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Inverse matrix estimations by iterative methods with weight functions and their stability analysis

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ABSTRACT

In this paper, we construct a parametric family of iterative methods to compute the inverse of a nonsingular matrix. This class is free of inverse operators. We prove the third-order of convergence under some conditions involving the parameter of the family. Moreover, a dynamical analysis is made for the first time to a matrix iterative method, finding intervals of stability, that include but are wider than those found in the convergence analysis. Numerical tests on large random matrices confirm the results found.

1. Introduction

Approximating matrix inverses is a problem with multiple applications in areas such as physics, statistics, engineering, linear programming, among others. See, for example [1,2] for reference texts in this area of research.

Let A be a nonsingular complex matrix of size $n \times n$. We want to design iterative methods to compute the inverse A^{-1} of A, i.e., iterative schemes to solve the nonlinear matrix equation $F(X) = X^{-1} - A = 0$, where $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is a nonlinear matrix function. For this purpose, we extend to the matrix context some iterative algorithms without memory, implemented with good results in the solution of the scalar equation $f(x) = 0$.

The Newton–Schulz method [2] is the most used iterative scheme to approximate A^{-1}; its iterative expression is:

$$X_{k+1} = X_k(2I - AX_k), \quad k = 0, 1, 2, \dots, \tag{1}$$

where I is the identity matrix of order n. In [2] it is proven that the scheme given in Eq. (1) converges quadratically. Another iterative algorithm without memory for computing A^{-1}, omitting the use of inverse operators, is Chebyshev's method, presented in [3], with convergence order 3 and whose iterative expression is:

$$X_{k+1} = 3X_k - 3X_kAX_k + X_kAX_kAX_k, \quad k = 0, 1, 2, \dots, \tag{2}$$

In the literature, several iterative schemes have been extended to estimate the inverse of a matrix, see for example [4–6]. Few of them define a class of iterative schemes and, as far as we know, there exist no stability analysis in this context that allows to select the best members of those families, in terms of wideness of the set of converging initial guesses.

In this study, we construct a parametric family of iterative methods without memory to compute the inverse of a nonsingular matrix, avoiding inverse operators in its iterative formula. We prove the convergence order of the family and apply real discrete dynamics techniques to the analyze the qualitative behavior of the proposed iterative schemes.

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Next, we recall some basic concepts about iterative methods and their dynamics, see for example [7]. Let $R: \mathbb{R} \to \mathbb{R}$ be a rational operator associated to a method on a function $f(x)$. $x_0 \in \mathbb{R}$ is called fixed point if $R(x_0) = x_0$. If x_0 is a fixed point of R but $f(x_0) \neq 0$, then it is called a strange fixed point. Thus, a fixed point x_0 can be an: attractor if $|R'(x_0)| < 1$, superattractor if $|R'(x_0)| = 0$, repulsor if $|R'(x_0)| > 1$, and parabolic or neutral if $|R'(x_0)| = 1$. A critical point of an operator R is a point x_0 where the derivative of R cancels out, that is: $R'(x_0) = 0$. Critical points different to the zeros of $f(x)$ are called free critical points.

In Section 2, we design the parametric class of matrix iterative schemes and prove its third-order of convergence. Section 3 is devoted to the dynamical analysis of the class and the finding of the intervals where stable and unstable members of the family of iterative methods can be found. In Section 4, some numerical tests are made that confirm the theoretical results and, finally, some conclusions are stated in Section 5.

2. Homeier-type methods with weight functions

For a scalar equation $f(x) = 0$, Homeier's method (see [8]) is a scheme without memory, which converges cubically. For the matrix equation $F(X) = X^{-1} - A$, Homeier's method can be expressed as

$$X_{k+1} = -\frac{1}{2} X_k \left(-7I + AX_k \left(9I + AX_k \left(-5I + AX_k\right)\right)\right), \quad k \geq 0. \tag{3}$$

It was extended to the computation of inverses of matrices by Li et al. in [9]; recently, Kansal et al. in [5] showed its convergence order three.

