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Existence of Traveling Waves of a Diffusive Susceptible–Infected–Symptomatic–Recovered Epidemic Model with Temporal Delay

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Abstract: The aim of this article is to investigate the existence of traveling waves of a diffusive model that represents the transmission of a virus in a determined population composed of the following populations: susceptible (S), infected (I), asymptomatic (A), and recovered (R). An analytical study is performed, where the existence of solutions of traveling waves in a bounded domain is demonstrated. We use the upper and lower coupled solutions method to achieve this aim. The existence and local asymptotic stability of the endemic (E_e) and disease-free (E_0) equilibrium states are also determined. The constructed model includes a discrete-time delay that is related to the incubation stage of a virus. We find the crucial basic reproduction number $\mathcal{R}_0$, which determines the local stability of the steady states. We perform numerical simulations of the model in order to provide additional support to the theoretical results and observe the traveling waves. The model can be used to study the dynamics of SARS-CoV-2 and other viruses where the disease evolution has a similar behavior.

Keywords: diffusive mathematical models; traveling wave solutions; SARS-CoV-2 virus; discrete-time delay

MSC: 34K05; 37N25; 92D30

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1. Introduction

Mathematical models have been extensively used to study the dynamics of many infectious diseases [1]. It is common to use differential equations to construct these models. The most well-known models are based on ordinary differential equations (ODEs), partial differential equations (PDEs), and delay differential equations (DDEs) [2–5]. The SIR (susceptible–infected–recovered) model is by far the most commonly used mathematical model based on differential equations [1]. Recently, the SAIR model, or its variants, has been used to address well-known asymptomatic cases [6–9]. In [10], a SIR model that incorporates awareness and a time delay to account for the latent stage was studied. These previous studies have used ODEs, as specific spatial effects were not considered.

Often in mathematical epidemiology, models that consider the spatial component, where individuals live, are usually described using partial differential equations. This is justified by the movement of individuals from one place to another, either within a region or through the emigration of the population to other regions through its borders. Another important variable in modeling infectious diseases is the delay in the appearance of the symptoms of the disease after individuals become infected. In [11], the authors proposed an epidemic SIR mathematical model with spatial diffusion and time delay. The authors investigated the existence of traveling waves in their proposed SIR model. In addition, they determined the stationary states and analyzed the existence of waves through the existence of upper and lower solutions. Other works have incorporated discrete-time delays in partial differential equations [12–14]. For instance, in [12], the authors analyzed the existence of wavefront solutions using a monotonous iteration scheme for the wave system. The authors proved that the wave system has a solution if the quasi-monotonicity (QM) condition is satisfied.

The incorporation of time delays can make the mathematical models more realistic, but then these models and their analysis become more complex. Thus, time delays can be incorporated into reaction-diffusion systems, but then classical techniques such as the phase plane cannot be applied to guarantee the existence of the solution. However, by applying the monotonous iteration technique, the existence of a traveling wave solution was demonstrated for a reaction-diffusion model with delays [13]. The monotonous iteration technique was demonstrated in [12,14]. Another interesting work was presented in [15]. The authors found and proved the existence of traveling waves when they analyzed an SEIR-type model represented by a system of reaction-diffusion equations. In addition, the authors computed the basic reproduction number, which is a threshold parameter for the long-term dynamics, and used it to prove the existence of traveling waves.

In this direction of epidemiological applications, in [16], the authors analyzed the role of population mobility in the transmission of coronavirus disease 2019 (COVID-19) using a nonlinear parabolic system. The authors presented alternatives to control the transmission of the virus by applying restrictions such as the closure of borders, reduction of travel, and interruption of human mobility. In another interesting work [17], the authors presented an epidemiological SEIR-type model with reaction-diffusion terms, nonlinear incidence, and distributive delay. The authors applied the Lyapunov function methodology and the basic reproduction number to analyze the stability of the stationary states.

In this paper, we present a SAIR-type epidemiological model based on a system of partial differential equations, incorporating a discrete-time delay to mimic the virus's incubation period within a host and the time it takes for a person to become infected [10,18–21]. The spatial effect is justified due to people's mobility and the spread of the virus [6,16,22,23]. Thus, the main contributions of this paper are the introduction of spatial effects in conjunction with the time delay due to the latent stage of individuals. Moreover, the model includes asymptomatic cases and considers the possibility of death for asymptomatic individuals. In addition, we find the crucial basic reproduction number $\mathcal{R}_0$, which determines the local stability of the steady states. In general, this theoretical result is more complex to achieve in mathematical models that include spatial effects and time delays compared to classical models based on ODEs.

There are different ways to introduce a discrete-time delay in an epidemiological mathematical model. This delay takes into account the fact that when a susceptible individual has effective contact with an infected individual, the susceptible person does not become infectious right away, i.e., they are unable to spread the virus [24–29]. In this article, we use a particular approach to introduce the time delay that has been used in several previous papers related to the mathematical modeling of epidemics [11,30,31]. We aim to analyze the existence and uniqueness of a traveling wave that connects the two equilibrium points of the system. In addition, we will analyze the stability of the stationary points of the model. The proposed mathematical model represents the transmission of a virus in a determined population composed of susceptible (S), infected (I), asymptomatic (A), and recovered (R) individuals. As with any real-world mathematical model, there are simplifications and assumptions. However, this type of study provides additional insights into the dynamics of different diseases [1,26,32]. We perform an analytical study, where the existence of solutions of traveling waves in a bounded domain is demonstrated. We use a method that is based on the upper and lower coupled solutions to prove this existence. The existence and local asymptotic stability of the endemic (E_e) and disease-free (E_0) equilibrium states are determined. It should be mentioned that the proposed mathematical model can be used to study the dynamics of SARS-CoV-2 and other viruses where the disease status evolution is similar.

This paper is organized as follows. Section 2 introduces the constructed model and Section 3 provides the mathematical analysis of the existence and uniqueness of the solution. The stability of the equilibrium points is then discussed in Section 4. In Section 5, numerical simulations of the solutions are presented, and finally, in Section 6, the conclusions of this study are given.

2. Materials and Methods

In this section, we present the most important elements that are often used to design new epidemiological mathematical models and the mathematical tools that can be used to demonstrate that the model is well posed.

2.1. Mathematical Model

In this subsection, we present a mathematical model that is based on a system of partial differential equations with a discrete-time delay, where the population has been subdivided into the following subpopulations: susceptible (S) people who do not have the virus but can be infected, infected (I) people with symptoms of the disease who have the ability to infect susceptible individuals, asymptomatic (A) infected people who can transmit the disease but do not present symptoms of the disease, and finally, recovered (R) people who have had the disease but do not transmit it.

Let us consider a bounded domain $\Omega \subset R$ and a time interval $[0, T]$. We assume that $N : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is a non-negative function representing the spatio-temporal density of the population. That is, $N(x,t)$ denotes the number of individuals over the point $xin\overline{\Omega}$ at the instant $tin[0, T]$. It is also assumed that $S, I, A, R : \overline{\Omega} \times [0, T] \to \mathbb{R}$ are non-negative functions such that $N(x,t) = S(x,t) + I(x,t) + A(x,t) + R(x,t)$, and they describe the densities of the subpopulations that interact in this model. These functions are given as follows:

- $S(x,t)$ is the population susceptible to the virus in space and time;
- $I(x,t)$ is the infected population presenting symptoms of the disease and transmitting the virus in space and time;
- $A(x,t)$ is the population infected by the virus that does not present symptoms, that is, it is asymptomatic and transmits the virus in space and time;
- $R(x,t)$ is the population that has recovered from the disease and does not present symptoms of the disease but is under medical treatment.

With the above considerations, we present a model governed by partial differential equations, where the interaction and flow of the subpopulations involved in this class of epidemiological models are shown in Figure 1. Thus, one has

$$\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= \Lambda - \beta S(x,t)[I(x,t-\tau) + A(x,t-\tau)] - \mu S(x,t) \\
&\quad + \nabla \cdot [\nu_S \nabla S(x,t)], \\
\frac{\partial I(x,t)}{\partial t} &= r\beta S(x,t)[I(x,t-\tau) + A(x,t-\tau)] - \mu I(x,t) - \eta I(x,t) \\
&\quad - \alpha I(x,t) + \nabla \cdot [\nu_I \nabla I(x,t)], \\
\frac{\partial A(x,t)}{\partial t} &= (1-r)\beta S(x,t)[I(x,t-\tau) + A(x,t-\tau)] - \mu A(x,t) \\
&\quad - \alpha A(x,t) + \nabla \cdot [\nu_A \nabla A(x,t)], \\
\frac{\partial R(x,t)}{\partial t} &= \alpha[I(x,t) + A(x,t)] - \mu R(x,t) + \nabla \cdot [\nu_R \nabla R(x,t)],
\end{aligned} \tag{1}$$

where $\nabla = \left(\frac{\partial}{\partial x}\right)$ is the vector differential operator; Λ represents the rate of new people entering the susceptible population; α is the recovery rate; η is the mortality rate caused by infection; β is the contagion rate; μ is the general natural mortality rate; r is the proportion of people who become infected and present symptoms; and ν_S, ν_A, ν_I, ν_R are diffusion parameters, indicating the space and time mobility of the susceptible, infected, and recovered individuals on those classes, respectively [33].

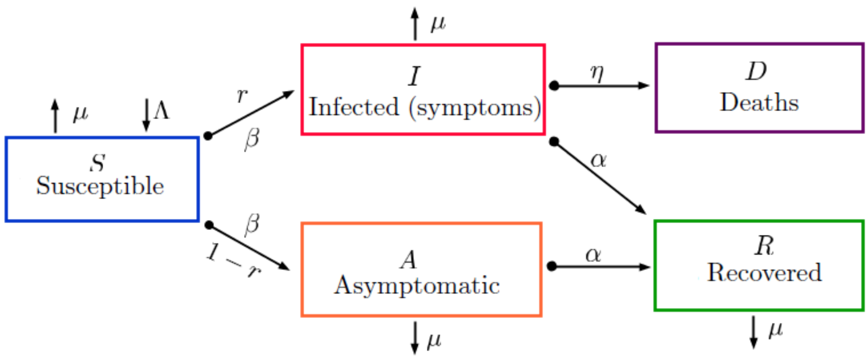

Figure 1. Diagram of the mathematical model (1) based on a system of partial differential equations with a discrete-time delay τ.

The mathematical model (1) is considered closed. Therefore, we consider Neumann-type boundary conditions. That is, we assume that $\mathbf{n}(x,t)$ is the unit normal vector of $\overline{\Omega}$ to the point $(x,t) \in \partial\Omega \times [0,T]$, where $\partial\Omega$ represents the border of Ω and $\mathcal{C} := C(\Omega \times [-\tau,0], \mathbb{R}^4)$ for $\tau > 0$ represents the set of continuous functions defined in $\Omega \times [-\tau,0]$. Due to the above, the initial and boundary conditions of the mathematical model (1) are as follows:

$$\begin{cases} \nabla u(x,t) \cdot \mathbf{n}(t) = \mathbf{0} & (x,t) \in \partial\Omega \times (0,T]; \\ u_0(x,\theta) = \varphi(x,\theta) & \varphi \in \mathcal{C}, \end{cases} \tag{2}$$

where $u = (S,I,A,R)^T$ and $\varphi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T$ such that φ_i is a non-negative real function for $i = 1,2,3,4$.

2.2. Existence of Equilibrium States

Now, we analyze the equations of system (1) with conditions (2) to determine the existence of a disease-free equilibrium and an endemic equilibrium. In this way, the stability of these steady states is shown. The Neumann boundary conditions are given in (2) and indicate that

$$\frac{\partial s(x,t)}{\partial n} = \frac{\partial I(x,t)}{\partial n} = \frac{\partial A(x,t)}{\partial n} = \frac{\partial R(x,t)}{\partial n} = 0 \ \forall (x,t) \in \partial\Omega \times [0,T],$$

where $\frac{\partial}{\partial n}$ denotes the normal outward derivative over $\partial\Omega$. These boundary conditions mean that subpopulations do not cross the boundary $\partial\Omega$.

The disease-free equilibrium E_0 is obtained by setting $A = I = R = 0$. Thus,

$$E_0\left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \tag{3}$$

Now, if we assume $I, A > 0$, the solution of the following system represents the endemic stationary state, i.e,

$$\begin{aligned} \Lambda - \beta S^e(I^e + A^e) - \mu S^e &= 0, \\ r\beta S^e(I^e + A^e) - (\mu + \alpha + \eta)I^e &= 0, \\ (1-r)\beta S^e(I^e + A^e) - (\mu + \alpha)A^e &= 0, \\ \alpha(I^e + A^e) - \mu R^e &= 0. \end{aligned} \tag{4}$$

From the second and third equations of (4) and dividing by $I^e > 0$, one obtains

$$r\beta S^e(1+\frac{A^e}{I^e})-(\mu+\alpha+\eta)=0, \tag{5}$$
$$(1-r)\beta S^e(1+\frac{A^e}{I^e})-(\mu+\alpha)\frac{A^e}{I^e}=0.$$

Next, from (5), one obtains

$$\beta S^e(1+\frac{A^e}{I^e})=\frac{\mu+\alpha+\eta}{r}. \tag{6}$$

We can obtain from (6) and the second equation of (5) that

$$(1-r)\left(\frac{\mu+\alpha+\eta}{r}\right)=(\mu+\alpha)\frac{A^e}{I^e}\Rightarrow\frac{(1-r)(\mu+\alpha+\eta)}{r(\mu+\alpha)}=\frac{A^e}{I^e}. \tag{7}$$

Thus,

$$\begin{aligned}&\beta S^e\left(1+\frac{(1-r)(\mu+\alpha+\eta)}{r(\mu+\alpha)}\right)=\frac{\mu+\alpha+\eta}{r},\\&\beta S^e\left(\frac{r(\mu+\alpha)+(1-r)(\mu+\alpha+\eta)}{r(\mu+\alpha)}\right)=\frac{\mu+\alpha+\eta}{r},\\&S^e=\frac{(\mu+\alpha)(\mu+\alpha+\eta)}{\beta(\mu+\alpha+(1-r)\eta)}.\end{aligned}$$

Now, we can replace the first equation of (4) and substitute S^e into the second and third equations of (4) to obtain I^e and A^e as

$$\begin{aligned}I^e&=\frac{r\Lambda}{\mu+\alpha+\eta}-\frac{r\mu(\mu+\alpha)}{\beta(\mu+\alpha+(1-r)\eta)}=\frac{r[\Lambda\beta(\mu+\alpha+(1-r)\eta)-\mu(\mu+\alpha)(\mu+\alpha+\eta)]}{\beta(\mu+\alpha+\eta)(\mu+\alpha+(1-r)\eta)}.\\A^e&=\frac{(1-r)\Lambda}{\mu+\alpha}-\frac{(1-r)\mu(\mu+\alpha+\eta)}{\beta(\mu+\alpha+(1-r)\eta)}\\&=\frac{(1-r)[\Lambda\beta(\mu+\alpha+(1-r)\eta)-\mu(\mu+\alpha)(\mu+\alpha+\eta)]}{\beta(\mu+\alpha)(\mu+\alpha+(1-r)\eta)}.\end{aligned}$$

Now, we define the following parameter

$$\mathcal{R}_0=\frac{\Lambda\beta(\mu+\alpha+(1-r)\eta)}{\mu(\mu+\alpha)(\mu+\alpha+\eta)}. \tag{8}$$

With the above results, we have the following proposition:

Proposition 1. *If $\mathcal{R}_0 > 1$, there is only one endemic state $E_e(S^e, I^e, A^e, R^e)$ of system (1) with the conditions given in (2), which is*

$$\begin{aligned}S^e&=\frac{(\mu+\alpha)(\mu+\alpha+\eta)}{\beta(\mu+\alpha+(1-r)\eta)}=\frac{\Lambda}{\mu\mathcal{R}_0},\\I^e&=\frac{r[\Lambda\beta(\mu+\alpha+(1-r)\eta)-\mu(\mu+\alpha)(\mu+\alpha+\eta)]}{\beta(\mu+\alpha+\eta)(\mu+\alpha+(1-r)\eta)}=\frac{\Lambda r[\mathcal{R}_0-1]}{\mathcal{R}_0(\mu+\alpha+\eta)},\\A^e&=\frac{(1-r)[\Lambda\beta(\mu+\alpha+(1-r)\eta)-\mu(\mu+\alpha)(\mu+\alpha+\eta)]}{\beta(\mu+\alpha)(\mu+\alpha+(1-r)\eta)}=\frac{\Lambda(1-r)[\mathcal{R}_0-1]}{\mathcal{R}_0(\mu+\alpha)},\\R^e&=\frac{\alpha}{\mu}(I^e+A^e)=\frac{\alpha[\mathcal{R}_0-1]}{\beta}.\end{aligned}$$

3. Existence and Uniqueness of the Solution

Since model (1) represents an epidemiological model, from a biological point of view, it is important to analyze the epidemic waves generated when solutions propagate as traveling waves that move at a certain speed. Here, we show the existence of a single solution of model (1) with the conditions given in (2) (see Appendix A). We assume that the diffusion coefficients are constant. We simplify the notation by setting for $(x,t) \in \overline{\Omega} \times [0,T]$, and $\tau > 0$ the following: $S = S(x,t)$, $I = I(x,t)$, $A = A(x,t)$, $R = R(x,t)$, and $N = N(x,t)$, $T_\tau = T(x, t-\tau)$. Then, system (1) can be written as

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \Lambda - \beta S I_\tau - \beta S A_\tau - \mu S + \nabla \cdot (\nu_S \nabla S), \\
\frac{\partial I}{\partial t} &= r\beta S I_\tau + r\beta S A_\tau - (\mu + \eta + \alpha) I + \nabla \cdot (\nu_I \nabla I), \\
\frac{\partial A}{\partial t} &= (1-r)\beta S I_\tau + (1-r)\beta S A_\tau - (\alpha + \mu) A + \nabla \cdot (\nu_A \nabla A), \\
\frac{\partial R}{\partial t} &= \alpha I + \alpha A - \mu R + \nabla \cdot (\nu_R \nabla R).
\end{aligned} \tag{9}$$

We set $M(x,t) = \frac{\Lambda}{\mu} - S(x,t)$. Then, system (9) is transformed into

$$\begin{aligned}
\frac{\partial M}{\partial t} &= -\mu M + \beta\left(\frac{\Lambda}{\mu} - M\right)(I_\tau + A_\tau) + \nu_M \Delta M, \\
\frac{\partial I}{\partial t} &= r\beta\left(\frac{\Lambda}{\mu} - M\right)(I_\tau + A_\tau) - (\alpha + \mu + \eta) I + \nu_I \Delta I, \\
\frac{\partial A}{\partial t} &= (1-r)\beta\left(\frac{\Lambda}{\mu} - M\right)(I_\tau + A_\tau) - (\alpha + \mu) A + \nu_A \Delta A, \\
\frac{\partial R}{\partial t} &= \alpha(I + A) - \mu R + \nu_R \Delta R.
\end{aligned} \tag{10}$$

System (10) has two stationary states. If $\mathcal{R}_0 < 1$ it is $\mathbf{0} = (0,0,0,0)^T$, and for $\mathcal{R}_0 > 1$, it is $\mathbf{k} = (k_1, k_2, k_3, k_4)^T$, where

$$k_1 = \frac{\Lambda}{\mu} - S^e, \quad k_2 = I^e, \quad k_3 = A^e, \quad k_4 = R^e, \quad k_i > 0, \quad i = 1,2,3,4. \tag{11}$$

Using the notation presented in [34] and substituting into (10), the solutions are given by

$$M(x,t) = \varphi_1\left(\frac{x}{c} + t\right), \quad I(x,t) = \varphi_2\left(\frac{x}{c} + t\right), \quad A(x,t) = \varphi_3\left(\frac{x}{c} + t\right), R(x,t) = \varphi_4\left(\frac{x}{c} + t\right),$$

and by setting $s = \frac{x}{c} + t$, one obtains the following system:

$$\begin{aligned}
\frac{d_1}{c^2}\phi_1''(s) - \phi_1'(s) + f_1(\phi_s) = 0, \\
\frac{d_2}{c^2}\phi_2''(s) - \phi_2'(s) + f_2(\phi_s) = 0, \\
\frac{d_3}{c^2}\phi_3''(s) - \phi_3'(s) + f_3(\phi_s) = 0, \\
\frac{d_3}{c^2}\phi_4''(s) - \phi_4'(s) + f_4(\phi_s) = 0,
\end{aligned} \tag{12}$$

where $d_1 = \nu_M$, $d_2 = \nu_I$, $d_3 = \nu_A$, $d_4 = \nu_R$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, $\phi_s \in C([-\tau, 0], \mathbb{R}^4)$ with $\phi_s = (\phi_1^s, \phi_2^s, \phi_3^s, \phi_4^s)^T$; $\phi_i^s(\theta) = \phi_i(s + \theta)$ and

$$f_1(\phi_s) = -\mu\phi_1^s(0) + \beta\left(\frac{\Lambda}{\mu} - \phi_1^s(0)\right)(\phi_2^s(-\tau) + \phi_3^s(-\tau)),$$
$$f_2(\phi_s) = r\beta\left(\frac{\Lambda}{\mu} - \phi_1^s(0)\right)[\phi_2^s(-\tau) + \phi_3^s(-\tau)] - (\alpha + \mu + \eta)\phi_2^s(0),$$
$$f_3(\phi_s) = (1-r)\beta\left(\frac{\Lambda}{\mu} - \phi_1^s(0)\right)[\phi_2^s(-\tau) + \phi_3^s(-\tau)] - (\alpha + \mu)\phi_3^s(0), \quad (13)$$
$$f_4(\phi_s) = \alpha(\phi_2^s(0) + \phi_3^s(0)) - \mu\phi_4^s(0),$$

where ϕ satisfies the following asymptotic boundary conditions:

$$\lim_{s\to-\infty}\phi(s) = \mathbf{0}, \ \lim_{s\to\infty}\phi(s) = \mathbf{k}. \quad (14)$$

Let us show that the function $f = (f_1, f_2, f_3, f_4)^T$ satisfies conditions $\mathbf{A_1}$, $\mathbf{A_2}$, and $\mathbf{A_3}$ given in (A4).

