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On shadowing and chain recurrence in linear dynamics

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ABSTRACT

In the present work we study the concepts of shadowing and chain recurrence in the setting of linear dynamics. We prove that shadowing and finite shadowing always coincide for operators on Banach spaces, but we exhibit operators on the Fréchet space $H(\mathbb{C})$ of entire functions that have the finite shadowing property but do not have the shadowing property. We establish a characterization of mixing for continuous maps with the finite shadowing property in the setting of uniform spaces, which implies that chain recurrence and mixing coincide for operators with the finite shadowing property on any topological vector space. We establish a characterization of dense distributional chaos for operators with the finite shadowing property on Fréchet spaces. As a consequence, we prove that if a Devaney chaotic (resp. a chain recurrent) operator on a Fréchet space (resp. on a Banach space) has the finite shadowing property, then it is densely distributionally chaotic. We obtain complete characterizations of chain recurrence for weighted shifts on Fréchet sequence spaces. We prove that generalized hyperbolicity implies periodic shadowing for operators on Banach spaces. Moreover, the concepts of shadowing and periodic shadowing coincide

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for unilateral weighted backward shifts, but these notions do not coincide in general, even for bilateral weighted shifts.

1. Introduction

Consider a metric space X with metric d and a map $f : X \to X$. Given $\delta > 0$, recall that a *δ-pseudotrajectory* of f is a finite or infinite sequence $(x_j)_{i<j<k}$ in X, where $-\infty \leq i < k \leq \infty$ and $k - i \geq 3$, such that

$$d(f(x_j), x_{j+1}) \leq \delta \quad \text{for all } i < j < k - 1.$$

A finite δ-pseudotrajectory of the form $(x_j)_{j=0}^{k}$ is also called a *δ-chain for f from x_0 to x_k* and the number k is its *length*. Recall that f has the *positive shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudotrajectory $(x_j)_{j\in\mathbb{N}_0}$ of f is *ε-shadowed* by a real trajectory of f, that is, there exists $x \in X$ such that

$$d(x_j, f^j(x)) < \varepsilon \quad \text{for all } j \in \mathbb{N}_0.$$

If f is bijective, then the *shadowing property* is defined by replacing the set $\mathbb{N}_0$ by the set $\mathbb{Z}$ in the above definition. Recall also that f is *chain recurrent* (resp. *chain transitive*) if for every $x \in X$ (resp. $x, y \in X$) and every $\delta > 0$, there is a δ-chain for f from x to itself (resp. from x to y). Moreover, f is *chain mixing* if for every $x, y \in X$ and every $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, there is a δ-chain for f from x to y with length k.

The notions of pseudotrajectory, shadowing and chain recurrence originated in the seminal works of Conley [20], Sinaĭ [43] and Bowen [15] in the early 1970's. These concepts play a fundamental role in the qualitative theory of dynamical systems and differential equations. We refer the reader to the books [4,22,29,39,40,42] for nice expositions on these important concepts and their applications.

In the last few years some interesting results on shadowing and chain recurrence were obtained in the setting of linear dynamics [1,3,9,11,16]. For instance, it has long been known that every invertible hyperbolic operator on a Banach space has the shadowing property [36,37] (an operator on a Banach space is said to be *hyperbolic* if its spectrum does not intersect the unit disc) and that the converse holds in the finite-dimensional setting [36,37] and for invertible normal operators on Hilbert spaces [33]. However, it remained open for a while whether this converse is always true or not. This problem was finally settled in [9], where the first examples of non-hyperbolic operators with the shadowing property were exhibited. In the present work we will continue this line of investigation by analyzing some problems on shadowing and chain recurrence for operators. Although our main goal is to investigate the dynamics of linear operators on Fréchet spaces and, in particular, on Banach spaces, some of our results will be established in much greater generality. Below we present the topics covered in the paper and its organization.

In Section 2 we will consider the finite shadowing property. This variation of the notion of shadowing is defined as the positive shadowing property but considering only finite pseudotrajectories (of arbitrary length) instead of infinite ones. From the computational point of view, it seems to be even more relevant than shadowing, since computer-generated trajectories are actually *finite* pseudotrajectories. It is well known that shadowing and finite shadowing coincide in the setting of compact metric spaces [40, Lemma 1.1.1], but this equivalence already fails on a certain locally compact subspace of $\mathbb{R}$ [21, Example 2.3.4]. We will investigate the validity of the equivalence between shadowing and finite shadowing in the setting of linear dynamics. Our main result asserts that these concepts always coincide for operators on Banach spaces (Theorem 1). Nevertheless, we will exhibit operators on the Fréchet space $H(\mathbb{C})$ of entire functions that have the finite shadowing property but do not have the shadowing property (Theorem 2).

In Section 3 we will investigate some chaotic behaviors of operators with the finite shadowing property. We will establish a characterization of mixing for continuous maps with the finite shadowing property in the setting of uniform spaces (Theorem 5), which will imply a very general theorem in linear dynamics (Theorem 7). We will also establish a characterization of dense distributional chaos for operators with the finite shadowing property on Fréchet spaces (Theorem 9). As applications, we will show that if a Devaney chaotic (resp. a chain recurrent) continuous linear operator on a Fréchet space (resp. on a Banach space) has the finite shadowing property, then it is densely distributionally chaotic (Theorems 11 and 12). In particular, the Devaney chaotic operators constructed by Menet [35] do not have the finite shadowing property, since they are not distributionally chaotic.

In Section 4 we will consider weighted shifts on Fréchet sequence spaces. Due to the importance of weighted shifts in the area of operator theory and its applications, the dynamics of these operators has been extensively investigated by many researchers (see the books [5,27] and the papers [6–9,11,19,26,41], for instance). Our goal in this section is to establish complete characterizations of chain recurrence for weighted shifts on Fréchet sequence spaces (Theorems 13, 14, 15 and 16), which extend previous results from [1] in the case of c_0 and ℓ^p spaces. Moreover, we will illustrate these characterizations by presenting concrete examples on some classical sequence spaces.