Let us now consider the following family of third-order scalar iterative schemes:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots, \\ x_{k+1} &= x_k - G\left(u_k\right) \frac{f(x_k)}{f'(y_k)}, \end{aligned} \tag{4}$$

where $u_k = \frac{f'(y_k)}{f'(x_k)}$, $G(1) = 1$, $G'(1) = \frac{1}{2}$. These conditions on weight function G ensure the convergence of the schemes of (4) with third-order. Among the infinite weight functions that meet these conditions, we consider the particular cases: $G(u_k) = 1 + \frac{1}{2}(u_k - 1) + \gamma(u_k - 1)^2$ and $G(u_k) = \dfrac{\gamma + u_k + (3\gamma + 1)u_k^2}{2u_k(1 + 2\gamma)}$. The first polynomial expression does not lead to an iterative formula free of inverse operators, while the second one does. The next result proves the convergence of the resulting class of iterative methods, and generalizes that obtained in [5] for Homeier's method.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. Let X_0 be an initial approximation such that $\|I - AX_0\| < 1$. Then, sequence $\{X_k\}$, obtained by*

$$X_{k+1} = \frac{1}{2} \frac{X_k(7I - B_k(9I + B_k(-5I + B_k)))}{1 + 2\gamma} + \frac{\gamma}{2} \frac{23I - B_k(51I + B_k(-56I + B_k(32I + B_k(-9I + B_k))))}{1 + 2\gamma}, \tag{5}$$

being $B_k = AX_k$ and I the identity matrix, converges to A^{-1} with convergence order $p = 3$, if $\gamma \in \left[0, \frac{1}{2}\right]$. In that case, the error equation, denoting $e_k = X_k - A^{-1}$, is $\|e_{k+1}\| \leq \|A^2\| \, \|e_k\|^3$.

Proof. Let us define $E_k := I - AX_k$, $k = 0, 1, 2, 3, \dots$. Then it can be proven that

$$E_{k+1} = \frac{1}{2 + 4\gamma} \left[(1 - 2\gamma) E_k^3 + (1 + 2\gamma) E_k^4 + 3\gamma E_k^5 + \gamma E_k^6\right].$$

By induction process, we prove that $\| E_k \| < 1, \ \forall k \in \mathbb{N}$. Therefore, $\|E_{k+1}\| \leq w^k \|E_0\|^{3^{k+1}}$, where $w = \frac{|1-2\gamma|+|1+2\gamma|+|3\gamma|+|\gamma|}{|2+4\gamma|}$. Let us remark that, by definition, $w \geq 1$ but to guarantee convergence, it must be imposed that $w \leq 1$; then, $w = 1$ and it follows that $\gamma \in \mathbb{R}$ and $0 \leq \gamma \leq \frac{1}{2}$. So, when $k \to +\infty$, $\|E_{k+1}\|$ tends to zero and sequence $\{X_k\}$ converges to A^{-1}. On the other hand, let $e_k = X_k - A^{-1}$ denote the error in the iterate k. Then,

$$\|I - AX_{k+1}\| \leq \|I - AX_k\|^3 \Leftrightarrow \|A(A^{-1} - X_{k+1})\| \leq \|A(A^{-1} - X_k)\|^3.$$

So,

$$\left\|A^{-1} \left[A\left(A^{-1} - X_{k+1}\right)\right]\right\| \leq \|A^{-1}\| \left\|A\left(A^{-1} - X_{k+1}\right)\right\| \leq \|A^2\| \, \|A^{-1} - X_k\|^3,$$

we can state that $\|e_{k+1}\| \leq \|A^2\|\|e_k\|^3$ and therefore $\{X_k\}$ converges to A^{-1} with order of convergence 3. □

In the next section, we study the qualitative performance of the members of this class of iterative methods and state the dependence between convergence and stability conditions.

3. Stability analysis

As far as we know, no kind of qualitative study on iterative schemes to approximate matrix inverses has been made in the literature. To afford the task of a qualitative analysis for estimating the inverse a nonsingular matrix, we make a different approach

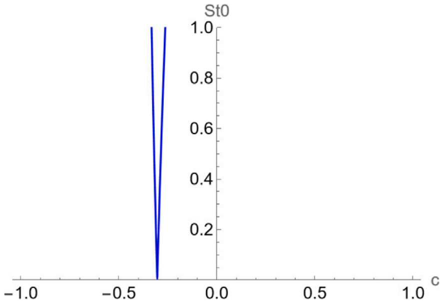
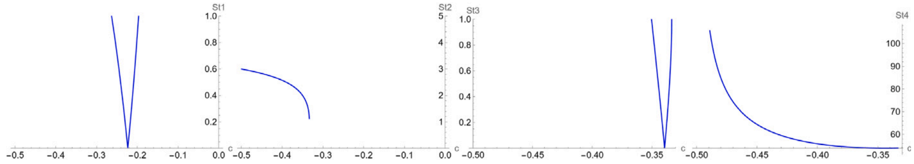


Fig. 1. $\left|R'_\gamma(0)\right|$.



