Int. J. Fuzzy Syst. (2024) 26(5):1417–1427
https://doi.org/10.1007/s40815-023-01676-x

On the Conditions for Total Orderings in Lexicographic Methods to Rank Fuzzy Numbers

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Received: 21 May 2023 / Revised: 14 November 2023 / Accepted: 25 December 2023 / Published online: 26 February 2024

Abstract Lexicographic methods to rank fuzzy numbers present the advantages of simplicity, consistency with human intuition, and power of discrimination. In this paper, we tackle the problem of finding the conditions for these methods to produce a rank in specific steps. Our main results are twofold. First, we prove that a necessary and sufficient condition for a ranking function to be a total order is that this function is either injective, surjective, or bijective. Second, we provide further insight into the required steps for a lexicographic order to rank same-type and different-type fuzzy numbers. A counterexample refutes a conjecture in the literature about the maximum number of steps needed to rank different-type fuzzy numbers.

Keywords Fuzzy numbers · Ranking · Lexicographic order · Total order

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1 Introduction

In many real-world situations, decision-makers face problems that are difficult to address because the uncertainties associated with the problem are vague or non-probabilistic. To deal with vagueness and imprecision in such situations, Zadeh [31] proposed the theory of fuzzy sets, which has been extensively used to address a wide range of decision-making situations [3, 6, 18]. The ranking of fuzzy quantities is a challenging problem that has attracted the interest of many researchers. The different approaches to ranking fuzzy numbers can be considered to fall into three main categories [28, 30]:

1. Ranking methods based on the distance to a reference set. These methods evaluate each fuzzy number by computing its distance to some reference set. Among this class are included the approaches focusing on maximizing sets [9, 17] and Fuzzy-TOPSIS methods [24–26].
2. Ranking methods based on pairwise comparisons. These ranking methods construct a fuzzy preference relation for ranking or choosing from a group of alternatives by comparing them against each other in pairs as in [2, 21, 30].
3. Ranking methods based on defuzzification. In this category, we find procedures relying on fuzzy numbers mapped to real numbers to derive a ranking as in [14, 29]. Also, in this group are included the ranking methods based on lexicographic orders [12, 27], which are particularly interesting because the final ranking is not based on a single defuzzified value but in a lexicographic order of different values.

While no method can be considered superior to any other, some scholars consider that ranking methods relying on lexicographic orders present the advantages of simplicity, consistency with human intuition, and power of discrimination [12]. They have been widely used in multiobjective optimization since their first introduction by [8] and represent valuable tools for ranking fuzzy numbers based on a strict structure of preferences. Several works [12, 23, 27] claim the ability of lexicographic methods to solve the shortcomings of alternative techniques. For instance, ranking functions that map fuzzy numbers to the real line, such as the one described in [14], present the following limitation: if two fuzzy numbers are mapped into the same real number, then these two fuzzy numbers are indistinguishable by the ranking function even though they may look different to a decision-maker. On the application side, lexicographic orders have been employed to solve fuzzy linear programming problems [11, 23], to evaluate the relative efficiency of a set of decision-making units [16], to solve matrix games [22], and in multi-attribute decision-making [13].

Focusing on specific lexicographic methods, [12] proposed to rank trapezoidal and triangular fuzzy numbers in four steps. Other authors proposed a lexicographic screening procedure to rank fuzzy numbers, claiming efficient computation and ease of understanding [27]. The authors proved that the described method could generate an ordered set of fuzzy numbers of the same type with membership functions defined by at most eight parameters. In addition, they suggested but did not prove that the maximum number of steps in a lexicographic method to rank two fuzzy numbers of a different type is given by the smallest number of defining parameters plus one (see Conjecture 1). However, there is a need to consider the underlying motives why lexicographic orders can solve a ranking problem in a limited number of steps.

Along the lines of previous research [12, 23, 27], our main research assumption is that lexicographic methods provide an efficient and robust way to rank fuzzy numbers. Instead of considering the advantages that some lexicographic methods may present compared to others, we focus on the conditions to produce a ranking and the required number of steps. As a result, our work gives two fundamental insights. Firstly, we provide novel theoretical results on the conditions to obtain a total ordering from an arbitrary set of fuzzy numbers based on lexicographic methods. For this purpose, we start from lexicographic methods of the same type as described in [12, 27]. The authors do not provide the conditions a lexicographic order must meet to produce a total order. We here show that a necessary and sufficient condition for a ranking function to be a total order on a set of fuzzy numbers is that this function is injective, surjective, or bijective. Secondly, we provide insight into the required steps for a lexicographic order to rank same-type and different-type fuzzy numbers. One of these results refutes Conjecture 1 by [27] about the maximum number of steps required to rank different-type fuzzy numbers.

In our research, we focus on the exploration of the conditions to establish an ordering system through the utilization of various lexicographic methods. Rather than advocating the superiority of one method over another, we offer a versatile approach applicable to a wide array of fuzzy numbers. In direct contrast to the works by [12, 27], our approach is not limited by the type or number of defining parameters, extending its utility to same-type and different-type fuzzy numbers. Our results not only enrich the existing landscape of lexicographic methods described but also provide a valuable framework for designing novel methods to rank fuzzy numbers.

In addition to this introduction, this paper includes Sect. 2, in which we provide useful background on fuzzy numbers and lexicographic orders. Section 3 is the central part of this paper describing novel theoretical results. In Sect. 4, we illustrate the theoretical results with numerical examples, and Sect. 5 concludes this paper by highlighting natural research extensions of this work.

2 Useful Background

In this section, we provide basic concepts on fuzzy numbers and lexicographic orders that will be useful to follow the main results of this paper.

2.1 Basic Definitions

Definition 1 Fuzzy set [31]. Let X be a non-empty set, the fuzzy set A is expressed as

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\} \tag{1}$$

where $\mu_A(x)$ is the degree of membership of element x to A, defined as a function $\mu_A : X \to [0, 1]$, with 0 and 1 representing the lowest and highest degrees of membership.

Definition 2 Fuzzy number [6]. A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex, and continuous membership function of bounded support.