In Section 5 we will investigate the so-called periodic shadowing property [30,38] for continuous linear operators on Banach spaces. Our main result asserts that generalized hyperbolicity implies periodic shadowing (Theorem 18 and Corollary 19). Next we will prove that positive shadowing and positive periodic shadowing coincide for unilateral weighted backward shifts on the classical Banach sequence spaces $\ell_p(\mathbb{N})$ $(1 \leq p < \infty)$ and $c_0(\mathbb{N})$ (Theorem 21). However, the notions of shadowing and periodic shadowing do not coincide in general, even for bilateral weighted shifts. In fact, we will obtain a class of operators with the periodic shadowing property (Theorem 22) that includes bilateral weighted shifts without the shadowing property (Corollary 23).

In the Appendix at the end of the paper we will establish several basic facts related to the notion of chain recurrence and the shadowing property for continuous linear operators on topological vector spaces. Our goal is to lay the foundations in great generality, complementing and extending previous basic results on this subject (see [1,3,9,11,40], for instance). Since these results are of a more elementary character, we decided to postpone them to the Appendix. However, some of these results will be used in previous sections, but properly referenced.

We will close the paper by proposing some open problems.

Throughout $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Moreover, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Whenever we consider a Fréchet space X, we will tacitly assume that we have already chosen an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms that induces its topology and that it is endowed with the compatible complete invariant metric given by

$$d(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x-y\|_k\} \qquad (x,y \in X). \tag{1}$$

We observe that the notions of shadowing and chain recurrence depend only on the underlying uniform structure of the space, and so they do not depend on the specific compatible invariant metric we choose.

2. Shadowing versus finite shadowing for operators

In this section we will investigate whether or not shadowing and finite shadowing coincide for operators on Fréchet spaces.

Given a metric space X, recall that a map $f : X \to X$ has the *finite shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each δ-chain $(x_j)_{j=0}^k$ of f, there exists $x \in X$ with

$$d(x_j, f^j(x)) < \varepsilon \quad \text{for all } j \in \{0,\ldots,k\}.$$

In this case, if f is bijective, $(x_j)_{j=-i}^k$ is a finite δ-pseudotrajectory of f ($i \geq 0$, $k \geq 1$) and we define $y_t := x_{t-i}$ for $t \in \{0,\ldots,i+k\}$, then $(y_t)_{t=0}^{i+k}$ is a δ-chain for f, and so there exists $y \in X$ with $d(y_t, f^t(y)) < \varepsilon$ for all $t \in \{0,\ldots,i+k\}$. Hence, $x := f^i(y) \in X$ satisfies

$$d(x_j, f^j(x)) < \varepsilon \quad \text{for all } j \in \{-i,\ldots,k\}.$$

This explains why we use the terminology “finite shadowing” instead of “positive finite shadowing” for the above notion.

It turns out that shadowing and finite shadowing always coincide for operators on Banach spaces.

Theorem 1. *For any invertible continuous linear operator T on any Banach space X, the following assertions are equivalent:*

(i) *T has the shadowing property;*
(ii) *T has the positive shadowing property;*
(iii) *T has the finite shadowing property.*

In the non-invertible case, (ii) and (iii) are equivalent.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (i): Given $\varepsilon > 0$, let $\delta > 0$ be associated to $\varepsilon/4$ according to the fact that T has the finite shadowing property. Let $(x_j)_{j\in\mathbb{Z}}$ be a δ-pseudotrajectory of T. The first step consists in replacing the δ-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ by a δ-pseudotrajectory $(y_j)_{j\in\mathbb{Z}}$ which is close to $(x_j)_{j\in\mathbb{Z}}$, in the sense that

$$\|y_j - x_j\| < \frac{\varepsilon}{4} \quad \text{for all } j \in \mathbb{Z}, \tag{2}$$

and has the following additional property:

$$\lim_{j\to\pm\infty} \|Ty_j - y_{j+1}\| = 0. \tag{3}$$

For this purpose, choose $m \in \mathbb{N}$ with $\varepsilon/m < \delta$ and define

$$n_k := \frac{k(k-1)m}{2} \quad \text{for all } k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, there exists $u_k \in X$ such that

$$\|x_j - T^j u_k\| < \frac{\varepsilon}{4} \quad \text{for all } j \in \{-n_{k+1}, \dots, n_{k+1}\}. \tag{4}$$

For each $k \in \mathbb{N}$ and each $j \in \{0, \dots, km-1\}$, define

$$y_{n_k+j} := \frac{km-j}{km}\, T^{n_k+j} u_k + \frac{j}{km}\, T^{n_k+j} u_{k+1}$$

and

$$y_{-n_k-j} := \frac{km-j}{km}\, T^{-n_k-j} u_k + \frac{j}{km}\, T^{-n_k-j} u_{k+1}.$$

It follows from (4) that (2) holds. Moreover, for each $k \in \mathbb{N}$ and each $j \in \{0, \dots, km-1\}$,

$$\begin{aligned}\|Ty_{n_k+j} - y_{n_k+j+1}\| &= \left\| \frac{1}{km} T^{n_k+j+1} u_k - \frac{1}{km} T^{n_k+j+1} u_{k+1} \right\| \\ &\leq \frac{1}{km} \left\| T^{n_k+j+1} u_k - x_{n_k+j+1} \right\| + \frac{1}{km} \left\| x_{n_k+j+1} - T^{n_k+j+1} u_{k+1} \right\| \\ &< \frac{\varepsilon}{2km} < \delta\end{aligned}$$

and, similarly,

$$\|Ty_{-n_k-j} - y_{-n_k-j+1}\| < \frac{\varepsilon}{2km} < \delta.$$

This shows that $(y_j)_{j\in\mathbb{Z}}$ is also a δ-pseudotrajectory of T and that (3) holds.

The second step consists in constructing inductively an increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers, a sequence $(v_k)_{k\in\mathbb{N}}$ of vectors in X and a sequence $((y_j^{(k)})_{j\in\mathbb{Z}})_{k\in\mathbb{N}}$ of pseudotrajectories of T satisfying the following conditions for each $k \in \mathbb{N}$:

(a) $(y_j^{(k)})_{j\in\mathbb{Z}}$ is a $\dfrac{\delta}{2^{k-1}}$-pseudotrajectory of T;
(b) $\lim\limits_{j\to\pm\infty} \|Ty_j^{(k)} - y_{j+1}^{(k)}\| = 0$;
(c) $\|Ty_j^{(k)} - y_{j+1}^{(k)}\| < \dfrac{\delta}{2^{k+1}}$ whenever $|j| \geq m_k$;
(d) $\|y_j^{(k)} - T^j v_k\| < \dfrac{\varepsilon}{2^{k+1}}$ whenever $|j| \leq m_k + p$;
(e) $y_0^{(k)} = v_{k-1}$ and $\|y_j^{(k)} - y_j^{(k-1)}\| < \dfrac{\varepsilon}{2^k}$ for all $j \in \mathbb{Z}$ (provided $k \geq 2$).