Proposition 2 ($\mathbf{A_1}$). *Suppose that $f = (f_1, f_2, f_3, f_4)^T$ and that $W(\Omega, \mathbb{R}^4)$ is like in (A3). Then, $f(\boldsymbol{0}) = f(\boldsymbol{k}) = \boldsymbol{0}$.*

Proof. The proof follows from (13). □

Proposition 3 ($\mathbf{A_2}$). *f_i satisfies the Lipschitz condition in $W([-\tau, 0], \mathbb{R}^4)$ for $i = 1, 2, 3, 4$.*

Proof. Set $\phi, \psi \in W([-\tau, 0], \mathbb{R}^4)$. Then,

$$\begin{aligned}
|f_1(\phi) - f_1(\psi)| = &\left|-\mu\phi_1(0) + \beta\left(\frac{\Lambda}{\mu} - \phi_1(0)\right)(\phi_2(-\tau) + \phi_3(-\tau)) + \mu\psi_1(0)\right. \\
&\left. - \beta\left(\frac{\Lambda}{\mu} - \psi_1(0)\right)[\psi_2(-\tau) + \psi_3(-\tau)]\right| \\
&\leq \mu|\psi_1(0) - \phi_1(0)| \\
&+ \beta\left|\left(\frac{\Lambda}{\mu} - \phi_1(0)\right)[\phi_2(-\tau) - \phi_3(-\tau)] - \left(\frac{\Lambda}{\mu} - \psi_1(0)\right)[\psi_2(-\tau) - \psi_3(-\tau)]\right| \\
&\leq \mu|\psi_1(0) - \phi_1(0)| + \frac{\beta\Lambda}{\mu}(|\phi_2(-\tau) - \psi_2(-\tau)| + |\phi_3(-\tau) - \psi_3(-\tau)|) \\
&\quad + \beta| - \phi_1(0)(\phi_2(-\tau) + \phi_3(-\tau)) + \psi_1(0)(\psi_2(-\tau) + \psi_3(-\tau))| \\
&\leq \mu|\psi_1(0) - \phi_1(0)| + \frac{\beta\Lambda}{\mu}(|\phi_2(-\tau) - \psi_2(-\tau)| + |\phi_3(-\tau) - \psi_3(-\tau)|) \\
&\quad + \beta| - \phi_1(0)\phi_2(-\tau) - \phi_1(0)\phi_3(-\tau) + \psi_1(0)\psi_2(-\tau) + \psi_1(0)\psi_3(-\tau)|.
\end{aligned}$$

Thus, one obtains

$$\begin{aligned}
|f_1(\phi) - f_1(\psi)| \leq &\ \mu|\psi_1(0) - \phi_1(0)| + \frac{\beta\Lambda}{\mu}(|\phi_2(-\tau) - \psi_2(-\tau)| + |\phi_3(-\tau) - \psi_3(-\tau)|) \\
&+ \beta m_1|\psi_2(-\tau) - \phi_2(-\tau)| + \beta m_2|\psi_1(0) - \phi_1(0)| \\
&+ \beta m_3|\psi_1(0) - \phi_1(0)| + \beta m_1|\psi_3(-\tau) - \phi_3(-\tau)|.
\end{aligned}$$

Therefore, $|f_1(\phi) - f_1(\psi)| \leq L_1\|\phi - \psi\|_W$, where $L_1 = \mu + \frac{2\beta\Lambda}{\mu} + 2\beta m_1 + \beta(m_2 + m_3)$. In a similar process, for all $\phi, \psi \in W([-\tau, 0], \mathbb{R}^4)$, we can find the constants L_i, con $i = 2, 3, 4$, such that $|f_i(\phi) - f_i(\psi)| \leq L_i\|\phi - \psi\|_W$. □

Proposition 4 ($\mathbf{A_3}$). *The function $f = (f_1, f_2, f_3, f_4)^T$ holds the partial conditions of quasi-monotonicity (PQM) given in Definition A4, and $W([-\tau, 0], \mathbb{R}^4)$.*

Proof. Consider $\psi, \varphi \in W([-\tau, 0], \mathbb{R}^4)$, with $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$, and $0 \leq \psi_i \leq \varphi_i \leq m_i$ for all $i = 1, 2, 3, 4$. Moreover, suppose that

$$\varphi^{2,3} = (\varphi_1, \psi_2, \psi_3, \varphi_4)^T \ \varphi^1 = (\psi_1, \varphi_2, \varphi_3, \varphi_4)^T, \ \varphi^4 = (\varphi_1, \varphi_2, \varphi_3, \psi_4)^T.$$

Set $\mathbf{m} = (m_1, m_2, m_3, m_4)^T > \mathbf{k}$ such that $\frac{\Lambda}{\mu} - m_i > 0$. Then, $\frac{\Lambda}{\mu} - \varphi_i(0) \geq \frac{\Lambda}{\mu} - m_i > 0$, and hence

$$\begin{aligned} f_1(\varphi) - f_1(\psi) &= -\mu\varphi_1(0) + \beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\varphi_2(-\tau) + \varphi_3(-\tau)] \\ &\quad + \mu\psi_1(0) - \beta\left(\frac{\Lambda}{\mu} - \psi_1(0)\right)[\psi_2(-\tau) + \psi_3(-\tau)] \\ &\geq -\mu\varphi_1(0) + \beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\psi_2(-\tau) + \psi_3(-\tau)] \\ &\quad + \mu\psi_1(0) - \beta\left(\frac{\Lambda}{\mu} - \psi_1(0)\right)[\psi_2(-\tau) + \psi_3(-\tau)] \\ &\geq \mu(\psi_1(0) - \varphi_1(0)) + \beta(\psi_1(0) - \varphi_1(0))[\psi_2(-\tau) + \psi_3(-\tau)]. \end{aligned}$$

Now, since $\psi_2 + \psi_3 \leq m_2 + m_3$ and $\psi_1(0) - \varphi_1(0) \leq 0$, this implies that

$$(\psi_1(0) - \varphi_1(0))(\psi_2(-\tau) + \psi_3(-\tau)) \geq (\psi_1(0) - \varphi_1(0))(m_2 + m_3).$$

Thus,

$$\begin{aligned} f_1(\varphi) - f_1(\psi) &\geq \mu(\psi_1(0) - \varphi_1(0)) + \beta(m_2 + m_3)(\psi_1(0) - \varphi_1(0)) \\ &\geq [\mu + \beta(m_2 + m_3)](\psi_1(0) - \varphi_1(0)). \end{aligned}$$

Consider $\beta_1 = \mu + \beta(m_2 + m_3) > 0$, then $f_1(\varphi) - f_1(\psi) + \beta_1(\varphi_1(0) - \psi_1(0)) \geq 0$. Now,

$$\begin{aligned} f_2(\varphi) - f_2(\varphi^{2,3}) &= r\beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\varphi_2(-\tau) + \varphi_3(-\tau)] - (\alpha + \mu + \eta)\varphi_2(0) \\ &\quad - r\beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\psi_2(-\tau) + \psi_3(-\tau)] + (\alpha + \mu + \eta)\psi_2(0) \\ &= r\beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\varphi_2(-\tau) - \psi_2(-\tau) + \varphi_3(-\tau) - \psi_3(-\tau)] \\ &\quad + (\alpha + \mu + \eta)(\psi_2(0) - \varphi_2(0)) \geq (\alpha + \mu + \eta)(\psi_2(0) - \varphi_2(0)). \end{aligned}$$

By setting $\beta_2 = \alpha + \mu + \eta > 0$, one obtains $f_2(\varphi) - f_2(\varphi^{2,3}) + \beta_2(\varphi_2(0) - \psi_2(0)) \geq 0$. On the other hand,

$$\begin{aligned} f_2(\varphi) - f_2(\varphi^1) &= r\beta\left(\frac{\Lambda}{\mu} - \varphi_1(0)\right)[\varphi_2(-\tau) + \varphi_3(-\tau)] - (\alpha + \mu + \eta)\varphi_2(0) \\ &\quad - r\beta\left(\frac{\Lambda}{\mu} - \psi_1(0)\right)[\varphi_2(-\tau) + \varphi_3(-\tau)] + (\alpha + \mu + \eta)\varphi_2(0) \\ &= r\beta(\psi_1(0) - \varphi_1(0))[\varphi_2(-\tau) + \varphi_3(-\tau)] \leq 0. \end{aligned}$$

Hence, $f_2(\varphi) - f_2(\varphi^1) \leq 0$. Since ϕ_4 is not in f_2, then $f_2(\varphi) - f_2(\varphi^4) = 0 \leq 0$. Similarly, for f_3, $\beta_3 = \alpha + \mu > 0$ is obtained such that $f_3(\varphi) - f_3(\varphi^{2,3}) + \beta_3(\varphi_3(0) - \psi_3(0)) \geq 0$, and $f_3(\varphi) - f_3(\varphi^i) \leq 0$, $i = 1, 4$. Furthermore,

$$f_4(\varphi) - f_4(\psi) = \alpha(\varphi_2(0) + \varphi_3(0)) - \mu\varphi_4(0) - \alpha(\psi_2(0) + \psi_3(0)) + \mu\psi_4(0) \geq \mu(\psi_4(0) - \varphi_4(0)).$$

If we set $\beta_4 = \mu > 0$, then $f_4(\varphi) - f_4(\psi) + \beta_4(\varphi_4(0) - \psi_4(0)) \geq 0$. Thus, we have found $\beta_1, \ \beta_2, \ \beta_3, \beta_4 > 0$, such that Definition A4 (PQM) holds. □

As a consequence of the above propositions, the function f satisfies the hypotheses $(\mathbf{A_1})$, $(\mathbf{A_2})$, and $(\mathbf{A_3})$. Next, we determine the upper and lower coupled solutions, i.e., the functions of the form presented in (A5). These functions are given by

$$\overline{\phi} = (\overline{\phi_1}, \overline{\phi_2}, \overline{\phi_3}, \overline{\phi_4})^T; \qquad \underline{\phi} = (\underline{\phi_1}, \underline{\phi_2}, \underline{\phi_3}, \underline{\phi_4})^T;$$
$$\overline{\phi}^{2,3} = (\underline{\phi_1}, \overline{\phi_2}, \overline{\phi_3}, \underline{\phi_4})^T; \qquad \underline{\phi}^{2,3} = (\overline{\phi_1}, \underline{\phi_2}, \underline{\phi_3}, \overline{\phi_4})^T,$$

which must meet the following conditions almost everywhere in $\mathbb{R}$:

$$\begin{aligned}
&\frac{d_1}{c^2}\overline{\phi}_1'' - \overline{\phi}_1' + f_1(\overline{\phi}_s) \le 0, &\quad &\frac{d_1}{c^2}\underline{\phi}_1'' - \underline{\phi}_1' + f_1(\underline{\phi}_s) \ge 0,\\
&\frac{d_2}{c^2}\overline{\phi}_2'' - \overline{\phi}_2' + f_2(\overline{\phi}_s^{2,3}) \le 0, &\quad &\frac{d_2}{c^2}\underline{\phi}_2'' - \underline{\phi}_2' + f_2(\underline{\phi}_s^{2,3}) \ge 0,\\
&\frac{d_3}{c^2}\overline{\phi}_3'' - \overline{\phi}_3' + f_3(\overline{\phi}_s^{2,3}) \le 0, &\quad &\frac{d_3}{c^2}\underline{\phi}_3'' - \underline{\phi}_3' + f_3(\underline{\phi}_s^{2,3}) \ge 0,\\
&\frac{d_4}{c^2}\overline{\phi}_4'' - \overline{\phi}_4' + f_4(\overline{\phi}_s) \le 0, &\quad &\frac{d_4}{c^2}\underline{\phi}_4'' - \underline{\phi}_4' + f_4(\underline{\phi}_s) \ge 0.
\end{aligned}$$

Set $D = \max\{d_i : i = 1,2,3,4\}$, $K = \min\{k_i : i = 1,2,3,4\}$, $m_i > k_i$ for $i = 1,\dots,4$, and we consider p_i, $i = 1,\dots,4$, which is given by

$$p_1 = \frac{\beta\Lambda}{m_1}(m_2+m_3),\ p_2 = \frac{\beta\Lambda}{\mu K}(m_2+m_3),\ p_3 = \frac{\alpha}{m_4}(m_2+m_3),\ p_4 = -(\alpha+\mu+\eta).$$

Then, the characteristic equation $\Delta_i(\lambda) = \frac{D}{c^2}\lambda^2 - \lambda + p_i$ has at least one positive real root if $c > 2\sqrt{D|p_i|}$. We define $c^* := \max\{2\sqrt{D|p_i|} : i = 1,2,3,4\}$. If we set $c > c^*$, there are constants $\lambda_i > 0$ ($i = 1,2,3,4$) such that

$$\begin{cases}
\frac{D}{c^2}\lambda_1^2 - \lambda_1 + \frac{\beta\Lambda}{\mu m_1}(m_2+m_3) & = 0,\\
\frac{D}{c^2}\lambda_2^2 - \lambda_2 + \frac{\beta\Lambda}{\mu K}(m_2+m_3) & = 0,\\
\frac{D}{c^2}\lambda_3^2 - \lambda_3 + \frac{\alpha}{m_4}(m_2+m_3) & = 0,\\
\frac{D}{c^2}\lambda_4^2 - \lambda_4 - (\alpha+\mu+\eta) & = 0, \quad \lambda_1, \lambda_3 < \lambda_2.
\end{cases} \tag{15}$$

Define the following continuous functions

$$\overline{\phi} = (\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3, \overline{\phi}_4)^T, \quad \underline{\phi} = (\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3, \underline{\phi}_4)^T,$$

such that

$$\overline{\phi}_1(t) = \begin{cases} k_1 e^{\lambda_1 t} & t \le t_1,\\ k_1 + \varepsilon_1 e^{-\lambda t} & t > t_1. \end{cases} \qquad \underline{\phi}_1(t) = \begin{cases} 0 & t \le t_1^*,\\ k_1 - \varepsilon_1^* e^{-\lambda t} & t > t_1^*. \end{cases}$$

$$\overline{\phi}_2(t) = \begin{cases} k_2 e^{\lambda_2 t} & t \le t_2,\\ k_2 + \varepsilon_2 e^{-\lambda t} & t > t_2. \end{cases} \qquad \underline{\phi}_2(t) = \begin{cases} 0 & t \le t_2^*,\\ k_2 e^{\lambda_4 t} - C_0 & t_2^* < t \le t_3^*\\ k_2 - \varepsilon_2^* e^{-\lambda t} & t > t_3^*. \end{cases}$$

$$\overline{\phi}_3(t) = \begin{cases} k_3 e^{\lambda_2 t} & t \le t_3,\\ k_3 + \varepsilon_3 e^{-\lambda t} & t > t_3. \end{cases} \qquad \underline{\phi}_3(t) = \begin{cases} 0 & t \le t_4^*,\\ k_3 - \varepsilon_3^* e^{-\lambda t} & t > t_4^*. \end{cases}$$

$$\overline{\phi}_4(t) = \begin{cases} k_4 e^{\lambda_3 t} & t \le t_4,\\ k_4 + \varepsilon_4 e^{-\lambda t} & t > t_4. \end{cases} \qquad \underline{\phi}_4(t) = \begin{cases} 0 & t \le t_5^*,\\ k_4 - \varepsilon_4^* e^{-\lambda t} & t > t_5^*, \end{cases}$$

where the constants $\varepsilon_i,\ \varepsilon_i^* > 0$ for $i = 1,2,3,4$ meet the following conditions:

$$\begin{cases} \mu(k_1+\varepsilon_1) - \beta\left(\frac{\Lambda}{\mu} - k_1 - \varepsilon_1\right)(m_2+m_3) > 0, \\ (\alpha+\mu+\eta)(k_2+\varepsilon_2) - r\beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^*\right)(m_2+m_3) > 0, \\ (\alpha+\mu+\eta)(k_3+\varepsilon_3) - r\beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^*\right)(m_3+m_2) > 0, \\ \mu(k_4+\varepsilon_4) - \alpha(m_2+m_3) > 0, \\ -\mu(k_1-\varepsilon_1^*) + \beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^*\right)(k_2-\varepsilon_2^*+k_3-\varepsilon_3^*) > 0, \\ r\beta K_0\left(\frac{\Lambda}{\mu} - m_1\right) - (\alpha+\mu+\eta)(k_2-\varepsilon_2^*) > 0, \\ r\beta\left(\frac{\Lambda}{\mu} - m_1\right)\left(k_2 e^{\lambda_4 t_3^*} - C_0\right) - (\alpha+\mu+\eta)(k_2-\varepsilon_2^*) > 0, \\ (1-r)\beta\left(\frac{\Lambda}{\mu} - m_1\right)(k_2-\varepsilon_2^*+k_3-\varepsilon_3^*) - (\alpha+\mu+\eta)(k_3-\varepsilon_3^*) > 0, \\ \alpha(k_2-\varepsilon_2^*+k_3-\varepsilon_3^*) - \mu(k_4-\varepsilon_4^*) > 0. \end{cases} \tag{16}$$

$\lambda > 0$ is a constant that is appropriately set later. The values of t_i and t_j^* are set such that $t_1^*, t_1, t_4 \le t_2 \le t_3$; $t_3 - \tau \le t_2$; $t_3^*, t_4^* \le t_1^* - \tau$, t_5^* ; and $t_4^* - \tau > t_3^*$. Moreover, $C_0 = k_2 e^{\lambda_4 t_2^*}$, $K_0 = k_2 e^{\lambda_4 (t_3^* - \tau)} - C_0 > 0$, $m_i := \sup_{t\in\mathbb{R}}\{\overline{\phi}_i\} > k_i$ for $i = 1,2,3,4$ (see [11]). If we set $\mathbf{m} = (m_1, m_2, m_3, m_4)^T$, the functions $\overline{\phi}$ and $\underline{\phi}$ verify the following conditions:

(i) $\mathbf{0} \le \underline{\phi} \le \overline{\phi} \le \mathbf{m}$;
(ii) $\lim_{t\to-\infty} \underline{\phi} = \mathbf{0}$ and $\lim_{t\to\infty} \overline{\phi} = \mathbf{k}$;
(iii) $\underline{\phi}_i'(t+) \ge \underline{\phi}_i'(t-)$ and $\overline{\phi}_i'(t+) \le \overline{\phi}_i'(t-)$ for all $t \in \mathbb{R}$ and $i = 1,2,3,4$.