Definition 3 LR fuzzy number [7, 23]. A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be an *LR* fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{a_2 - x}{a_2 - a_1}\right), & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ R\left(\dfrac{x - a_3}{a_4 - a_3}\right), & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where L and R are shape functions $[0,1] \to [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, which are non-increasing, continuous mappings.

If $L(x) = R(x) = 1 - x$, then $\tilde{A}$ is called a trapezoidal fuzzy number. If $a_2 = a_3$, trapezoidal fuzzy number $\tilde{A}$ reduces to a triangular fuzzy number.

Definition 4 Support [12]. The support $\mathrm{Supp}(\tilde{A})$ of fuzzy number $\tilde{A}$ is defined as the crisp set:

$$\mathrm{Supp}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}. \tag{3}$$

Definition 5 Core [10]. The core $\mathrm{Core}(\tilde{A})$ of fuzzy number $\tilde{A}$ is defined as the crisp set:

$$\mathrm{Core}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) = 1\}. \tag{4}$$

2.2 Lexicographic Orders

Before defining lexicographic orders, let us consider the following preliminary definitions.

Definition 6 Type of fuzzy numbers. A type (or class) of fuzzy numbers is a set of all fuzzy numbers whose membership functions share a group of features that can be summarized in a vector of parameters of the same size.

For instance, triangular fuzzy numbers share standard features usually summarized in a three-dimensional vector, $\tilde{A} = (a_1, a_2, a_3)$, with $a_1 < a_2 < a_3$, where a_1 refers to the left end-point of $\mathrm{Supp}(\tilde{A})$, a_2 is the modal value of $\tilde{A}$ in which $\mu(\tilde{A}) = 1$, and a_3 refers to the right end-point of $\mathrm{Supp}(\tilde{A})$. Furthermore, symmetric triangular fuzzy numbers are a subtype of triangular fuzzy numbers whose membership function can be summarized using a two-dimensional vector $\tilde{A} = (a_1, a_2)$, where a_1 is the modal value and a_2 is the left and right spread from a_1, equivalent to $\tilde{A} = (a_1 - a_2, a_1, a_1 + a_2)$. However, symmetric triangular fuzzy numbers can also be expressed using a three-dimensional vector to be considered general triangular fuzzy numbers for ranking purposes.

Definition 7 Same-type fuzzy numbers. Two fuzzy numbers $\tilde{M}_1$ and $\tilde{M}_2$, described by vectors of parameters $\tilde{M}_1 = (\sigma_i, i = 1, \ldots, m_1)$ and $\tilde{M}_2 = (\tau_i, i = 1, \ldots, m_2)$, where m_1 and m_2 are the total number of defining parameters, are of the same type if and only if $m_1 = m_2$, and $\mu(\tilde{M}_1) = \mu(\tilde{M}_2)$, when $\sigma_i = \tau_i$, $\forall i$.

Definition 8 Different-type fuzzy numbers. Two fuzzy numbers are said to be different-type when they are not same-type fuzzy numbers.

Then, two triangular fuzzy numbers $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are of the same-type because when $a_i = b_i, \forall i \in \{1, 2, 3\}$, it necessary implies that $\mu(\tilde{A}) = \mu(\tilde{B})$. Similarly, triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ and trapezoidal $\tilde{B} = (b_1, b_2, b_3, b_4)$ are not same-type fuzzy numbers because $m_1 = 3$ is not equal to $m_2 = 4$. Furthermore, triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ and trapezoidal fuzzy number $\tilde{B} = (b_1, b_2, b_3, b_4)$, with $b_2 = b_3$, are of the same type because $\tilde{B}$ is indeed a triangular fuzzy number that can be rewritten as $\tilde{B} = (b_1, b_2, b_4)$. Finally, symmetric triangular fuzzy number $\tilde{A} = (a_1, a_2)$ and Gaussian fuzzy number $\tilde{B} = (b_1, b_2)$ are not of the same type because $a_i = b_i, \forall i \in \{1, 2\}$ does not imply that $\mu(\tilde{A}) = \mu(\tilde{B})$, according to the definition of a Gaussian fuzzy number with membership function $\mu_{\tilde{B}}(x) = e^{-(\frac{x - b_1}{b_2})^2}$.

Definition 9 Strict lexicographic order [12]. For $x, y \in \mathbb{R}^n$, the strict lexicographic order $x \succ_{lex} y$ holds, if and only if there is $1 \leq i \leq n$ so that $x_j = y_j$, for $j < i$, and $x_i > y_i$.

Definition 10 Weak lexicographic order [12]. For $x, y \in \mathbb{R}^n$, the weak lexicographic order $x \succeq_{lex} y$ holds, if and only if, $x \succ_{lex} y$, or $x = y$.

At this point, it is essential to distinguish between a vector of defining parameters and a function defined on a vector of parameters. For instance, a trapezoidal fuzzy number is characterized by a vector of parameters denoted by $\tilde{A} = (a_1, a_2, a_3, a_4)$. On the other hand, function $g : \tilde{A} \to \mathbb{R}$ is a function mapping the vector of parameters to the real line that can be used for comparison purposes.

The lexicographic procedure proposed in [27] is based on eight functions for sigmoid fuzzy numbers with eight parameters $(c_l, c_r, b_1, b_2, \delta_1, \delta_2, \beta_1, \beta_2)$, where c_l and c_r are the left and right modal values, b_1 and b_2 are the membership function values of the left and right inflection points, δ_1 and δ_2 are the first and second left spreads, and β_1 and β_2 are the first and second right spreads. In this case, the mode is $(c_l + c_r)/2$ and the membership function is given by

$$\mu(x) = \begin{cases} b_1 - \dfrac{b_1}{\delta_1}\sqrt{\delta_1^2 - (x - (c_l - \delta_1 - \delta_2))^2}, & c_l - \delta_1 - \delta_2 \leq x \leq c_l - \delta_2 \\ b_1 + \dfrac{1 - b_1}{\delta_2}\sqrt{\delta_2^2 - (c_l - x)^2}, & c_l - \delta_2 \leq x \leq c_l \\ 1, & c_l \leq x \leq c_r \\ b_2 + \dfrac{1 - b_2}{\beta_1}\sqrt{\beta_1^2 - (x - c_r)^2}, & c_r \leq x \leq c_r + \beta_1 \\ b_2 - \dfrac{b_2}{\beta_2}\sqrt{\beta_2^2 - (c_r + \beta_1 + \beta_2 - x)^2}, & c_r + \beta_1 \leq x \leq c_r + \beta_1 + \beta_2 \end{cases} \tag{5}$$