The number p is a fixed positive integer greater than ε/δ. We begin by defining

$$y_j^{(1)} := y_j \quad \text{for all } j \in \mathbb{Z}.$$

By (3), we can choose an $m_1 \in \mathbb{N}$ such that

$$\|Ty_j^{(1)} - y_{j+1}^{(1)}\| < \frac{\delta}{2^2} \quad \text{whenever } |j| \geq m_1.$$

By finite shadowing, there exists $v_1 \in X$ such that

$$\|y_j^{(1)} - T^j v_1\| < \frac{\varepsilon}{2^2} \quad \text{whenever } |j| \leq m_1 + p.$$

Hence, (a), (b), (c) and (d) hold with $k := 1$. Suppose that m_k, v_k and $(y_j^{(k)})_{j\in\mathbb{Z}}$ have already been chosen for $k \in \{1, \ldots, t\}$ so that all the desired properties hold. Define

$$y_j^{(t+1)} := \begin{cases} T^j v_t & \text{if} \quad |j| \leq m_t \\ \frac{m_t+p-|j|}{p} T^j v_t + \frac{|j|-m_t}{p} y_j^{(t)} & \text{if} \quad m_t < |j| < m_t + p \\ y_j^{(t)} & \text{if} \quad |j| \geq m_t + p \end{cases}.$$

Some elementary computations show that (a) and (e) hold with $k := t + 1$. Since

$$\lim_{j\to\pm\infty} \|Ty_j^{(t+1)} - y_{j+1}^{(t+1)}\| = \lim_{j\to\pm\infty} \|Ty_j^{(t)} - y_{j+1}^{(t)}\| = 0,$$

we have that (b) holds with $k := t+1$. Moreover, we can choose an $m_{t+1} > m_t$ such that (c) hold with $k := t+1$. Finally, by finite shadowing, there exists a vector $v_{t+1} \in X$ such that (d) holds with $k := t+1$. This completes our induction process.

Let us now complete the proof. Since

$$\|v_k - v_{k+1}\| = \|y_0^{(k+1)} - T^0 v_{k+1}\| < \frac{\varepsilon}{2^{k+2}} \quad \text{for all } k \in \mathbb{N},$$

we have that $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X. By completeness, there exists

$$v := \lim_{k\to\infty} v_k \in X.$$

Moreover,

$$\|T^j v_k - y_j\| \le \|T^j v_k - y_j^{(k)}\| + \sum_{t=2}^{k} \|y_j^{(t)} - y_j^{(t-1)}\| < \frac{\varepsilon}{2^{k+1}} + \sum_{t=2}^{k} \frac{\varepsilon}{2^t} < \frac{\varepsilon}{2},$$

whenever $k \in \mathbb{N}$ and $|j| \le m_k + p$. By fixing $j \in \mathbb{Z}$ and letting $k \to \infty$, we obtain

$$\|T^j v - y_j\| \le \frac{\varepsilon}{2} \quad \text{for all } j \in \mathbb{Z}. \tag{5}$$

By (2) and (5), the δ-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ is ε-shadowed by the trajectory of v, proving that T has the shadowing property.

In the non-invertible case, the arguments are analogous. □

One important fact in the previous proof was that the ε-δ association can be selected to be linear if we have a Banach space, essential for applying the induction process in the second step of the proof to get (d) with $\frac{\varepsilon}{2^{k+1}}$ from finite shadowing when we have a $\frac{\delta}{2^{k-1}}$-pseudotrajectory. This is something that we cannot do in general with an F-norm for a Fréchet space. Actually, the equivalence between shadowing and finite shadowing may fail for operators on Fréchet spaces.

Theorem 2. *Let $H(\mathbb{C})$ be the Fréchet space of all entire functions endowed with the compact-open topology. For each $\lambda \in \mathbb{C}$ with $|\lambda| \notin \{0, 1\}$, the multiplication operator*

$$M_\lambda : f \in H(\mathbb{C}) \mapsto \lambda f \in H(\mathbb{C})$$

has the finite shadowing property but does not have the shadowing property.

Proof. For each $k \in \mathbb{N}$, let

$$D_k := \{z \in \mathbb{C} : |z| < k\} \quad \text{and} \quad \|f\|_k := \sup_{z \in D_k} |f(z)| \ \text{ for } f \in H(\mathbb{C}).$$

The sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms induces the compact-open topology on $H(\mathbb{C})$. Consider $H(\mathbb{C})$ endowed with its canonical metric given by (1). For each $k \in \mathbb{N}$, let $A(D_k)$ be the "disk algebra on the disk D_k", that is,

$$A(D_k) := \{g : \overline{D_k} \to \mathbb{C} : g \text{ is continuous on } \overline{D_k} \text{ and analytic on } D_k\}$$

endowed with the norm $\|\cdot\|_k$, which is a Banach space (actually, a Banach algebra). In view of Proposition 33 in the Appendix, it is enough to consider the case $|\lambda| > 1$, since $M_{\lambda^{-1}} = (M_\lambda)^{-1}$. Thus, fix $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and let $T := M_\lambda$.