The following two lemmas show the results for the upper and lower solutions.

Lemma 1. *The function* $\overline{\phi}$ *is an upper solution of system* (10)*.*

Proof. Let us show that $\frac{d_1}{c^2}\overline{\phi}_1'' - \overline{\phi}_1' + f(\overline{\phi}_t) \le 0$. If $t \le t_1$, $\overline{\phi}_1(t) = k_1 e^{\lambda_1 t}$, $\overline{\phi}_2(t) = k_2 e^{\lambda_2 (t-\tau)}$, $\overline{\phi}_3(t) = k_3 e^{\lambda_2 (t-\tau)}$, it is obtained that

$$\begin{aligned} \frac{d_1}{c^2}\overline{\phi}_1''(t) - \overline{\phi}_1'(t) + f_1(\overline{\phi}_t) &= \frac{d_1}{c^2}\overline{\phi}_1''(t) - \overline{\phi}_1'(t) - \mu\overline{\phi}_1(t) \\ &\quad + \beta\left(\frac{\Lambda}{\mu} - \overline{\phi}_1(t)\right)[\overline{\phi}_2(t-\tau) + \overline{\phi}_3(t-\tau)] \\ &\le \frac{d_1 k_1 \lambda_1^2}{c^2} e^{\lambda_1 t} - k_1\lambda_1 e^{\lambda_1 t} + \frac{\beta\Lambda}{\mu}\left(k_2 e^{\lambda_2(t-\tau)} + k_3 e^{\lambda_2(t-\tau)}\right) \\ &\le \frac{d_1 k_1 \lambda_1^2}{c^2} e^{\lambda_1 t} - k_1\lambda_1 e^{\lambda_1 t} + \frac{\beta\Lambda}{\mu}\left(k_2 e^{\lambda_2 t} + k_3 e^{\lambda_2 t}\right). \end{aligned}$$

Since $t \le t_1 \le t_2, t_3$, and $\lambda_1 \le \lambda_2$, then

$$\begin{aligned} &e^{\lambda_1(t_1-t)} \le e^{\lambda_2(t_2-t)},\ \ e^{\lambda_1(t_1-t)} \le e^{\lambda_2(t_3-t)},\ e^{\lambda_2 t} \le \frac{e^{\lambda_2 t_2}}{e^{\lambda_1 t_1}} e^{\lambda_1 t},\ \ e^{\lambda_2 t} \le \frac{e^{\lambda_2 t_3}}{e^{\lambda_1 t_1}} e^{\lambda_1 t}, \\ &k_2 e^{\lambda_2 t} \le \frac{k_2 e^{\lambda_2 t_2}}{k_1 e^{\lambda_1 t_1}} k_1 e^{\lambda_1 t},\ \ k_3 e^{\lambda_2 t} \le \frac{k_3 e^{\lambda_2 t_3}}{k_1 e^{\lambda_1 t_1}} k_1 e^{\lambda_1 t}, \\ &k_2 e^{\lambda_2 t} \le \frac{m_2}{m_1} k_1 e^{\lambda_1 t},\ \ k_3 e^{\lambda_2 t} \le \frac{m_3}{m_1} k_1 e^{\lambda_1 t}. \end{aligned}$$

With the above and using (15), we have

$$\frac{d_1}{c^2}\overline{\phi}_1''(t) - \overline{\phi}_1'(t) + f_1(\overline{\phi}_t) \le k_1 e^{\lambda_1 t}\left[\frac{D}{c^2}\lambda_1^2 - \lambda_1 + \frac{\beta\Lambda}{m_1\mu}(m_2+m_3)\right] = 0.$$

But, if $t > t_1$, $\overline{\phi}_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, then

$$\frac{d_1}{c^2}\overline{\phi}_1'' - \overline{\phi}_1' + f_1(\overline{\phi}_t) = \frac{d_1\varepsilon_1\lambda_1^2}{c^2}e^{-\lambda t} + \varepsilon_1\lambda e^{-\lambda t} - \mu(k_1 + \varepsilon_1 e^{-\lambda t}) + \beta\left(\frac{\Lambda}{\mu} - k_1 - \varepsilon_1 e^{-\lambda t}\right)[\overline{\phi}_2(t-\tau) + \overline{\phi}_3(t-\tau)] \leq g_1(\lambda).$$

where

$$g_1(\lambda) = \frac{d_1\varepsilon_1\lambda_1^2}{c^2}e^{-\lambda t} + \varepsilon_1\lambda e^{-\lambda t} - \mu(k_1 + \varepsilon_1 e^{-\lambda t}) + \beta\left(\frac{\Lambda}{\mu} - k_1 - \varepsilon_1 e^{-\lambda t}\right)[m_2 + m_3],$$

and from condition (16), it follows that $g_1(0) < 0$. Next, since g_1 is continuous, there is $\lambda_1^* > 0$ such that if $\lambda \in (0, \lambda_1^*)$, then $\frac{d_1}{c^2}\overline{\phi}_1'' - \overline{\phi}_1' + f(\overline{\phi}_t) \leq g_1(\lambda) < 0$. We consider $\frac{d_2}{c^2}\overline{\phi}_2''(t) - \overline{\phi}_2'(t) + f_2(\overline{\phi}_t^{2,3})$. If $t \leq t_2$, $\overline{\phi}_2 = k_2 e^{\lambda_2 t}$, then using (15), it is deduced that

$$\begin{aligned}
&\frac{d_2}{c^2}\overline{\phi}_2'' - \overline{\phi}_2' + r\beta\left(\frac{\Lambda}{\mu} - \underline{\phi}_1\right)[\overline{\phi}_2(t-\tau) + \overline{\phi}_3(t-\tau)] - (\alpha + \mu + \eta)\overline{\phi}_2 \\
&\leq k_2 e^{\lambda_2 t}\left[\frac{d_2\lambda_2^2}{c^2} - \lambda_2 + \frac{r\beta\Lambda}{\mu K}(k_2 + k_3)\right] \\
&\leq k_2 e^{\lambda_2 t}\left[\frac{D\lambda_2^2}{c^2} - \lambda_2 + \frac{\beta\Lambda}{\mu K}(m_2 + m_3)\right] = 0,
\end{aligned}$$

for $t > t_2$, $\overline{\phi}_1(t) = k_2 + \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}_1(t) = k_1 - \varepsilon_1^* e^{-\lambda t}$. Hence,

$$\frac{d_2}{c^2}\overline{\phi}_2''(t) - \overline{\phi}_2'(t) + f_2(\overline{\phi}_t^{2,3}) \leq g_2(\lambda),$$

where

$$g_2(\lambda) = \left(\frac{d_2\lambda^2\varepsilon_2}{c^2} + \lambda\varepsilon_2\right)e^{-\lambda t} + r\beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^* e^{-\lambda t}\right)(m_2 + m_3) - (\alpha + \mu + \eta)(k_2 + \varepsilon_2 e^{-\lambda t}).$$

From the conditions in (16), it is deduced that

$$g_2(0) = r\beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^*\right)(m_2 + m_3) - (\alpha + \mu + \eta)(k_2 + \varepsilon_2) < 0.$$

Thus, there exists a $\lambda_2^* > 0$ such that if $\lambda \in (0, \lambda_2^*)$, then

$$\frac{d_2}{c^2}\overline{\phi}_2''(t) - \overline{\phi}_2'(t) + f_2(\overline{\phi}_t^{2,3}) \leq 0.$$

Similarly, there is $\lambda_3^* > 0$ such that if $\lambda \in (0, \lambda_3^*)$, one obtains

$$\frac{d_3}{c^2}\overline{\phi}_3''(t) - \overline{\phi}_3'(t) + f_3(\overline{\phi}_t^{2,3}) \leq 0.$$

Next, we consider $\frac{d_4}{c^2}\overline{\phi}_4''(t) - \overline{\phi}_4'(t) + f_4(\overline{\phi}_t)$. For $t \leq t_4$, it holds that $\overline{\phi}_4(t) = k_4 e^{\lambda_3 t}$, and hence

$$\begin{aligned}
\frac{d_4}{c^2}\overline{\phi}_4''(t) - \overline{\phi}_4'(t) + f_4(\overline{\phi}_t) &= \frac{d_4}{c^2}\overline{\phi}_4''(t) - \overline{\phi}_4'(t) + \alpha(\overline{\phi}_2(t) + \overline{\phi}_3(t)) - \mu\overline{\phi}_4(t) \\
&\leq \frac{d_4 k_4\lambda_3^2}{c^2}e^{\lambda_3 t} - k_4\lambda_3 e^{\lambda_3 t} + \alpha(k_2 e^{\lambda_2 t} + k_3 e^{\lambda_2 t}).
\end{aligned}$$

Now, since $t_4 < t_2, t_3$ and $\lambda_3 \le \lambda_2$, when $t \le t_1$ and from (15), it follows that

$$\frac{d_4}{c^2}\overline{\phi}_4''(t) - \overline{\phi}_4'(t) + f_4(\overline{\phi}_t) \le \frac{d_4 k_4 \lambda_3^2}{c^2} e^{\lambda_3 t} - k_4 \lambda_3 e^{\lambda_3 t} + k_4 e^{\lambda_3 t} \alpha \left(\frac{m_2}{m_4} + \frac{m_3}{m_4} \right)$$
$$\le k_4 e^{\lambda_3 t} \left[\frac{D\lambda_3^2}{c^2} - \lambda_3 + \frac{\alpha}{m_4}(m_2 + m_3) \right] = 0.$$

For $t > t_4$, it is deduced that $\overline{\phi}_4 = k_4 + \varepsilon_4 e^{-\lambda t}$. Therefore,

$$\frac{d_4}{c^2}\overline{\phi}_4''(t) - \overline{\phi}_4'(t) + \alpha(\overline{\phi}_2(t) + \overline{\phi}_3(t)) - \mu\overline{\phi}_4 \le g_4(\lambda),$$

where

$$g_4(\lambda) = \varepsilon_4 e^{-\lambda t}\left(\frac{d_4}{c^2}\lambda^2 + \lambda\right) + \alpha(m_2 + m_3) - \mu(k_4 + \varepsilon_4)e^{-\lambda t}.$$

From (16), it is obtained that $g_4(0) = \alpha(m_2 + m_3) - \mu(k_4 + \varepsilon_4) < 0$. Then, there exists $\lambda_4^* > 0$ such that if $\lambda \in (0, \lambda_4^*)$, we obtain

$$\frac{d_4}{c^2}\overline{\phi}_4'' - \overline{\phi}_4' + f(\overline{\phi}^4, \overline{\phi}_\tau^4) \le 0.$$

By choosing $\lambda \in (0, \lambda^*)$ with $\lambda^* = \min\{\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*\}$, the desired result is obtained. □

Lemma 2. *The function $\underline{\phi}$ is a lower solution of system (10).*

Proof. Let $\frac{d_1}{c^2}\underline{\phi}_1''(t) - \underline{\phi}_1'(t) + f_1(\underline{\phi}_t)$, $t \le t_1^*$, and $\underline{\phi}_1(t) = 0$, then

$$\frac{d_1}{c^2}\underline{\phi}_1''(t) - \underline{\phi}_1'(t) + f_1(\underline{\phi}_t) = \frac{\Lambda}{\mu}(\underline{\phi}_2(t-\tau) + \underline{\phi}_3(t-\tau)) \ge 0.$$

Thus, if $t > t_1^*$, then $\underline{\phi}_1(t) = k_1 - \varepsilon_1^* e^{-\lambda t}$, and since $t - \tau > t_1^* - \tau > t_3^*, t_4^*$, $\underline{\phi}_2(t-\tau) = k_2 - \varepsilon_2^* e^{-\lambda(t-\tau)}$, $\underline{\phi}_3(t-\tau) = k_3 - \varepsilon_3^* e^{-\lambda(t-\tau)}$. Similarly,

$$\frac{d_1}{c^2}\underline{\phi}_1''(t) - \underline{\phi}_1'(t) + f_1(\underline{\phi}_t) = g_5(\lambda),$$

where

$$g_5(\lambda) = -\frac{d_1}{c^2}\varepsilon_1^*\lambda e^{-\lambda t} - \varepsilon_1^*\lambda e^{-\lambda t} - \mu(k_1 - \varepsilon_1^* e^{-\lambda t})$$
$$+ \beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^* e^{-\lambda t}\right)\left(k_2 - \varepsilon_2^* e^{-\lambda(t-\tau)} + k_3 - \varepsilon_3^* e^{-\lambda(t-\tau)}\right).$$

From (16), we have

$$g_5(0) = -\mu(k_1 - \varepsilon_1^*) + \beta\left(\frac{\Lambda}{\mu} - k_1 + \varepsilon_1^*\right)(k_2 - \varepsilon_2^* + k_3 - \varepsilon_3^*) > 0,$$

and thus there exists a $\lambda_5^* > 0$ such that $\lambda \in (0, \lambda_5^*)$. It is concluded that

$$\frac{d_1}{c^2}\underline{\phi}_1''(t) - \underline{\phi}_1'(t) + f_1(\underline{\phi}_t) \ge 0.$$

Now, for $\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3})$, if $t \le t_2^*$, $\underline{\phi}_2(t) = 0$, and therefore,

$$\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3}) = r\beta\left(\frac{\Lambda}{\mu} - \overline{\phi}_1 e\right)\underline{\phi}_3(t-\tau) \ge 0.$$

Suppose that $t_2^* < t \leq t_3^*$, $\underline{\phi}_2 = k_2 e^{\lambda_4 t} - C_0$. So, from (15) it follows that

$$\begin{aligned}&\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + r\beta\left(\frac{\Lambda}{\mu} - \overline{\phi}_1(t)\right)(\underline{\phi}_2(t-\tau) + \underline{\phi}_3(t-\tau)) - (\alpha+\mu+\eta)\underline{\phi}_2(t)\\ &\geq k_2 e^{-\lambda_4 t}\left(\frac{d_2}{c^2}\lambda_4 - \lambda_4 - (\alpha+\mu+\eta)\right) = 0.\end{aligned}$$

For $t > t_3^*$, we have $\underline{\phi}_2 = k_2 - \varepsilon_2^* e^{-\lambda t}$. Since $t - \tau > t_3 - \tau > t_2^*$, then $t - \tau \leq t_3^*$. Cinsequently, after $\underline{\phi}_2(t-\tau) = k_2 e^{\lambda_4(t-\tau)} - C_0 > K_0$, and thus

$$\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3}) \geq g_6(\lambda),$$

with

$$g_6(\lambda) = -\left(\frac{d_2}{c^2}\varepsilon_2^*\lambda^2 + \varepsilon_2^*\lambda\right)e^{-\lambda t} + r\beta\left(\frac{\Lambda}{\mu} - m_1\right)K_0 - (\mu+\alpha+\eta)(k_2 - \varepsilon_2^* e^{-\lambda t}).$$

From (16), $g_6(0) = r\beta\left(\frac{\Lambda}{\mu} - m_1\right)K_0 - (\mu+\alpha+\eta)(k_2 - \varepsilon_2^*) > 0$. Therefore, there exists a $\lambda_6^* > 0$. For instance, when $\lambda \in (0, \lambda_6^*)$, this implies that

$$\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3}) \geq 0.$$

Now, in another case, with $t - \tau > t_3^*$, we obtain

$$\underline{\phi}_2(t-\tau) = k_2 - \varepsilon_2^* e^{-\lambda(t-\tau)} > k_2 e^{\lambda_4 t_3^*} - C_0 > 0,$$

which implies that

$$\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3}) \geq g_7(\lambda),$$

with

$$g_7(\lambda) = -\left(\frac{d_2}{c^2}\varepsilon_2^*\lambda^2 + \varepsilon_2^*\lambda\right)e^{-\lambda t} - (\mu+\alpha+\eta)(k_2 - \varepsilon_2^* e^{-\lambda t}) + r\beta\left(\frac{\Lambda}{\mu} - m_1\right)\left(k_2 e^{\lambda_4 t_3^*} - C_0\right).$$

Thus, from (16), it is deduced that

$$g_7(0) = r\beta\left(\frac{\Lambda}{\mu} - m_1\right)\left(k_2 e^{\lambda_4 t_3^*} - C_0\right) - (\mu+\alpha+\eta)(k_2 - \varepsilon_2^*) > 0.$$

Thus, there exists a $\lambda_7^* > 0$, with $\lambda \in (0, \lambda_7^*)$, which implies that

$$\frac{d_2}{c^2}\underline{\phi}_2''(t) - \underline{\phi}_2'(t) + f_2(\underline{\phi}_t^{2,3}) \geq 0.$$

Reasoning in the same way, we can obtain λ_8^* and λ_9^* such that if $\lambda^* = \min\{\lambda_5^*, \lambda_6^*, \lambda_7^*, \lambda_8^*, \lambda_9^*\}$, for $\lambda \in (0, \lambda^*)$, $\underline{\phi}$ is a lower solution of (10). □

With the above results, the following conclusion is obtained, which ensures the existence of a traveling wave solution with speed $c > 0$ of system (1) with the conditions given in (2).