Then, the authors proposed the following lexicographic procedure:

1. The larger the mode $d = (c_l + c_r)/2$ is, the larger is the fuzzy number.
2. The larger the right spread s_r with respect to the mode, the larger the fuzzy number.
3. The larger the total area M below the membership, the larger the fuzzy number.
4. The larger the upper modal value $c_r = \sup\{x \in \mathrm{Supp}(\tilde{A}); \mu_{\tilde{A}}(x) = 1\}$ is, the larger is the fuzzy number.
5. The larger the first right spread β_1 is, the larger the fuzzy number.
6. The larger the membership value of the right inflection point b_2 is, the larger the fuzzy number.
7. The larger the second left spread δ_2 is, the larger the fuzzy number.
8. The larger the membership value of the left inflection point b_1 is, the larger the fuzzy number.

As a result, the strict lexicographic order $\tilde{A} \succ_{lex} \tilde{B}$ holds if and only if $Z(\tilde{A}) \succ_{lex} Z(\tilde{B})$, where vector Z is constructed by

$$Z(\tilde{A}) = (d(\tilde{A}), s_r(\tilde{A}), M(\tilde{A}), c_r(\tilde{A}), \beta_1(\tilde{A}), b_2(\tilde{A}), \delta_2(\tilde{A}), b_1(\tilde{A})) \tag{6}$$

$$Z(\tilde{B}) = (d(\tilde{B}), s_r(\tilde{B}), M(\tilde{B}), c_r(\tilde{B}), \beta_1(\tilde{B}), b_2(\tilde{B}), \delta_2(\tilde{B}), b_1(\tilde{B})). \tag{7}$$

The authors also apply this lexicographic procedure to fuzzy numbers with fewer parameters by focusing on the first three steps for triangular fuzzy numbers and the first four for trapezoidal fuzzy numbers.

Similarly, [12] established a lexicographic rule to rank fuzzy numbers based on the following functions defined on four-dimensional vectors of parameters for *LR* fuzzy numbers:

Definition 11 Let $\tilde{A}$ be a fuzzy number. Define:

1. Lower modal value: $C(\tilde{A}) = \inf\{x \in \mathrm{Supp}\ (\tilde{A}); \mu_{\tilde{A}}(x) = 1\}$
2. Left end-point of support: $L(\tilde{A}) = \inf\{\mathrm{Supp}(\tilde{A})\}$.
3. Length of support $W(\tilde{A}) = |\mathrm{Supp}(\tilde{A})|$.
4. Area under the membership function $M(\tilde{A}) = \int \mu_{\tilde{A}} dx$.
5. Ordered set: $V(\tilde{A}) = (C(\tilde{A}), L(\tilde{A}), W(\tilde{A}), M(\tilde{A}))$.

Then, the author proposed the following ranking function for two *LR* fuzzy numbers $\tilde{A}, \tilde{B}$:

- Step 1. Compare $C(\tilde{A})$ and $C(\tilde{B})$. If $C(\tilde{A}) = C(\tilde{B})$, then go to Step 2. Otherwise, stop. The larger $C(*)$ is, the larger the corresponding fuzzy number $*$ is.
- Step 2. Compare $L(\tilde{A})$ and $L(\tilde{B})$. If $L(\tilde{A}) = L(\tilde{B})$, then go to Step 3. Otherwise, stop. The larger $L(*)$ is, the larger the corresponding fuzzy number $*$ is.
- Step 3. Compare $W(\tilde{A})$ and $W(\tilde{B})$. If $W(\tilde{A}) = W(\tilde{B})$, then go to Step 4. Otherwise, stop. The larger $W(*)$ is, the larger the corresponding fuzzy number $*$ is.
- Step 4. Compare $M(\tilde{A})$ and $M(\tilde{B})$. If $M(\tilde{A}) = M(\tilde{B})$, then $\tilde{A} \sim \tilde{B}$. Otherwise, stop. The larger $M(*)$ is, the larger the corresponding fuzzy number $*$ is.

As a result, the strict lexicographic order $\tilde{A} \succ_{lex} \tilde{B}$ holds if and only if $V(\tilde{A}) \succ_{lex} V(\tilde{B})$. Similarly, the weak lexicographic order $\tilde{A} \succeq_{lex} \tilde{B}$ holds if and only if $V(\tilde{A}) \succ_{lex} V(\tilde{B})$, or $V(\tilde{A}) = V(\tilde{B})$. Note that the lexicographic order $\succeq_{lex}$ derived from Definition 11 is a total order for same-type *LR* fuzzy numbers because it presents the properties for a total order [23]:

Definition 12 **Total order** [4]. Given set S, a total order is a binary relation $\succeq$ on S with the following properties:

1. Reflexivity: $\tilde{A}_i \succeq \tilde{A}_i, \forall \tilde{A}_i \in S$.
2. Anti-symmetry: $\tilde{A}_i \succeq \tilde{A}_j$ and $\tilde{A}_j \succeq \tilde{A}_i$, implies $\tilde{A}_i = \tilde{A}_j$, $\forall \tilde{A}_i, \tilde{A}_j \in S$.
3. Transitivity: $\tilde{A}_i \succeq \tilde{A}_j$ and $\tilde{A}_j \succeq \tilde{A}_k$, implies $\tilde{A}_i \succeq \tilde{A}_k$, $\forall \tilde{A}_i, \tilde{A}_j, \tilde{A}_k \in S$.
4. Comparability: $\tilde{A}_i \succeq \tilde{A}_j$ or $\tilde{A}_j \succeq \tilde{A}_i, \forall \tilde{A}_i, \tilde{A}_j \in S$.