(a) $\left|R'_\gamma(ex_1(\gamma))\right|$ (b) $\left|R'_\gamma(ex_2(\gamma))\right|$ (c) $\left|R'_\gamma(ex_3(\gamma))\right|$ (d) $\left|R'_\gamma(ex_4(\gamma))\right|$

Fig. 2. Stability regions of strange fixed points $ex_i(\gamma)$, $i = 1, 2, 3, 4$.

from the usual quadratic polynomial. Let $R_\gamma : \mathbb{R} \to \mathbb{R}$ be the rational operator, obtained by applying (5) on $f(x) = \dfrac{1}{x} - 1$, $R_\gamma(x) = \dfrac{x(-7 + 9x - 5x^2 + x^3 + \gamma(-23 + 51x - 56x^2 + 32x^3 - 9x^4 + x^5))}{2(1 + 2\gamma)}$.

Let us remark that $R_\gamma(x)$ is not defined for $\gamma = -\frac{1}{2}$. The amount of fixed points, their asymptotic behavior and the existence of critical points and their respective basins of attraction are the main points of the analysis made, with their respective sketch of proofs.

Theorem 3.1. *Let us consider $R_\gamma : D \subset \mathbb{R} \to \mathbb{R}$, $\gamma \in \mathbb{R} \setminus \{-\frac{1}{2}\}$, the rational function associated with the family of iterative methods (5) on $f(x) = 1/x - 1$, whose only zero is $x = 1$. Then, $x = 0$ is the only strange fixed point of $R_\gamma(x)$ if $\gamma \in \left]-\infty, -\frac{1}{2}\right[\cup [0, +\infty[$. Moreover,*

(a) If $\gamma \in \left]-\frac{1}{2}, -\frac{1}{3}\right]$, then the strange fixed points of $R_\gamma(x)$ are $x = 0$ and the four (real) zeros of the polynomial $p(x) = \gamma x^4 - 8\gamma x^3 + (1 + 24\gamma)x^2 + (-4 - 32\gamma)x + 5 + 19\gamma$, denoted by $ex_j(\gamma)$, $j = 1, 2, 3, 4$. If $\gamma = -\frac{1}{3}$, $ex_1(\gamma) = ex_2(\gamma) = 2$ and $ex_{3,4}(\gamma) = 2 \pm \sqrt{3}$.

(b) If $\gamma \in \left]-\frac{1}{3}, 0\right[$, then the strange fixed points of $R_\gamma(x)$ are $x = 0$ (double) and two other (real) zeros of the polynomial $p(x)$. In particular, if $\gamma = -\frac{5}{19}$, then the strange fixed points are $x = 0$ (triple) and $x = 4$.

Proof. The result is straightforward from solving the equation $R_\gamma(x) = x$, as $\frac{(x-1)x\left(\gamma\left(x^4-8x^3+24x^2-32x+19\right)+x^2-4x+5\right)}{4\gamma+2} = 0$, must be satisfied and the intervals where there exist real roots of $p(x) = \gamma x^4 - 8\gamma x^3 + (1 + 24\gamma)x^2 + (-4 - 32\gamma)x + 5 + 19\gamma$ define the strange fixed points, in addition to $x = 0$. □

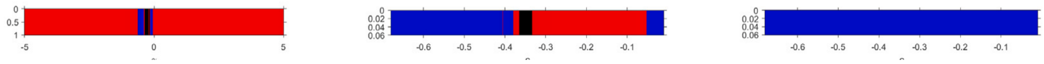
Theorem 3.2. *If $\gamma \neq -\frac{1}{2}$, the strange fixed point $x = 0$ of the operator $R_\gamma(x)$ is an attractor for $\gamma \in \left]-\frac{1}{3}, -\frac{5}{19}\right[$, superattractor for $\gamma = -\frac{7}{23}$, parabolic when $\gamma = -\frac{1}{3}$ or $\gamma = -\frac{5}{19}$ and repulsor at all other real values of γ.*

Proof. The stability area of $x = 0$ can be seen in Fig. 1. The graph shows that in the real interval $\left]-\frac{1}{3}, -\frac{5}{19}\right[$ the fixed point is an attractor ($|R'_\gamma(0)| < 1$), it is parabolic at $\gamma = -\frac{1}{3}$ or $\gamma = -\frac{5}{19}$, and outside this interval it is a repulsor.