Theorem 1. *Suppose that $\mathcal{R}_0 > 1$ and $c > c^*$. Then, system (1) with the conditions in (2) has a traveling wave solution with speed $c > 0$, which does not depend on $\tau \geq 0$ and connects the stationary points E_0 and E_E.*

Proof. Since $\mathcal{R}_0 > 1$, there are only stationary points $\mathbf{0}$ and $\mathbf{k}$ of system (12) with the conditions A_1, A_2, and A_3 given in (14). From Lemmas 1 and 2, $\overline{\phi}$ and $\underline{\phi}$ are the upper and lower solutions. Then, from Theorem A1, system (1) with the conditions in (2) has a unique traveling wave solution with speed $c > 0$ connecting points E_0 and E_E (see [35]). $\square$

4. Local Stability of Equilibrium States

In this section, we examine the local stability of the endemic equilibrium E_e and the disease-free equilibrium E_0 of system (1) with the initial and homogeneous Neumann boundary conditions given in (2). From system (9), we set $u = (S, I, A, R)^T, u_\tau = u(x, t-\tau)$, and $f = (f_1, f_2, f_3, f_4)^T$ with

$$\begin{aligned} f_1(u, u_\tau) &= \Lambda - \beta S(I_\tau + A_\tau) - \mu S, \\ f_2(u, u_\tau) &= r\beta S(I_\tau + A_\tau) - (\alpha + \mu + \eta) I, \\ f_3(u, u_\tau) &= (1-r)\beta S(I_\tau + A_\tau) - (\alpha + \mu) A, \\ f_4(u, u_\tau) &= \alpha(I + A) - \mu R. \end{aligned}$$

Next, we have

$$\frac{\partial f}{\partial u}(E_e) = \begin{bmatrix} -\beta(I^e + A^e) - \mu & 0 & 0 & 0 \\ r\beta(I^e + A^e) & -(\alpha+\mu+\eta) & 0 & 0 \\ (1-r)\beta(I^e + A^e) & 0 & -(\alpha+\mu) & 0 \\ 0 & \alpha & \alpha & -\mu \end{bmatrix},$$

$$\frac{\partial f}{\partial u_\tau}(E_e) = \begin{bmatrix} 0 & -\beta S^e & -\beta S^e & 0 \\ 0 & r\beta S^e & r\beta S^e & 0 \\ 0 & (1-r)\beta S^e & (1-r)\beta S^e & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\frac{\partial f}{\partial u}(E_e)$ is the Jacobian matrix. We set $Q = \frac{\partial f}{\partial u}(E_e)$ and $G = \frac{\partial f}{\partial u_\tau}(E_e)$. Then, the linearization of system (9) at the equilibrium point E_e is given by

$$Lu = V\Delta u + Qu + Gu_\tau, \tag{17}$$

where $V = Diag(\nu_S, \nu_I, \nu_A, \nu_R)$. Suppose that $0 = \rho_1 < \rho_2 < \rho_3 \ldots$, are eigenvalues of operator $-\Delta$ on Ω with Neumann conditions, and consider E_{ρ_i} the eigenspace corresponding to eigenvalue ρ_i in $C^1(\Omega)$. We denote $n_i := dim(E_{\rho_i})$ and $\mathcal{M} := [C^1(\Omega)]^3$. Set $B_i := \{v_{ij} : j = 1, 2, 3, \ldots, n_i\}$ as an orthonormal basis of E_{ρ_i} and $\mathcal{M}_{ij} := \{cv_{ij} : v_{ij} \in B_i;\ c \in \mathbb{R}^3\}$. Next, the direct sum of the vector subspaces is $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \oplus \ldots$, where $\mathcal{M}_i = \mathcal{M}_{i1} \oplus \mathcal{M}_{i2} \oplus \mathcal{M}_{i3} \oplus \ldots \oplus \mathcal{M}_{in_i}$. Thus, M_i is invariant under the operator L for each $i \geq 1$. In addition, λ is an eigenvalue of L if and only if it is the solution of the characteristic equation

$$\det(\lambda I_4 - L_\lambda) = 0\,, \tag{18}$$

with $L_\lambda := -\rho_i V + Q + e^{-\lambda\tau} G$ for some $i \geq 1$, in which case, there is an eigenvector in $\mathcal{M}_i$ (see [11]). We suppose that $\nu_I, \nu_S, \nu_R, \nu_A = v$, and from (18), we have

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ 0 & c_{42} & c_{43} & c_{44} \end{vmatrix} = 0\,, \tag{19}$$

where

$c_{11} = \lambda + \rho_i v + \beta(I^e + A^e) + \mu,\ c_{12} = c_{13} = \beta S^e e^{-\lambda\tau},\ c_{21} = -r\beta(I^e + A^e)\,,$
$c_{22} = \lambda + \rho_i v + \alpha + \mu + \eta - r\beta S^e e^{-\lambda\tau},\ c_{23} = -r\beta S^e e^{-\lambda\tau},\ c_{31} = -(1-r)\beta(I^e + A^e),$
$c_{32} = -(1-r)\beta S^e e^{-\lambda\tau},\ c_{33} = \lambda + \rho_i v + \alpha + \mu - (1-r)\beta S^e e^{-\lambda\tau},\ c_{42} = c_{43} = -\alpha,$
$c_{44} = \lambda + \rho_i v + \mu\,.$

Therefore, from (19), it follows that

$$c_{44}[c_{11}(c_{22}c_{33} - c_{32}c_{23}) - c_{12}(c_{21}c_{33} - c_{31}c_{23}) + c_{13}(c_{21}c_{32} - c_{31}c_{22})] = 0.$$

Now, by replacing the values c_{ij} in $-c_{12}(c_{21}c_{33} - c_{31}c_{23}) + c_{13}(c_{21}c_{32} - c_{31}c_{22})$, we obtain

$$\begin{aligned}
&- \beta S^e e^{-\lambda\tau}\left[-r\beta(I^e + A^e)\left(\lambda + \rho_i v + \alpha + \mu - (1-r)\beta e^{-\lambda\tau}\right) - r(1-r)\beta^2 S^e (I^e + A^e) e^{-\lambda\tau}\right] \\
&+ \beta S^e e^{-\lambda\tau}\left[r(1-r)\beta^2 S^e (I^e + A^e) e^{-\lambda\tau} + (1-r)\beta(I^e + A^e)\left(\lambda + \rho_i v + \alpha + \mu + \eta - r\beta S^e e^{-\lambda\tau}\right)\right] \\
&= \beta S^e e^{-\lambda\tau}\left[\beta(I^e + A^e)\left(\lambda + \rho_i v + \alpha + \mu + (1-r)\eta - 2r(1-r)\beta S^e e^{-\lambda\tau}\right)\right. \\
&\left. + 2r(1-r)\beta^2 S^e (I^e + A^e) e^{-\lambda\tau}\right] \\
&= \beta^2 S^e (I^e + A^e)[\lambda + \rho_i v + \alpha + \mu + (1-r)\eta] e^{-\lambda\tau}.
\end{aligned} \tag{20}$$

Similarly, by simplifying $c_{22}c_{33}$, it follows that

$$\begin{aligned}
&\left(\lambda + \rho_i v + \alpha + \mu + \eta - r\beta S^e e^{-\lambda\tau}\right)\left(\lambda + \rho_i v + \alpha + \mu - (1-r)\beta S^e e^{-\lambda\tau}\right) \\
&= \lambda^2 + (2\rho_i v + 2\alpha + 2\mu + \eta - \beta S^e e^{-\lambda\tau})\lambda + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) \\
&+ [-r(\rho_i v + \alpha + \mu) - (\rho_i v + \alpha + \mu)(1-r) - (1-r)\eta]\beta S^e e^{-\lambda\tau} + r(1-r)\beta^2 (S^e)^2 e^{-2\lambda\tau} \\
&= \lambda^2 + (2\rho_i v + 2\alpha + 2\mu + \eta)\lambda + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) \\
&\qquad + [-\beta S^e \lambda - \beta S^e(\rho_i v + \alpha + \mu + (1-r)\eta)] e^{-\lambda\tau} + r(1-r)\beta^2 (S^e)^2 e^{-2\lambda\tau}.
\end{aligned} \tag{21}$$

Now, $-c_{32}c_{23} = -r(1-r)\beta^2 (S^e)^2 e^{-2\lambda\tau}$ and thus the expression $c_{22}c_{33} - c_{32}c_{23}$ is

$$\begin{aligned}
&\lambda^2 + (2\rho_i v + 2\alpha + 2\mu + \eta)\lambda + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) \\
&- \beta S^e[\lambda + \rho_i v + \alpha + \mu + (1-r)\eta] e^{-\lambda\tau}.
\end{aligned} \tag{22}$$

Then, by multiplying (22) by $c_{11} = (\lambda + \rho_i v + \mu) + \beta(I^e + A^e)$, we obtain

$$\begin{aligned}
&(\lambda + \rho_i v + \mu)\left[\lambda^2 + (2\rho_i v + 2\alpha + 2\mu + \eta)\lambda + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu)\right. \\
&\left. - \beta S^e[\lambda + \rho_i v + \alpha + \mu + (1-r)\eta] e^{-\lambda\tau}\right] + \beta(I^e + A^e)[\lambda^2 + (2\rho_i v + 2\alpha + 2\mu + \eta)\lambda \\
&+ (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu)] - \beta^2 S^e (I^e + A^e)[\lambda + \rho_i v + \alpha + \mu + (1-r)\eta] e^{-\lambda\tau}.
\end{aligned} \tag{23}$$

Finally, from (23), (20), and c_{44}, we obtain

$$(\lambda + \rho_i v + \mu)\left[\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 - C(\lambda^2 + q_1\lambda + q_0) e^{-\lambda\tau}\right] = 0, \tag{24}$$

where

$$\begin{aligned}
p_0 &= [\rho_i v + \mu + \beta(I^e + A^e)](\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu), \\
p_1 &= [p_i v + \mu + \beta(I^e + A^e)](2p_i v + 2\alpha + 2\mu + \eta) + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu), \\
p_2 &= 3p_i v + 2\alpha + 3\mu + \eta + \beta(I^e + A^e), \\
q_0 &= (\rho_i v + \mu)[\rho_i v + \alpha + \mu + (1-r)\eta], \\
q_1 &= 2\rho_i v + 2\mu + \alpha + (1-r)\eta, \\
C &= \beta S^e.
\end{aligned}$$

For the case E_0, $A^e = I^e = 0$ and hence from (19) and (23), Equation (24) is

$$(\lambda + \rho_i v + \mu)^2[\lambda^2 + b_1\lambda + b_0 + (a_1\lambda + a_0) e^{-\lambda\tau}] = 0, \tag{25}$$

with

$$b_0 = (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu),\ b_1 = 2p_i v + 2\alpha + 2\mu + \eta,$$
$$a_0 = -\frac{\beta\Lambda}{\mu}[\rho_i v + \alpha + \mu + (1-r)\eta],\ a_1 = -\frac{\beta\Lambda}{\mu}.$$

Thus, for all $i \geq 1$ $\lambda = -(\rho_i v + \mu)$. The other roots are in

$$\lambda^2 + b_1\lambda + b_0 + (a_1\lambda + a_0)e^{-\lambda\tau} = 0. \tag{26}$$

We define $W_i(\lambda) = \lambda^2 + b_1\lambda + b_0 + (a_1\lambda + a_0)e^{-\lambda\tau}$. The following cases are given:

Case I: If $\mathcal{R}_0 > 1$ and $\lambda \in \mathbb{R}$, since $\rho_1 = 0$, then

$$\begin{aligned} W_1(0) = b_0 + a_0 &= (\rho_1 v + \alpha + \mu + \eta)(\rho_1 v + \alpha + \mu) - \frac{\beta\Lambda}{\mu}[\rho_1 v + \alpha + \mu + (1-r)\eta] \\ &= (\alpha + \mu + \eta)(\alpha + \mu) - \frac{\beta\Lambda}{\mu}[\alpha + \mu + (1-r)\eta] \\ &= (\alpha + \mu + \eta)(\alpha + \mu)\left[1 - \frac{\beta\Lambda(\alpha + \mu + (1-r)\eta)}{\mu(\alpha + \mu + \eta)(\alpha + \mu)}\right] \\ &= (\alpha + \mu + \eta)(\alpha + \mu)(1 - \mathcal{R}_0) < 0, \end{aligned}$$

moreover, $\lim_{\lambda \to +\infty} W_1(\lambda) = +\infty$. Thus, (26) has a positive root. Therefore, there is a λ feature root with a positive real part in the operator spectrum L (17). This implies that $E_0 = (\Lambda/\mu, 0, 0, 0)$ is unstable while $\mathcal{R}_0 > 1$.

Case II: Suppose that $\mathcal{R}_0 < 1$. If $\tau = 0$, then $W_i(\lambda) = \lambda^2 + (b_1 + a_1)\lambda + (b_0 + a_0)$. Now, we can see that

$$\begin{aligned} b_0 + a_0 &= (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) - \frac{\beta\Lambda}{\mu}[\rho_i v + \alpha + \mu + (1-r)\eta] \\ &\geq (\alpha + \mu + \eta)(\alpha + \mu) - \frac{\beta\Lambda}{\mu}[\alpha + \mu + (1-r)\eta] \\ &\geq (\alpha + \mu + \eta)(\alpha + \mu)\left[1 - \frac{\beta\Lambda(\alpha + \mu + (1-r)\eta)}{\mu(\alpha + \mu + \eta)(\alpha + \mu)}\right] \\ &\geq (\alpha + \mu + \eta)(\alpha + \mu)(1 - \mathcal{R}_0) > 0, \end{aligned}$$

and as $\mathcal{R}_0 < 1$, it follows that

$$\frac{\beta\Lambda(\alpha + \mu + (1-r)\eta)}{\mu(\alpha + \mu + \eta)(\alpha + \mu)} < 1, \text{ implies that } \frac{1}{\mu + \alpha} - \frac{\beta\Lambda}{\mu(\alpha + \mu + \eta)(\alpha + \mu)} > 0.$$

Hence, we obtain

$$\begin{aligned} b_1 + a_1 = 2\rho_i v + 2\alpha + 2\mu + \eta - \frac{\beta\Lambda}{\mu} &= 2\rho_i v + (\alpha + \mu) + (\alpha + \mu + \eta) - \frac{\beta\Lambda}{\mu} \\ &= 2\rho_i v + (\alpha + \mu)(\alpha + \mu + \eta)\left[\frac{1}{\alpha + \mu} + \frac{1}{\alpha + \mu + \eta} - \frac{\beta\Lambda}{\mu(\alpha + \mu + \eta)(\alpha + \mu)}\right] \\ &> 0. \end{aligned}$$

The above allows us to conclude that the equation $W_i(\lambda) = 0$ has two roots with a negative real part for all i. Now, if we suppose that $\tau > 0$, and ωj $(\omega > 0)$ is the solution of $W_i(\lambda) = 0$, where $j = \sqrt{-1}$. Then, by solving $W_i(\omega j) = 0$ and separating the real and imaginary parts, it follows that

$$\begin{cases} a_1 \sin(\omega\tau)\omega + a_0 \cos(\omega\tau) & = \omega^2 - b_0, \\ -a_1 \cos(\omega\tau)\omega + a_0 \sin(\omega\tau) & = b_1\omega. \end{cases} \tag{27}$$

By squaring both sides of each equation in (27) and then adding the two equations, we obtain

$$\omega^4 + (b_1^2 - 2b_0 - a_1^2)\omega + b_0^2 - a_0^2 = 0.$$

Set $z = \omega^2$. Next,

$$z^2 + (b_1^2 - 2b_0 - a_1^2)z + (b_0^2 - a_0^2) = 0. \tag{28}$$

Thus,

$$\begin{aligned} b_1^2 &= [(\rho_i v + \alpha + \mu + \eta) + (\rho_i v + \alpha + \mu)]^2 \\ &= (\rho_i v + \alpha + \mu + \eta)^2 + (\rho_i v + \alpha + \mu)^2 + 2(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu). \end{aligned}$$

Therefore,

$$\begin{aligned} b_1^2 - 2b_0 + a_1^2 &= (\rho_i v + \alpha + \mu + \eta)^2 + (\rho_i v + \alpha + \mu)^2 - \left[\frac{\beta\Lambda}{\mu}\right]^2 \\ &> (\alpha + \mu + \eta)^2 + (\alpha + \mu)^2 - \left[\frac{\beta\Lambda}{\mu}\right]^2 \\ &> (\alpha + \mu + \eta)^2(\alpha + \mu)^2\left[\frac{1}{(\alpha + \mu + \eta)^2} + \frac{1}{(\alpha + \mu)^2} - \frac{\beta^2\Lambda^2}{\mu^2(\alpha + \mu)^2(\alpha + \mu + \eta)^2}\right] \\ &> 0, \end{aligned}$$

because $\mathcal{R}_0 < 1$ implies that

$$\begin{aligned} &\frac{\beta^2\Lambda^2(\alpha + \mu + (1 - r)\eta)^2}{\mu^2(\alpha + \mu)^2(\alpha + \mu + \eta)^2} < 1 \\ &\Rightarrow - \frac{\beta^2\Lambda^2}{\mu^2(\alpha + \mu)^2(\alpha + \mu + \eta)^2} > -\frac{1}{(\alpha + \mu + (1 - r)\eta)^2} \\ &\Rightarrow \frac{1}{(\alpha + \mu)^2} - \frac{\beta^2\Lambda^2}{\mu^2(\alpha + \mu)^2(\alpha + \mu + \eta)^2} > \frac{1}{(\alpha + \mu)^2} - \frac{1}{(\alpha + \mu + (1 - r)\eta)^2} > 0. \end{aligned}$$

Moreover, $b_0^2 - a_0^2 = (b_0 + a_0)(b_0 - a_0) > 0$. Consequently, Equation (28) has no positive real roots. Therefore, if $\mathcal{R}_0 < 1$, E_0 is asymptotically stable locally, for all $\tau > 0$.

In the endemic point $E^* = E_e(S^e, I^e, A^e, R^e)$, for $\mathcal{R}_0 > 1$, we obtain

$$S^e = \frac{\Lambda}{\mu\mathcal{R}_0}, \; I^e = \frac{r\Lambda(\mathcal{R}_0 - 1)}{(\mu + \alpha + \eta)\mathcal{R}_0}, \; A^e = \frac{(1 - r)\Lambda(\mathcal{R}_0 - 1)}{(\mu + \alpha)\mathcal{R}_0}, \; R^e = \frac{\alpha}{\mu}(I^e + A^e) = \frac{\alpha(\mathcal{R}_0 - 1)}{\beta}.$$

Thus, the characteristic Equation (24) for E_e is

$$(\lambda + \rho_i v + \mu)\left[\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 - C(\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau}\right] = 0, \tag{29}$$

with

$$\begin{aligned} p_0 &= (\rho_i v + \mu\mathcal{R}_0)(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu), \\ p_1 &= (p_i v + \mu\mathcal{R}_0)(2p_i v + 2\alpha + 2\mu + \eta) + (\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu), \\ p_2 &= 3p_i v + 2\alpha + 2\mu + \eta + \mu\mathcal{R}_0, \\ q_0 &= (\rho_i v + \mu)[\rho_i v + \alpha + \mu + (1 - r)\eta], \\ q_1 &= 2\rho_i v + 2\mu + \alpha + (1 - r)\eta, \\ C &= \frac{\beta\Lambda}{\mu\mathcal{R}_0}. \end{aligned}$$

Is clear that for all $i \geq 1$, $-(\rho_i v + \mu)$ is a real negative root of (29). The other roots are in $F_i(\lambda) = 0$, where

$$F_i(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 - C(\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau}.$$

To demonstrate that the roots of the characteristic equation $F_i(\lambda) = 0$ have negative real parts for $\tau \geq 0$, we use the following result presented in [36].

Proposition 5. *Suppose that*

$$P(\lambda) + Y(\lambda)e^{-r\lambda} = 0,$$

where P and Y are polynomials with real coefficients. If this holds true, then:

1. $P(\lambda) \neq 0$, *si* $\Re(\lambda) \geq 0$.
2. $|Y(wj)| < |P(wj)|$, *para* $0 \leq w < \infty$.
3. $\lim_{|\lambda|\to\infty,\, \Re(\lambda)\geq 0} |Q(\lambda)/P(\lambda)| = 0$.

   Then, $\Re(\lambda) < 0$ *for all roots* λ *and* $r \geq 0$.