Furthermore, the specific order $V(\tilde{A}) = (C(\tilde{A}), L(\tilde{A}), W(\tilde{A}), M(\tilde{A}))$, in which functions in $V(\tilde{A})$ are compared to rank *LR* fuzzy numbers, reflects the importance of the information provided by each of the functions in deciding the ranking [23].

On the other hand, [27] provided the following result derived from their lexicographic ranking method:

Theorem 1 *The lexicographic screening procedure* [27] *generates a totally ordered set of fuzzy numbers with* ***same type*** *of membership functions defined by at most eight parameters.*

In addition, [27] suggested, but did not prove, that the maximum number of steps in a lexicographic order to rank different-type fuzzy numbers is given by the following conjecture:

Conjecture 1 [27] *Given two fuzzy numbers of different types denoted by* $\tilde{M}_1 = (\sigma_i, i = 1, \ldots, m_1)$ *and* $\tilde{M}_2 = (\tau_i, i = 1, \ldots, m_2)$ *where* m_1, m_2 *are numbers of defining parameters, respectively, the maximal number of ranking steps is* $\min(m_1, m_2) + 1$.

From the previous background on lexicographic orders, we consider the following research questions: 1) what are the conditions to obtain a total order on a set of fuzzy numbers? and 2) how many steps are necessary to produce a total order by following a lexicographic algorithm? Sect. 3 addresses these questions.

3 Conditions for Total Orderings Based on Lexicographic Approaches

In this section, we provide the conditions to produce a total order over an arbitrary set of different fuzzy numbers defined by m parameters through decision rules or classification functions such as lexicographic methods. We consider the case of sets with the same type of fuzzy numbers and those with different types of fuzzy numbers.

Let us consider an arbitrary set $S = \{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\}$ populated with different fuzzy numbers such that $\tilde{A}_i \neq \tilde{A}_j$, for all $i \neq j$ in range $[1, n]$. Consider ordered set $Y = \{1, 2, \ldots, n\}$ with ordinal numbers representing a ranking in decreasing order of preference, and ranking function $f : S \rightarrow Y$ mapping elements of S to ranking Y.

Definition 13 **Injective function** [20]. A function $f : S \rightarrow Y$ is injective if and only if for all $\tilde{A}_i, \tilde{A}_j \in S$, $\tilde{A}_i \neq \tilde{A}_j$ implies that $f(\tilde{A}_i) \neq f(\tilde{A}_j)$.

Definition 14 **Surjective function** [20]. A function $f : S \rightarrow Y$ is surjective if and only if for every $y \in Y$, there exists $\tilde{A}_i \in S$ with $y = f(\tilde{A}_i)$.

Definition 15 **Bijective function** [20]. A function $f : S \rightarrow Y$ is bijective if and only if f is both injective and surjective, or equivalently, if and only if for every $y \in Y$, there exists a unique $\tilde{A}_i \in S$ with $y = f(\tilde{A}_i)$.

Theorem 2 ***Injectivity and surjectivity*** [20]. *If S and Y are finite sets with the same number of elements, then function $f : S \rightarrow Y$ is injective if and only if f is surjective.*

Proof Suppose S and Y both have n elements, and write $S = \{\tilde{A}_i, \ldots, \tilde{A}_n\}$. Assume that function $f : S \rightarrow Y$ is injective. Then, set $T = \{f(\tilde{A}_i), \ldots, f(\tilde{A}_n)\}$ is a subset of Y consisting of n distinct elements. Since Y has n elements, this subset must be all of Y. This means that every $y \in Y$ has the form $f(\tilde{A}_i)$ for some $\tilde{A}_i \in S$, so that f is surjective. Conversely, assume that function $f : S \rightarrow Y$ is not injective. Then there exists $i \neq j$ with $f(\tilde{A}_i) = f(\tilde{A}_j)$. It follows that the set T contains fewer than n elements because it contains at least one duplicate. Thus, T is a proper subset of Y. Letting y be any element of a set difference $Y \backslash T$ ($y \in Y \backslash T$ if and only if $y \in Y$ and $y \notin T$), we see that y does not have the form $f(\tilde{A}_i)$ for any $\tilde{A}_i \in S$. Therefore f is not surjective. □

Based on the properties of injective, surjective, and bijective functions, we derive the following results:

Lemma 1 *Given set S of different fuzzy numbers, totally ordered set Y of the same size of S, and ranking function $f : S \rightarrow Y$, a necessary and sufficient condition to rank every fuzzy number in S is that function f is injective.*

Proof To prove that injection is a necessary condition, it is sufficient to prove that if f is not injective, then f does not produce a total order. If f is not injective, then there are at least two elements $\tilde{A}_i, \tilde{A}_j$ in S with the same ranking $f(\tilde{A}_i) = f(\tilde{A}_j)$, and the anti-symmetry property is not guaranteed because $\tilde{A}_i \neq \tilde{A}_j$. As a result, if f is not injective, then f does not produce the total order on S. To prove that injection is a sufficient condition, we need to prove that the reflexivity, anti-symmetry, transitivity, and comparability properties are guaranteed by an injective function f. Reflexivity $\tilde{A}_i \succeq \tilde{A}_i$ is guaranteed because there is only one ranking $f(\tilde{A}_i)$, for every $\tilde{A}_i \in S$. Anti-symmetry holds because if $f(\tilde{A}_i) = f(\tilde{A}_j)$, then $\tilde{A}_i \succeq \tilde{A}_j$ and $\tilde{A}_j \succeq \tilde{A}_i$, and it implies that $\tilde{A}_i = \tilde{A}_j$ because for every ranking in Y, there is a unique element $\tilde{A}_i \in S$. Transitivity is ensured because an injective function for three arbitrary elements $\tilde{A}_i, \tilde{A}_j, \tilde{A}_k$, such that $f(\tilde{A}_i) < f(\tilde{A}_j)$ and $f(\tilde{A}_j) < f(\tilde{A}_k)$, necessarily implies that $f(\tilde{A}_i) < f(\tilde{A}_k)$ because of the order in Y, meaning that $\tilde{A}_i \succeq \tilde{A}_k$. Finally, comparability is certified because there exist $f(\tilde{A}_i)$ and $f(\tilde{A}_j)$ for all $\tilde{A}_i, \tilde{A}_j \in S$, and either $f(\tilde{A}_i) < f(\tilde{A}_j)$ or $f(\tilde{A}_i) > f(\tilde{A}_j)$ hold. □

Remark 1 Sets S and Y are finite sets with the same number of elements. Then, function $f : S \rightarrow Y$ is injective if and only if f is surjective because of Theorem 2. As a result, a necessary and sufficient condition to rank every fuzzy number in S is that function f is either injective, surjective, or bijective.