Let us prove that T has the finite shadowing property. For this purpose, fix $\varepsilon > 0$ and choose $\ell \in \mathbb{N}$ such that

$$d(f, 0) < \varepsilon \quad \text{whenever } f \in H(\mathbb{C}) \text{ and } \|f\|_\ell < \frac{\varepsilon}{2} \cdot$$

Let

$$S : g \in A(D_\ell) \mapsto \lambda g \in A(D_\ell).$$

Since S is a proper dilation on the Banach space $A(D_\ell)$ (i.e., $\|S^{-1}\| < 1$), S is a hyperbolic operator. Hence, S has the shadowing property. Let $\eta > 0$ be such that every η-pseudotrajectory of S is $(\varepsilon/2)$-shadowed by a real trajectory of S. Choose $\delta > 0$ such that

$$\|f\|_\ell \leq \eta \quad \text{whenever } f \in H(\mathbb{C}) \text{ and } d(f, 0) \leq \delta.$$

If $(f_j)_{j=0}^k$ is a δ-chain for T, then $(f_j|_{\overline{D_\ell}})_{j=0}^k$ is an η-chain for S, and so there exists $g \in A(D_\ell)$ such that

$$\|f_j|_{\overline{D_\ell}} - S^j g\|_\ell < \frac{\varepsilon}{2} \quad \text{for all } j \in \{0, \ldots, k\}.$$

By the density of the polynomials in $A(D_\ell)$, there is a polynomial f so close to g in $A(D_\ell)$ that we have

$$\|f_j - T^j f\|_\ell = \|f_j|_{\overline{D_\ell}} - S^j f\|_\ell < \frac{\varepsilon}{2} \quad \text{for all } j \in \{0, \ldots, k\}.$$

Thus, $d(f_j, T^j f) < \varepsilon$ for all $j \in \{0, \ldots, k\}$, as it was to be shown.

Now, suppose that T has the shadowing property. Let $\delta > 0$ be associated to $\varepsilon := 1/2$ according to this property. Choose $\ell \in \mathbb{N}$ such that

$$d(f,0) < \delta \quad \text{whenever } f \in H(\mathbb{C}) \text{ and } \|f\|_\ell < \frac{\delta}{2} \cdot$$

Choose a function $g \in A(D_\ell)$ that cannot be extended to an entire function. We shall construct inductively a sequence $(f_j)_{j\in\mathbb{N}_0}$ of polynomials such that:

(A) $\|\lambda^j g - f_j\|_\ell < \dfrac{\delta}{2|\lambda|}$ for all $j \in \mathbb{N}_0$;

(B) $\|\lambda f_{j-1} - f_j\|_\ell < \dfrac{\delta}{2}$ for all $j \in \mathbb{N}$.

We begin by choosing a polynomial f_0 with $\|g - f_0\|_\ell < \frac{\delta}{2|\lambda|}$. Assume $k \in \mathbb{N}_0$ and $f_0, \dots, f_k$ already chosen with the desired properties. Since

$$\|\lambda^{k+1} g - \lambda f_k\|_\ell = |\lambda| \|\lambda^k g - f_k\|_\ell < \frac{\delta}{2},$$

there is a polynomial p_k so close to $\lambda^{k+1} g - \lambda f_k$ in $A(D_\ell)$ that we have

$$\|p_k\|_\ell < \frac{\delta}{2} \quad \text{and} \quad \|\lambda^{k+1} g - \lambda f_k - p_k\|_\ell < \frac{\delta}{2|\lambda|} \cdot$$

Hence, it is enough to define $f_{k+1} := \lambda f_k + p_k$. By (B), $(f_j)_{j\in\mathbb{N}_0}$ is a δ-pseudotrajectory of T. Therefore, there exists $f \in H(\mathbb{C})$ such that

$$d(f_j, T^j f) < \frac{1}{2} \quad \text{for all } j \in \mathbb{N}_0.$$

This implies that

$$\|f_j - \lambda^j f\|_1 < 1 \quad \text{for all } j \in \mathbb{N}_0.$$

By (A), we obtain

$$\|\lambda^j g - \lambda^j f\|_1 < 1 + \frac{\delta}{2|\lambda|} \quad \text{for all } j \in \mathbb{N}_0.$$

Thus, $g = f$ on D_1. By the principle of analytic continuation, $g = f$ on $\overline{D_\ell}$. This contradicts our choice of g as an element of $A(D_\ell)$ that cannot be extended to an entire function. Our conclusion is that T does not have the shadowing property. □

Remark 3.

(a) The above proof actually shows that M_λ does not have the positive shadowing property whenever $|\lambda| > 1$. Thus, the notions of finite shadowing and positive shadowing do not coincide in general for invertible operators on Fréchet spaces.

(b) For $0 < |\lambda| < 1$, the operator M_λ has the positive shadowing property. Indeed, this follows easily from the fact that if $\ell \in \mathbb{N}$ and if a sequence $(f_j)_{j\in\mathbb{N}_0}$ in $H(\mathbb{C})$ satisfies

$$\|M_\lambda f_j - f_{j+1}\|_\ell \leq \delta \quad \text{for all } j \in \mathbb{N}_0,$$

then

$$\|f_j - (M_\lambda)^j f_0\|_\ell \leq \frac{\delta}{1-|\lambda|} \quad \text{for all } j \in \mathbb{N}_0.$$

Thus, the notions of shadowing and positive shadowing do not coincide in general for invertible operators on Fréchet spaces. Since $(M_\lambda)^{-1} = M_{\lambda^{-1}}$ does not have the positive shadowing property, this also shows that we cannot replace shadowing by positive shadowing in Proposition 33.

Remark 4. We observe that completeness is essential for the validity of Theorem 1. For instance, let X be the vector space of all sequences $(x_n)_{n\in\mathbb{N}}$ of scalars with finite support endowed with any ℓ_p-norm ($1 \leq p \leq \infty$) and let $T \in GL(X)$ be twice the identity operator on X. Since twice the identity operator on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ ($1 \leq p < \infty$) has the shadowing property (because it is hyperbolic), it follows that T has the finite shadowing property. However, T does not have the positive shadowing property. In fact, given any $\delta > 0$, consider the sequence $(x^{(j)})_{j\in\mathbb{N}_0}$ in X given by

$$x^{(0)} := 0 \quad \text{and} \quad x^{(j)} := (2^{j-1}\delta, 2^{j-2}\delta, \ldots, 2\delta, \delta, 0, 0, \ldots) \text{ for } j \geq 1.$$

Then $(x^{(j)})_{j\in\mathbb{N}_0}$ is a δ-pseudotrajectory of T, but it cannot be 1-shadowed by a trajectory of T since each element of X has finite support.

3. Chaotic behaviors in the presence of the finite shadowing property

Our goal in this section is to investigate some types of chaotic behavior for operators with the finite shadowing property.