   Consider $F_i(\lambda) = P(\lambda) + Y(\lambda)e^{-\lambda\tau}$. Let us consider the following:

1. $P(\lambda) \neq 0$, if $\Re(\lambda) \geq 0$. Indeed, set $\lambda = a + bj$ with $a \geq 0$. It follows that

$$\begin{aligned} P(\lambda) &= (a+bj)^3 + p_2(a+bj)^2 + p_1(a+bj) + p_0 \\ &= a^3 + 3a^2bj - 3ab^2 - b^3j + p_2a^2 + 2p_2abj - p_2b^2 + p_1a + p_1bj + p_0, \\ &= (a^3 + p_2a^2 + p_1a + p_0 - 3ab^2 - p_2b^2) + (3a^2b - b^3 + 2p_2ab + p_1b)j. \end{aligned}$$

Now, if $P(\lambda) = 0$, then we have

$$a^3 + p_2a^2 + p_1a + p_0 = 3ab^2 + p_2b^2, \tag{30}$$

$$3a^2b + 2p_2ab + p_1b = b^3. \tag{31}$$

Notice that $b \neq 0$, since if $b = 0$ in Equation (30) one obtains

$$a^3 + p_2a^2 + p_1a + p_0 = 0,$$

which is impossible, since $a \geq 0$ and $p_2, p_1, p_0 > 0$ imply that that the equation is positive. After Equation (31), it follows that $b_2 = 3a^2 + 2p_2a + p_1$. By replacing this in Equation (30), we obtain

$$\begin{aligned} a^3 + p_2a^2 + p_1a + p_0 &= 3a(3a^2 + 2p_2a + p_1) + p_2(3a^2 + 2p_2a + p_1), \\ a^3 + p_2a^2 + p_1a + p_0 &= 9a^3 + 9p_2a^2 + (3p_1 + 2p_2^2)a + p_1p_2, \end{aligned}$$

which is also a contradiction, since $a \geq 0$, $9a^3 \geq a^3$, $9p_2a^2 \geq p_2a^2$, $(3p_1 + 2p_2^2)a \geq p_1a$, and $p_1p_2 > p_0$.

2. $|Y(wj)| < |P(wj)|$, if w is real and $w \geq 0$. Indeed, given that

$$Y(\lambda) = -C\lambda^2 - Cq_1\lambda - Cq_0 \text{ y } P(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0,$$

we have

$$\begin{aligned} Y(wj) &= Cw^2 - Cq_1wj - Cq_0 = (Cw^2 - Cq_0) - Cq_1wj, \\ |Y(wj)|^2 &= (Cw^2 - Cq_0)^2 + C^2q_1^2w^2, \\ |Y(wj)|^2 &= C^2w^4 - 2C^2q_0w^2 + C^2q_0^2 + C^2q_1^2w^2. \end{aligned}$$

$$\begin{aligned} P(wj) &= -w^3j - p_2w^2 + p_1wj + p_0 = (p_0 - p_2w^2) + (p_1w - w^3)j, \\ |P(wj)|^2 &= (w^3 - p_1w)^2 + (p_2w^2 - p_0)^2, \\ |P(wj)|^2 &= w^6 - 2p_1w^4 + p_1^2w^2 + p_2^2w^4 - 2p_2p_0w^2 + p_0^2. \end{aligned}$$

Consequently,

$$|P(wj)|^2 - |Y(wj)|^2 = (p_2^2 - 2p_1 - C^2)w^4 + (p_1^2 + 2C^2q_0 - 2p_0p_2 - C^2q_1^2)w^2 + w^6 + (p_0^2 - C^2q_0^2).$$

The following expression $p_2^2 - 2p_1 - C^2$ is expanded as

$$(3\rho_i v + 2\alpha + 2\mu + \eta + \mu\mathcal{R}_0)^2 - 2(\rho_i v + \mu\mathcal{R}_0)(2\rho_i v + 2\alpha + 2\mu + \eta) - 2(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}.$$

Notice that

$$\begin{aligned}(3\rho_i v + 2\alpha + 2\mu + \eta + \mu\mathcal{R}_0)^2 &= [(2\rho_i v + 2\alpha + 2\mu + \eta) + (\rho_i v + \mu\mathcal{R}_0)]^2 \\ &= [(\rho_i v + \alpha + \mu) + (\rho_i v + \alpha + \mu + \eta)]^2 \\ &\quad + 2(2\rho_i v + 2\alpha + 2\mu + \eta)(\rho_i v + \mu\mathcal{R}_0) + (\rho_i v + \mu\mathcal{R}_0)^2 \\ &= (\rho_i v + \alpha + \mu)^2 + 2(\rho_i v + \alpha + \mu)(\rho_i v + \alpha + \mu + \eta) \\ &\quad + (\rho_i v + \alpha + \mu + \eta)^2 + (\rho_i v + \mu\mathcal{R}_0)^2 \\ &\quad + 2(2\rho_i v + 2\alpha + 2\mu + \eta)(\rho_i v + \mu\mathcal{R}_0).\end{aligned}$$

Thus,

$$\begin{aligned}p_2^2 - 2p_1 - C^2 &= (\rho_i v + \alpha + \mu + \eta)^2 + (\rho_i v + \mu\mathcal{R}_0)^2 + (\rho_i v + \alpha + \mu)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2} \\ &> (\rho_i v + \mu\mathcal{R}_0)^2 + (\alpha + \mu + \eta)^2 + (\alpha + \mu)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2} \\ &> (\rho_i v + \mu\mathcal{R}_0)^2 + (\alpha + \mu + \eta)^2(\alpha + \mu)^2\Bigg[\frac{1}{(\alpha + \mu + \eta)^2} + \frac{1}{(\alpha + \mu)^2} \\ &\quad - \frac{\beta^2\Lambda^2}{\mu^2(\mu + \alpha)^2(\mu + \alpha + \eta)^2\mathcal{R}_0^2}\Bigg].\end{aligned}$$

Since

$$\frac{\beta^2\Lambda^2}{\mu^2(\mu + \alpha)^2(\mu + \alpha + \eta)^2\mathcal{R}_0^2} = \frac{1}{(\mu + \alpha + (1 - r)\eta)^2},$$

and

$$\frac{1}{(\alpha + \mu)^2} - \frac{1}{(\mu + \alpha + (1 - r)\eta)^2} > 0,$$

then $p_2^2 - 2p_1 - C^2 > 0$. Next, for $p_1^2 + 2C^2q_0 - 2p_0p_2 - C^2q_1^2$, one obtains

$$\begin{aligned}p_1^2 &= (\rho_i + \mu\mathcal{R}_0)^2(2p_i v + 2\alpha + 2\mu + \eta)^2 + (\rho_i v + \alpha + \mu + \nu)^2(\rho_i v + \alpha + \mu)^2 \\ &\quad + 2(\rho_i v + \mu\mathcal{R}_0)(2\rho_i + 2\alpha + 2\mu + \eta)(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) \\ &= (\rho_i + \mu\mathcal{R}_0)^2(\rho_i v + \alpha + \mu)^2 + 2(\rho_i + \mu\mathcal{R}_0)^2(\rho_i v + \alpha + \mu)(\rho_i v + \alpha + \mu + \nu) \\ &\quad + (\rho_i + \mu\mathcal{R}_0)^2(\rho_i v + \alpha + \mu + \nu)^2 + (\rho_i v + \alpha + \mu + \nu)^2(\rho_i v + \alpha + \mu)^2 \\ &\quad + 2(\rho_i v + \mu\mathcal{R}_0)(2\rho_i + 2\alpha + 2\mu + \eta)(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu).\end{aligned} \tag{32}$$

$$\begin{aligned}-2p_0p_2 &= -2(\rho_i v + \mu\mathcal{R}_0)(\rho_i v + \alpha + \mu)(\rho_i v + \alpha + \mu + \eta)(3\rho_i v + 2\alpha + 2\mu + \eta) \\ &= -2(\rho_i v + \mu\mathcal{R}_0)(\rho_i v + \alpha + \mu)(\rho_i v + \alpha + \mu + \eta)(2\rho_i v + 2\alpha + 2\mu + \eta) \\ &\quad - 2(\rho_i v + \mu\mathcal{R}_0)^2(\rho_i v + \alpha + \mu)(\rho_i v + \alpha + \mu + \eta).\end{aligned} \tag{33}$$

$$2C^2q_0 = 2\frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)(\rho_i v + \alpha + \mu + (1 - r)\eta). \tag{34}$$

$$\begin{aligned}
-C^2q_1^2 &= -\frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(2\rho_i v + 2\mu + \alpha + (1-r)\eta)^2 \\
&= -\frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)^2 - 2\frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)(\rho_i v + \alpha + \mu + (1-r)\eta) \\
&\quad - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \alpha + \mu + (1-r)\eta)^2.
\end{aligned} \tag{35}$$

Then, from (32)–(35), one obtains

$$\begin{aligned}
p_1^2 + 2C^2q_0 - 2p_0p_2 - C^2q_1^2 = {} & (\rho_i v + \mu\mathcal{R}_0)^2(\rho_i v + \mu + \alpha)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)^2 \\
& + (\rho_i + \mu + \alpha + \eta)^2(\rho_i v + \mu\mathcal{R}_0)^2 \\
& + (\rho_i v + \mu + \alpha)^2(\rho_i + \mu + \alpha + \eta)^2 \\
& - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \alpha + \mu + (1-r)\eta)^2.
\end{aligned}$$

As $\mathcal{R}_0 > 1$, then $\mu\mathcal{R}_0 > \mu$, after

$$\begin{aligned}
p_1^2 + 2C^2q_0 - 2p_0p_2 - C^2q_1^2 > {} & (\rho_i v + \mu)^2(\rho_i v + \mu + \alpha)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)^2 \\
& + (\rho_i v + \mu + \alpha)^2(\rho_i v + \mu + \alpha + \eta)^2 \\
& + (\rho_i + \mu + \alpha + \eta)^2(\rho_i v + \mu)^2 \\
& - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \alpha + \mu + (1-r)\eta)^2.
\end{aligned}$$

Now,

$$\begin{aligned}
&(\rho_i v + \mu)^2(\rho_i v + \mu + \alpha)^2 + (\rho_i + \mu + \alpha + \eta)^2(\rho_i v + \mu)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \mu)^2 \\
&> (\rho_i v + \mu)^2\left[(\mu + \alpha)^2 + (\mu + \alpha + \eta)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}\right] > 0,
\end{aligned}$$

and

$$\begin{aligned}
&(\rho_i v + \mu + \alpha)^2(\rho_i v + \mu + \alpha + \eta)^2 - \frac{\beta^2\Lambda^2}{\mu^2\mathcal{R}_0^2}(\rho_i v + \alpha + \mu + (1-r)\eta)^2 \\
&= \left[(\rho_i v + \mu + \alpha)(\rho_i v + \mu + \alpha + \eta) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v + \alpha + \mu + (1-r)\eta)\right] \\
&\times \left[(\rho_i v + \mu + \alpha)(\rho_i v + \mu + \alpha + \eta) + \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v + \alpha + \mu + (1-r)\eta)\right].
\end{aligned}$$

Now, recall that

$$\frac{\beta\Lambda(\mu + \alpha + (1-r)\eta)}{\mu(\mu + \alpha)(\mu + \alpha + \eta)} = \mathcal{R}_0, \quad \Rightarrow \frac{\beta\Lambda(\mu + \alpha + (1-r)\eta)}{\mu\mathcal{R}_0} = (\mu + \alpha)(\mu + \alpha + \eta),$$

then

$$\begin{aligned}
&(\rho_i v + \mu + \alpha)(\rho_i v + \mu + \alpha + \eta) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v + \alpha + \mu + (1-r)\eta) \\
&= \rho_i^2 v^2 + \rho_i v(\alpha + \mu + \eta + \alpha + \mu) + (\alpha + \mu + \eta)(\alpha + \mu) \\
&- \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\alpha + \mu + (1-r)\eta) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v) \\
&= \rho_i^2 v^2 + \rho_i v(\alpha + \mu + \eta + \alpha + \mu) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v) \\
&= \rho_i^2 v^2 + \rho_i v\left(\alpha + \mu + \eta + \alpha + \mu - \frac{\beta\Lambda}{\mu\mathcal{R}_0}\right) \\
&= \rho_i^2 v^2 + \rho_i v(\alpha + \mu + \eta)(\alpha + \mu)\left(\frac{1}{\alpha + \mu + \eta} + \frac{1}{\alpha + \mu} - \frac{\beta\Lambda}{\mu(\alpha + \mu)(\alpha + \mu + \eta)\mathcal{R}_0}\right) > 0.
\end{aligned}$$

Thus, $p_1^2 + 2C^2q_0 - 2p_0p_2 - C^2q_1^2 > 0$. Finally, let us consider $p_0^2 - C^2q_0^2 = (p_0 - Cq_0)(p_0 + Cq_0)$, where $p_0 - Cq_0$ is given by

$$(\rho_i v + \mu\mathcal{R}_0)(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i + \mu)(\rho_i v + \alpha + \mu + (1-r)\eta)$$
$$> (\rho_i v + \mu)(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v + \mu)(\rho_i v + \alpha + \mu + (1-r)\eta)$$
$$> (\rho_i v + \mu)\left[(\rho_i v + \alpha + \mu + \eta)(\rho_i v + \alpha + \mu) - \frac{\beta\Lambda}{\mu\mathcal{R}_0}(\rho_i v + \alpha + \mu + (1-r)\eta)\right] > 0.$$

Hence, $p_0^2 - C^2q_0^2 > 0$. Thus, $|P(wj)| > |Y(wj)|$ for all $w \geq 0$.

3. Let us show that $\lim_{|z|\to\infty,\, \Re(z)\geq 0} |Y(z)/P(z)| = 0$. Indeed, suppose that $F_i(z) = 0$, with $\Re(z) \geq 0$. Then, $P(z) \neq 0$, and if $z = r(\cos\theta + \sin\theta j)$ $(r > 0)$, we have

$$\begin{aligned} Y(z) &= r^2(\cos 2\theta + \sin 2\theta j) + q_1 r(\cos\theta + \sin\theta j) + q_0, \\ |Y(z)|^2 &= (r^2\cos 2\theta + q_1 r\cos\theta + q_0) + (r^2\sin 2\theta + q_1 r\sin\theta)^2, \\ P(z) &= r^3(\cos 3\theta + \sin 3\theta j) + p_2 r^2(\cos 2\theta + \sin 2\theta j) + p_1 r(\cos\theta + \sin\theta j) + p_0, \\ |P(z)|^2 &= (r^3\cos 3\theta + p_2 r^2\cos 2\theta + p_1 r\cos\theta + p_0)^2 + (r^3\sin 3\theta + p_2 r^2\sin 2\theta + p_1 r\sin\theta)^2. \end{aligned}$$

However, $\left|\frac{Y(z)}{P(z)}\right|^2$ yields

$$\frac{r^4\left(\cos 2\theta + \frac{q_1}{r}\cos\theta + \frac{q_0}{r}\right)^2 + r^4\left(\sin 2\theta + \frac{q_1}{r}\sin\theta\right)^2}{r^6\left(\cos 3\theta + \frac{p_2}{r}\cos 2\theta + \frac{p_1}{r^2}\cos\theta + \frac{p_0}{r^2}\right)^2 + r^6\left(\sin 3\theta + \frac{p_2}{r}\sin 2\theta + \frac{p_1}{r^2}\sin\theta\right)^2}.$$

Therefore, if $\Re(z) \geq 0$,

$$\lim_{|z|\to+\infty}\left|\frac{Y(z)}{P(z)}\right|^2 = \lim_{r\to+\infty}\frac{1}{r^2}\cdot\frac{(\cos 2\theta)^2 + (\sin 2\theta)^2}{(\cos 3\theta)^2 + (\sin 3\theta)^2} = 0.$$

Thus, we arrive at the following theoretical result:

Theorem 2. *For system (1) with the conditions in (2), we have the following:*

1. *If $\mathcal{R}_0 \leq 1$, the infection-free state of E_0 is locally asymptotically stable.*
2. *If $\mathcal{R}_0 > 1$, the endemic balance E_e is locally asymptotically stable, if $\tau > 0$.*

The next section is devoted to performing numerical simulations to support the previous theoretical results.

5. Numerical Simulations of the Mathematical Model

In this section, we present numerical simulations of the mathematical model (1), which is based on a system of partial differential equations with a discrete-time delay τ. The numerical simulations allow us to examine the dynamics of a simplified scenario of the COVID-19 pandemic. In addition, they allow us to support the theoretical findings obtained in the previous subsections.

For the numerical simulations, we used available approximated real data where possible to test the theoretical results. Some parameter values are not known in the scientific literature, making it extremely difficult to find accurate values. There are many scientific works devoted to dealing with uncertainties in mathematical models for the COVID-19 pandemic and other epidemics. In addition, there are specific works devoted to the uncertainties associated with some parameter values. For instance, in [37–41], this topic is investigated as one of the main aims. In our work, one of the main aims was the mathematical analysis of the model (1). Nevertheless, we used approximated realistic values for most of the parameters. In particular, we used approximated initial conditions from the early phase of the COVID-19 pandemic in Colombia. For the history functions, we assumed constant values while accounting for proportions in the early phase of the COVID-19 situation in Colombia. This is a common approach used in similar theoretical works that deal with discrete-time delays. The time delay, which takes into account the incubation time, was set at seven days, which is slightly longer than in other works but still plausible. For asymptomatic cases, we assumed that 30% of cases were asymptomatic, which is one of the scenarios assumed by the CDC. For the transmission rate, we chose a parameter value (beta) that allows us to verify the theoretical results and also aligns with reported basic reproduction numbers from other studies.

We simulated a specific scenario where $\mathcal{R}_0 > 1$. Based on the theoretical results, COVID-19 did not become extinct. We present the numerical simulations, where the dynamics of the model (1) can be observed in space and time. The main idea behind this simulation is to use a specific value of the basic reproduction number $\mathcal{R}_0$ since this threshold parameter is crucial for the dynamics of the disease. The simulations provide further insights, such as the transient and long-term behavior of the disease for a specific simplified scenario.

We executed the simulations using the IMEX method [42]. This method combines the explicit and implicit Euler methods. In addition, we relied on the ghost-node technique to discretize the model (1) [43,44]. The numerical simulations were implemented using Matlab version R2023a. These simulations allow us to illustrate and support the theoretical results. The general basic idea of the numerical method is to divide the domain $[0, L] \times [0, T]$ using a grid of points. The space-step size is given by $h = \frac{L}{n}$ and the time-step size is given by $k = \frac{T}{m}$ for $m, n \in \mathbb{N}$. In addition, the value of k was chosen such that $\tau = pk$ for some integer number p such that $p \geq 1$.

For the numerical simulations, we used some illustrative numerical values for the parameters of the mathematical model (1). These values were as follows: $\Lambda = (180 \times 365)/50352943$, $\mu = 0.0001523835 * 365$, $\eta = 0.00006586199 \times 365$, $\beta = 0.4 \times 365$, $\alpha = 0.65$, $r = 0.3$, $\nu_S = 3.75 + 10^{-5} \times 365$, $\nu_I = 0.75 \times 10^{-10} \times 365$, $\nu_A = 0.75 \times 10^{-3} \times 365$, and $\nu_R = 3.75 + 10^{-5} \times 365$. For the domain, we used $[0, 1] \times [0, T]$, where T is the simulation time. The initial conditions, that is, the history functions, were assumed to be constant and given by $S(x, 0) = 46054839/50352943$, $A(x, 0) = 35005/50352943$, $I(x, 0) = 10500/50352943$, and $R(x, 0) = 4160000/50352943$ for $\theta \in [-\tau, 0]$. The boundary conditions were Neumann-type, as explained in the previous section. Figure 2 shows the numerical solution of the mathematical model (1) with a discrete-time delay $\tau = 7/365$ and $\mathcal{R}_0 = 1.27 > 1$ for $t \in [0, 1]$. It can be seen that the susceptible population $S(x, t)$ becomes extinct, whereas the infected $I(x, t)$ and asymptomatic $A(x, t)$ populations are approaching an endemic steady state through traveling waves. On the left-hand side of Figure 3, we can see the numerical solution of the mathematical model (1) at $x = 1$. Again, it can be observed that the susceptible population becomes extinct, whereas the infected and asymptomatic populations are approaching an endemic steady state, as expected. On the right-hand side of Figure 3, we can see the numerical solution of the mathematical model (1) at the final time of $t = 1$ for different values of x. It can be seen that the disease is approaching an endemic equilibrium state since $\mathcal{R}_0 = 1.27 > 1$. This numerical result supports the theoretical results, despite the fact that the model (1) was designed for the early phase of the COVID-19 pandemic. A more complex model is needed for the current pandemic to account for vaccinated individuals and the waning of immunity [8,45–47].