The stability of the rest of strange fixed points can be deduced in a similar way. In Fig. 2, the repulsive character of $ex_2(\gamma)$ and $ex_4(\gamma)$ is clear, as well as the small intervals where $ex_1(\gamma)$ and $ex_3(\gamma)$ are attracting. As each basin of attraction is directly related with a critical point (see [7]), their existence is a key fact to detect all the unstable performances to be avoided.

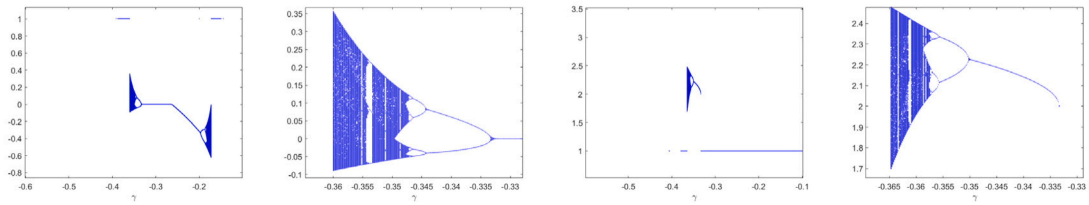
Theorem 3.3. *If $\gamma \neq -\frac{1}{2}$, $x = 1$ is a critical point of the operator $R_\gamma(x)$. For $\gamma = \frac{1}{2}$, there are no free critical points. Moreover,*

(a) If $\gamma \in \left]-\infty, c^*\right[\cup \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, +\infty\right[$, there is only one free critical point, $cr_1(\gamma)$, the only real zero of the polynomial $q(x) = 6\gamma x^3 - 33\gamma x^2 + (4 + 56\gamma)x - 7 - 23\gamma$ and $c^* \approx -0.683408$ (only real zero of the polynomial $r(x) = 2470x^3 + 795x^2 - 423x + 128$).



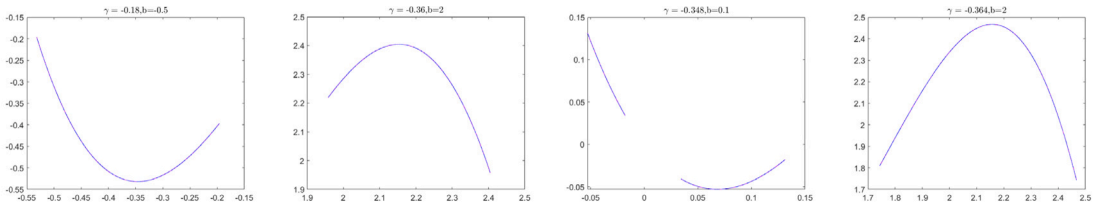
(a) $cr_1(\gamma)$ (b) $cr_1(\gamma)$ (c) $cr_1(\gamma)$

Fig. 3. Parameter lines for real free critical points.



(a) $cr_1(\gamma)$ (b) $cr_1(\gamma)$ detail (c) $cr_2(\gamma)$ (d) $cr_2(\gamma)$ detail

Fig. 4. Feigenbaum diagrams of free critical points.



(a) $\gamma = -0.18$ (b) $\gamma = -0.36$ (c) $\gamma = -0.348$ (d) $\gamma = -0.364$

Fig. 5. Strange attractors of $R_\gamma(x)$.

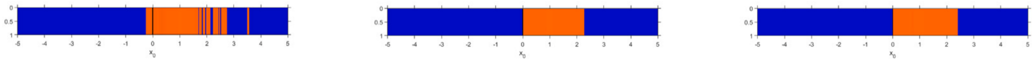
(b) If $\gamma \in \left]c^*, -\frac{1}{2}\right[\cup \left]-\frac{1}{2}, 0\right[$, there are three free critical points, $cr_i(\gamma)$, $i = 1, 2, 3$, zeros of $q(x)$. In particular, if $\gamma = c^*$, one of the three zeros of $q(x)$ is double: $cr_1(c^*) \approx 0.316537$, $cr_2(c^*) = cr_3(c^*) \approx 2.59173$.