We begin by recalling some notions of chaotic behavior. Let X be a topological space and $f : X \to X$ a map. Given sets $A, B \subset X$, the *return set of f from A to B* is defined by

$$N_f(A,B) := \{n \in \mathbb{N}_0 : f^n(A) \cap B \neq \emptyset\}.$$

Recall that f is *topologically transitive* (resp. *topologically ergodic*, *topologically mixing*) if for any pair A, B of nonempty open subsets of X, the return set $N_f(A,B)$ is nonempty (resp. syndetic, cofinite), where a set $I := \{n_1 < n_2 < \cdots\} \subset \mathbb{N}_0$ is *syndetic* when it has bounded gaps, that is, $\sup_k(n_{k+1} - n_k) < \infty$. Moreover, f is *topologically weakly mixing* if $f \times f$ is topologically transitive, that is, $N_f(A_1,B_1) \cap N_f(A_2,B_2) \neq \emptyset$ for any

4-tuple A_1, A_2, B_1, B_2 of nonempty open subsets of X. In the sequel we will omit the word "topologically" from these notions. If X is a second countable Baire space without isolated points and f is continuous, then *Birkhoff's transitivity theorem* asserts that f is transitive if and only if it admits a dense orbit, that is, there exists a point $x \in X$ whose orbit $\mathrm{Orb}(x, f) := \{f^n(x) : n \in \mathbb{N}_0\}$ is dense in X.

In the setting of linear dynamics, the existence of a dense orbit is known under the name of *hypercyclicity*. Hence, a continuous linear operator T on a topological vector space X is *hypercyclic* if it admits a dense orbit. Recall also that T is *Devaney chaotic* if it is transitive and has a dense set of periodic points. Hypercyclic and Devaney chaotic operators have been extensively studied during the last 30 years. We refer the reader to the books [5,27] for an overview of the area of linear dynamics up to 2010.

Our first theorem in this section will give us a characterization of mixing for continuous maps with finite shadowing in the setting of uniform spaces. In order to state and prove the theorem, let us first recall the notion of chain transitivity and the finite shadowing property in this more general setting. We refer the reader to [14, Chapter II] for the basics on uniform spaces.

Consider a uniform space X with uniformity $\mathcal{U}$ and a map $f : X \to X$. Given $V \in \mathcal{U}$, a *V-chain for f* is a finite sequence $(x_j)_{j=0}^k$ in X satisfying

$$(f(x_j), x_{j+1}) \in V \quad \text{for all } n \in \{0, \dots, k-1\}.$$

In this case, we also say that $(x_j)_{j=0}^k$ is a *V-chain for f from x_0 to x_k*. The map f has the *finite shadowing property* if for every $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that for each U-chain $(x_j)_{j=0}^k$ for f, there exists $x \in X$ with

$$(x_j, f^j(x)) \in V \quad \text{for all } j \in \{0, \dots, k\}.$$

A point $x \in X$ is a *chain recurrent point* of f if for every $V \in \mathcal{U}$, there is a V-chain for f from x to itself. The set $CR(f)$ of all chain recurrent points of f is called the *chain recurrent set* of f and f is said to be *chain recurrent* if $CR(f) = X$. Moreover, f is said to be *chain transitive* if for every $x, y \in X$ and every $V \in \mathcal{U}$, there is a V-chain for f from x to y. We recall that for each point $x \in X$, the sets of the form

$$V(x) := \{y \in X : (x, y) \in V\},$$

as V runs through the uniformity $\mathcal{U}$, constitute a fundamental system of neighborhoods of x in X. We also recall that $A \subset X$ is a *V-small set* if $A \times A \subset V$.

Theorem 5. *Consider a uniform space X with uniformity $\mathcal{U}$ and a continuous map $f : X \to X$. If f has the finite shadowing property, then f is mixing if and only if the following conditions hold:*

(I) *f is chain transitive;*

(II) *For each $V \in \mathcal{U}$, there is a V-small set $A \subset X$ with $N_f(A, A)$ cofinite.*

Proof. Since the necessity of the conditions is clear, let us prove their sufficiency. Let A and B be nonempty open sets in X. Choose points $x \in A$ and $y \in B$, and let $V \in \mathcal{U}$ be such that

$$V(x) \subset A \quad \text{and} \quad V(y) \subset B.$$

Let $U \in \mathcal{U}$ be associated to V according to the definition of the finite shadowing property, and let $W \in \mathcal{U}$ satisfy

$$W \circ W := \{(a, c) : (a, b) \in W \text{ and } (b, c) \in W \text{ for some } b \in X\} \subset U.$$

By condition (II), there is a W-small set $Z \subset X$ such that $N_f(Z, Z)$ is cofinite. Choose a point $z \in Z$ and let $m \in \mathbb{N}$ be such that $n \in N_f(Z, Z)$ for all $n \geq m$. By condition (I), there exist W-chains $(x_j)_{j=0}^{k}$ and $(y_j)_{j=0}^{\ell}$ for f from x to z and from z to y, respectively. Given $n \geq m$, there exists $z' \in Z$ such that $f^n(z') \in Z$. Hence,

$$(u_j)_{j=0}^{k+n+\ell} := (x_0, x_1, \ldots, x_{k-1}, z', f(z'), \ldots, f^{n-1}(z'), y_0, y_1, \ldots, y_\ell)$$

is a U-chain for f from x to y, and so there exists $u \in X$ such that

$$(u_j, f^j(u)) \in V \quad \text{for all } j \in \{0, \ldots, k + n + \ell\}.$$

In particular, $u \in A$ and $f^{k+n+\ell}(u) \in B$. This proves that

$$f^t(A) \cap B \neq \emptyset \quad \text{for all } t \geq t_0,$$

where $t_0 := k + m + \ell$. □

Remark 6. Condition (II) in Theorem 5 cannot be omitted in general. For instance, consider $X := \{0, 1\}$ endowed with its discrete uniformity and let $f : X \to X$ be given by $f(0) := 1$ and $f(1) := 0$. Then f is chain transitive and has the shadowing property, but it is not mixing.

As an application of the above theorem, we obtain the following result on linear dynamics.