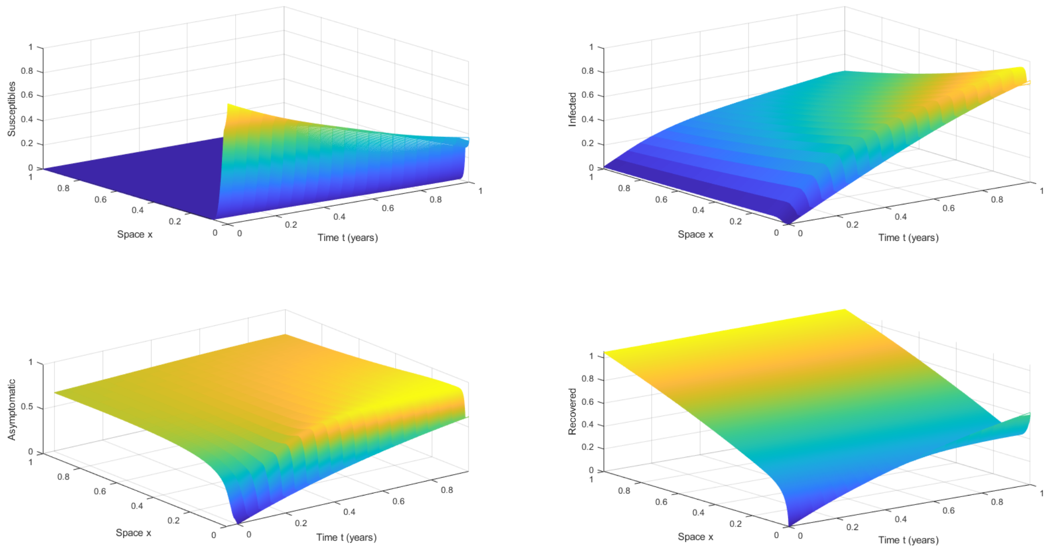

Figure 2. Numerical solution of the mathematical model (1) with a discrete-time delay $\tau = 7/365$, $t \in [0, 1]$, and $\mathcal{R}_0 = 1.27 > 1$.

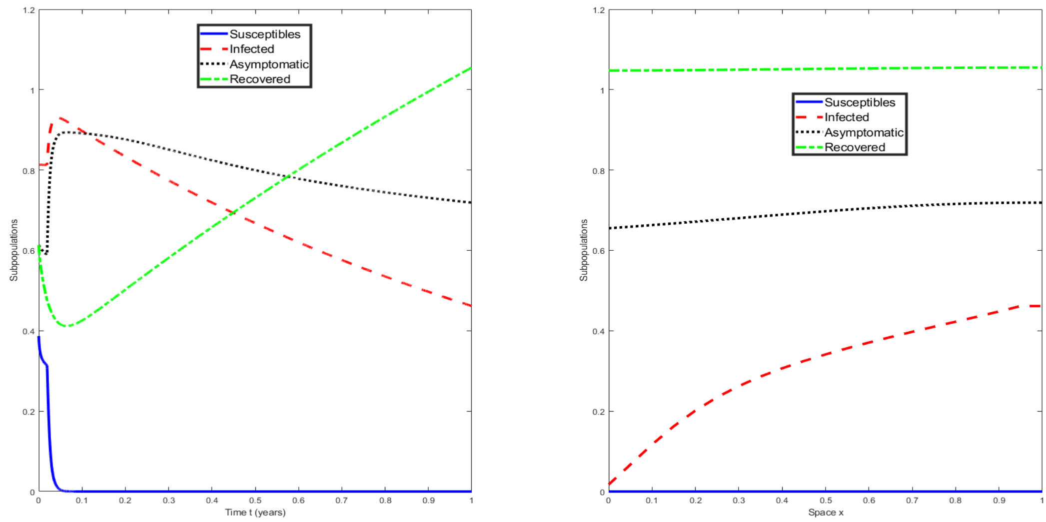

Figure 3. Numerical solution of the mathematical model (1) at $t = 1$ with a discrete-time delay $\tau = 7/365$, $t = 1$, and $\mathcal{R}_0 = 1.27 > 1$.

6. Conclusions

In this article, we studied the existence of traveling waves in a diffusive mathematical model representing the transmission of a virus within a population composed of susceptible (S), infected (I), asymptomatic (A), and recovered (R) individuals. The constructed model is a simplification of a complex real-world situation, but it provides further insight into the dynamics of different diseases, such as COVID-19. Furthermore, it demonstrates a mathematical modeling approach that considers spatial effects and time delays in the biological process related to SARS-CoV-2 virus transmission among the human population. An analytical study was performed, where the existence of solutions of traveling waves within a bounded domain was demonstrated using the upper and lower coupled solutions method. The existence and local asymptotic stability of both endemic (E_e) and disease-free (E_0) equilibrium states were also determined. We presented numerical simulations of the solutions in order to provide additional support to the theoretical results. In addition, we provided further simulations in order to illustrate the effects of discrete-time delays on the dynamics of the traveling waves.

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Appendix A

In this appendix, we present some results and definitions of a type of partial differential equation with delay, which is applied to our proposed model. For more information, see [11–13,34,48]. $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$. $\|\cdot\|_\infty$ denotes the space supreme norm $C_b(\mathbb{R},\mathbb{R}^4)$ of the continuous and bounded functions of $\mathbb{R}$ in $\mathbb{R}^4$. $\mathcal{C} = C([-\tau,0];\mathbb{R}^4)$ for $\tau > 0$. Consider the reaction-diffusion system with delay given by

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + f(u_t(x)) \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{A1}$$

where $u := (u_1,u_2,u_3,u_4)^T \in \mathbb{R}^4$, $D = diag(d_1,d_2,d_3,d_4)$ with $d_i > 0$ for $i = 1,2,3,4$, $f : \mathcal{C} \to \mathbb{R}^4$ is a continuous function, and for $t \geq 0$, $x \in \mathbb{R}$, $u_t(x) \in \mathcal{C}$ is

$$u_t(x)(\theta) = u(x,t+\theta), \quad \theta \in [-\tau,0].$$

When $f = (f_1,f_2,f_3,f_4)^T$, the Lipschitz condition for each component f_i is defined, as in the definition below.

Definition A1 (Lipschitz)**.** *If $u,v \in \mathcal{C}$ and $f_i : \mathcal{C} \to \mathbb{R}$ is a function, then we say that f_i is* ***Lipschitz*** *if there is $L_i > 0$ such that*

$$|f_i(u) - f_i(v)| \leq L_i\|u - v\|_\infty, \quad \textit{for} \quad i = 1,2,3,4.$$

Definition A2. *Let $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^4$ be with $\boldsymbol{p} = (p_1,p_2,p_3,p_4)^T$ and $\boldsymbol{q} = (q_1,q_2,q_3,q_4)^T$. Then, we define the following:*

1. *$\boldsymbol{q} \leq \boldsymbol{p}$ si $q_i \leq p_i$ for $i = 1,2,3,4$.*
2. *$\boldsymbol{q} < \boldsymbol{p}$ si $q_i < p_i$ and $q_i \neq p_i$ for $i = 1,2,3,4$.*
3. *$[\boldsymbol{q},\boldsymbol{p}] = \{\boldsymbol{r} \in \mathbb{R}^4 : \boldsymbol{q} \leq \boldsymbol{r} \leq \boldsymbol{p}\}$.*

Definition A3. *A traveling wave solution of* (A1) *is a special invariant translation function of the form $u(x,t) = \phi(\frac{x}{c} + t)$, where $\phi \in C^2(\mathbb{R},\mathbb{R}^4)$ is bounded and $c > 0$ is a constant known as the wave speed.*

Suppose that $\phi = (\phi_1,\phi_2,\phi_3,\phi_4)^T$ and $u_i(x,t) = \phi_i(\frac{x}{c} + t) \in \mathbb{R}$ for $i = 1,2,3,4$. Let $\theta \in [-\tau,0]$, and we make the change of variable $s = \frac{x}{c} + t$. Thus, we obtain

$$\frac{\partial u_i(x,t)}{\partial t} = \phi_i'(s)\frac{\partial s}{\partial t} = \phi_i'(s),\ \frac{\partial u_i(x,t)}{\partial x} = \phi_i'(s)\frac{\partial s}{\partial x} = \frac{1}{c}\phi_i'(s),$$
$$\frac{\partial^2 u_i(x,t)}{\partial x^2} = \frac{1}{c}\phi_i''(s)\frac{\partial s}{\partial x} = \frac{1}{c^2}\phi_i''(s),$$
$$f_i(u_t) = f_i(u(x,t+\theta)) = f_i\Big(\phi\Big(\frac{x}{c}+t+\theta\Big)\Big) = f_i(\phi(s+\theta)).$$

Therefore,
$$\frac{\partial u_i(x,t)}{\partial t} = d_i\frac{\partial^2 u_i}{\partial x^2} + f_i(u_t) \Leftrightarrow \phi_i'(s) = \frac{d_i}{c^2}\phi_i''(s) + f_i(\phi_s),$$

i.e.,
$$\frac{D}{c^2}\phi''(s) - \phi'(s) + f(\phi_s) = \mathbf{0},\ \text{para } s\in\mathbb{R},\quad \phi_s(\theta) = \phi(s+\theta). \tag{A2}$$

If for $c > 0$, system (A2) has a solution ϕ defined on $\mathbb{R}$ such that $\lim_{s\to-\infty}\phi(s) = u_-$ and $\lim_{s\to\infty}\phi(s) = u_+$, where u_- and u_+ are stationary states of (A1), then there exists $u(x,t) = \phi(\frac{x}{c}+t)$, which is called a traveling wave, with a velocity of $c > 0$.

Without loss of generality, suppose $\mathbf{k}\in\mathbb{R}^4$ such that $\mathbf{k} = (k_1,k_2,k_3,k_4)^T$, $\mathbf{k} > \mathbf{0}$ and

$$u_- = \mathbf{0},\ u_+ = \mathbf{k}.$$

Let $\mathbf{m} = (m_1,m_2,m_3,m_4)\in\mathbb{R}^4$ such that $\mathbf{m} > \mathbf{k}$ and for $\Omega\subseteq\mathbb{R}$, consider the following set:

$$W(\Omega;\mathbb{R}^4) = \{\phi\in C_b(\Omega,\mathbb{R}^4) : 0\le\phi_i(s)\le m_i,\quad i=1,2,3,4;\ \ s\in\Omega\}. \tag{A3}$$

Next, let $\psi,\varphi\in W([-\tau,0];\mathbb{R}^4)$, with

$$\psi = (\psi_1,\psi_2,\psi_3,\psi_4)^T,\ \ \varphi = (\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T,$$

and $\psi\le\varphi$. Moreover, suppose that

$$\varphi^1 = (\psi_1,\varphi_2,\varphi_3,\varphi_4)^T,\ \varphi^{2,3} = (\varphi_1,\psi_2,\psi_3,\varphi_4)^T,\ \varphi^4 = (\varphi_1,\varphi_2,\varphi_3,\psi_4)^T.$$

Definition A4. *The partial conditions of quasi-monotonicity (PQM) for $f = (f_1,f_2,f_3,f_4)^T$ are given if there are constants $\beta_1,\ \beta_2,\ \beta_3,\beta_4 > 0$ such that*

$$\begin{aligned}
f_1(\varphi) - f_1(\psi) + \beta_1(\varphi_1(0)-\psi_1(0)) &\ge 0,\\
f_2(\varphi) - f_2(\varphi^{2,3}) + \beta_2(\varphi_2(0)-\psi_2(0)) &\ge 0,\\
f_3(\varphi) - f_3(\varphi^{2,3}) + \beta_3(\varphi_3(0)-\psi_3(0)) &\ge 0,\\
f_4(\varphi) - f_4(\psi) + \beta_4(\varphi_4(0)-\psi_4(0)) &\ge 0,\\
f_2(\varphi) - f_2(\varphi^1) &\le 0,\\
f_2(\varphi) - f_2(\varphi^4) &\le 0,\\
f_3(\varphi) - f_2(\varphi^1) &\le 0,\\
f_3(\varphi) - f_2(\varphi^4) &\le 0.
\end{aligned}$$

We assume the following hypothesis:

$(\mathbf{A_1})f(\mathbf{0}) = f(\mathbf{k}) = \mathbf{0}$, and there exists $\mathbf{u} = (u_1,u_2,u_3,u_4)\in\mathbb{R}^4$ with $\mathbf{0} < \mathbf{u} < \mathbf{k}$ such that $f(\mathbf{u}) \ne \mathbf{0}$.

$(\mathbf{A_2})f_i$ satisfies Lipschitz's condition in $W([-\tau,0];\mathbb{R}^4), i = 1,2,3,4.$ (A4)

$(\mathbf{A_3})f_i$ fulfills the conditions of PQM, $i = 1,2,3,4.$

We use the following notation:

- A function has a certain property in **almost everywhere** (a.e) in $\mathbb{R}$ if it has the property in $\mathbb{R}$ except on a set of measure zero (see [49], Chapter 11).
- The notation
$$\phi_i(s+) = \lim_{t\to s^+} \phi_i(s), \quad \phi_i(s-) = \lim_{t\to s^-} \phi_i(s),$$
indicates that $\phi_i(s+)$ and $\phi_i(s-)$ exist for $i = 1,2,3,4$.

Definition A5. *Let $\underline{\phi}$, $\overline{\phi} : \mathbb{R} \to \mathbb{R}^4$ be boundless and continuous functions such that $\overline{\phi} \geq \underline{\phi}$. If there are constants $T_1, T_2, T_3, \ldots, T_N$ such that $\underline{\phi}$, $\overline{\phi} \in C^2(\mathbb{R} - \{T_1, T_2, T_3, \ldots, T_N\})$, and for*

$$\underline{\phi} = (\underline{\phi_1}, \underline{\phi_2}, \underline{\phi_3}, \underline{\phi_4})^T, \ \underline{\phi}^{2,3} = (\overline{\phi_1}, \underline{\phi_2}, \underline{\phi_3}, \overline{\phi_4})^T,$$
$$\overline{\phi}^{2,3} = (\underline{\phi_1}, \overline{\phi_2}, \overline{\phi_3}, \underline{\phi_4})^T, \ \overline{\phi} = (\overline{\phi_1}, \overline{\phi_2}, \overline{\phi_3}, \overline{\phi_4})^T$$

it satisfies the following properties

$$\frac{d_i}{c^2}\overline{\phi}_i''(s) - \overline{\phi}_i'(s) + f_i(\overline{\phi}_s) \leq 0, \qquad \frac{d_i}{c^2}\underline{\phi}_i''(s) - \underline{\phi}_i'(s) + f_i(\underline{\phi}_s) \geq 0 \ \text{ a.e. in } \mathbb{R},$$
$$\frac{d_j}{c^2}\overline{\phi}_j''(s) - \overline{\phi}_j'(s) + f_j(\overline{\phi}_s^{2,3}) \leq 0, \qquad \frac{d_j}{c^2}\underline{\phi}_j''(s) - \underline{\phi}_j'(s) + f_j(\underline{\phi}_s^{2,3}) \geq 0 \ \text{ a.e. in } \mathbb{R},$$

for $i = 1,4$ and $j = 2,3$. Then, $\underline{\phi}, \overline{\phi}$ are called lower and upper coupled solutions, respectively, of (A2).

If Equation (A2) has an upper solution $\overline{\phi}$ and a lower solution $\underline{\phi}$, they must satisfy the following hypotheses:

(i) $\mathbf{0} \leq \underline{\phi} \leq \overline{\phi} \leq \mathbf{m}$.
(ii) $\lim_{s\to-\infty} \underline{\phi}(s) = \mathbf{0}, \quad \lim_{s\to\infty} \overline{\phi}(s) = \mathbf{k}$.
(iii) $\underline{\phi}_i'(s+) \geq \underline{\phi}_i'(s-)$ and $\overline{\phi}_i'(s+) \leq \overline{\phi}_i'(s-)$ for all $s \in \mathbb{R}$ and $i = 1,2,3,4$.

Using the constants β_i given in Definition A4, we define the operator $H : C(\mathbb{R}, \mathbb{R}^4) \to C(\mathbb{R}, \mathbb{R}^4)$ given by

$$H(\varphi)(s) = f(\varphi_s) + \beta\varphi(s), \quad \varphi \in C(\mathbb{R}, \mathbb{R}^4), \quad \forall s \in \mathbb{R}, \tag{A5}$$

where $H = (H_1, H_2, H_3, H_4)$, $\beta = diag(\beta_1, \beta_2, \beta_3, \beta_4)$, and $H_i(\varphi)(s) = f_i(\varphi_s) + \beta_i\varphi_i(s)$. This operator holds the following properties:

Lemma A1. *Let $\phi, \psi, \ \phi^i, \phi^{2,3}$ be for $i = 1,4$, as in Definition A4, and suppose that* $(\mathbf{A_3})$ *holds. Then, for $j = 2,3$, and $s \in \mathbb{R}$ one obtains*

$$H_j(\phi)(s) \leq H_j(\phi^i)(s), \ H_j(\phi^{2,3})(s) \leq H_j(\phi)(s).$$

Proof. From the hypothesis and the definition of the operator H for $j = 2,3$ and $s \in \mathbb{R}$ we have

$$H_j(\phi)(s) - H_j(\phi^i)(s) = f_j(\phi_s) + \beta_j\phi_j - f_j(\phi_s^i) - \beta_j\phi_j = f_j(\phi_s) - f_j(\phi_s^i) \leq 0.$$

For $\theta \in [-\tau, 0]$, it follows that

$$\begin{aligned} H_j(\phi)(s) - H_j(\phi^{2,3})(s) &= f_j(\phi_s(\theta)) + \beta_j\phi_j(s) - f_j(\phi_s^{2,3}(\theta)) - \beta_j\psi_j(s) \\ &= f_j(\phi(s+\theta)) - f_j(\phi^{2,3}(s+\theta)) + \beta_j(\phi_j(s+0) - \psi_j(s+0)) \\ &= f_j(\phi_s(\theta)) - f_j(\phi_s^{2,3}(\theta)) + \beta_j((\phi_j)_s(0) - (\psi_j)_s(0)) \geq 0. \end{aligned}$$

□

Lemma A2. *Suppose that* $(\mathbf{A_1})$ *and* $(\mathbf{A_3})$ *are fulfilled. If $\phi, \psi \in W(\mathbb{R}, \mathbb{R}^4)$, then the following statements hold:*

(*i*) $H_j(\phi)(s) \geq 0$ *for* $j = 1, 4$ *and* $s \in \mathbb{R}$.
(*ii*) $H_j(\psi)(s) \leq H_j(\phi)(s)$ *for* $j = 1, 4$, $\psi \leq \phi$, *and* $s \in \mathbb{R}$.

