLB)-spaces consists in discussing whether they are bornological/barrelled. In the case that $X = \mathrm{proj}_n (X_n)'_\beta$ is a reduced (PLB)-space of strong duals of reflexive Fréchet spaces X_n, if the (LF)-space $\mathrm{ind}_n X_n$ is weakly acyclic, then X is bornological, as shown by Vogt in [41, Lemma 4.2]. Under the assumption that the steps X_n are Fréchet-Montel spaces, Wengenroth [43, Theorem 3.5] proved that $X = \mathrm{proj}_n (X_n)'_\beta$ is bornological if, and only if, the (LF)-space $\mathrm{ind}_n X_n$ satisfies the condition (wQ). We refer to [40, 41, 43] for further results.

For $1 \le p \le \infty$, we write $\|\cdot\|_p$ for the standard norm in ℓ_p. For $1 < p < \infty$ we define

$$\mathrm{ces}(p) = \{x \in \omega : \quad \|x\|_{\mathrm{ces}(p)} = \|C_1(|x|)\|_p < \infty\}.$$

The Banach spaces $\mathrm{ces}(p)$, $1 < p < \infty$, have been deeply studied in Bennett [12] (see also [10, 19, 25, 32]).

For $1 < p < \infty$, the dual Banach spaces $(\mathrm{ces}(p))'$ are rather complicated (see [29]). An isomorphic identification of $(\mathrm{ces}(p))'$ is given in [12, Corollary 12.17], that is, it is shown there that

$$d(p) = \left\{x \in \ell_\infty : \hat{x} := \left(\sup_{k\ge n} |x_k|\right)_{n\in\mathbb{N}_0} \in \ell_p\right\}$$

is a Banach space, with the norm

$$\|x\|_{d(p)} = \left\|\hat{x}\right\|_p, \qquad x \in d(p),$$

and that it is isomorphic to $(\mathrm{ces}(p'))'$, where $1 < p' < \infty$ satisfies $1/p + 1/p' = 1$.

For $1 < p, q < \infty$, we have that $p \le q$ if, and only if, $\ell_p \subseteq \ell_q$, $\mathrm{ces}(p) \subseteq \mathrm{ces}(q)$ [5, Proposition 3.2(iii)], $d(p) \subseteq d(q)$ [14, Proposition 5.1(iii)], $\ell_p \subseteq \mathrm{ces}(q)$ ([5, Proposition 3.2(ii)] and [19, Remark 2.2(ii)], where Hardy's inequality [28, Theorem 326] is used), $d(p) \subseteq \ell_q$ [14, Proposition 2.7(v) and 5.1(ii)], and $d(p) \subseteq \mathrm{ces}(q)$ [14, Proposition 5.1(i)] with continuous inclusions. We have that ℓ_p, $\mathrm{ces}(p)$ (and hence $d(p)$) ($1 < p < \infty$) are reflexive ([12, p. 61] and [29, Proposition 2]), and also they are separable. They contain $\{e_n\}_{n\in\mathbb{N}_0}$ (here, $e_n = (\delta_{nk})_{k\in\mathbb{N}_0}$) as an unconditional basis. The spaces ℓ_p, $\mathrm{ces}(q)$, and $d(r)$ are not isomorphic for $1 < p, q, r < \infty$ (see [15, Proposition 2.2] for references).

We now recall some properties of ω, of the Fréchet spaces defined in (1.3), and of the (LB)-spaces defined in (1.4). First of all, the definition of the Fréchet spaces above ((LB)-spaces above) is independent of the choice of the sequence $p_n \downarrow p$

$(p_n \uparrow p)$. The spaces $\ell(p+)$ are studied in, among others, [21, 35]. The spaces $\ell(p+)$, for $1 \leq p < \infty$, are reflexive and separable, but not Montel. We refer to [4] (see also [7]) for the properties of the spaces $\mathrm{ces}(p+)$. The spaces $\mathrm{ces}(p+)$, for $1 \leq p < \infty$, are separable, Fréchet–Schwartz (hence Montel) [4, Theorem 3.1 and Corollary 3.2], but not nuclear [4, Proposition 3.5(ii)]. Clearly, $\ell_p \subseteq \ell(p+) \subseteq \omega$ and $\mathrm{ces}(p) \subseteq \mathrm{ces}(p+) \subseteq \omega$, for $1 \leq p < \infty$, with continuous inclusions. We also know [4, Proposition 3.5(iii)] that $\ell(p+)$ is not isomorphic to any $\mathrm{ces}(q+)$, $1 \leq q < \infty$. The spaces $d(p+)$, for $1 \leq p < \infty$, are also separable, Fréchet–Schwartz, not nuclear, and not isomorphic to any $\mathrm{ces}(q+)$, $1 \leq q < \infty$ [15, Theorem 4.7].

As in the Banach case, we have that $1 \leq p \leq q < \infty$ if, and only if, $\ell(p+) \subseteq \ell(q+)$ [7, Proposition 26(ii)], $\mathrm{ces}(p+) \subseteq \mathrm{ces}(q+)$ ([7, Proposition 26(iii)]), $d(p+) \subseteq d(q+)$ (see for instance [15, (4.2)]), $\ell(p+) \subseteq \mathrm{ces}(q+)$ [7, Proposition 26(ii)], $d(p+) \subseteq \ell(q+)$ and $d(p+) \subseteq \mathrm{ces}(q+)$ (see for instance [15, (4.4)]), with continuous inclusions.

On the other hand, it is known (see again [21, 35]) that the space $\ell(p-)$, for $1 < p \leq \infty$, is complete, reflexive, but not Montel. Furthermore, $\ell(p-)$ is isomorphic to $(\ell(p'+))'_\beta$, and $(\ell(p-))'_\beta$ is isomorphic to $\ell(p'+)$, where $1 \leq p' < \infty$ satisfies $1/p + 1/p' = 1$. In [6], the spaces $\mathrm{ces}(p-)$ are studied. There, it is shown that $\mathrm{ces}(p-)$ (and also $d(p-)$), with $1 < p \leq \infty$, are reflexive, separable, Montel (see also [13, pp. 61–62]) and not nuclear. Furthermore, by [6, Proposition 5.1], we have that the inclusions $\ell(p-) \subseteq \ell(q-)$, $\mathrm{ces}(p-) \subseteq \mathrm{ces}(q-)$, $d(p-) \subseteq d(q-)$, $\ell(p-) \subseteq \mathrm{ces}(q-)$, $d(p-) \subseteq \ell(q-)$, $d(p-) \subseteq \mathrm{ces}(q-)$ are continuous provided $1 < p \leq q \leq \infty$. By [15, Proposition 4.3 and Remark 4.4], it holds that

$$(d(p+))'_\beta = \mathrm{ces}(p'-), \quad 1 \leq p < \infty \qquad \text{and} \qquad (d(p-))'_\beta = \mathrm{ces}(p'+), \quad 1 < p \leq \infty$$

isomorphically, where p' satisfies $1/p + 1/p' = 1$. Therefore, the spaces $\mathrm{ces}(p-)$ and $d(p-)$, for $1 < p \leq \infty$, are (DFS)-spaces.

3 (LF) and (PLB) sequence spaces

For the Fréchet spaces $\ell(p+)$, $\mathrm{ces}(p+)$, $d(p+)$ given in (1.3), we have that $\ell(p+) \subseteq \ell(q+)$, $\mathrm{ces}(p+) \subseteq \mathrm{ces}(q+)$, and $d(p+) \subseteq d(q+)$ continuously if and only if $1 \leq p \leq q < \infty$ (the continuous inclusions have dense range because $\{e_n\}_{n\in\mathbb{N}_0}$ is a basis in each of the spaces $\ell(p+)$, $\mathrm{ces}(p+)$, $d(p+)$). Thus, for given $1 < p \leq \infty$ and $\{p_n\}_n$ a sequence satisfying $p_n < p_{n+1} < p$, for $n \in \mathbb{N}_0$, with $p_n \uparrow p$, we can define the (LF)-spaces

$$L(p-) := \bigcup_{n\in\mathbb{N}_0} \ell(p_n+), \qquad C(p-) := \bigcup_{n\in\mathbb{N}_0} \mathrm{ces}(p_n+), \qquad D(p-) := \bigcup_{n\in\mathbb{N}_0} d(p_n+).$$

On the other hand, we also have that the (LB)-spaces given in (1.4) satisfy that $\ell(p-) \subseteq \ell(q-)$, $\mathrm{ces}(p-) \subseteq \mathrm{ces}(q-)$, and $d(p-) \subseteq d(q-)$ continuously, provided $1 < p \leq q \leq \infty$ (the continuous inclusions have dense range because $\{e_n\}_{n\in\mathbb{N}_0}$ is a basis in each of the spaces $\ell(p-)$, $\mathrm{ces}(p-)$, $d(p-)$). Thus, for given $1 \leq p < \infty$ and $\{p_n\}_n$ a sequence satisfying $p < p_{n+1} < p_n$, for $n \in \mathbb{N}_0$, and $p_n \downarrow p$, we can define the (PLB)-spaces

$$L(p+) := \bigcap_{n\in\mathbb{N}_0} \ell(p_n-), \qquad C(p+) := \bigcap_{n\in\mathbb{N}_0} \mathrm{ces}(p_n-), \qquad D(p+) := \bigcap_{n\in\mathbb{N}_0} d(p_n-).$$

The definition of the (LF)-spaces above ((PLB)-spaces above) is independent of the choice of the sequence $p_n \uparrow p$ ($p_n \downarrow p$).