Theorem 7. *Suppose that a continuous linear operator T on a topological vector space X has the finite shadowing property. Then the following assertions are equivalent:*

(i) *T is chain recurrent;*
(ii) *T is transitive;*

(iii) *T is ergodic;*
(iv) *T is weakly mixing;*
(v) *T is mixing.*

Moreover, T is Devaney chaotic if and only if T has a dense set of periodic points.

Proof. Recall that a basis for the uniformity of X is given by the sets

$$\widetilde{V} := \{(x, y) \in X \times X : x - y \in V\},$$

as V runs through the set of all neighborhoods of 0 in X.

In order to prove the equivalences from (i) to (v), it is enough to show that (i) implies (v). This follows from Theorem 5, because condition (II) is automatically true in the present case (note that $N_T(A, A) = \mathbb{N}_0$ whenever $0 \in A$) and Proposition 24 in the Appendix gives condition (I).

Now, suppose that T has a dense set of periodic points. Since the set $CR(T)$ of all chain recurrent points of T is closed in X (Proposition 26), we have that T is chain recurrent. Hence, by (i) ⇒ (ii), T is transitive, and so it is Devaney chaotic. □

The above theorem extends [3, Theorem 3.3] and [3, Corollary 3.9] from normed spaces and separable Banach spaces, respectively, to arbitrary topological vector spaces, but we observe that the arguments in [3] also work in the more general context.

Our next goal is to establish a characterization of dense distributional chaos for operators with the finite shadowing property in the setting of Fréchet spaces. For this purpose, let us recall the notion of distributional chaos in metric spaces and some related concepts in the context of linear dynamics.

Given a metric space X, recall that $f : X \to X$ is said to be *distributionally chaotic* if there exist an uncountable set $\Gamma \subset X$ and an $\varepsilon > 0$ such that each pair (x, y) of distinct elements of Γ is an *ε-distributionally chaotic pair* for f, in the sense that

$$\overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) \geq \varepsilon\} = 1$$

and

$$\overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \delta\} = 1 \text{ for all } \delta > 0,$$

where $\overline{\text{dens}}(I)$ stands for the *upper density* of the subset I of $\mathbb{N}$, that is,

$$\overline{\text{dens}}(I) := \limsup_{n\to\infty} \frac{\text{card}(I \cap [1, n])}{n} \cdot$$

If X is a Fréchet space whose topology is induced by an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms and $T : X \to X$ is a continuous linear operator, recall that $x \in X$ is called a *distributionally irregular vector* for T if there exist $m \in \mathbb{N}$ and $I, J \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(I) = \overline{\mathrm{dens}}(J) = 1$ such that

$$\lim_{n \in I} T^n x = 0 \quad \text{and} \quad \lim_{n \in J} \|T^n x\|_m = \infty.$$

It was proved in [7] that:

T is distributionally chaotic $\Leftrightarrow$ T admits a distributionally irregular vector.

We shall follow the terminology of [25] and say that T is *densely distributionally chaotic* if it admits a dense set of distributionally irregular vectors. It follows from results in [7] that this is equivalent to the existence of a *residual set* of distributionally irregular vectors.

Before stating our theorem, let us introduce the following notations: For each continuous linear operator T on a Fréchet space X, we define the sets

$$I_0(T) := \{x \in X : \text{for each } \delta > 0, \text{ there is a } \delta\text{-chain for } T \text{ from } x \text{ to } 0\},$$
$$O_0(T) := \{x \in X : \text{for each } \delta > 0, \text{ there is a } \delta\text{-chain for } T \text{ from } 0 \text{ to } x\}.$$

Lemma 8. *The sets $I_0(T)$ and $O_0(T)$ are T-invariant closed subspaces of X.*

As a consequence, if both $I_0(T)$ and $O_0(T)$ are dense in X, then T is chain recurrent. The converse is also true, because the notions of chain recurrence and chain transitivity coincide for linear operators (Proposition 24). We leave the proof of the above lemma to the reader.

Theorem 9. *Suppose that a continuous linear operator T on a Fréchet space X has the finite shadowing property. Then T is densely distributionally chaotic if and only if the following conditions hold:*

(I) *$I_0(T)$ is dense in X;*
(II) *There exists $\gamma > 0$ such that for every $\delta > 0$, there is a δ-chain for T from 0 to a vector $x \in X$ satisfying*

$$\overline{\mathrm{dens}}\{j \in \mathbb{N}_0 : d(T^j x, 0) \geq \gamma\} = 1.$$

If X is a Banach space, then we can replace condition (II) by the following weaker condition:

(II') *There exists $\gamma > 0$ such that for every $\delta > 0$, there is a δ-chain for T from 0 to a vector $x \in X$ satisfying*

$$\overline{\mathrm{dens}}\{j \in \mathbb{N}_0 : d(T^j x, 0) \geq \gamma\} \geq \gamma.$$

Proof. Suppose that T is densely distributionally chaotic. Since T admits a dense set of vectors whose trajectories have subsequences converging to 0, condition (I) holds. Moreover, if y is a distributionally irregular vector for T, then there exists $\gamma > 0$ such that

$$\overline{\mathrm{dens}}\{j \in \mathbb{N}_0 : d(T^j y, 0) \geq \gamma\} = 1.$$

Given $\delta > 0$, there exists $n \in \mathbb{N}$ such that $d(T^n y, 0) < \delta$, and so $(0, T^n y)$ is a δ-chain for T from 0 to the vector $x := T^n y$ which satisfies $\overline{\mathrm{dens}}\{j \in \mathbb{N}_0 : d(T^j x, 0) \geq \gamma\} = 1$, proving that condition (II) also holds.