Proof. If $(\mathbf{A_1})$ and $(\mathbf{A_3})$ hold, then $f(\mathbf{0}) = \mathbf{0}$. For $\gamma, \alpha \in W([-\tau, 0], \mathbb{R}^4)$, with $\gamma \leq \alpha$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, one obtains

$$f_j(\alpha) - f_j(\gamma) + \beta_j(\alpha_j(0) - \gamma_j(0)) \geq 0, \quad j = 1, 4.$$

Consider $\phi, \psi \in W(\mathbb{R}, \mathbb{R}^4)$ such that $\psi \leq \phi$ and $\rho = (\rho_1, \rho_2, \rho_3, \rho_4) \equiv \mathbf{0} \in W(\mathbb{R}, \mathbb{R}^4)$. We now show the following:

(*i*) If $s \in \mathbb{R}$ and $j = 1, 4$, we have

$$\begin{aligned} H_j(\phi)(s) &= f_j(\phi_s) + \beta_j \phi_j(s) = f_j(\phi_s) + 0 + \beta_j[(\phi_j)_s(0) - 0] \\ &= f_j(\phi_s) - f_j(\rho_s) + \beta_j[(\phi_j)_s(0) - (\rho_j)_s(0)] \geq 0. \end{aligned}$$

(*ii*) For $s \in \mathbb{R}$ and $j = 1, 4$, we obtain

$$\begin{aligned} H_j(\phi)(s) - H_j(\psi)(s) &= f_j(\phi_s) - f_j(\psi_s) + \beta_j(\phi_j(s) - \psi_j(s)) \\ &= f_j(\phi_s) - f_j(\psi_s) + \beta_j[(\phi_j)_s(0) - (\psi_j)_s(0)] \geq 0. \end{aligned}$$

□

Therefore, system (A2) is equivalent to the system

$$-\frac{d_i}{c^2}\varphi_i''(s) + \varphi_i'(s) + \beta_i \varphi_i(s) = H_i(\varphi)(s), \; i = 1, 2, 3, 4, \; s \in \mathbb{R}. \tag{A6}$$

Thus, we have the characteristic equation for the part of the homogeneous equation as

$$\frac{d_i}{c^2}\lambda^2 - \lambda - \beta_i = 0, \quad i = 1, 2, 3, 4,$$

which has the roots

$$\lambda_{1i} = \frac{c^2(1 - \sqrt{1 + 4\beta_i d_i / c^2})}{2d_i}, \quad \lambda_{2i} = \frac{c^2(1 + \sqrt{1 + 4\beta_i d_i / c^2})}{2d_i}.$$

Since $\lambda_{2i} > 0$ and $\lambda_{1i}\lambda_{2i} = \frac{c^4}{4d_i^2}\left[1 - \left(1 + 4\frac{\beta_i d_i}{c^2}\right)\right] = \frac{c^4}{4d_i^2}\left(\frac{-4\beta_i d_i}{c^2}\right) < 0$, then $\lambda_{1i} < 0$. Hence, $\lambda_{2i} - \lambda_{1i} > 0$. Now, we define the application

$$F = (F_1, F_2, F_3, F_4) : W(\mathbb{R}, \mathbb{R}^4) \to C(\mathbb{R}, \mathbb{R}^4),$$

given by

$$F_i(\phi_i)(s) = \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)} H_i(\phi)(p)dp + \int_{s}^{\infty} e^{\lambda_{2i}(s-p)} H_i(\phi)(p)dp\right],$$

for $i = 1, 2, 3, 4$.

The following statement holds:

$$\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}dp + \int_{s}^{\infty} e^{\lambda_{2i}(s-p)}dp = \frac{1}{\lambda_{2i}} - \frac{1}{\lambda_{1i}} > 0. \tag{A7}$$

Indeed,

$$\begin{aligned} \int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}dp &= e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}dp = e^{\lambda_{1i}s}\left[-\frac{1}{\lambda_{1i}}(e^{-\lambda_{1i}s} - \lim_{r \to -\infty} e^{-\lambda_{1i}r})\right] \\ &= e^{\lambda_{1i}s}\left[-\frac{1}{\lambda_{1i}}(e^{-\lambda_{1i}s} - 0)\right] = -\frac{1}{\lambda_{1i}}, \end{aligned}$$

and
$$\int_s^{\infty} e^{\lambda_{2i}(s-p)}dp = e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}dp = e^{\lambda_{2i}s}\left[-\frac{1}{\lambda_{2i}}(\lim_{r\to\infty} e^{-\lambda_{2i}r} - e^{-\lambda_{2i}s})\right]$$
$$= e^{\lambda_{2i}s}\left[-\frac{1}{\lambda_{2i}}(-e^{-\lambda_{2i}s})\right] = \frac{1}{\lambda_{2i}}.$$

Proposition A1. *The F application is well defined and satisfies* (A6).

Proof. Since $f = (f_1, f_2, f_3, f_4)$ is continuous and satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$, then for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in W(\mathbb{R}, \mathbb{R}^4)$, by the Lipschitz condition of f_i in $W([-\tau, 0], \mathbb{R}^4)$ and $f_i(\mathbf{0}) = 0$, if we set $\psi \equiv \mathbf{0} \in W([-\tau, 0], \mathbb{R}^4)$, there is $L_i > 0$ such that for all $s \in \mathbb{R}$, it holds that
$$|f_i(\phi_s) - f_i(\psi)| \le L_i\|\phi_s - \psi\|_\infty \Rightarrow |f_i(\phi_s)| \le L_i\|\phi_s\|_\infty \le L_i\|\phi\|_\infty.$$
Thus, for $s \in \mathbb{R}$, we have
$$|H_i(\phi)(s)| \le |f_i(\phi_s)| + \beta_i|\phi_i(s)| \le L_i\|\phi\|_\infty + \beta_i\|\phi\|_\infty.$$
Next, considering $K_i = \|\phi\|_\infty(L_i + \beta_i)$, it is deduced that $|H_i(\phi)(s)| \le K_i$ for all $s \in \mathbb{R}$. Hence, from (A7) and for $s \in \mathbb{R}$ and $i = 1, 2, 3, 4$, it follows that
$$|F_i(\phi)(s)| \le \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}|H_i(\phi)(p)|dp + \int_s^{\infty} e^{\lambda_{2i}(s-p)}|H_i(\phi)(p)|dp\right]$$
$$\le \frac{c^2K_i}{d_i(\lambda_{2i} - \lambda_{1i})}\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}dp + \int_s^{\infty} e^{\lambda_{2i}(s-p)}dp\right] < \infty.$$
It is concluded that for $\phi \in W(\mathbb{R}, \mathbb{R}^n)$, one obtains $F(\phi) \in C_b(\mathbb{R}, \mathbb{R}^4)$, i.e., F is well defined. Let us assume that F satisfies (A6). Indeed, for $i = 1, 2, 3, 4$, we have $\gamma_i = \frac{c^2}{d_i(\lambda_{2i} - \lambda_{1i})}$. Then,
$$F_i(\phi)(s) = \gamma_i\left[e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp + e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp\right],$$
$$\frac{d}{ds}F_i(\phi)(s) = \gamma_i\Big[\lambda_{1i}e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp + e^{\lambda_{1i}s}e^{-\lambda_{1i}s}H_i(\phi)(s)$$
$$+ \lambda_{2i}e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp - e^{\lambda_{2i}s}e^{-\lambda_{2i}s}H_i(\phi)(s)\Big]$$
$$= \gamma_i\left[\lambda_{1i}e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp + \lambda_{2i}e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp\right].$$
$$\frac{d^2}{ds^2}F_i(\phi)(s) = \gamma_i\Big[\lambda_{1i}^2e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp + \lambda_{1i}e^{\lambda_{1i}s}e^{-\lambda_{1i}s}H_i(\phi)(s)$$
$$+ \lambda_{2i}^2e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp - \lambda_{2i}e^{\lambda_{2i}s}e^{-\lambda_{2i}s}H_i(\phi)(s)\Big]$$
$$= \gamma_i\Big[\lambda_{1i}^2e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp + \lambda_{2i}^2e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp$$
$$+ (\lambda_{1i} - \lambda_{2i})H_i(\phi)(s)\Big].$$
Now, $I_i(s) = e^{\lambda_{1i}s}\int_{-\infty}^{s} e^{-\lambda_{1i}p}H_i(\phi)(p)dp$ and $J_i(s) = e^{\lambda_{2i}s}\int_s^{\infty} e^{-\lambda_{2i}p}H_i(\phi)(p)dp$, and then
$$\beta_iF_i(\phi)(s) = \beta_i\gamma_iI_i(s) + \beta_i\gamma_iJ_i(s),$$
$$F_i'(\phi)(s) = \lambda_{1i}\gamma_iI_i(s) + \lambda_{2i}\gamma_iJ_i(s),$$
$$-\frac{d_i}{c^2}F_i''(\phi)(s) = -\frac{d_i}{c^2}\lambda_{1i}^2\gamma_iI_i(s) - \frac{d_i}{c^2}\lambda_{2i}^2\gamma_iJ_i(s) + H_i(\phi)(s).$$

Since $-\frac{d_i}{c^2}\lambda_{1i}^2 + \lambda_{1i} + \beta_i = 0$ and $-\frac{d_i}{c^2}\lambda_{2i}^2 + \lambda_{2i} + \beta_i = 0$, it is clear that

$$-\frac{d_i}{c^2}F_i''(\phi)(s) + F_i'(\phi)(s) + \beta_i F_i(\phi)(s) = H(\phi)(s).$$

□

Now, for $\rho > 0$ $\rho < \min\{-\lambda_{1i}, \lambda_{2i} : i = 1,2,3,4\}$, we define the following set:

$$B_\rho(\mathbb{R}, \mathbb{R}^4) = \left\{\phi \in C_b(\mathbb{R}, \mathbb{R}^4) : \sup_{s\in\mathbb{R}} e^{-\rho|s|}\|\phi(s)\| < \infty\right\},$$

and $\|\phi\|_\rho := \sup_{s\in\mathbb{R}} e^{-\rho|s|}\|\phi(s)\|$ (see [34], pag. 7). Let $\underline{\phi}$ and $\overline{\phi}$ be the lower and upper coupled solutions, respectively, of (A2). We define the set

$$\Gamma = \{\phi \in W(\mathbb{R}, \mathbb{R}^4) : \underline{\phi} \le \phi \le \overline{\phi}\}.$$

Proposition A2. *Γ is a non-empty set that is closed, bounded, and convex.*

Proof. From (A5) and the functions, it holds that $\underline{\phi}$ and $\overline{\phi}$, then $\underline{\phi}, \overline{\phi} \in \Gamma$. This implies that $\Gamma \neq$. Γ is bounded. Indeed, let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ be such that $\phi, \psi \in \Gamma$. Since $\phi, \psi \in W(\mathbb{R}, \mathbb{R}^4)$, then

$$0 \le \phi_i \le m_i, \quad 0 \le \psi_i \le m_i, \quad i = 1,2,3,4.$$

Let $m = \max\{m_i : i = 1,2,3,4\}$. Then, $|\phi_i - \psi_i| < 2m$ for $i = 1,2,3,4$. Hence,

$$\|\phi - \psi\|^2 = \sum_{i=1}^{4} |\phi_i - \psi_i|^2 \le 4(2m)^2,$$

$$\Rightarrow \|\phi - \psi\| < 4m \Rightarrow \|\phi - \psi\|_\infty \le 4m.$$

Next, Γ is closed. Indeed, let $\{\phi_n\} \subset \Gamma$ be a sequence and $\psi \in C_b(\mathbb{R}, \mathbb{R}^4)$ be a cluster point of $\{\phi_n\}$. There is a subsequence $\{\psi_k\}$ of $\{\phi_n\}$ that converges to ψ. Suppose that $\psi_k = (\psi_{k1}, \psi_{k2}, \psi_{k3}, \psi_{k4})^T$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. From the definition of Γ, for $i = 1,2,3,4$, we have

$$\underline{\phi}_i \le \psi_{ki} \le \overline{\phi}_i \Rightarrow \underline{\phi}_i \le \lim_{k\to\infty} \psi_{ki} \le \overline{\phi}_i \Rightarrow \underline{\phi}_i \le \psi_i \le \overline{\phi}_i.$$

Thus, $\underline{\phi} \le \psi \le \overline{\phi}$, and therefore, $\psi \in \Gamma$, i.e., Γ is closed. Next, for $t \in [0,1]$, let $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ such that $\phi, \psi \in \Gamma$. Next, for $i = 1,2,3,4$, we have

$$\underline{\phi}_i \le \phi_i, \ \psi_i \le \overline{\phi}_i.$$

Consequently,

$$t\underline{\phi}_i \le t\phi_i \le t\overline{\phi}_i \ \text{ and } \ (1-t)\underline{\phi}_i \le (1-t)\psi_i \le (1-t)\overline{\phi}_i.$$

Summing both inequalities, one obtains

$$\underline{\phi}_i \le t\phi_i + (1-t)\psi_i \le \overline{\phi}_i \Rightarrow \underline{\phi} \le t\phi + (1-t)\psi \le \overline{\phi},$$

which implies that $[t\phi + (1-t)\psi] \in \Gamma$, and therefore, Γ is convex. □

Proposition A3. *$B_\rho(\mathbb{R}, \mathbb{R}^4)$ con $\|\cdot\|_\rho$ is a Banach space.*

Proof. It is clear that $B_\rho(\mathbb{R},\mathbb{R}^4)$ is a vectorial space and $\|\cdot\|_\rho$ is a norm. Let $\{\phi_n\}$ be a Cauchy sequence in $B_\rho(\mathbb{R},\ \mathbb{R}^n)$. Fix $s_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then, there is $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n \geq n_0$, we have

$$\|\phi_n - \phi_m\|_\rho < \frac{\varepsilon}{2} e^{-\rho|s_0|}.$$

In particular, for s_0 we have

$$\|\phi_n(s_0) - \phi_m(s_0)\| e^{-\rho|s_0|} < \|\phi_n - \phi_m\|_\rho < \frac{\varepsilon}{2} e^{-\rho|s_0|} \Rightarrow \|\phi_n(s_0) - \phi_m(s_0)\| < \frac{\varepsilon}{2}.$$

Now, s_0 is arbitrary, Then, $\|\phi_n(s) - \phi_m(s)\| < \frac{\varepsilon}{2}$ for all $s \in \mathbb{R}$, which implies $\|\phi_n - \phi_m\|_\infty < \varepsilon$. Thus, $\{\phi_n\}$ is a Cauchy sequence in the Banach Space $C_b(\mathbb{R},\mathbb{R}^4)$. Then, there exists $\phi \in C_b(\mathbb{R},\mathbb{R}^4)$ such that $\phi_n \to \phi$. Thus, for $\varepsilon_0 > 0$, there is $m_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq m_0$, we have

$$\|\phi_n - \phi\|_\infty < \frac{\varepsilon_0}{2}.$$

It follows that for $s \in \mathbb{R}$, we have

$$\begin{aligned} \|\phi_n(s) - \phi(s)\| \leq \|\phi_n - \phi\|_\infty < \frac{\varepsilon_0}{2}, \\ \Rightarrow \|\phi_n(s) - \phi(s)\| e^{-\rho|s|} < \frac{\varepsilon_0}{2} e^{-\rho|s|} < \frac{\varepsilon_0}{2}, \\ \Rightarrow \|\phi_n - \phi\|_\rho \leq \frac{\varepsilon_0}{2} < \varepsilon_0. \end{aligned}$$

The above implies that $\{\phi_n\}$ converges in $B_\rho(\mathbb{R},\mathbb{R}^4)$ with the norm $\|\cdot\|_\rho$. □

The following result for F is deduced from (A1) and (A2).

Lemma A3. *Suppose that conditions (A_1) and (A_3) hold. Let $\phi, \psi, \phi^i, \phi^{2,3}$ be such that $i = 1,4$, as in (A4). Then, we can conclude the following:*

(i) $F_i(\psi) \leq F_i(\phi)$ *for* $i = 1,4$.
(ii) $F_j(\phi) \leq F_j(\phi^i)$ *for* $j = 2,3$, $i = 1,4$.
(iii) $F_j(\phi^{2,3}) \leq F_j(\phi)$ *for* $j = 2,3$.

Now, the application F is continuous.

Lemma A4. *Suppose that (A_2) holds. Then, $F = (F_1, F_2, F_3, F_4)^T$ is continuous with respect to the norm $\|\cdot\|_\rho$ in $B_\rho(\mathbb{R},\mathbb{R}^4)$.*

Proof. Fix $\varepsilon > 0$ and consider $\delta > 0$ such that $\delta < \varepsilon/(L_i e^{\rho\tau} + \beta_i.)$ Then, for $s \in \mathbb{R}$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4), \psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in B_\rho(\mathbb{R},\mathbb{R}^4)$ with $\|\phi - \psi\|_\rho < \delta$, we have

$$\begin{aligned} |H_i(\phi)(s) - H_i(\psi)(s)| e^{-\rho|s|} &\leq |f_i(\phi_s) - f_i(\psi_s)| e^{-\rho|s|} + \beta_i |\phi_i(s) - \psi_i(s)| e^{-\rho|s|} \\ &\leq L_i \|\phi_s - \psi_s\|_\infty e^{-\rho|s|} + \beta_i \|\phi(s) - \psi(s)\| e^{-\rho|s|} \\ &\leq L_i \sup_{\theta\in[-\tau,0]} \|\phi(s+\theta) - \psi(s+\theta)\| e^{-\rho|s|} + \beta_i \|\phi - \psi\|_\rho \\ &\leq L_i \sup_{\theta\in[-\tau,0]} \|\phi(s+\theta) - \psi(s+\theta)\| e^{-\rho|s+\theta|} \sup_{\theta\in[-\tau,0]} e^{\rho|s+\theta|} e^{-\rho|s|} \\ &+ \beta_i \|\phi - \psi\|_\rho \\ &\leq L_i \|\phi - \psi\|_\rho e^{\rho|s|} e^{\rho\tau} e^{-\rho|s|} + \beta_i \|\phi - \psi\|_\rho \\ &\leq (L_i e^{\rho\tau} + \beta_i) \|\phi - \psi\|_\rho < \varepsilon. \end{aligned}$$

Now, for all $s > 0$ if $\gamma_i = \frac{c^2}{d_i(\lambda_{2i}-\lambda_{1i})}$, we have

$$\begin{aligned}|F_i(\phi)(s) - F_i(\psi)(s)|e^{-\rho|s|} &\leq \gamma_i e^{-\rho|s|}\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}|H_i(\phi)(p) - H_i(\psi)(p)|dp\right.\\ &\quad \left. + \int_{s}^{\infty} e^{\lambda_{2i}(s-p)}|H_i(\phi)(p) - H_i(\psi)(p)|dp\right]\\ &\leq \gamma_i e^{-\rho s}\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)+\rho|p|}|H_i(\phi)(p) - H_i(\psi)(p)|e^{-\rho|p|}dp + \right.\\ &\quad \left.\int_{s}^{\infty} e^{\lambda_{2i}(s-p)+\rho|p|}|H_i(\phi)(p) - H_i(\psi)(p)|e^{-\rho|p|}dp\right]\\ &\leq \gamma_i e^{-\rho s}\varepsilon\left[e^{\lambda_{1i}s}\int_{-\infty}^{0} e^{-(\lambda_{1i}+\rho)p}dp + e^{\lambda_{1i}s}\int_{0}^{s} e^{(-\lambda_{1i}+\rho)p}dp\right.\\ &\quad \left. e^{\lambda_{2i}s}\int_{-\infty}^{s} e^{(-\lambda_{2i}+\rho)p}dp\right].\end{aligned}$$