3.1 Same-Type Fuzzy Numbers

The following results show that both the minimum and the maximum number of steps to produce a total order on S with different fuzzy numbers of the same type are determined by the condition that function f is injective:

Lemma 2 *The number of steps for a lexicographic order to rank same-type fuzzy numbers is equal to the level of the lexicographic order in which an injective function is found.*

Proof Assume that function $f_1 : S \rightarrow Y$ exists and is used to rank fuzzy numbers on the first level of a lexicographic order. Suppose that $f_1(\tilde{A}_i) \neq f_1(\tilde{A}_j)$ for all $\tilde{A}_i$ and $\tilde{A}_j$ in S. Then, f_1 is injective, and it can derive a total order, and the number of steps required to rank elements in S is one. Otherwise, if there is at least a pair $\tilde{A}_i, \tilde{A}_j$ with $f_1(\tilde{A}_i) = f_1(\tilde{A}_j)$, then f_1 is not injective, and it is necessary to go the following level of the lexicographic order. Assume that $f_2 : S \rightarrow Y$ exists and is used to rank fuzzy numbers on the second level of a lexicographic order. Suppose that $f_2(\tilde{A_i}) \neq f_2(\tilde{A_j})$ for all $\tilde{A_i}$ and $\tilde{A_j}$ in S. Then, f_2 is injective, and it can derive a total order, and the number of steps required to rank elements in S is two. Otherwise, if there is at least a pair $\tilde{A_i}, \tilde{A_j}$ with $f_2(\tilde{A_i}) = f_2(\tilde{A_j})$, then f_2 is not injective, and it is necessary to go the following level of the lexicographic order. Assume that $f_n : S \rightarrow Y$ exists and is used to rank fuzzy numbers on the n-th level of a lexicographic order. Suppose that $f_n(\tilde{A_i}) \neq f_n(\tilde{A_j})$ for all $\tilde{A_i}$ and $\tilde{A_j}$ in S. Then, f_n is injective, and it can derive a total order, and the number of steps required to rank elements in S is n. Otherwise, if there is at least a pair $\tilde{A_i}, \tilde{A_j}$ with $f_n(\tilde{A_i}) = f_n(\tilde{A_j})$, then f_n is not injective, and it is necessary to go the following level of the lexicographic order. As a result, the number of steps required for a lexicographic order to rank same-type fuzzy numbers is equal to the level of the lexicographic order in which an injective function is found. $\square$

Now, we focus on the maximum number of steps for a lexicographic order to rank same-type and different-type sets of fuzzy numbers, hence generalizing Theorem 1 in [27]. Assuming that the values of the defining parameters of fuzzy numbers cannot be observed before the ranking procedure is applied:

Theorem 3 *The maximum number of steps for a lexicographic order to rank same-type fuzzy numbers equals the number of defining parameters.*

Proof Assume that m is equal to the number of defining parameters of fuzzy numbers $\tilde{A_i}$ and $\tilde{A_j}$ in S, and use $f_m : S \rightarrow Y$ to rank fuzzy numbers on the m-th level of a lexicographic order. If there exists at least a vector of functions $f = (f_1, f_2, \ldots, f_m)$ that constructs a lexicographic injective ranking function, then the maximum number of steps for a lexicographic order to rank same-type fuzzy numbers is m according to Lemma 2. Consider fuzzy number $\tilde{A_i} = (a_1, a_2, \ldots, a_m)$, and functions $f_1(\tilde{A_i}) = a_1$, $f_2(\tilde{A_i}) = a_2$ and $f_m(\tilde{A_i}) = a_m$ to rank fuzzy numbers in a lexicographic order such that $f_1, f_2, \ldots, f_m$ are, respectively, used to rank fuzzy numbers at levels 1, 2, and m. Then, lexicographic order f is injective at most at the m-th level because S contains only different same-type fuzzy numbers, hence ensuring that at least one defining parameter is different for any pair $\tilde{A_i}$ and $\tilde{A_j}$ in S. $\square$

Remark 2 Suppose we relax the assumption of ignorance of the value of defining parameters. In that case, there is always an injective function to rank two different fuzzy numbers of the same type in one step by comparing the parameter that is not equal.

Next, we prove that the lexicographic order by [12] is an injective function.

Theorem 4 *Given set S populated with different LR fuzzy numbers of the same-type, the lexicographic order $\succ_{lex}$ by [12] based on ordered set $V(\tilde{A}) = (C(\tilde{A}), L(\tilde{A}), W(\tilde{A}), M(\tilde{A}))$, described in Definition 11, is an injective function.*

Proof The ranking function $f : S \rightarrow Y$ derived from the lexicographic order $\succ_{lex}$ by [12] is a total order because the properties of reflexivity, anti-symmetry, transitivity, and comparability are satisfied. By Lemma 1, the ranking function is injective because if it is a total order, it is injective. $\square$

3.2 Different-Type Fuzzy Numbers

In what follows, we focus on the minimum and maximum steps required to rank sets of different-type fuzzy numbers. Let us consider again set S populated with different-type fuzzy numbers according to Definition 7 and denoted by $\tilde{A_i} = (\tau_j, j = 1, 2, \ldots, m_i)$, where m_i is a natural number expressing the size of the set of the defining parameters for each element in S.