Conversely, suppose that conditions (I) and (II) hold. For each $k \in \mathbb{N}$, let

$$A_k := \{x \in X : \exists n \in \mathbb{N} \text{ with } \mathrm{card}\{1 \leq j \leq n : d(T^j x, 0) < k^{-1}\} \geq n(1 - k^{-1})\}.$$

It is clear that each A_k is open in X. Fix $k \in \mathbb{N}$, $x \in X$ and $\varepsilon > 0$. Let $\eta := \min\{\varepsilon, k^{-1}\}$ and let $\delta > 0$ be associated to η according to the fact that T has the finite shadowing property. By condition (I), there is a δ-chain $(x_j)_{j=0}^t$ for T from x to 0. Choose $n \in \mathbb{N}$ such that $n - t > n(1 - k^{-1})$ and define $x_j := 0$ for all $j \in \{t+1, \ldots, n\}$. Since $(x_j)_{j=0}^n$ is a δ-chain for T, there exists $y \in X$ such that

$$d(x_j, T^j y) < \eta \quad \text{for all } j \in \{0, \ldots, n\}.$$

Hence, $y \in A_k$ and $d(y, x) < \varepsilon$. This proves that A_k is dense in X. It follows that the set R_1 of all $x \in X$ for which there exists $I \subset \mathbb{N}$ with

$$\overline{\mathrm{dens}}(I) = 1 \quad \text{and} \quad \lim_{n \in I} T^n x = 0$$

is residual in X.

Now, let $\varepsilon := \gamma/2$ and, for each $k \in \mathbb{N}$, let $\delta_k > 0$ be associated to ε/k according to the finite shadowing property. By condition (II), for each $k \in \mathbb{N}$, there is a δ_k-chain $(x_{k,j})_{j=0}^{t_k}$ for T from 0 to a vector $x_k \in X$ satisfying

$$\overline{\mathrm{dens}}(I_k) = 1, \quad \text{where } I_k := \{j \in \mathbb{N}_0 : d(T^j x_k, 0) \geq \gamma\}.$$

For each $k \in \mathbb{N}$, consider a natural number $N_k > t_k$ and define

$$x_{k,j} := T^{j - t_k} x_k \quad \text{for all } j \in \{t_k + 1, \ldots, N_k\}.$$

Since $(x_{k,j})_{j=0}^{N_k}$ is a δ_k-chain for T, there exists $y_k \in X$ such that

$$d(x_{k,j}, T^j y_k) < \varepsilon/k \quad \text{for all } j \in \{0, \ldots, N_k\}.$$

If $j \in \{t_k + 1, \ldots, N_k\}$ and $j - t_k \in I_k$, then

$$d(T^j y_k, 0) \geq d(x_{k,j}, 0) - d(x_{k,j}, T^j y_k) > \gamma - \varepsilon/k \geq \varepsilon.$$

Since $\overline{\text{dens}}(I_k) = 1$, we can choose N_k as large as we want so that

$$\text{card}\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\} > N_k(1 - k^{-1}). \tag{6}$$

Hence, we can choose $N_1 < N_2 < N_3 < \cdots$ so that (6) holds for all $k \in \mathbb{N}$. Thus,

$$\lim_{k\to\infty} y_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{N_k} \text{card}\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\} = 1.$$

By [7, Proposition 8], the set R_2 of all $x \in X$ for which there are $m \in \mathbb{N}$ and $J \subset \mathbb{N}$ with

$$\overline{\text{dens}}(J) = 1 \quad \text{and} \quad \lim_{n\in J} \|T^n x\|_m = \infty$$

is residual in X. Therefore, the set $R_1 \cap R_2$ is also residual in X. Since each element of this set is a distributionally irregular vector for T, we conclude that T is densely distributionally chaotic.

Let us now assume that X is a Banach space and that conditions (I) and (II') hold. Let R_1 and R_2 be as above. Since condition (I) was not changed, R_1 is residual in X. By following the arguments used in the previous paragraph with $\overline{\text{dens}}(I_k) \geq \gamma$ instead of $\overline{\text{dens}}(I_k) = 1$, we see that we can choose N_k as large as we want so that

$$\text{card}\{1 \leq j \leq N_k : d(T^j y_k, 0) > \varepsilon\} > \varepsilon N_k. \tag{7}$$

Hence, we can choose $N_1 < N_2 < N_3 < \cdots$ so that (7) holds for all $k \in \mathbb{N}$. By [7, Proposition 8] in the case of Banach spaces, we conclude that R_2 is residual in X. □

Remark 10. If T is a proper contraction (respectively, a proper dilation) on a Banach space X, then T has the shadowing property and condition (I) (respectively, condition (II)) holds, but T is not distributionally chaotic. This shows that each one of the conditions in Theorem 9 is essential for its validity.

As applications of the previous theorem, we obtain the following results.

Theorem 11. *If a Devaney chaotic continuous linear operator T on a Fréchet space X has the finite shadowing property, then it is densely distributionally chaotic.*

Proof. Actually, it is not difficult to check that Devaney chaos implies conditions (I) and (II) of the previous theorem. □

Theorem 12. *If a chain recurrent continuous linear operator T on a Banach space X has the finite shadowing property, then it is densely distributionally chaotic.*

Proof. Since T is chain transitive (Proposition 24), condition (I) holds. By Theorem 1, T has the positive shadowing property. Let $\eta > 0$ be associated to $\varepsilon := 1$ according to this property. Choose a vector $y \in X$ with $\|y\| \geq 2$ and let $(y_j)_{j=0}^k$ be an η-chain for T from y to itself. Since

$$(y_0, y_1, \ldots, y_k, y_1, \ldots, y_k, y_1, \ldots, y_k, \ldots)$$

is an η-pseudotrajectory of T, there exists $x \in X$ such that

$$\|y - T^{nk}x\| < 1 \quad \text{for all } n \in \mathbb{N}_0.$$

Thus,

$$\overline{\text{dens}}\{j \in \mathbb{N}_0 : \|T^j x\| \geq 1\} \geq 1/k.$$

Since T is chain transitive, we conclude that condition (II') holds with $\gamma := 1/k$. □

4. Chain recurrent weighted shifts on Fréchet sequence spaces

Our goal in this section is to characterize the notion of chain recurrence for weighted shifts on Fréchet sequence spaces.