Clearly, the (LF)-spaces $L(p-)$, $C(p-)$ and $D(p-)$ are continuously included in ω, as well as the (PLB)-spaces $L(p+)$, $C(p+)$ and $D(p+)$.

Proposition 2 *For $1 < p \le \infty$, the spaces $L(p-)$, $C(p-)$, and $D(p-)$ are acyclic and hence, complete.*

Proof By Theorem 1, it is enough to show that these spaces satisfy the condition (Q). To this end, we will use a well-known interpolation estimate: for $1 < p < q < r$,

$$\|\cdot\|_q \le \|\cdot\|_r^{\theta}\, \|\cdot\|_p^{1-\theta}\,, \tag{3.1}$$

with $\theta = \frac{r(q-p)}{q(r-p)} \in (0,1)$, where $\|\cdot\|_s$ denotes the ℓ_s-norm (see, f.i., [33, Proposition 1.d.2(ii). p.43]). In order to apply (3.1), we observe that for every $\theta \in (0,1)$ and $x, y \ge 0$ we have

$$x^{\theta} y^{1-\theta} \le x + y,$$

as it is easy to verify. Furthermore, we have for every $x, y \ge 0$ and $\varepsilon > 0$ that

$$x^{\theta} y^{1-\theta} = \frac{1}{\varepsilon^{1-\theta}} x^{\theta} \varepsilon^{1-\theta} y^{1-\theta} = \left(\frac{1}{\varepsilon^{(1-\theta)/\theta}} x\right)^{\theta} \cdot (\varepsilon y)^{1-\theta}.$$

Therefore, we deduce for every $\varepsilon > 0$, $\theta \in (0,1)$ and $x, y \ge 0$ that

$$x^{\theta} y^{1-\theta} \le \frac{1}{\varepsilon^{(1-\theta)/\theta}} x + \varepsilon y.$$

Thus, by (3.1) it follows for every $1 < p < q < r$ and $\varepsilon > 0$ that

$$\|\cdot\|_q \le \frac{1}{\varepsilon^{(1-\theta)/\theta}}\, \|\cdot\|_r + \varepsilon\, \|\cdot\|_p\,. \tag{3.2}$$

Now, for a fixed $1 < p \le \infty$, let $\{p_n\}_{n\in\mathbb{N}}$ be any strictly increasing sequence satisfying $p_n \uparrow p$. Then for any $n \in \mathbb{N}$, we choose a strictly decreasing sequence $\{p_{n,l}\}_{l\in\mathbb{N}}$ satisfying $p_n < p_{n,l} < p_{n+1}$, for $l \in \mathbb{N}$, and $p_{n,l} \downarrow p_n$, and denote by $\|\cdot\|_{n,l}$ the $\ell_{p_{n,l}}$-norm for every $l \in \mathbb{N}$. Then $\{\|\cdot\|_{n,l}\}_{l\in\mathbb{N}}$ is a fundamental system of seminorms of the Fréchet space $\ell(p_n+)$ for every $n \in \mathbb{N}$.

By (3.2), for every $n \in \mathbb{N}$ there exists $m = n+1$ such that for every $k > m$, $N, M, K \in \mathbb{N}$ and $\varepsilon > 0$ there exists $S = \frac{1}{\varepsilon^{(1-\theta)/\theta}} > 0$, where $\theta = \frac{p_{k,K}(p_{n+1,M} - p_{n,N})}{p_{n+1,M}(p_{k,K} - p_{n,N})}$, such that

$$\|x\|_{m,M} \le S\, \|x\|_{k,K} + \varepsilon\, \|x\|_{n,N}\,, \qquad \forall x \in \ell(p_n+). \tag{3.3}$$

This implies that $L(p-)$ satisfies the condition (Q). So, by Theorem 1, the (LF)-space $L(p-)$ satisfies condition (M), i.e., it is acyclic and hence, complete.

To show the acyclicity of the (LF)-space $C(p-)$ $(D(p-))$, it suffices to replace x in (3.3) by $C_1(|x|)$ (by $\hat{x}$). Indeed, proceeding in this way, we obtain that for every $n \in \mathbb{N}$ there exists $m = n + 1$ such that for every $k > m$, $N, M, K \in \mathbb{N}$ and $\varepsilon > 0$ there exists $S = \frac{1}{\varepsilon^{(1-\theta)/\theta}} > 0$, where $\theta = \frac{p_{k,K}(p_{n+1,M} - p_{n,N})}{p_{n+1,M}(p_{k,K} - p_{n,N})}$, such that

$$\|C_1(|x|)\|_{m,M} \leq S\,\|C_1(|x|)\|_{k,K} + \varepsilon\,\|C_1(|x|)\|_{n,N}\,, \qquad \forall x \in \mathrm{ces}(p_n+)$$
$$\left(\left\|\hat{x}\right\|_{m,M} \leq S\left\|\hat{x}\right\|_{k,K} + \varepsilon\left\|\hat{x}\right\|_{n,N}, \qquad \forall x \in d(p_n+)\right),$$

that is,

$$\|x\|_{\mathrm{ces}(p_{m,M})} \leq S\,\|x\|_{\mathrm{ces}(p_{k,K})} + \varepsilon\,\|x\|_{\mathrm{ces}(p_{n,N})}\,, \qquad \forall x \in \mathrm{ces}(p_n+)$$
$$\left(\|x\|_{d(p_{m,M})} \leq S\,\|x\|_{d(p_{k,K})} + \varepsilon\,\|x\|_{d(p_{n,N})}\,, \qquad \forall x \in d(p_n+)\right).$$

Since $\{\|\cdot\|_{\mathrm{ces}(p_{n,l})}\}_{l\in\mathbb{N}}(\{\|\cdot\|_{d(p_{n,l})}\}_{l\in\mathbb{N}})$ is a fundamental system of seminorms of the Fréchet space $\mathrm{ces}(p_n+)$ $(d(p_n+))$ for every $n \in \mathbb{N}$, we deduce that the (LF)-space $C(p-)$ $(D(p-)))$ satisfies the condition (Q) and hence, by Theorem 1 it is acyclic and necessarily complete. □

Proposition 2 clearly implies that

Corollary 3 *Let $1 < p \leq \infty$. Then the following properties are satisfied.*

(i) *The (LF)-space $L(p-)$ is reflexive.*
(ii) *The (LF)-spaces $C(p-)$ and $D(p-)$ are Montel and hence, reflexive.*

Proof (i) Since $L(p-)$ is an (LF)-space, it is clearly ultrabornological and hence, bornological and barrelled. So, to conclude that $L(p-)$ is reflexive, it suffices to show that the sets $B \in \mathcal{B}(L(p-))$ are relatively $\sigma(L(p-), (L(p-))')$-compact. So, we fix $B \in \mathcal{B}(L(p-))$. Since by Proposition 2 the (LF)-space $L(p-)$ is complete and hence regular, there exists $n \in \mathbb{N}$ such that B is contained and bounded in $\ell(p_n+)$. But, $\ell(p_n+)$ is a reflexive Fréchet space and so, B is relatively $\sigma(\ell(p_n+), (\ell(p_n+))')$-compact. This implies that B is necessarily relatively $\sigma(L(p-), (L(p-))')$-compact.

(ii) The (LF)-space $C(p-)$ $(D(p-))$ is ultrabornological and hence, bornological and barrelled. So, to conclude that $C(p-)$ $(D(p-))$ is Montel, it suffices to show that the sets $B \in \mathcal{B}(C(p-))$ $(B \in \mathcal{B}(D(p-)))$ are relatively compact. So, we fix $B \in \mathcal{B}(C(p-))$ $(B \in \mathcal{B}(D(p-)))$. Since by Proposition 2 the (LF)-space $C(p-)$ $(D(p-))$ is complete and hence regular, there exists $n \in \mathbb{N}$ such that B is contained and bounded in $\mathrm{ces}(p_n+)$ $(d(p_n+))$. But, by [4, Theorem 3.1 and Corollary 3.2] (by [15, Theorem 4.7]) $\mathrm{ces}(p_n+)$ $(d(p_n+))$ is a Fréchet–Schwartz space and so, B is relatively compact in $\mathrm{ces}(p_n+)$ $(d(p_n+))$. This implies that B is necessarily relatively compact in $C(p-)$ $(D(p-))$. □

A further immediate consequence of Proposition 2 above combined with [41, Lemma 4.2] is the following result.

Proposition 4 *Let $1 \leq p < \infty$. Then the following properties are satisfied.*

(i) *The (PLB)-spaces $L(p+)$, $C(p+)$ and $D(p+)$ are bornological.*
(ii) *The (PLB)-space $L(p+)$ is reflexive, whereas the (PLB)-spaces $C(p+)$ and $D(p+)$ are Montel.*

Proof (i) It suffices to give the proof only for the (PLB)-space $L(p+)$. The other cases follow in a similar way.

Let $1 < p' \leq \infty$ satisfy $1/p + 1/p' = 1$. Then $L(p'-) = \bigcup_{n\in\mathbb{N}} \ell(p'_n+)$, with $1 < p'_n \uparrow p'$, is a reflexive, acyclic (hence, complete and weakly acyclic) (LF)-space (by Corollary 3 and Proposition 2). So, it follows that its strong dual $(L(p'-))'_\beta$ is canonically isomorphic to the projective limit of the strong duals of the spaces $\ell(p'_n+)$, i.e., $(L(p'-))'_\beta = \bigcap_{n\in\mathbb{N}}(\ell(p'_n+))'_\beta$ and that by [41, Lemma 4.2] the strong dual $(L(p'-))'_\beta$ is bornological. But, if for every $n \in \mathbb{N}$, we take $1 \leq p_n < \infty$ satisfying $1/p_n + 1/p'_n = 1$, then $p_n \downarrow p$ and $(\ell(p'_n+))'_\beta = \ell(p_n-)$. Therefore, we deduce that

$$L(p+) = (L(p'-))'_\beta$$

and hence, it is bornological.