Proof. As critical points satisfy $R'_\gamma(x) = 0$, we solve equation $\dfrac{(-1+x)^2(-7+4x+c(-23+x(56-33x+6x^2)))}{2+4c} = 0$, and find the critical point $x = 1$ as its double root and the zeros of $q(x)$, at the intervals where they are real. Let us remark that c^* corresponds to the value of γ that forces polynomial $q(x)$ to have two real roots, on of them double. □

Parameter lines (see [10]) are graphs made from the free critical points of the rational function associated with the family of iterative methods (5). They allow visualizing methods with similar dynamical behavior and selecting the best ones in terms of stability. Fig. 3(a) shows the parameter line corresponding to the free critical point $cr_1(\gamma)$. The intervals defined by the red colored regions correspond to the elements of the family whose behavior is stable as $cr_1(\gamma)$ lay in the basin of the root $x = 1$; in black color an area of unstable behavior is shown, that corresponds to $cr_1(\gamma)$ converging to an attracting element (fixed or periodic points), and in blue color the intervals where divergence appear. In stable (red) regions, the only free critical point is $cr_1(\gamma)$ and converges to the root. That is, in these regions $cr_2(\gamma)$ and $cr_3(\gamma)$ are not free critical points. Parameter lines for $cr_2(\gamma)$ and $cr_3(\gamma)$ are shown in Fig. 3(b) and (c). So, almost the entire real line is of stable behavior.

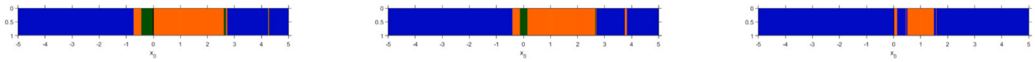
In order to analyze the behavior of $R_\gamma(x)$ when we select an element in the black area of the parameter line, we use bifurcation diagrams, able to detect strange attractors. In Fig. 4, we see the bifurcation diagrams corresponding to $cr_1(\gamma)$ and $cr_2(\gamma)$ for parameter values located in the black zone, with $-5 < \gamma < 5$ ó $-0.68 < \gamma < -0.01$. In these intervals there are attracting strange fixed points ($x = 0$, $ex_1(\gamma)$ or $ex_3(\gamma)$), and when they change their character and become repulsive, they bifurcate into double period orbits that finally give rise to chaotic behavior, see Fig. 5. In them, the iteration is trapped in a region of phase space.

To detail the study on the stability of the members of the uniparametric family given in Eq. (5), we use dynamical lines, which help us to visualize more accurately the information obtained in the parameter lines. Fig. 6 shows the dynamical lines for some specific values of γ in the stability zone. The orange color corresponds to the $x = 1$ basin of attraction, the blue color to the divergence basin. It is observed that convergent behavior can be found with good properties of stability outside the interval where Theorem 2.1 assures it. On the other hand, Fig. 7 shows the dynamical lines for values of γ in the instability zone, in which three basins of attraction are observed: the basin of $x = 1$ in orange, the divergence basin in blue and the basin of an attracting strange fixed point in green. Black areas of no convergence to $x = 1$ are also observed and, therefore, intervals of specially bad performance can be expected.



(a) $\gamma = -0.38$ (b) $\gamma = -4$ (c) $\gamma = 1.5$

Fig. 6. Dynamical lines for methods in the stability region.



(a) $\gamma = -0.2$ (b) $\gamma = -0.3$ (c) $\gamma = -0.6$

Fig. 7. Dynamical lines for methods outside the stability region.

Table 1
Results obtained by approximating the inverse of a random matrix of order $n = 100$.

Method	n	COC	iter	$\| X_{k+1} - X_k \|_2$	$\| I - AX_{k+1} \|_2$	$e - time$
Newton–Schulz	100	2.0000	26	0.002288	3.80×10^{-09}	0.156803
Chebyshev	100	1.7231	17	0.0001474	2.48×10^{-12}	0.135422
Block 1: Convergence conditions are met ($0 \leq \gamma \leq 1/2$)						
FH $\gamma = 0.25$	100	3.0489	13	0.01025	3.52×10^{-12}	0.1414
FH $\gamma = 0.40$	100	3.4933	12	0.6139	2.89×10^{-07}	0.1422
FH $\gamma = 0.5$	100	4.0475	12	0.1416	1.06×10^{-10}	0.0832
Block 2: $\gamma \notin [0, 1/2]$ and in the stability zone						
FH $\gamma = -4$	100	2,2827	10	0.09308	1.02×10^{-08}	0.1392
FH $\gamma = -0.38$	100	2.7729	26	3.48×10^{-05}	8.23×10^{-14}	0.2395
FH $\gamma = 3$	100	2.7951	11	0.1418	2.00×10^{-08}	0.1398
Block 3: $\gamma \notin [0, 1/2]$ and in instability zone						
FH $\gamma = -0.2$	100	3.0049	24	0.01259	4.56×10^{-11}	0.163646
FH $\gamma = -0.3$	100	–	–	–	–	–
FH $\gamma = -0.6$	100	3.0049	24	0.01259	4.56×10^{-11}	0.189989