On the one hand, we know from Lemma 2 that the number of steps for a lexicographic order to rank same-type fuzzy numbers is equal to the level of the lexicographic order in which an injective function is found. This result is still valid for the subset of fuzzy numbers in S of the same type. Then, we conclude that the minimum number of steps for a lexicographic order to rank different-type fuzzy numbers equals one. On the other hand, to derive the maximum number of steps, we need to focus on the subset with the maximum m_i among the different types of fuzzy numbers in S. As a result, if set S contains more than one same-type fuzzy number, then the maximum number of steps for a lexicographic order to rank different-type fuzzy numbers is equal to $\max(m_i)$, because according to Theorem 3, the maximum number of steps for a lexicographic order to rank same-type fuzzy numbers is equal to the number of defining parameters.

Recall now that Conjecture 1, proposed by [27], states that the maximum number of steps to rank a pair of different-type fuzzy numbers with the number of parameters m_1 and m_2, respectively, is equal to $\min(m_1, m_2) + 1$. The following reasoning shows that this conjecture is false. Assuming that the values of the defining parameters of fuzzy numbers cannot be observed before the ranking procedure is applied, consider a set with a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, and a trapezoidal fuzzy number

$\tilde{B} = (b_1, b_2, b_3, b_4)$. According to Conjecture 1, the maximum number of steps is $\min(3,4) + 1 = 4$. However, the following method ranks these two elements in two steps at most:

1. Define $g_1(\tilde{A}) = \inf\ \{x \in \mathrm{Supp}(\tilde{A}); \mu_{\tilde{A}}(x) = 1\}$ as a comparing function. If $g_1(\tilde{A}) > g_1(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$. If $g_1(\tilde{A}) < g_1(\tilde{B})$, then $\tilde{B} \succ \tilde{A}$. Otherwise, $g_1(\tilde{A}) = g_1(\tilde{B})$ and $\tilde{B} \succeq \tilde{A}$. Go to step 2.
2. Define $g_2(\tilde{A}) = \sup\ \{x \in \mathrm{Supp}(\tilde{A}); \mu_{\tilde{A}}(x) = 1\}$ as a comparing function. If $g_2(\tilde{A}) < g_2(\tilde{B})$, then $\tilde{B} \succ \tilde{A}$. If $g_2(\tilde{A}) = g_2(\tilde{B})$, then $\tilde{A} = \tilde{B}$.

Note that when $\tilde{A}$ is a triangular fuzzy number and $\tilde{B}$ is a trapezoidal fuzzy number, if $g_1(\tilde{A}) = g_1(\tilde{B})$ in step 1, then $a_2 = b_2$ and $a_2 \leq b_3$. Furthermore, if $b_2 < b_3$ for a proper trapezoidal fuzzy number $\tilde{B}$, then it follows that $\tilde{B} \succ \tilde{A}$. Only when $b_2 = b_3$ for a trapezoidal, which is indeed a triangular fuzzy number, can we find the case $\tilde{B} \sim \tilde{A}$ if $a_2 = b_2 = b_3$. We conclude that functions g_1 and g_2 are sufficient to rank triangular and trapezoidal fuzzy numbers in two steps. This counterexample invalidates Conjecture 1.

Remark 3 Suppose we relax the assumption of ignorance of the value of defining parameters. In that case, there is always an injective function to rank two different rank fuzzy numbers of different-type in one step by focusing on the membership function value, which is different for the two fuzzy numbers under study.

The main implications derived from the results described in this section are twofold. From the theoretical standpoint, any claim of an improved lexicographic method can now be contrasted with the theoretical results of the minimum and maximum steps described in this paper. From the practical application perspective, establishing an upper bound on the required steps for lexicographic methods to rank fuzzy can be valuable in various decision-making contexts where ranking involves a mixture of same-type and different-type fuzzy criteria. It can help decision-makers make more informed and balanced choices, considering various factors and uncertainties. For instance, in supply chain management, companies often must compare and rank suppliers based on multiple criteria. Our results can be applied to rank suppliers when the evaluation is made using fuzzy numbers of the same type (e.g., for cost and quality) and fuzzy numbers of different types (e.g., for delivery time and customer service). This insight can aid supplier selection and evaluation, ensuring the most suitable suppliers are chosen. It can potentially improve the accuracy and robustness of decision support systems in several application domains.

4 Numerical Examples

In this section, we describe several numerical examples that illustrate the theoretical results presented in this paper. To illustrate Lemma 1, let us first consider the following set of triangular and trapezoidal fuzzy numbers:

$$S_1 = \{\tilde{A}_1 = (3,4,6,7), \tilde{A}_2 = (2,4.5,5.5,7), \tilde{A}_3 = (3,5,5,7)\}. \tag{8}$$

If we compute the magnitude for the fuzzy numbers in S according to the ranking method proposed by [14], we obtain the results summarized in Table 1. In addition, we compute function $C(\tilde{A}_i)$ to derive a ranking from the lexicographic order proposed by [12]. Even though $\tilde{A}_1$ and $\tilde{A}_3$ are different, the magnitude method fails to be an injective function, preventing us from obtaining a total order. However, the first step of the lexicographic order results in a different value for function $C(\tilde{A}_i)$, hence producing a total order. This result is guaranteed by Theorem 4, which states that the lexicographic order by [12] is an injective function.

The results in Table 1 also show that the required number of steps for the lexicographic order by [12] to rank fuzzy numbers in S_1 is one because function $C(\tilde{A}_i)$ is sufficient to discriminate among the elements in S_1.