(ii) As it follows from the proof of point (i), we have that

$$L(p+) = (L(p'-))'_\beta, \quad C(p+) = (D(p'-))'_\beta, \quad D(p+) = (C(p'-))'_\beta,$$

with $1 < p' \leq \infty$ satisfying $1/p + 1/p' = 1$. Accordingly, as $L(p+)$ ($C(p+)$ and $D(p+)$) is the strong dual of a reflexive (of a Montel) lcHs, it is reflexive (they are Montel). □

As an immediate consequence of Corollary 3 and Proposition 4, we obtain that

Corollary 5 (i) *Let $1 < p \leq \infty$. If $1 \leq p' < \infty$ satisfies $1/p + 1/p' = 1$, then*

$$(L(p-))'_\beta = L(p'+), \quad (C(p-))'_\beta = D(p'+), \quad (D(p-))'_\beta = C(p'+).$$

(ii) *Let $1 \leq p < \infty$. If $1 < p' \leq \infty$ satisfies $1/p + 1/p' = 1$, then*

$$(L(p+))'_\beta = L(p'-), \quad (C(p+))'_\beta = D(p'-), \quad (D(p+))'_\beta = C(p'-).$$

We conclude this section with some results regarding the validity of some inclusions between the spaces introduced above. For this, we recall a characterization for the continuity of an operator in (LF)-spaces (see [27, p. 147]):

Lemma 6 *Let $X = \mathrm{ind}_n X_n$ and $Y = \mathrm{ind}_n Y_n$ be two (LF)-spaces. Let $T : X \to Y$ be a linear operator. Then $T \in \mathcal{L}(X, Y)$ if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(X_n) \subseteq Y_m$ and the restriction $T : X_n \to Y_m$ is continuous.*

Applying Lemma 6 we immediately obtain that

Proposition 7 *We have* $L(p-) \subseteq L(q-)$, $C(p-) \subseteq C(q-)$, $D(p-) \subseteq D(q-)$, $L(p-) \subseteq C(q-)$, $D(p-) \subseteq L(q-)$, *and* $D(p-) \subseteq C(q-)$ *with continuous inclusions if, and only if,* $1 < p \leq q \leq \infty$.

Proof It suffices to give the proof only for the first inclusion. The other cases follow in a similar way.

So, let $1 < p < q < \infty$. If $1 < p_n \uparrow p$ and $p \leq q_n \uparrow q$, then $p_n < q_n$ for every $n \in \mathbb{N}$. Accordingly, $\ell(p_n+) \subseteq \ell(q_n+)$ with continuous inclusion for every $n \in \mathbb{N}$. Since $L(p-) = \bigcup_{n\in\mathbb{N}} \ell(p_n+)$ and $L(q-) = \bigcup_{n\in\mathbb{N}} \ell(q_n+)$, the continuity of the inclusion $L(p-) \subseteq L(q-)$ follows from Lemma 6.

Suppose that $L(p-) \subseteq L(q-)$ with continuous inclusion for some $1 < p, q < \infty$. If $L(p-) = \bigcup_{n\in\mathbb{N}} \ell(p_n+)$ and $L(q-) = \bigcup_{n\in\mathbb{N}} \ell(q_n+)$ with $p_n \uparrow p$ and $q_n \uparrow q$ respectively, then by Lemma 6 for each $n \in \mathbb{N}$ there exists $m(n) \in \mathbb{N}$ such that $\ell(p_n+) \subseteq \ell(q_{m(n)}+)$ with continuous inclusion. But, for any $n \in \mathbb{N}$, $\ell(p_n+) \subseteq \ell(q_{m(n)}+)$ with continuous inclusion if, and only if, $p_n \leq q_{m(n)} < q$. Letting $n \to \infty$, it follows that $p \leq q$. □

For analogous inclusions in the (PLB)-spaces considered, we state a characterization for the continuity in (PLB)-spaces given in [9, Proposition 2] (cf. [22, Lemma 4]).

Lemma 8 *Let* $X = \mathrm{proj}_n X_n$ *and let* $Y = \mathrm{proj}_n Y_n$ *be (PLB)-spaces such that* $X \subseteq X_n$ *has dense range for all* $n \in \mathbb{N}$, *and each* Y_n *is a complete (LB)-space. Let* $T : X \to Y$ *be a linear operator. We have that* $T \in \mathcal{L}(X, Y)$ *if and only if for all* $n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that* T *admits a unique continuous extension* $T : X_m \to Y_n$.

Proposition 9 *We have* $L(p+) \subseteq L(q+)$, $C(p+) \subseteq C(q+)$, $D(p+) \subseteq D(q+)$, $L(p+) \subseteq C(q+)$, $D(p+) \subseteq L(q+)$, *and* $D(p+) \subseteq C(q+)$ *with continuous inclusions if, and only if,* $1 \leq p \leq q < \infty$.

Proof The proof is similar to the one of Proposition 7. Actually, it suffices to apply Lemma 8, after having observed that each (LB)-space in (1.4) is complete and that each (PLB)-space is dense in its steps as it contains the set $\{e_n\}_{n\in\mathbb{N}_0}$. □

4 Spectra of generalized Cesàro operators in (LF) and (PLB)-spaces

Let X be a lcHs and $T \in \mathcal{L}(X)$. The *resolvent set* $\rho(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The *spectrum of* T is defined by $\sigma(T; X) := \mathbb{C} \setminus \rho(T; X)$. The *point spectrum* $\sigma_{pt}(T; X) \subseteq \sigma(T; X)$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. The elements in $\sigma_{pt}(T; X)$ are called *eigenvalues*. An eigenvalue $\lambda \in \mathbb{C}$ is called *simple* if $\dim\ker(\lambda I - T) = 1$. Waelbroeck [42] considered the set $\rho^*(T; X) (\subseteq \rho(T; X))$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disk $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; X)$ and $\{R(\mu; T) : \mu \in B(\lambda, \delta)\}$ is an equicontinuous subset in $\mathcal{L}(X)$. Then $\sigma^*(T; X) := \mathbb{C} \setminus \rho^*(T; X)$ is a closed set in $\mathbb{C}$, and satisfies $\overline{\sigma(T; X)} \subseteq \sigma^*(T; X)$. Note that if X is a Banach space, then $\sigma(T; X)$ coincides with $\sigma^*(T; X)$. For the classical spectral theory of compact operators in lcHs, we refer to [23, 27]. We know (see for example [8, Corollary 2.2]) that if X is a complete and barrelled lcHs and $T \in \mathcal{L}(X)$, then

$$\rho(T; X) = \rho(T'; X'_\beta), \qquad \sigma(T; X) = \sigma(T'; X'_\beta), \qquad \sigma^*(T'; X'_\beta) \subseteq \sigma^*(T; X). \tag{4.1}$$

The proof of this result is along the lines of that for (LB)-spaces given in [3, Lemma 5.2]:

Lemma 10 *Let $X = \text{ind}_n X_n$ be an (LF)-space. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$, the restriction T_n of T to X_n maps X_n into itself and $T_n \in \mathcal{L}(X_n)$.*

Then, the following properties are satisfied:

(i) $\sigma_{pt}(T; X) = \bigcup_{n=1}^{\infty} \sigma_{pt}(T_n; X_n)$.
(ii) $\sigma(T; X) \subseteq \bigcap_{m\in\mathbb{N}} \Big(\bigcup_{n=m}^{\infty} \sigma(T_n; X_n)\Big)$.
(iii) *If* $\bigcup_{n=m}^{\infty} \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$ *for some $m \in \mathbb{N}$, then* $\sigma^*(T; X) = \overline{\sigma(T; X)}$.

Observe that in the proof it is used the open mapping theorem (see for example [34]) which is valid in the setting of (LF)-spaces as they have a web and are ultrabornological.

Results regarding the spectra of (PLB)-spaces are stated and shown below (compare them with Lemma 10). To that aim, we need some preparation.

Lemma 11 *Let X be a lcHs, $T \in \mathcal{L}(X)$ and $\lambda \in \rho^*(T; X)$. If $\overline{B(\lambda, \varepsilon)} \subset \rho^*(T; X)$ for some $\varepsilon > 0$, then the set $\{R(z, T) : z \in \overline{B(\lambda, \varepsilon)}\}$ is equicontinuous.*

Proof Since $\overline{B(\lambda, \varepsilon)} \subset \rho^*(T; X)$, for every $\mu \in \overline{B(\lambda, \varepsilon)}$, there exists $\varepsilon(\mu) > 0$ such that $B(\mu, \varepsilon(\mu)) \subset \rho(T; X)$ and the set $\{R(z, T) : z \in B(\mu, \varepsilon(\mu))\}$ is equicontinuous. Therefore,

$$\overline{B(\lambda, \varepsilon)} \subset \bigcup_{\mu \in \overline{B(\lambda,\varepsilon)}} B(\mu, \varepsilon(\mu)).$$

Since $\overline{B(\lambda, \varepsilon)}$ is a compact subset of $\mathbb{C}$, there exist $\mu_1, \ldots, \mu_k \in \overline{B(\lambda, \varepsilon)}$ such that

$$\overline{B(\lambda, \varepsilon)} \subset \bigcup_{i=1}^{k} B(\mu_i, \varepsilon_i), \tag{4.2}$$

with $\varepsilon_i := \varepsilon(\mu_i)$ for $1 \leq i \leq k$.