Since $\rho < -\lambda_{1i}, \lambda_{2i}$, we have $\rho + \lambda_{1i}, \rho - \lambda_{2i} < 0$. Then,

$$\begin{aligned}e^{\lambda_{1i}s}\int_{-\infty}^{0} e^{-(\lambda_{1i}+\rho)p}dp &= e^{\lambda_{1i}s}\left(-\frac{1}{\lambda_{1i}+\rho} + \frac{1}{\lambda_{1i}+\rho}\lim_{q\to-\infty} e^{-(-\lambda_{1i}+\rho)q}\right) = -\frac{e^{\lambda_{1i}s}}{\lambda_{1i}+\rho},\\ e^{\lambda_{1i}s}\int_{0}^{s} e^{(-\lambda_{1i}+\rho)p}dp &= e^{\lambda_{1i}s}\left(\frac{e^{(-\lambda_{1i}+\rho)s}}{-\lambda_{1i}+\rho} - \frac{1}{-\lambda_{1i}+\rho}\right) = \frac{e^{\rho s}}{-\lambda_{1i}+\rho} - \frac{e^{\lambda_{1i}s}}{-\lambda_{1i}+\rho},\\ e^{\lambda_{2i}s}\int_{s}^{\infty} e^{(-\lambda_{2i}+\rho)p}dp &= e^{\lambda_{2i}s}\left(\frac{1}{-\lambda_{2i}+\rho}\lim_{q\to\infty} e^{(-\lambda_{2i}+\rho)q} - \frac{e^{(-\lambda_{2i}+\rho)s}}{-\lambda_{2i}+\rho}\right) = -\frac{e^{\rho s}}{-\lambda_{2i}+\rho}.\end{aligned}$$

This implies that

$$\begin{aligned}|F_i(\phi)(s) - F_i(\psi)(s)|e^{-\rho|s|} &\leq \gamma_i e^{-\rho s}\varepsilon\left(-\frac{e^{\lambda_{1i}s}}{\lambda_{1i}+\rho} + \frac{e^{\rho s}}{-\lambda_{1i}+\rho} - \frac{e^{\lambda_{1i}s}}{-\lambda_{1i}+\rho} - \frac{e^{\rho s}}{-\lambda_{2i}+\rho}\right)\\ &\leq \gamma_i\varepsilon\left(\frac{2\rho}{\lambda_{1i}^2-\rho^2}e^{(\lambda_{1i}-\rho)s} + \frac{\lambda_{1i}-\lambda_{2i}}{(\rho-\lambda_{2i})(\rho-\lambda_{1i})}\right)\\ &< \gamma_i\varepsilon\left(\frac{2\rho}{\lambda_{1i}^2-\rho^2} + \frac{\lambda_{1i}-\lambda_{2i}}{(\rho-\lambda_{2i})(\rho-\lambda_{1i})}\right).\end{aligned}$$

In the same way for $s < 0$, we have

$$|F_i(\phi)(s) - F_i(\psi)(s)|e^{-\rho|s|} < \gamma_i\varepsilon\left(\frac{2\rho}{\lambda_{1i}^2-\rho^2} + \frac{\lambda_{1i}-\lambda_{2i}}{(\rho+\lambda_{2i})(\rho+\lambda_{1i})}\right).$$

Then, $F_i : B_\rho(\mathbb{R},\mathbb{R}^4) \to B_\rho(\mathbb{R},\mathbb{R}^4)$ is continuous with respect to the norm $\|\cdot\|_\rho$ in $B_\rho(\mathbb{R},\mathbb{R}^4)$. Thus, F is continuous. □

Lemma A5. *We consider the closed and convex set $\Gamma = \{\phi \in W(\mathbb{R},\mathbb{R}^4) : \underline{\phi} \leq \phi \leq \overline{\phi}\}$, where $\overline{\phi}$ and $\underline{\phi}$ are the upper and lower coupled solutions of (A2) such that $\overline{\phi}_i'$, $\overline{\phi}_i''$, $\underline{\phi}_i'$, and $\underline{\phi}_i''$ for $i = 1,2,3,4$, are essentially bounded. Then, $F(\Gamma) \subseteq \Gamma$.*

Proof. Let $\phi \in \Gamma$ be such that $\phi = (\phi_1,\phi_2,\phi_3,\phi_4)$. From Lemma (A3), for $i = 1,4$ and $j = 2,3$, one obtains

$$F_i(\underline{\phi}) \leq F_i(\phi) \leq F_i(\overline{\phi}) \qquad F_j(\overline{\phi}^{2,3}) \leq F_j(\phi) \leq F_j(\underline{\phi}^{2,3}). \tag{A8}$$

We need to check that

$$\underline{\phi}_i \leq F_i(\underline{\phi}) \qquad\qquad F_i(\overline{\phi}) \leq \overline{\phi}_i,$$
$$\underline{\phi}_j \leq F_j(\overline{\phi}^{2,3}) \qquad\qquad F_j(\underline{\phi}^{2,3}) \leq \overline{\phi}_j.$$

Indeed, from Definition (A5), we have the following equations in terms of H a.e. in $\mathbb{R}$ for $i = 1, 4$ and $j = 2, 3$.

$$\begin{aligned}
&\frac{d_i}{c^2}\overline{\phi}_i'' - \overline{\phi}_i' - \beta_i\overline{\phi}_i + H_i(\overline{\phi})(s) \leq 0, \qquad \frac{d_i}{c^2}\underline{\phi}_i'' - \underline{\phi}_i' - \beta_i\underline{\phi}_i + H_i(\underline{\phi})(s) \geq 0,\\
&\frac{d_j}{c^2}\overline{\phi}_j'' - \overline{\phi}_j' - \beta_i\overline{\phi}_j + H_j(\overline{\phi}^{2,3})(s) \leq 0, \qquad \frac{d_j}{c^2}\underline{\phi}_j'' - \underline{\phi}_j' - \beta_j\underline{\phi}_j + H_j(\underline{\phi}^{j})(s) \geq 0.
\end{aligned} \tag{A9}$$

Without loss of generality, we assume that $\overline{\phi}$ and $\underline{\phi}$ are $C^2(\mathbb{R} - \{T_1, T_2, \ldots, T_N\})$, with $-\infty < T_1 < T_2 < \ldots < T_N < \infty$. We denote $T_0 = -\infty$ and $T_{N+1} = \infty$. Then, from the definition of F and (A9) for all $s \in (T_n, T_{n+1})$, $n = 0, 1, 2, \ldots, N$, and considering γ_i as before, for $i = 1, 4$ we have

$$\begin{aligned}
F_i(\overline{\phi})(s) &= \gamma_i\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)} H_i(\overline{\phi})(p)dp + \int_s^{\infty} e^{\lambda_{2i}(s-p)} H_i(\overline{\phi})(p)dp\right]\\
&\leq \gamma_i\left[\int_{-\infty}^{s} e^{\lambda_{1i}(s-p)}\left(-\frac{d_i}{c^2}\overline{\phi}_i''(p) + \overline{\phi}_i'(p) + \beta_i\overline{\phi}_i(p)\right)dp\right.\\
&\quad\left. + \int_s^{\infty} e^{\lambda_{2i}(s-p)}\left(-\frac{d_i}{c^2}\overline{\phi}_i''(p) + \overline{\phi}_i'(p) + \beta_i\overline{\phi}_i(p)\right)dp\right].
\end{aligned}$$

Using integration by parts for $k = 1, 2$, it follows that

$$\begin{aligned}
-\frac{d_i}{c^2}\int e^{\lambda_{ki}(s-p)}\overline{\phi}_i''(p)dp &= -\frac{d_i}{c^2}e^{\lambda_{ki}(s-p)}\overline{\phi}_i'(p) - \frac{d_i\lambda_{ki}}{c^2}e^{\lambda_{ki}(s-p)}\overline{\phi}_i(p) - \frac{d_i\lambda_{ki}^2}{c^2}\int e^{\lambda_{ki}(s-p)}\overline{\phi}_i(p)dp\\
\int e^{\lambda_{ki}(s-p)}\overline{\phi}_i'(p)dp &= e^{\lambda_{ki}(s-p)}\overline{\phi}_i(p) + \lambda_{ki}\int e^{\lambda_{ki}(s-p)}\overline{\phi}_i(p)dp.
\end{aligned}$$

If

$$Y_{ki}(p) = e^{\lambda_{ki}(s-p)}\left(-\frac{d_i}{c^2}\overline{\phi}_i''(p) + \overline{\phi}_i'(p) + \beta_i\overline{\phi}_i(p)\right),$$

and since $-\frac{d_i}{c^2}\lambda_{ki}^2 + \lambda_{ki} + \beta_i = 0$, then

$$\int Y_{ki}(p)dp = e^{\lambda_{ki}(s-p)}\left(-\frac{d_i}{c^2}\overline{\phi}_i'(p) - \frac{\lambda_{ki}d_i}{c^2}\overline{\phi}_i(p) + \overline{\phi}_i(p)\right).$$

Now, $s \in (T_n, T_{n+1})$ and $\overline{\phi}_i, \overline{\phi}_i', \overline{\phi}_i''$ are essentially bounded, and it follows that

$$\begin{aligned}
\int_{-\infty}^{s} Y_{1i}(p)dp + \int_s^{\infty} Y_{2i}(p)dp &= \int_{-\infty}^{T_1} Y_{1i}(p)dp + \int_{T_1}^{T_2} Y_{1i}(p)dp + \ldots + \int_{T_n}^{s} Y_{1i}(p)dp\\
&\quad + \int_s^{T_{n+1}} Y_{2i}(p)dp + \ldots + \int_{T_N}^{\infty} Y_{2i}(p)dp,
\end{aligned}$$

where

$$\int_{-\infty}^{T_1} Y_{1i}(p)dp = -\frac{d_i}{c^2}e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i'(T_1-) - \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i(T_1) + e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i(T_1),$$

$$\int_{T_1}^{T_2} Y_{1i}(p)dp = -\frac{d_i}{c^2}e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i'(T_2-) - \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i(T_2) + e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i(T_2)$$
$$+ \frac{d_i}{c^2}e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i'(T_1+) + \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i(T_1) - e^{\lambda_{1i}(s-T_1)}\overline{\phi}_i(T_1),$$

$$\int_{T_2}^{T_3} Y_{1i}(p)dp = -\frac{d_i}{c^2}e^{\lambda_{1i}(s-T_3)}\overline{\phi}_i'(T_3-) - \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_3)}\overline{\phi}_i(T_3) + e^{\lambda_{1i}(s-T_3)}\overline{\phi}_i(T_3)$$
$$+ \frac{d_i}{c^2}e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i'(T_2+) + \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i(T_2) - e^{\lambda_{1i}(s-T_2)}\overline{\phi}_i(T_2),$$

$$\vdots$$

$$\int_{T_n}^{s} Y_{1i}(p)dp = -\frac{d_i}{c^2}\overline{\phi}_i'(s) - \frac{\lambda_{1i}d_i}{c^2}\overline{\phi}_i(s) + \overline{\phi}_i(s) + \frac{d_i}{c^2}e^{\lambda_{1i}(s-T_n)}\overline{\phi}_i'(T_n+)$$
$$+ \frac{\lambda_{1i}d_i}{c^2}e^{\lambda_{1i}(s-T_n)}\overline{\phi}_i(T_n) - e^{\lambda_{1i}(s-T_n)}\overline{\phi}_i(T_n),$$

$$\int_{s}^{T_{n+1}} Y_{2i}(p)dp = -\frac{d_i}{c^2}e^{\lambda_{2i}(s-T_{n+1})}\overline{\phi}_i'(T_{n+1}-) - \frac{\lambda_{2i}d_i}{c^2}e^{\lambda_{2i}(s-T_{n+1})}\overline{\phi}_i(T_{n+1})$$
$$+ e^{\lambda_{2i}(s-T_{n+1})}\overline{\phi}_i(T_{n+1}) + \frac{d_i}{c^2}\overline{\phi}_i'(s) + \frac{\lambda_{2i}d_i}{c^2}\overline{\phi}_i(s) - \overline{\phi}_i(s),$$

$$\vdots$$

$$\int_{T_N}^{\infty} Y_{2i}(p)dp = \frac{d_i}{c^2}e^{\lambda_{2i}(s-T_N)}\overline{\phi}_i'(T_N+) + \frac{\lambda_{2i}d_i}{c^2}e^{\lambda_{2i}(s-T_N)}\overline{\phi}_i(T_N) - e^{\lambda_{1i}(s-T_N)}\overline{\phi}_i(T_N).$$

From the above, it follows that

$$\int_{-\infty}^{s} Y_{1i}(p)dp + \int_{s}^{\infty} Y_{2i}(p)dp = \frac{d_i}{c^2}(\lambda_{2i}-\lambda_{1i})\overline{\phi}_i(s) + \frac{d_i}{c^2}\sum_{j=1}^{n} e^{\lambda_{1i}(s-Tj)}\left(\overline{\phi}_i'(T_j+) - \overline{\phi}_i'(T_j-)\right)$$
$$+ \frac{d_i}{c^2}\sum_{j=n+1}^{N} e^{\lambda_{2i}(s-Tj)}\left(\overline{\phi}_i'(T_j+) - \overline{\phi}_i'(T_j-)\right).$$

Which implies that

$$F_i(\overline{\phi})(s) \le \gamma_i\left(\int_{-\infty}^{s} Y_{1i}(p)dp + \int_{s}^{\infty} Y_{2i}(p)dp\right)$$
$$F_i(\overline{\phi})(s) \le \overline{\phi}_i(s) + \frac{1}{\lambda_{2i}-\lambda_{1i}}\left[\sum_{j=1}^{n} e^{\lambda_{1i}(s-Tj)}\left(\overline{\phi}_i'(T_j+) - \overline{\phi}_i'(T_j-)\right)\right.$$
$$\left. + \sum_{j=n+1}^{N} e^{\lambda_{2i}(s-Tj)}\left(\overline{\phi}_i'(T_j+) - \overline{\phi}_i'(T_j-)\right)\right].$$

From Property (iii) in (A5), we have $\overline{\phi}_i(t+) \le \overline{\phi}_i(t-)$. Thus, one obtains

$$F_i(\overline{\phi})(s) \le \overline{\phi}_i(s), \quad \forall\, s \in \mathbb{R}.$$

In the same form, $\underline{\phi}_i \le F_i(\underline{\phi})$, $\underline{\phi}_j \le F_j(\overline{\phi}^{2,3})$ and $F_j(\underline{\phi}^{2,3}) \le \overline{\phi}_j$. □

In the next lemma, we show that $F : \Gamma \to \Gamma$ is a compact application.

Lemma A6. *Suppose that* $(A3)$ *and* Γ *are as in Lemma A5. Then, the application* $F : \Gamma \to \Gamma$ *is compact with respect to the norm* $\|\cdot\|_\rho$.

Proof. Suppose that PQM holds. From Lemma A5, the application $F(\Gamma)$ is bounded uniformly. F is equicontinuous on $W(\mathbb{R}, \mathbb{R}^4)$. Indeed, let $\gamma_i = \frac{c^2}{d_i(\lambda_{2i}-\lambda_{1i})}$, and from Proposition A1, for $\phi \in W(\mathbb{R}, \mathbb{R}^4)$ and $i = 1, 2, 3, 4$, it holds that

$$\frac{d}{ds}F_i(\phi)(s) = \gamma_i \left[\lambda_{1i} \int_{-\infty}^{s} e^{\lambda_{1i}(s-p)} H_i(\phi)(p)dp + \lambda_{2i} \int_{s}^{\infty} e^{\lambda_{2i}(s-p)} H_i(\phi)(p)dp \right].$$

Then, by a similar argument as shown in (A1) and using (A7), we have

$$\begin{aligned} \left| \frac{d}{ds}F_i(\phi)(s) \right| &\leq \gamma_i \left[|\lambda_{1i}| \int_{-\infty}^{s} e^{\lambda_{1i}(s-p)} |H_i(\phi)(p)|dp + |\lambda_{2i}| \int_{s}^{\infty} e^{\lambda_{2i}(s-p)} |H_i(\phi)(p)|dp \right] \\ &\leq \gamma_i K_i \left[-\lambda_{1i} \int_{-\infty}^{s} e^{\lambda_{1i}(s-p)} dp + \lambda_{2i} \int_{s}^{\infty} e^{\lambda_{2i}(s-p)} dp \right] \\ &\leq 2\gamma_i K_i. \end{aligned}$$

Thus, for $i = 1, 2, 3, 4$,

$$\left| \frac{d}{ds}F_i(\phi)(s) \right| e^{-\rho|s|} \leq 2\gamma_i K_i e^{-\rho|s|} \leq 2\gamma_i K_i \Rightarrow \left| \frac{d}{ds}F_i(\phi)(s) \right|_{\rho} \leq 2\gamma_i K_i.$$

Therefore, F is equicontinuous on $W(\mathbb{R}, \mathbb{R}^4)$.

$F : \Gamma \to \Gamma$ is compact. We define the sequence operator given by $F^{(n)} : W(\mathbb{R}, \mathbb{R}^4) \to W(\mathbb{R}, \mathbb{R}^4)$ and

$$F^{(n)}(\varphi)(s) = \begin{cases} F(\varphi)(-n) & , s \in (-\infty, -n), \\ F(\varphi)(s) & , s \in [-n, n], \\ F(\varphi)(n) & , s \in (n, \infty). \end{cases}$$

Then, for all $n \geq 1$, $F^{(n)}$ is bounded uniformly and equicontinuous. By using the Arzela–Ascoli theorem, it is deduced that $F^{(n)}$ is compact in the interval $[-n, n]$. On the other hand, since

$$\begin{aligned} |F_i^{(n)} - F_i|_{\rho} &= \sup_{s \in \mathbb{R}} |F_i^{(n)}(\varphi)(s) - F_i(\varphi)(s)| e^{-\rho|s|} \\ &= \sup_{s \in (-\infty, -n) \cup (n, \infty)} |F_i^{(n)}(\varphi)(s) - F_i(\varphi)(s)| e^{-\rho|s|} \\ &\leq 2k_i e^{-\rho n} \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $\varphi \in W(\mathbb{R}, \mathbb{R}^4)$. Hence, the sequence $\{F^{(n)}\}$ converges to F in Γ with respect to the norm $\|\cdot\|_{\rho}$. From Proposition 2.1 in [50], $F : \Gamma \to \Gamma$ is compact. □

Thus, we have the following theorem:

Theorem A1. *We assume that A1, A2, and A3 are true. Suppose that $\overline{\phi}, \underline{\phi} \in W(\mathbb{R}, \mathbb{R}^4)$ are the upper and lower solutions of* (A2) *and*

$$\lim_{s \to -\infty} \overline{\phi}(s) = \mathbf{0}, \quad \lim_{s \to \infty} \underline{\phi} = \mathbf{k},$$

Then, (A2) *and* (A6) *have a solution. Thus,* (A1) *has a solution.*

Proof. Consider the set $\Gamma = \{\varphi \in W(\mathbb{R}, \mathbb{R}^4) : \underline{\phi} \leq \varphi \leq \overline{\phi}\}$ and the operator $F : \Gamma \to \Gamma$ as in Lemma A5. Next, from Lemma A4, Lemma A5, and Lemma A6, F is continuous and compact with respect to the norm $\|\cdot\|_{\rho}$. From the fixed-point theorem, there is $\varphi^* \in \Gamma$ such that $F(\varphi^*) = \varphi^*$. Then, φ^* is a solution of (A6) and, therefore, a solution of (A2). Now, from Lemma A5, $\underline{\phi} \leq \varphi^* \leq \overline{\phi}$. And thus,

$$\mathbf{0} \leq \lim_{s \to -\infty} \varphi^*(s) \leq \lim_{s \to -\infty} \overline{\phi}(s) = \mathbf{0},$$
$$\mathbf{k} \leq \lim_{s \to \infty} \varphi^*(s) \leq \lim_{s \to \infty} \overline{\phi}(s) = \mathbf{k},$$

i.e.,

$$\lim_{s \to -\infty} \varphi^*(s) = \mathbf{0}, \quad \lim_{s \to \infty} \varphi^*(s) = \mathbf{k}.$$

Hence, φ^* satisfies the asymptotic boundary conditions. Thus, φ^* is a solution of (A1). □

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