In Table 2, we compare the ranking derived from applying different ranking methods to the fuzzy numbers in S_1. Although we pay particular attention to lexicographic methods, we consider the magnitude-based methods described in [1, 14, 15]. In addition, we include the fuzzy TOPSIS method [19] based on the following preference relation:

$$\tilde{A}_i \succeq \tilde{A}_j \iff K_i \geq K_j\,, \tag{9}$$

where K_i is the usual closeness measure defined as

$$K_i = \frac{d_i^-}{d_i^- + d_i^+} \tag{10}$$

where d_i^- is the Euclidean distance of parameters $\tilde{A}_i = (a_{i1}, a_{i4}, a_{i3}, a_{i4})$ to the anti-ideal point (0, 0, 0, 0), and d_i^+ is the Euclidean distance of vector of parameters $\tilde{A}_i = (a_{i1}, a_{i4}, a_{i3}, a_{i4})$ to the ideal point (10, 10, 10, 10). Note

Table 1 An injective and a non-injective ranking method

Ranking function	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	Ranking
$Mag(\tilde{A}_i)$ [14]	5,82	5,85	5,82	$\tilde{A}_2 \succ \tilde{A}_1 \sim \tilde{A}_3$
$C(\tilde{A}_i)$ [12]	4,00	4,50	5,00	$\tilde{A}_3 \succ \tilde{A}_2 \succ \tilde{A}_1$

Table 2 A comparative study of different ranking methods

Ranking method	Ranking
Magnitude [1]	$\tilde{A_3} \sim \tilde{A_1} \succ \tilde{A_2}$
Magnitude [15]	$\tilde{A_3} \succ \tilde{A_2} \succ \tilde{A_1}$
Magnitude [14]	$\tilde{A_2} \succ \tilde{A_1} \sim \tilde{A_3}$
Fuzzy TOPSIS [19]	$\tilde{A_1} \succ \tilde{A_3} \succ \tilde{A_2}$
Lexicographic [12]	$\tilde{A_3} \succ \tilde{A_2} \succ \tilde{A_1}$
Lexicographic [27]	$\tilde{A_1} \succ \tilde{A_2} \succ \tilde{A_3}$
Lexicographic Supp	$\tilde{A_3} \sim \tilde{A_1} \succ \tilde{A_2}$
Lexicographic Core	$\tilde{A_3} \succ \tilde{A_2} \succ \tilde{A_1}$
Lexicographic Param	$\tilde{A_3} \succ \tilde{A_1} \succ \tilde{A_2}$

that similar results can be obtained by selecting other ideal and anti-ideal points.

We consider the lexicographic methods proposed by [12, 27] and propose three additional lexicographic methods for illustrative purposes. First, we consider a lexicographic method based on the interval values of the support of fuzzy numbers relying on the concept of admissible order [5]:

$$\tilde{A_i} \succeq \tilde{A_j} \iff (a_{i1} > a_{j1}) \vee (a_{i1} = a_{j1} \wedge a_{i4} \geq a_{j4}). \tag{11}$$

Similarly, we propose a new lexicographic method based on the interval values of the core of fuzzy numbers as follows:

$$\tilde{A_i} \succeq \tilde{A_j} \iff (a_{i2} > a_{j2}) \vee (a_{i2} = a_{j2} \wedge a_{i3} \geq a_{j3}). \tag{12}$$

Finally, we propose a new lexicographic method based on the whole vector of defining parameters for fuzzy numbers:

$$\begin{aligned}\tilde{A_i} \succeq \tilde{A_j} \iff & (a_{i1} > a_{j1}) \vee (a_{i1} = a_{j1} \wedge a_{i2} > a_{j2}) \vee \\ & (a_{i1} = a_{j1} \wedge a_{i2} = a_{j2} \wedge a_{i3} > a_{j3}) \vee \\ & (a_{i1} = a_{j1} \wedge a_{i2} = a_{j2} \wedge a_{i3} = a_{j3} \wedge a_{i4} \geq a_{j4}).\end{aligned} \tag{13}$$

The results of this comparative study summarized in Table 2 show the variability of the results derived from applying different ranking methods. However, we must highlight an important insight derived from the theoretical results described in this paper. Depending on the definition of the magnitude of a fuzzy number, magnitude-based methods can produce a tie between fuzzy numbers when they are different by definition. This situation is more unlikely, but not impossible when using distance methods such as Fuzzy TOPSIS. However, we can ensure a ranking without ties when using lexicographic methods, provided we select a method with the appropriate number of steps. For instance, the lexicographic method based on the support comparison could not strictly rank $\tilde{A_3}$ and $\tilde{A_1}$ because the support is precisely the same for both fuzzy numbers. Luckily, the lexicographic method based on comparing the core could strictly rank $\tilde{A_3}$ and $\tilde{A_1}$ because the core interval eventually differed within the fuzzy numbers in set S_1. However, obtaining the same results with a different set of fuzzy numbers is not guaranteed. On the contrary, lexicographic methods described in [12, 27] and the last one proposed in this paper were able to rank without ties all the fuzzy numbers in the set because they are 4-step lexicographic methods.

Consider now the following set of trapezoidal fuzzy numbers:

$$S_2 = \{\tilde{B_1} = (1,4,6,7), \tilde{B_2} = (1,4,5.5,7), \tilde{B_3} = (1,4,5,7)\}. \tag{14}$$

In this case, the results obtained for each ranking function derived from the lexicographic order by [12] are shown in Table 3. We observe that four steps are necessary to rank the fuzzy numbers in S_2 because it is only at the fourth step that we find different values for each of the ranking functions included in the lexicographic method. The resulting ranking is $\tilde{B_1} \succ \tilde{B_2} \succ \tilde{B_3}$.

The results summarized in Table 3 allow us to illustrate Theorem 3. When considering a set of different same-type fuzzy numbers, the maximum number of steps to rank all of them equals the number of defining parameters. There is no other possibility because if, in our example, four steps were insufficient to order the elements in S_2, it would be because the defining parameters are equal for at least two elements in the set.

To illustrate the rebuttal of Conjecture 1 for different-type fuzzy numbers, consider the following set of triangular and trapezoidal fuzzy numbers:

$$S_3 = \{\tilde{C_1} = (3,5,6,7), \tilde{C_2} = (3,5,7)\}. \tag{15}$$

According to Conjecture 1, the maximum number of steps is $\min(3,4) + 1 = 4$. However, following the ranking method described in Sect. 3.2, we define the following functions for ranking purposes:

1. $g_1(\tilde{C_i}) = \inf\{x \in \mathrm{Supp}(\tilde{C_i}); \mu_{\tilde{C_i}}(x) = 1\}$.