4. Numerical experiments

In this section, we present numerical tests of nine different methods of the family given by Eq. (5), denoted by FH, designed to approximate the inverse of a nonsingular matrix A, by comparing them with the classical Newton–Schulz and Chebyshev methods. The numerical tests were carried out in Matlab R2023b, using an Intel Core i7-1065G7 processor up to 3.9 GHz, 16 GB DDR4 RAM. The stopping criterion used is $\|X_{k+1} - X_k\|_2 < 10^{-6}$ or $\|F(X_{k+1})\|_2 = \|I - AX_{k+1}\|_2 < 10^{-6}$. Tables 1 and 2 show the results obtained by approximating the inverse of nonsingular random matrices of size n, where $n = 100$ and $n = 500$, respectively. The initial estimate used for each method is $X_0 = \frac{A^T}{\|A\|^2}$, satisfying the convergence hypothesis of Theorem 2.1. Moreover, in each table, the selected values of γ correspond to $\gamma = 0.25$, $\gamma = 0.4$ and $\gamma = 0.5$ ($0 \leq \gamma \leq 1/2$), where convergence conditions are met; also $\gamma \notin [0, 1/2]$ but in the stability intervals of the dynamical analysis (in particular, $\gamma = -4$, $\gamma = -0.38$ and $\gamma = 3$). Finally, values of $\gamma \notin [0, 1/2]$ lying in instability intervals are used, $\gamma = -0.2$, $\gamma = -0.3$ and $\gamma = -0.6$.

To check the theoretical convergence order p, we use the approximate computational convergence order (COC), introduced by Jay (see [11]) and defined as: $p \approx COC = \dfrac{\ln\left(\|F(X_{k+1})\|_2/\|F(X_k)\|_2\right)}{\ln\left(\|F(X_k)\|_2/\|F(X_{k-1})\|_2\right)}$. Also, the execution time is provided by a mean of 50 consecutive runs using cputime command.

It is observed in both tables that the best results are obtained for those schemes in the stability area, lying or not in the interval $[0, 1/2]$, where the convergence is analytically assured. This statement is valid when comparing execution times, the number of iterations needed or the accuracy of the results.

5. Conclusions

In this research, we have constructed a parametric family of iterative methods to compute inverses of nonsingular matrices, with interesting computational properties. By imposing smooth conditions, convergence is guaranteed, under certain conditions on the parameter and the initial estimate. The study of the associated discrete dynamical system has allowed us to detect regions of stable behavior for certain values of the parameter, extending the possibilities of convergence. The numerical tests made show better results of some members of the new class than classical Newton and Chebyshev schemes in terms of execution time, number of iterations and residual error. In future works, we will analyze the behavior of this class for approximating generalized inverses.

Table 2
Results obtained by approximating the inverse of a random matrix of order $n = 500$.

Method	n	COC	iter	$\| X_{k+1} - X_k \|_2$	$\| I - AX_{k+1} \|_2$	$e - time$
Newton–Schulz	500	2.0000	42	0.06178	1.12×10^{-09}	3.275433
Chebyshev	500	3.0000	27	0.0124	5.87×10^{-08}	2.409281
Block 1: Convergence conditions are met ($0 \le \gamma \le 1/2$)						
FH $\gamma = 0.25$	500	1.3494	21	0.004824	1.15×10^{-11}	2.5544
FH $\gamma = 0.40$	500	2.0336	20	0.08249	1.16×10^{-11}	2.7980
FH $\gamma = 0.5$	500	4.0855	18	28.9	3.06×10^{-8}	2.1600
Block 2: $\gamma \notin [0, 1/2]$ and in the stability zone						
FH $\gamma = -4$	500	1.8401	17	0.02621	1.21×10^{-11}	2.2214
FH $\gamma = -0.38$	500	1.457	26	3.15×10^{-06}	1.06×10^{-11}	3.4440
FH $\gamma = 3$	500	2.1790	18	0.08965	1.18×10^{-11}	2.3306
Block 3: $\gamma \notin [0, 1/2]$ and in instability zone						
FH $\gamma = -0.2$	500	2.8349	40	0.2803	1.09×10^{-11}	5.268755
FH $\gamma = -0.3$	500	–	–	–	–	–
FH $\gamma = -0.6$	500	2.8349	40	0.2803	1.09×10^{-11}	5.286839

Data availability

No data was used for the research described in the article.

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