Since the set $\{R(z, T) : z \in B(\mu_i, \varepsilon_i)\}$ is equicontinuous for every $1 \leq i \leq k$, fixed $p \in \Gamma_X$, for each $i = 1, \ldots, k$ there exist $q_i \in \Gamma_X$ and $M_i > 0$ such that

$$p(R(z, T)x) \leq M_i q_i(x), \qquad z \in B(\mu_i, \varepsilon_i),\ x \in X.$$

Now, there exists $q \in \Gamma_X$ such that $\max\{q_i(x) : i = 1, \ldots, k\} \leq q(x)$ for all $x \in X$. So, set $M := \max\{M_i : i = 1, \ldots, k\}$, it follows that

$$p(R(z, T)x) \leq Mq(x), \qquad z \in \bigcup_{i=1}^{k} B(\mu_i, \varepsilon_i), \ x \in X.$$

In view of (4.2), this shows that $\{R(z, T) : z \in \overline{B(\lambda, \varepsilon)}\}$ is equicontinuous. □

We show the following result (cf. [2, Lemma 2.1] for Fréchet spaces):

Lemma 12 *Let $X = \bigcap_{n=1}^{\infty} X_n$ be a barrelled (PLB)-space. Let $T \in \mathcal{L}(X)$ satisfy the following property:*

(A') *For every $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that $T_n|_X = T$ and $T_n|_{X_{n+1}} = T_{n+1}$.*

Then:

(i) $\sigma(T; X) \subseteq \bigcup_{n=1}^{\infty} \sigma(T_n; X_n)$ *and* $\sigma_{pt}(T; X) \subseteq \bigcap_{n\in\mathbb{N}} \sigma_{pt}(T_n; X_n)$.
(ii) *For all $\lambda \in \bigcap_{n=1}^{\infty} \rho(T_n; X_n)$ the resolvent $R(\lambda, T)$ of T coincides with the restriction of $R(\lambda, T_n)$ of T_n to X for each $n \in \mathbb{N}$.*
(iii) *If $\bigcup_{n=1}^{\infty} \sigma^*(T_n; X_n) \subseteq \overline{\sigma(T; X)}$, then $\sigma^*(T; X) = \overline{\sigma(T; X)}$.*
(iv) *If $\dim\ker(\lambda I - T_m) = 1$ for each $\lambda \in \bigcap_{n\in\mathbb{N}} \sigma_{pt}(T_n; X_n)$ and for each $m \in \mathbb{N}$, then $\sigma_{pt}(T; X) = \bigcap_{n\in\mathbb{N}} \sigma_{pt}(T_n; X_n)$.*

Proof The proof of points (i) and (ii) is along the lines of [2, Lemma 2.1]. Indeed, we take $\lambda \in \bigcap_{n=1}^{\infty} \rho(T_n; X_n)$ and we show that $\lambda \in \rho(T; X)$. We see that $\lambda I - T : X \to X$ is injective: if $(\lambda I - T)x = 0$ for some $x \in X$, then by (A') we have $(\lambda I - T_1)x = 0$ in X_1. Since $\lambda \in \rho(T_1; X_1)$, we have $x = 0$. To show that $\lambda I - T$ is surjective, we fix $y \in X$. Since $\lambda I - T_n$ is surjective for each $n \in \mathbb{N}$, it follows that for every $n \in \mathbb{N}$ there exists $x_n \in X_n$ satisfying $(\lambda I - T_n)x_n = y$ in X_n for every $n \in \mathbb{N}$. By condition (A') we have that $T_n|_{X_{n+1}} = T_{n+1}$. Hence, $y = (\lambda I - T_n)x_n = (\lambda I - T_{n+1})x_{n+1} = (\lambda I - T_n)x_{n+1}$ in $X_{n+1} \subseteq X_n$. Since $\lambda \in \rho(T_n; X_n)$, we obtain $x_n = x_{n+1}$ for every $n \in \mathbb{N}$. So, $x_1 \in X$ and $y = (\lambda I - T)x_1$. Hence, there exists the inverse operator $(\lambda I - T)^{-1} : X \to X$. It remains to show that $(\lambda I - T)^{-1} \in \mathcal{L}(X)$, thereby implying that $R(\lambda, T) = (\lambda I - T)^{-1}$. So, we observe that the proof above implies that the resolvent $R(\lambda, T)$ of T coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$. Since $R(\lambda, T_n) \in \mathcal{L}(X_n)$ for each $n \in \mathbb{N}$, by Lemma 8 it then follows that $R(\lambda, T) \in \mathcal{L}(X)$. Accordingly, $\lambda \in \rho(T; X)$ as desired. Since $\lambda \in \bigcap_{n=1}^{\infty} \rho(T_n; X_n)$ is arbitrary, we conclude that $\bigcap_{n=1}^{\infty} \rho(T_n; X_n) \subseteq \rho(T; X)$ and hence, $\sigma(T; X) \subseteq \bigcup_{n=1}^{\infty} \sigma(T_n; X_n)$.

Finally, the proof of $\sigma_{pt}(T; X) \subseteq \bigcap_{n\in\mathbb{N}} \sigma_{pt}(T_n; X_n)$ is similar to that in [8, Lemma 2.5].

(iii) We have that $\overline{\sigma(T; X)} \subseteq \sigma^*(T; X)$. If $\overline{\sigma(T; X)} = \mathbb{C}$, then there is nothing to prove. So, we suppose $\mathbb{C}\backslash\overline{\sigma(T; X)} \neq \emptyset$ and we take $\lambda \in \mathbb{C}\backslash\overline{\sigma(T; X)}$. Then, there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon) \cap \overline{\sigma(T; X)} = \emptyset$. By assumption, we have $B(\lambda, \varepsilon) \cap \sigma^*(T_n; X_n) = \emptyset$ for every $n \in \mathbb{N}$, that is, $\overline{B(\lambda, \varepsilon)} \subseteq \rho^*(T_n; X_n)$ for every $n \in \mathbb{N}$. By Lemma 11, we have that $\{R(\mu, T_n) : \mu \in \overline{B(\lambda, \varepsilon)}\}$ is equicontinuous in $\mathcal{L}_s(X_n)$ for every $n \in \mathbb{N}$. We claim that $\lambda \in \rho^*(T; X)$. Since we know that $\overline{B(\lambda, \varepsilon)} \cap \sigma(T, X) = \emptyset$, we have $B(\lambda, \varepsilon) \subseteq \overline{B(\lambda, \varepsilon)} \subseteq \rho(T; X)$. So, to show the claim, it is enough to see that $\{R(\mu, T)x : \mu \in B(\lambda, \varepsilon)\}$ is bounded for every $x \in X$, as X is barrelled. By contradiction, we assume there exists $x \in X$ such that $\{R(\mu, T)x : \mu \in \overline{B(\lambda, \varepsilon)}\}$ is an unbounded subset of X. Then, there is $n_0 \in \mathbb{N}$ such that the set $\{R(\mu, T_{n_0})x : \mu \in \overline{B(\lambda, \varepsilon)}\}$ is unbounded in X_{n_0}. This contradicts the fact that $\{R(\mu, T_{n_0}) : \mu \in \overline{B(\lambda, \varepsilon)}\}$ is equicontinuous in $\mathcal{L}_s(X_{n_0})$ by Lemma 11.

The proof of point (iv) follows as in [8, Lemma 2.5]. □

4.1 Spectra of generalized Cesàro operators C_t ($0 \leq t < 1$)

The aim of this subsection is to study the spectra of the generalized Cesàro operators C_t, for $0 \leq t < 1$, acting in the (LF)-spaces $L(p-)$, $C(p-)$, and $D(p-)$ $(1 < p \leq \infty)$ and in the (PLB)-spaces $L(p+)$, $C(p+)$, and $D(p+)$ $(1 \leq p < \infty)$. In order to do this, we first observe that

Proposition 13 *Let* $0 \leq t < 1$ *and let* X *belong to* $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$ *or to* $\{L(p+), C(p+), D(p+); 1 \leq p < \infty\}$. *Then,* $C_t \in \mathcal{L}(X)$.

Proof We first consider the case $X \in \{L(p-), C(p-), D(p-)\}$, with $1 < p \leq \infty$. So, we take a strictly increasing sequence $\{p_k\}_{k\in\mathbb{N}}$ such that $1 < p_k \uparrow p$ and set $X_k := \ell(p_k+)$ if $X = L(p-)$ or $X_k := \mathrm{ces}(p_k+)$ $(X_k := d(p_k+))$ if $X = C(p-)$ (if $X = D(p-)$), for any $k \in \mathbb{N}$. Then by [8, Proposition 4.4] we have $C_t \in \mathcal{L}(X_k)$ for every $k \in \mathbb{N}$. By Lemma 6 it necessarily follows that $C_t \in \mathcal{L}(X)$.

We now pass to consider the case that $X \in \{L(p+), C(p+), D(p+)\}$, with $1 \leq p < \infty$. So, we take a strictly decreasing sequence $\{p_k\}_{k\in\mathbb{N}}$ such that $1 < p_k \downarrow p$ and set $X_k := \ell(p_k-)$ if $X = L(p+)$ or $X_k := \mathrm{ces}(p_k-)$ $(X_k := d(p_k-))$ if $X = C(p+)$ (if $X = D(p+)$), for any $k \in \mathbb{N}$. Then by [8, Proposition 5.2] we have $C_t \in \mathcal{L}(X_k)$ for every $k \in \mathbb{N}$. By Lemma 8 it necessarily follows that $C_t \in \mathcal{L}(X)$. □

We now turn our attention to the study of the spectra of C_t.