Table 3 The number of steps required to rank fuzzy numbers

Step	Ranking function	$\tilde{B_1}$	$\tilde{B_2}$	$\tilde{B_3}$
1	$C(\tilde{A_i})$	4,00	4,00	4,00
2	$L(\tilde{A_i})$	1,00	1,00	1,00
3	$W(\tilde{A_i})$	6,00	6,00	6,00
4	$M(\tilde{A_i})$	4,00	3,75	3,50

2. $g_2(\tilde{C}_i) = \sup\ \{x \in \mathrm{Supp}(\tilde{C}_i); \mu_{\tilde{C}_i}(x) = 1\}$.

It is not difficult to show that $g_1(\tilde{C}_1) = g_1(\tilde{C}_2)$, but $g_2(\tilde{C}_1) > g_2(\tilde{C}_2)$. Then $\tilde{C}_1 \succ \tilde{C}_2$, proving that Conjecture 1 is false.

In the last numerical example, we illustrate Remark 3. Consider set S_4 with two different-type fuzzy numbers:

$$S_4 = \{\tilde{D}_1 = (0,3), \tilde{D}_2 = (0,1)\} \quad (16)$$

populated with a symmetric triangular fuzzy number $\tilde{D}_1 = (0,3)$, where the first defining parameter is the modal value, the second one is the left and right spread from the modal value, and a Gaussian fuzzy number $\tilde{D}_2 = (0,1)$, with membership function $\mu_{\tilde{D}_2}(x) = e^{-x^2}$, as shown in Fig. 1.

In this case, Conjecture 1 states that the maximum number of required steps is $\min(2,2) + 1 = 3$. However, assuming that we can observe the set of fuzzy numbers before defining any ranking method, we can easily rank fuzzy numbers in S_4 in a single step by computing the area under membership functions $\mu_{\tilde{D}_1}(x)$ and $\mu_{\tilde{D}_2}(x)$ to obtain the ranking $\tilde{D}_1 \succ \tilde{D}_2$.

5 Concluding Remarks

In this paper, we focus on the underlying motives why lexicographic orders can solve a ranking problem for fuzzy numbers in a limited number of steps. The main findings of our work are twofold. We first show that a necessary and sufficient condition for a ranking function to be a total order on a set of fuzzy numbers is that this function is injective, surjective, or bijective. This result allows us to bridge the gap between theory and practice by connecting the practical side of lexicographic methods described in [12, 27] with essential properties of ranking methods.

As an extension of this first result, we also provide insight into the required steps for a lexicographic order to rank same-type and different-type fuzzy numbers. Revealing the minimum and maximum number of steps implies setting levels or goals for benchmarking purposes. For example, any claim of an improved lexicographic method can now be contrasted with the theoretical results of the minimum and maximum steps described in this paper. For same-type fuzzy numbers, we connect the required steps with the number of defining parameters as critical information for designing new ranking methods. This result implies that lexicographic methods requiring more than four steps to rank trapezoidal fuzzy numbers are inefficient. For different-type fuzzy numbers, we also prove that a Conjecture 1 by [27] about the maximum number of steps required to rank different-type fuzzy numbers is false, hence contributing to extending the knowledge about lexicographic methods.

As a consequence of the main research assumption stated in the introduction, namely, lexicographic methods provide an efficient and robust way of ranking fuzzy numbers, a limitation of our work is the empirical comparison of alternative lexicographic methods in terms of efficiency and robustness. Thus, we believe that a natural extension of this work is the search for simplified methods for ranking fuzzy numbers based on lexicographic orders. In this sense, the theoretical results of the required number of steps represent a good benchmark. In addition, analyzing the robustness of specific lexicographic methods is also an interesting future line of work. Both efficiency and

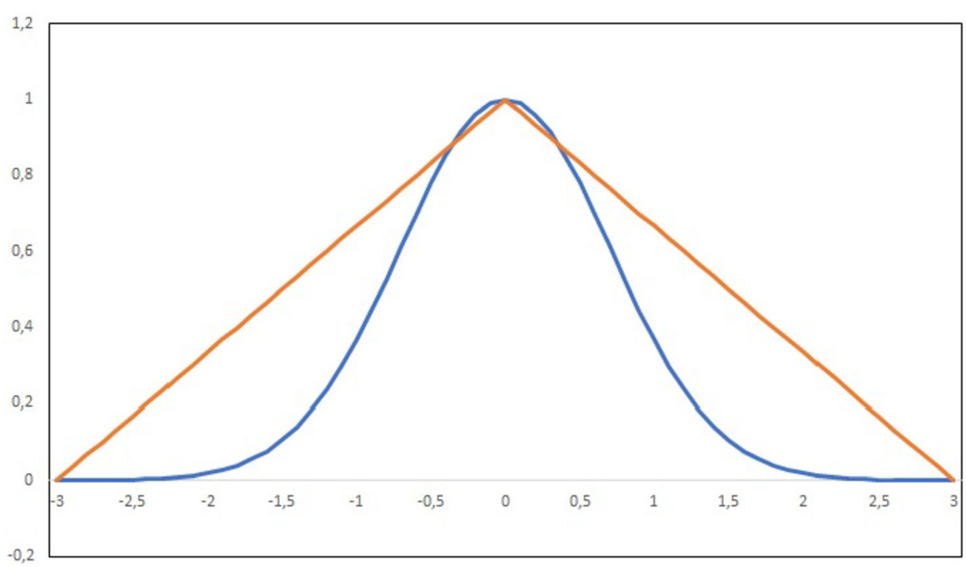

Fig. 1 A symmetric triangular and a Gaussian fuzzy number

robustness are desirable attributes that could be addressed by relying on multiple criteria decision-making.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare no conflict of interest.

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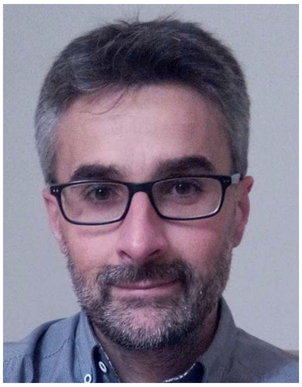

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