Lemma 14 *Let* $0 \leq t < 1$ *and let* X *belong to* $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$ *or to* $\{L(p+), C(p+), D(p+); 1 \leq p < \infty\}$. *Then* $0 \in \sigma(C_t; X)$.

Proof Let $0 \leq t < 1$ be fixed. By [8, Proposition 3.2] there exists the inverse operator $C_t^{-1} : \omega \to \omega$ and is given (see formula (3.5) in [8]) by

$$C_t^{-1}(x) = \left(x_0, \left((n+1)x_n - ntx_{n-1}\right)_{n\in\mathbb{N}}\right), \qquad x = (x_0, x_1, \ldots) \in \omega. \tag{4.3}$$

Since $X \subseteq \omega$ with continuous inclusion, to show that $0 \in \sigma(C_t; X)$ it suffices to establish that $C_t^{-1}(X)$ does not contain X.

We first consider the case that $X \in \{L(p-), C(p-), D(p-)\}$, with $1 < p \leq \infty$ fixed. So, we take a strictly increasing sequence $\{p_k\}_{k\in\mathbb{N}}$ such that $1 < p_k \uparrow p$ and strictly decreasing sequences $\{p_{k,l}\}_{l\in\mathbb{N}}$ such that $p_k < p_{k,l} < p_{k+1}$, for $k, l \in \mathbb{N}$, and $p_{k,l} \downarrow p_k$. Now, we observe that the sequence

$$\varphi = (\varphi_n)_{n\in\mathbb{N}_0} = \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}_0}$$

belongs to $L(p-)$. Indeed, for every $k, l \in \mathbb{N}$ we have

$$\|\varphi\|_{p_{k,l}} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^{p_{k,l}} < \infty,$$

and hence, $\varphi \in \ell(p_k+) \subseteq L(p-)$. However, from (4.3) it follows that

$$C_t^{-1}(\varphi) = \left(\varphi_0, \left(\frac{n+1}{n+1} - \frac{nt}{n}\right)_{n\in\mathbb{N}}\right) = (1, 1-t, 1-t, \ldots). \tag{4.4}$$

Accordingly, for every $k, l \in \mathbb{N}$ we have

$$\sum_{n=0}^{\infty} |(C_t^{-1}(\varphi))_n|^{p_{k,l}} = 1 + \sum_{n=1}^{\infty} (1-t)^{p_{k,l}} = +\infty,$$

which implies that $C_t^{-1}(\varphi) \notin L(p-)$.

Since φ and $C_t^{-1}(\varphi)$ are decreasing sequences, we have that $\hat{\varphi} = \varphi$ and $\widehat{C_t^{-1}(\varphi)} = C_t^{-1}(\varphi)$. So, the same argument shows that $\varphi \in d(p_k+) \subseteq D(p-)$ (for every $k \in \mathbb{N}$), but, $C_t^{-1}(\varphi) \notin D(p-)$.

Since $D(p-) \subseteq C(p-)$ (see Proposition 7) and $\varphi \in D(p-)$, we also have that $\varphi \in C(p-)$. On the other hand, from (4.4) it follows that

$$C_1(C_t^{-1}(\varphi)) = \left(1, \left(1 - \frac{n}{n+1}t\right)_{n\in\mathbb{N}}\right) = \left(1, 1 - \frac{1}{2}t, 1 - \frac{2}{3}t, \ldots\right)$$

and hence,

$$\sum_{n=0}^{\infty} |(C_1(C_t^{-1}(\varphi)))_n|^{p_{k,l}} = \sum_{n=0}^{\infty} \left(1 - \frac{n}{n+1}t\right)^{p_{k,l}} = \infty,$$

as $\left(1 - \frac{n}{n+1}t\right)^{p_{k,l}} \to (1-t)^{p_{k,l}} \neq 0$. This means that $C_t^{-1}(\varphi) \notin C(p-)$.

We now pass to consider the case that $X \in \{L(p+), C(p+), D(p+)\}$, with $1 \leq p < \infty$ fixed. So, we take a strictly decreasing sequence $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k \downarrow p$ and strictly increasing sequences $\{p_{k,l}\}_{l\in\mathbb{N}}$ such that $p_{k+1} < p_{k,l} < p_k$, for $k, l \in \mathbb{N}$, and $p_{k,l} \uparrow p_k$. Arguing as above, we obtain that $\varphi \in X$, but $C_t^{-1}(\varphi) \notin X$. □

We denote

$$\Lambda := \left\{\frac{1}{n+1} \ n \in \mathbb{N}_0\right\} \quad \text{and} \quad \Lambda_0 := \Lambda \cup \{0\}.$$

Theorem 15 *Let $0 \leq t < 1$ and let X belong to $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$ or $\{L(p+), C(p+), D(p+); 1 \leq p < \infty\}$. Then*

$$\sigma_{pt}(C_t; X) = \Lambda \qquad \text{and} \qquad \sigma(C_t; X) = \sigma^*(C_t; X) = \Lambda_0.$$

Moreover, every $\lambda \in \Lambda$ *is a simple eigenvalue.*

Proof Let $X \in \{L(p-), C(p-), D(p-)\}$ be fixed, with $1 < p \leq \infty$. Then, for every $k \in \mathbb{N}$ we denote by X_k the Fréchet space $\ell(p_k+)$ if $X = L(p-)$ or $\operatorname{ces}(p_k+)$ $(d(p_k+))$ if $X = C(p-)$ (if $D(p-)$), where $\{p_k\}_{k\in\mathbb{N}}$ is any strictly increasing sequence satisfying $p_k \uparrow p$. By [8, Theorem 4.5] we have that

$$\sigma_{pt}(C_t; X_k) = \Lambda \quad \text{and} \quad \sigma(C_t; X_k) = \sigma^*(C_t; X_k) = \Lambda_0. \tag{4.5}$$

Since the restriction of C_t to X_k maps X_k into itself for every $k \in \mathbb{N}$, we can apply Lemma 10(i)–(ii) to obtain that

$$\Lambda = \sigma_{pt}(C_t; X) \subseteq \sigma(C_t; X) \subseteq \Lambda_0.$$

On the other hand, by Lemma 14 we have that $0 \in \sigma(C_t; X)$. Therefore, it follows that $\sigma(C_t; X) = \Lambda_0$. Moreover, the assumption in Lemma 10(iii) is fulfilled, and hence

$$\sigma^*(C_t; X) = \overline{\sigma(C_t; X)} = \Lambda_0.$$

If $\lambda \in \sigma_{pt}(C_t; X)$, then $\{0\} \neq \ker(\lambda I - C_t) \subseteq \ker(\lambda I - C_t^{\omega})$ (here, C_t^{ω} denotes the operator C_t acting in ω), as $X \subseteq \omega$. Accordingly, $0 < \dim\ker(\lambda I - C_t) \leq \dim\ker(\lambda I - C_t^{\omega}) = 1$ (see [8, Lemma 3.4(i)]). It follows that $\dim\ker(\lambda I - C_t) = 1$, i.e., λ is a simple eigenvalue.

Now, we suppose $X \in \{L(p+), C(p+), D(p+)\}$, with $1 \leq p < \infty$. Then for every $k \in \mathbb{N}$ we denote by X_k the (LB)-space $\ell(p_k-)$ if $X = L(p+)$ or $\operatorname{ces}(p_k-)$ $(d(p_k-))$ if $X = C(p+)$ (if $X = D(p+)$), where $\{p_k\}_{k\in\mathbb{N}}$ is any strictly decreasing sequence satisfying $p_k \downarrow p$. By [8, Theorem 5.3] we have that (4.5) is valid also in this case. Moreover, by [8, Theorem 5.3] we also know that $\dim\ker(\frac{1}{n+1}I - C_t) = 1$ in X_k, for each $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Since C_t maps X_k into itself for every $k \in \mathbb{N}$, we can apply Lemma 12(iv) to conclude that

$$\sigma_{pt}(C_t; X) = \bigcap_{k\in\mathbb{N}} \sigma_{pt}(C_t; X_k) = \Lambda.$$

By Lemma 12(i) we also obtain that

$$\sigma_{pt}(C_t; X) \subseteq \sigma(C_t; X) \subseteq \bigcup_{k=1}^{\infty} \sigma(C_t; X_k) = \Lambda_0.$$

Since by Lemma 14 we have that $0 \in \sigma(C_t; X)$, it then follows that $\sigma(C_t; X) = \Lambda_0$. Finally, by Lemma 12(ii)–(iii) we conclude that

$$\sigma^*(C_t; X) = \overline{\sigma(C_t; X)} = \Lambda_0.$$

The proof that every $\lambda \in \sigma_{pt}(C_t; X)$ is a simple eigenvalue follows as in the case of (LF)-spaces. ◻

4.2 Spectra of the Cesàro operator

We are now concerned about the study of the operator C_1 and of its spectra. To do this, we first observe that

Proposition 16 *Let X be one of the spaces in $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$ or in $\{L(p+), C(p+), D(p+); 1 \leq p < \infty\}$. Then $C_1 \in \mathcal{L}(X)$.*

Proof The proof is along the lines of the proof of Proposition 13, after having observed that the operator C_1 acts continuously in the Fréchet and (LB)-spaces considered in this paper, see [2, Sect. 2] for $\ell(p+)$, [4] for $\operatorname{ces}(p+)$, [6, Proposition 5.3] for $\ell(p-)$ and $\operatorname{ces}(p-)$, and [15, Proposition 4.9] for $d(p+)$ and $d(p-)$. □

In the literature, the spectra of C_1 are analyzed, among others, when C_1 acts in the Fréchet and (LB)-spaces defined in (1.3) and (1.4). We state and refer to these results below. For $1 \leq p < \infty$, we write

$$B(p/2, p/2) = \left\{z \in \mathbb{C} \,\left|\, \left|z - \frac{p}{2}\right| < \frac{p}{2}\right.\right\}.$$

Lemma 17 *Let X belong to $\{\ell(p+), \operatorname{ces}(p+), d(p+); 1 \leq p < \infty\}$. If $1 < p < \infty$, then*

$$\begin{aligned}
&\sigma_{pt}(C_1; X) = \emptyset \quad \text{and} \quad B(p'/2, p'/2) \subseteq \sigma_{pt}(C_1', X_\beta');\\
&\sigma(C_1; X) = B(p'/2, p'/2) \cup \{0\};\\
&\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)},
\end{aligned}$$

where $1 < p' < \infty$ satisfies $1/p + 1/p' = 1$. On the other hand, if $p = 1$, then

$$\begin{aligned}
&\sigma_{pt}(C_1; X) = \emptyset \quad \text{and} \quad \{z \in \mathbb{C} : Re z > 0\} \subseteq \sigma_{pt}(C_1'; X_\beta');\\
&\sigma(C_1; X) = \{z \in \mathbb{C} : Re z > 0\} \cup \{0\};\\
&\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \{z \in \mathbb{C} : Re z \geq 0\}.
\end{aligned}$$

Moreover, every $0 \neq \lambda \in \sigma(C_1; X)$ is a simple eigenvalue for C_1' acting in X_β'.

For the proof of the results in Lemma 17, we refer the reader to [2, Theorem 2.2] for $\ell(p+)$ $(1 < p < \infty)$, to [2, Theorem 2.4] for $\ell(1+)$, to [7, Theorem 2.3 and Proposition 2.4] for $\operatorname{ces}(p+)$ $(1 \leq p < \infty)$, and to [16, Theorem 3.2 and Proposition 3.3] for $d(p+)$ $(1 \leq p < \infty)$.

Lemma 18 *Let X belong to $\{\ell(p-), \operatorname{ces}(p-), d(p-); 1 < p \leq \infty\}$. Then*

$$\begin{aligned}
&\sigma_{pt}(C_1; X) = \emptyset \quad \text{and} \quad B(p'/2, p'/2) \subseteq \sigma_{pt}(C_1'; X_\beta');\\
&B(p'/2, p'/2) \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)};\\
&\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)}.
\end{aligned}$$

where $1 \leq p' < \infty$ satisfies $1/p + 1/p' = 1$.

The proof of Lemma 18 for $\ell(p-)$ is given in [8, Proposition 5.5], for $\mathrm{ces}(p-)$ is given in [6, Propositions 3.1, 3.2, 3.3], and for $d(p-)$ in [16, Theorem 3.6].

Let us begin the study of the spectra by considering $X \in \{L(p-), C(p-), D(p-)\}$, with $1 < p \leq \infty$.

Lemma 19 *Let X belong to $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$. Then $0 \in \sigma(C_1; X)$.*

Proof The formula in (4.3) is valid also for $t = 1$, i.e., the inverse operator $C_1^{-1} : \omega \to \omega$ exists in $\mathcal{L}(\omega)$ and it is given by

$$C_1^{-1}(x) = \big(x_0, \big((n+1)x_n - nx_{n-1}\big)_{n\in\mathbb{N}}\big), \qquad x = (x_n)_n \in \omega.$$

We consider the following sequence as in the proof of [6, Proposition 3.2]:

$$x = (x_n)_n = \Big(\frac{1-(-1)^{n+1}}{2(n+1)}\Big)_{n\in\mathbb{N}_0} = (1, 0, 1/3, 0, 1/5, \ldots). \tag{4.6}$$

Therefore,

$$\hat{x} = (1, 1/3, 1/3, 1/5, 1/5, \ldots). \tag{4.7}$$

Now, let $X \in \{L(p-), C(p-), D(p-)\}$ be fixed, with $1 < p \leq \infty$, and let $\{p_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence satisfying $1 < p_k \uparrow p$. Then for every $k \in \mathbb{N}$ we denote by X_k the k-th step of the inductive spectrum defining X as done in the previous subsection (i.e., X_k is one of the Fréchet spaces $\ell(p_k+)$, $\mathrm{ces}(p_k+)$, $d(p_k+)$). So, for every $k \in \mathbb{N}$ we have the sequence x defined in (4.6) satisfies $x \in d(p_k+) \subseteq \ell(p_k+) \subseteq \mathrm{ces}(p_k+)$ (see (4.7)) and hence, $x \in X_k \subseteq X$. However, the n-th entry, for $n \in \mathbb{N}$, of $C_1^{-1}(x)$ is given by

$$(n+1)\frac{(1-(-1)^{n+1})}{2(n+1)} - n\frac{(1-(-1)^n)}{2n} = \frac{(-1)^n + (-1)^n}{2} = (-1)^n.$$

Since $x_0 = 1$, we get that $C_1^{-1}(|x|) = (|(-1)^n|)_{n\in\mathbb{N}_0}$, which does not belong to $\mathrm{ces}(p_k+)$ for all $k \in \mathbb{N}$, as $C_1(|C_1^{-1}(x)|) = (1, 1, 1, 1, 1, \ldots)$. Accordingly, $C_1^{-1}(|x|)$ does not belong to either $d(p_k+)$ or to $\ell(p_k+)$ for all $k \in \mathbb{N}$. Therefore, $C_1^{-1}(|x|) \notin X$. Thus, we obtain $0 \in \sigma(C_1; X)$ as we wanted. ◻

In the following result, we see that the behaviour of the spectra of C_1 in the (LF)-spaces considered in this paper is similar to that in the (LB)-spaces in (1.4) (cf. Lemma 18):

Theorem 20 *Let X belong to $\{L(p-), C(p-), D(p-); 1 < p \leq \infty\}$. Then*

$$\sigma_{pt}(C_1; X) = \emptyset \quad \textit{and} \quad B(p'/2, p'/2) \subseteq \sigma_{pt}(C_1'; X_\beta');$$
$$B(p'/2, p'/2) \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)};$$

$$\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)},$$

where $1 \leq p' < \infty$ *satisfies* $1/p + 1/p' = 1$.

Proof Let $X \in \{L(p-), C(p-), D(p-)\}$ be fixed, with $1 < p \leq \infty$. So, let $\{p_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence satisying $1 < p_k \uparrow p$. Then for every $k \in \mathbb{N}$ we denote by X_k the k-th step of the inductive spectrum defining X as done in the previous subsection (i.e., X_k is one of the Fréchet spaces $\ell(p_k+)$, $\mathrm{ces}(p_k+)$, $d(p_k+)$). We observe that the conjugate exponents p'_k form a strictly decreasing sequence such that $p'_k \downarrow p'$, where $1 \leq p' < \infty$ satisfies $1/p + 1/p' = 1$. We point out that also in the case $p = \infty$, we have $1 < p_k < \infty$ for every $k \in \mathbb{N}$. Since the assumptions in Lemma 10 are satisfied, we can apply Lemma 10(i)–(ii) combined with Lemma 17 to deduce that $\sigma_{pt}(C_1; X) = \emptyset$ and

$$\sigma(C_1; X) \subseteq \bigcap_{m\in\mathbb{N}} \Big(\bigcup_{k=m}^{\infty} \sigma(C_1; X_k) \Big) = \bigcap_{m\in\mathbb{N}} \big(B(p'_m/2, p'_m/2) \cup \{0\} \big)$$
$$= \overline{B(p'/2, p'/2)}. \tag{4.8}$$

Accordingly,

$$\overline{\sigma(C_1; X)} \subseteq \overline{B(p'/2, p'/2)}.$$

We now show that $B(p'/2, p'/2) \subseteq \sigma_{pt}(C'_1; X'_\beta)$. To this end, we recall that by Corollaries 3 and 5 we have that the strong dual X'_β of X is reflexive (hence barrelled) and it is given by $X'_\beta = \bigcap_{k\in\mathbb{N}} (X_k)'_\beta$, where each $(X_k)'_\beta$ is one of the (LB)-spaces $\ell(p'_k-)$, $\mathrm{ces}(p'_k-)$ or $d(p'_k-)$. On the other hand, by Lemma 17 we have that $B(p'_k/2, p'_k/2) \subseteq \sigma_{pt}(C'_1; (X_k)'_\beta)$ for every $k \in \mathbb{N}$. Since the assumptions in Lemma 12 are clearly satisfied with $T = C'_1$ and every element in $B(p'/2, p'/2)$ is a simple eigenvalue for C'_1, we can apply Lemma 12(iv) to deduce that

$$B(p'/2, p'/2) \subseteq \bigcap_{k\in\mathbb{N}} B(p'_k/2, p'_k/2) \subseteq \bigcap_{k\in\mathbb{N}} \sigma_{pt}(C'_1; (X_k)'_\beta) = \sigma_{pt}(C'_1; X'_\beta).$$

So, it follows via (4.1) that

$$B(p'/2, p'/2) \subseteq \sigma_{pt}(C'_1; X'_\beta) \subseteq \sigma(C'_1; X'_\beta) = \sigma(C_1; X).$$

Since $0 \in \sigma(C_1; X)$ by Lemma 19, it follows by (4.8) that

$$B(p'/2, p'/2) \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)}.$$

Since $\sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)}$ and $\overline{\sigma(C_1; X)} \subseteq \sigma^*(C_1; X)$, we can argue as the proof of [6, Proposition 3.3] to conclude that

$$\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)}.$$

□

We now pass to study the spectra of C_1 acting in the (PLB)-spaces $L(p+)$, $C(p+)$, and $D(p+)$, for $1 \le p < \infty$. We first show that the analogous of Lemma 19 holds also in this case. The argument is similar.

Lemma 21 *Let X belong to $\{L(p+), C(p+), D(p+); 1 \le p < \infty\}$. Then $0 \in \sigma(C_1; X)$.*

Proof Let $X \in \{L(p+), C(p+), D(p+)\}$ be fixed, with $1 \le p < \infty$ and let $\{p_k\}_{k\in\mathbb{N}}$ be a strictly decreasing sequence satisfying $p < p_k \downarrow p$. Then for every $k \in \mathbb{N}$ we denote by X_k the k-th step of the projective spectrum defining X as done in the previous subsection (i.e., X_k is one of the (LB)-spaces $\ell(p_k-)$, $\mathrm{ces}(p_k-)$, $d(p_k-)$ with $p_k > p \ge 1$).

We now observe that the sequence x defined in (4.6) satisfies $x \in d(p_k-) \subseteq \ell(p_k-) \subseteq \mathrm{ces}(p_k-)$ (see (4.7)) for all $k \in \mathbb{N}$, as each $p_k > 1$, and hence, $x \in X_k$ for all $k \in \mathbb{N}$. Accordingly, $x \in X$. But, $C_1^{-1}(|x|) = (1)_{n\in\mathbb{N}_0}$ (see the proof of Lemma 19) and so, $C_1^{-1}(|x|)$ does not belong to $\mathrm{ces}(p_k-)$ for all $k \in \mathbb{N}$, as $C_1(|C_1^{-1}(x)|) = (1)_{n\in\mathbb{N}}$. This implies that $C_1^{-1}(x) \notin X$. Therefore, we deduce that $0 \in \sigma(C_1, X)$. □

We are ready to study the spectra of C_1 in the (PLB) sequence spaces considered.

Theorem 22 *Let X belong to $\{L(p+), C(p+), D(p+); 1 \le p < \infty\}$. If $1 < p < \infty$, then*

$$\sigma_{pt}(C_1; X) = \emptyset \quad \text{and} \quad B(p'/2, p'/2) \subseteq \sigma_{pt}(C_1'; X_\beta');$$
$$B(p'/2, p'/2) \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)};$$
$$\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)},$$

where $1 < p' < \infty$ satisfies $1/p + 1/p' = 1$. On the other hand, if $p = 1$, then

$$\sigma_{pt}(C_1; X) = \emptyset \quad \text{and} \quad \{z \in \mathbb{C} : \ Re z > 0\} \subseteq \sigma_{pt}(C_1'; X_\beta');$$
$$\{z \in \mathbb{C} : \ Re z > 0\} \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \{z \in \mathbb{C} : \ Re z \ge 0\};$$
$$\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \{z \in \mathbb{C} : \ Re z \ge 0\}.$$

Proof Let $X \in \{L(p+), C(p+), D(p+)\}$ be fixed, with $1 \le p < \infty$ and let $\{p_k\}_{k\in\mathbb{N}}$ be a strictly decreasing sequence satisfying $p_k \downarrow p$, with $p < p_k$ for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ we denote by X_k the k-th step of the projective spectrum defining X as done in the previous subsection (i.e., X_k is one of the (LB)-spaces $\ell(p_k-)$, $\mathrm{ces}(p_k-)$, $d(p_k-)$ with $p_k > p \ge 1$). We observe that the conjugate exponent p_k' forms a strictly increasing sequence such that $p_k' \uparrow p'$ with $1 < p_k' < p'$, where $1 < p' \le \infty$.

Fix $p \ne 1$. Since the assumptions in Lemma 12 are clearly satisfied, we can apply Lemma 12(i) combined with Lemma 18 to deduce that

$$\sigma_{pt}(C_1; X) = \bigcap_{k\in\mathbb{N}} \sigma_{pt}(C_1; X_k) = \emptyset$$

and that

$$\sigma(C_1; X) \subseteq \bigcup_{k\in\mathbb{N}} \sigma(C_1; X_k) \subseteq \bigcup_{k\in\mathbb{N}} \overline{B(p'_k/2, p'_k/2)} \subseteq \overline{B(p'/2, p'/2)}.$$

We now show that $B(p'/2, p'/2) \subseteq \sigma_{pt}(C'_1; X'_\beta)$. To this end, we recall that by Proposition 4 and Corollary 5, the (PLB)-space X is reflexive and its strong dual X'_β is given by $X'_\beta = \bigcup_{k\in\mathbb{N}}(X_k)'_\beta$, where each $(X_k)'_\beta$ is one of the Fréchet spaces $\ell(p'_k+)$, ces(p'_k+) or $d(p'_k+)$. On the other hand, by Lemma 18 we have that $B(p'_k/2, p'_k/2) \subseteq \sigma_{pt}(C'_1; (X_k)'_\beta)$ for every $k \in \mathbb{N}$. Since the assumption in Lemma 10 is clearly satisfied with $T = C'_1$, we can apply Lemma 10(i) to deduce that

$$B(p'/2, p'/2) = \bigcup_{k\in\mathbb{N}} B(p'_k/2, p'_k/2) \subseteq \bigcup_{k\in\mathbb{N}} \sigma_{pt}(C'_1; (X_k)'_\beta) = \sigma_{pt}(C'_1; X'_\beta).$$

So, it follows by (4.1) that

$$B(p'/2, p'/2) \subseteq \sigma_{pt}(C'_1; X'_\beta) \subseteq \sigma(C'_1; X'_\beta) = \sigma(C_1; X).$$

Since $0 \in \sigma(C_1; X)$ by Lemma 21, we conclude that

$$B(p'/2, p'/2) \cup \{0\} \subseteq \sigma(C_1; X) \subseteq \overline{B(p'/2, p'/2)}$$

and hence,

$$\overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)}.$$

Since by Lemma 18 we have $\sigma^*(C_1; X_k) = \overline{B(p'_k/2, p'_k/2)}$ for every $k \in \mathbb{N}$, it follows that

$$\bigcup_{k\in\mathbb{N}} \sigma^*(C_1; X_k) = \bigcup_{k\in\mathbb{N}} \overline{B(p'_k/2, p'_k/2)} \subseteq \overline{B(p'/2, p'/2)} = \overline{\sigma(C_1; X)}.$$

We can then apply Lemma 12(iii) to conclude that $\sigma^*(C_1; X) = \overline{\sigma(C_1; X)} = \overline{B(p'/2, p'/2)}$.

Now, consider $p = 1$. In such a case, the result follows by arguing as above and observing that

$$\bigcup_{k\in\mathbb{N}} B(p'_k/2, p'_k/2) = \{z \in \mathbb{C} : \ Re z > 0\},$$

as $p'_k \to \infty$. □

5 Dynamics of generalized Cesàro operators in (LF) and (PLB)-spaces

Let X be a lcHs. For $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}$, we write $T^n = T \circ \cdots \circ T$. The Cesàro means of T are denoted by

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m.$$

We say that T is

(1) *power bounded* if $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous in $\mathcal{L}(X)$;
(2) *(uniformly) mean ergodic* if $\{T_{[n]}\}_{n\in\mathbb{N}}$ converges in $\mathcal{L}_s(X)$ (in $\mathcal{L}_b(X)$).

For a separable lcHs X, we say that T is

(3) *hypercyclic* if there exists $x \in X$ whose orbit $\{T^n x \ \ n \in \mathbb{N}_0\}$ is dense in X;
(4) *supercyclic* if there exists $x \in X$ such that the projective orbit $\{\lambda T^n x \ \ \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$ is dense in X.

We refer the reader to [11, 26] for general textbooks.

5.1 Dynamics of generalized Cesàro operators C_t ($0 \le t < 1$)

Proposition 23 *Let $0 \le t < 1$ and let X belong to $\{L(p-), C(p-), D(p-); 1 < p \le \infty\}$ or to $\{L(p+), C(p+), D(p+); 1 \le p < \infty\}$. Then the generalized Cesàro operator C_t is not supercyclic in X.*

Proof Since the operator C_t is not supercyclic in ω by [8, Theorem 6.1(iii)], it follows that C_t cannot be supercyclic in X, as C_t continuously maps X into itself and X is dense in ω. □

To study the power boundedness and the mean ergodicity of C_t acting in the (LF)-spaces (in the (PLB)-spaces) considered in this paper, we first establish some results on continuous linear operators to compare their ergodic properties with the ones of their inductive spectrum (projective spectrum).

We first consider the case of operators acting in (LF)-spaces.

Theorem 24 *Let $X = \operatorname{ind}_k X_k = \bigcup_{k\in\mathbb{N}} X_k$ be an (LF)-space such that the inclusion $X_k \subseteq X_{k+1}$ is continuous, for $k \in \mathbb{N}$, and let $T \in \mathcal{L}(X)$ satisfy assumption (A) of Lemma 10. Then the following properties are satisfied.*

(i) *If $T_k := T|_{X_k}$ is power bounded in X_k for every $k \in \mathbb{N}$, then T is power bounded in X.*
(ii) *If T_k is mean ergodic in X_k for every $k \in \mathbb{N}$, then T is mean ergodic in X.*
(iii) *If T_k is uniformly mean ergodic in X_k for every $k \in \mathbb{N}$ and X is regular, then T is uniformly mean ergodic in X.*

Proof (i) Let $x \in X$ be fixed. Then there exists $k \in \mathbb{N}$ so that $x \in X_k$. Since $T(X_k) = T_k(X_k) \subseteq X_k$, we have that $T^n x \in X_k$ for every $n \in \mathbb{N}$. But, $T = T_k$

is power bounded in X_k. So, we have that $\{T^n x : n \in \mathbb{N}\}$ is bounded in X_k, and hence, it is in X. Since $x \in X$ is arbitrary, and X is barrelled, we conclude that $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous, i.e., T is power bounded in X.

(ii) Let $x \in X$ be fixed. Then there exists $k \in \mathbb{N}$ such that $x \in X_k$. Since $T = T_k \colon X_k \to X_k$ is mean ergodic, we have that $\{T_{[n]}x\}$ is convergent in X_k and hence, $\{T_{[n]}x\}$ is also convergent in X, as the inclusion $X_k \subseteq X$ is continuous. Since $x \in X$ is arbitrary, we can conclude that T is mean ergodic in X.

(iii) We assume that X is regular and $T_k = T|_{X_k}$ is uniformly mean ergodic in each X_k. By (ii), we have that $T_{[n]}$ converges to some $P \in \mathcal{L}(X)$ in $\mathcal{L}_s(X)$. We fix $B \in \mathcal{B}(X)$. Then, there is $k \in \mathbb{N}$ so that $B \in \mathcal{B}(X_k)$. By assumption on $T \in \mathcal{L}(X_k)$, we have that for every $s \in \Gamma_k = \Gamma_{X_k}$,

$$\sup_{x\in B} s(T_{[n]}x - Px) \to 0 \qquad \text{as } n \to \infty.$$

Now, we take $r \in \Gamma_X$. Since $X_k \subseteq X$ with continuous inclusion, there exists $C > 0$ and $s \in \Gamma_k = \Gamma_{X_k}$ so that

$$r(x) \le Cs(x), \qquad x \in X_k.$$

Therefore,

$$\sup_{x\in B} r(T_{[n]}x - Px) \le C \sup_{x\in B} s(T_{[n]} - Px),$$

from which it follows

$$\sup_{x\in B} r(T_{[n]}x - Px) \to 0, \qquad \text{as } n \to \infty.$$

This completes the proof of point (iii). □

Corollary 25 *Let $0 \le t < 1$ and let X belong to $\{L(p-), C(p-), D(p-); 1 < p \le \infty\}$. Then the generalized Cesàro operator C_t is power bounded and uniformly mean ergodic in X.*

Proof The result follows by Theorem 24 and [8, Theorem 6.6]. □

We now pass to the case of operators acting in (PLB)-spaces.

Theorem 26 *Let $X = \mathrm{proj}_k X_k = \bigcap_{k\in\mathbb{N}} X_k$ be a (PLB)-space and let $T \in \mathcal{L}(X)$ satisfy the assumption (A') in Lemma 12, i.e., for every $k \in \mathbb{N}$ there exists $T_k \in \mathcal{L}(X_k)$ such that $T_k|_X = T$ and $T_k|_{X_{k+1}} = T_{k+1}$. If T_k is power bounded ((uniformly) mean ergodic) in X_k for every $k \in \mathbb{N}$, then T is power bounded ((uniformly) mean ergodic) in X.*

Proof Let us first prove the power boundedness. We write Γ_k to denote a fundamental system of seminorms in X_k, for $k \in \mathbb{N}$, and we set

$$\Gamma := \Big\{ \max_{i=1,\dots,k} r_i : k \in \mathbb{N},\ r_i \in \Gamma_i,\ i = 1, \dots, k \Big\}.$$

Then Γ is a fundamental system of seminorms in X.

Fixed $r \in \Gamma$, there exist $k \in \mathbb{N}$ and $r_i \in \Gamma_i$ for $i = 1, \dots, k$ such that $r = \max_{i=1,\dots,k} r_i$. Since T_i is power bounded in X_i, for $i = 1, \dots, k$, we have that for every $i = 1, \dots, k$, there exist $C_i > 0$ and $s_i \in \Gamma_i$ such that

$$r_i(T_i^n x) \le C_i s_i(x), \qquad x \in X_i,\ n \in \mathbb{N}.$$

Thus, for $C := \max_{i=1,\dots,k} C_i$, we have

$$r(T^n x) = \max_{i=1,\dots,k} r_i(T_i^n x) \le \max_{i=1,\dots,k} C_i s_i(x) \le C \max_{i=1,\dots,k} s_i(x)$$

for every $x \in X$ and $n \in \mathbb{N}$, as $T_i|_X = T$ for $i = 1, \dots, k$. Therefore, T is power bounded in X.

We now pass to consider the mean ergodicity. So, we fix $x \in X$. Then, $x \in X_k$ for every $k \in \mathbb{N}$. Since $T_k : X_k \to X_k$ is mean ergodic for every $k \in \mathbb{N}$, $(T_k)_{[n]}x$ converges to some y_k in X_k for every $k \in \mathbb{N}$. For $k = 2$, we have that $(T_2)_{[n]}x$ converges to y_2 in X_2, and also in X_1, as $X_2 \subseteq X_1$ with continuous inclusion. But, $(T_2)_{[n]}x = (T_1)_{[n]}x$ (as $x \in X \subseteq X_k$ for every k) converges to y_1 in X_1. Thus $y_2 = y_1$. Proceeding inductively, we can see that all the y_k coincide, and denoting it by y, we have that $y \in X$ and $T_{[n]}x$ converges to y in X.

Finally, we consider the uniformly mean ergodicity. So, we fix $B \in \mathcal{B}(X)$. Then $B \in \mathcal{B}(X_k)$ for every $k \in \mathbb{N}$. Since $T_k : X_k \to X_k$ is uniformly mean ergodic for every $k \in \mathbb{N}$, there exists $P_k \in \mathcal{L}(X_k)$ such that for every $r_k \in \Gamma_k$ we have (for every $k \in \mathbb{N}$ we have $T = T_k$ on B as $B \subseteq X$)

$$\sup_{x \in B} r_k(T_{[n]}x - P_k x) = \sup_{x \in B} r_k((T_k)_{[n]}x - P_k x) \to 0.$$

This yields that $T : X \to X$ is uniformly mean ergodic in X and $P_k|_X = P_{k+1}|_X$ for every $k \in \mathbb{N}$. In particular, the operator $P : X \to X$ defined by $Px := P_1 x$ for $x \in X$ belongs to $\mathcal{L}(X)$ and $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$. □

Corollary 27 *Let $0 \le t < 1$ and let X belong to $\{L(p+), C(p+), D(p+); 1 \le p < \infty\}$. Then the generalized Cesàro operator C_t is power bounded and uniformly mean ergodic in X.*

Proof The result follows as an immediate application of Theorem 26 and [8, Theorem 6.6]. □

5.2 Dynamics of the Cesàro operator

It is known that the Cesàro operator C_1 is neither power bounded nor mean ergodic in ℓ_p $(1 < p < \infty)$ (see [1, Proposition 4.2]), and it cannot be supercyclic since $\sigma_{pt}(C_1'; \ell_{p'})$ contains too many elements (see [11, Proposition 1.26]). Furthermore, the same characteristics hold for ces(p) $(1 < p < \infty)$ (see [5, Proposition 3.7(ii)]) and for $d(p)$ $(1 < p < \infty)$ (see [14, Propositions 3.10 and 3.11]).

For the Fréchet spaces and (LB)-spaces defined in (1.3) and (1.4), we have that C_1 is not mean ergodic nor power bounded nor supercyclic in $\ell(p+)$ $(1 \le p < \infty)$ [2, Theorems 2.3 and 2.5], in $\mathrm{ces}(p+)$ $(1 \le p < \infty)$ [7, Proposition 5], in $d(p+)$ $(1 \le p < \infty)$ [16, Proposition 3.5], in $\ell(p-)$ $(1 < p \le \infty)$ [8, Proposition 6.10], in $\mathrm{ces}(p-)$ $(1 < p \le \infty)$ [6, Propositions 3.4, 3.5], and in $d(p-)$ $(1 < p \le \infty)$ [16, Proposition 3.8].

On the other hand, the dynamics of C_1 in ω are the same as the ones for C_t $(0 \le t < 1)$ in ω. Indeed, by [7, Theorem 6.1] and [2, Proposition 4.3], we have that C_1 is power bounded, mean ergodic, and not supercyclic in ω.

The proof of the following result is similar to the ones from the references above.

Proposition 28 *Let X belong to $\{L(p-), C(p-), D(p-), L(q+), C(q+), D(q+);\ 1 < p \le \infty,\ 1 \le q < \infty\}$. Then C_1 is neither power bounded nor mean ergodic nor supercyclic in X.*

Proof By Theorems 20 and 22 we have that $B(p'/2, p'/2)$, for $1 < p \le \infty$ ($B(q'/2, q'/2)$, for $1 < q < \infty$ and $\{z \in \mathbb{C}\ Re z > 0\}$ for $q = 1$) are included in $\sigma_{pt}(C_1'; X_\beta')$. So, by [11, Proposition 1.26] we conclude that C_1 cannot be supercyclic in X. Furthermore, for $p, q < \infty$,

$$\sigma_{pt}(C_1'; X_\beta') \cap \{\lambda \in \mathbb{C} :\ |\lambda| > 1\} \ne \emptyset.$$

Thus, by [6, Lemma 3.2], C_1 is neither power bounded nor mean ergodic. The proof for $p = \infty$ is the same as in [6, Proposition 3.4]. $\square$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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