Recall that a *Fréchet sequence space* is a Fréchet space X which is a vector subspace of the product space $\mathbb{K}^{\mathbb{N}}$ such that the inclusion map $X \to \mathbb{K}^{\mathbb{N}}$ is continuous, that is, convergence in X implies coordinatewise convergence. Given a sequence $w := (w_n)_{n\in\mathbb{N}}$ of nonzero scalars, it follows from the closed graph theorem that the *unilateral weighted backward shift*

$$B_w(x_1, x_2, x_3, \ldots) := (w_2 x_2, w_3 x_3, w_4 x_4, \ldots)$$

is a continuous linear operator on X as soon as it maps X into itself. The *canonical vectors* of $\mathbb{K}^{\mathbb{N}}$ are the vectors e_n, $n \in \mathbb{N}$, where the n^{th} coordinate of e_n is 1 and the other coordinates of e_n are 0. The sequence $(e_n)_{n\in\mathbb{N}}$ is a *basis* of X if each e_n belongs to X and

$$x = \sum_{n=1}^{\infty} x_n e_n \quad \text{for all } x := (x_n)_{n\in\mathbb{N}} \in X.$$

We will also consider Fréchet sequence spaces consisting of bilateral sequences. A *Fréchet sequence space over* $\mathbb{Z}$ is a Fréchet space X which is a vector subspace of the product space $\mathbb{K}^{\mathbb{Z}}$ such that the inclusion map $X \to \mathbb{K}^{\mathbb{Z}}$ is continuous. As before, if $w := (w_n)_{n\in\mathbb{Z}}$ is a sequence of nonzero scalars, then the *bilateral weighted backward shift*

$$B_w((x_n)_{n\in\mathbb{Z}}) := (w_{n+1} x_{n+1})_{n\in\mathbb{Z}}$$

is a continuous linear operator on X as soon as it maps X into itself. By abuse of language, we also denote the *canonical vectors* of $\mathbb{K}^{\mathbb{Z}}$ by e_n, $n \in \mathbb{Z}$. The sequence $(e_n)_{n\in\mathbb{Z}}$ is a *basis* of X if each e_n belongs to X and

$$x = \sum_{n=-\infty}^{\infty} x_n e_n \quad \text{for all } x := (x_n)_{n\in\mathbb{Z}} \in X.$$

In the results below, we are adopting the convention that $c/0 = \infty$ whenever $c \in (0, \infty)$.

We begin by characterizing chain recurrence for bilateral (unweighted) backward shifts.

Theorem 13. *Suppose that X is a Fréchet sequence space over $\mathbb{Z}$ in which the sequence $(e_n)_{n\in\mathbb{Z}}$ of canonical vectors is a basis, $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is an increasing sequence of seminorms that induces the topology of X, and the bilateral backward shift*

$$B : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (x_{n+1})_{n\in\mathbb{Z}} \in X$$

is a well-defined operator. Then B is chain recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\|e_{-n}\|_k} = \sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} = \infty \quad \textit{for all } k \in \mathbb{N}. \tag{8}$$

Proof. Consider X endowed with its canonical metric given by (1).

Suppose that (8) holds. In view of Lemma 8, the chain recurrence of B follows from the two claims below.

Claim 1. *$e_i \in O_0(B)$ for all $i \in \mathbb{N}$.*

Indeed, fix $i \in \mathbb{N}$ and $\delta > 0$. Choose $\ell \in \mathbb{N}$ such that

$$d(x, 0) < \delta \quad \text{whenever } x \in X \text{ and } \|x\|_\ell < \delta/2. \tag{9}$$

By hypothesis,

$$\sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} = \infty \quad \text{for all } k \in \mathbb{N}. \tag{10}$$

Suppose that there exists $n_k \in \mathbb{N}$ with $\|e_{n_k}\|_k = 0$, for each $k \in \mathbb{N}$. Choose $k \geq \ell$ such that $n_k > i$. By (9),

$$0, e_{n_k}, e_{n_k - 1}, \ldots, e_i$$

is a δ-chain for B from 0 to e_i, proving that $e_i \in O_0(B)$.

Now, suppose that there exists $k_0 \in \mathbb{N}$ such that $\|e_n\|_{k_0} \neq 0$ for all $n \in \mathbb{N}$. Choose $k \in \mathbb{N}$ with $k \geq \max\{k_0, \ell\}$. By (10), there exists $m \in \mathbb{N}$ such that

$$t := \sum_{n=i+1}^{i+m} \frac{1}{\|e_n\|_k} > \frac{2}{\delta}.$$

Define

$$x_1 := \frac{e_{i+m}}{t\|e_{i+m}\|_k} \quad \text{and} \quad x_{j+1} := Bx_j + \frac{e_{i+m-j}}{t\|e_{i+m-j}\|_k} \text{ for } 1 \leq j < m.$$

Note that $x_m = e_{i+1}$. Hence, it follows from (9) that

$$0, x_1, x_2, \ldots, x_m, e_i$$

is a δ-chain for B from 0 to e_i, proving that $e_i \in O_0(B)$.

Claim 2. *$e_{-i} \in I_0(B)$ for all $i \in \mathbb{N}$.*

Indeed, let i, δ and ℓ be as in the proof of Claim 1. By hypothesis,

$$\sum_{n=1}^{\infty} \frac{1}{\|e_{-n}\|_k} = \infty \quad \text{for all } k \in \mathbb{N}. \tag{11}$$

If there exists $n_k \in \mathbb{N}$ with $\|e_{-n_k}\|_k = 0$, for each $k \in \mathbb{N}$, then we choose $k \geq \ell$ with $n_k > i$ and apply (9) to conclude that

$$e_{-i}, e_{-i-1}, \ldots, e_{-n_k+1}, 0$$

is a δ-chain for B from e_{-i} to 0, proving that $e_{-i} \in I_0(B)$.

If there exists $k_0 \in \mathbb{N}$ such that $\|e_{-n}\|_{k_0} \neq 0$ for all $n \in \mathbb{N}$, then we choose $k \in \mathbb{N}$ with $k \geq \max\{k_0, \ell\}$. By (11), there exists $m \in \mathbb{N}$ such that

$$t := \sum_{n=-i-m}^{-i-1} \frac{1}{\|e_n\|_k} > \frac{2}{\delta}.$$

Define

$$x_0 := e_{-i} \quad \text{and} \quad x_j := Bx_{j-1} - \frac{e_{-i-j}}{t\|e_{-i-j}\|_k} \text{ for } 1 \leq j \leq m.$$

Note that $x_m = 0$. Hence, it follows from (9) that

$$x_0, x_1, x_2, \ldots, x_m$$

is a δ-chain for B from e_{-i} to 0, proving that $e_{-i} \in I_0(B)$.

Conversely, suppose that B is chain recurrent. Let us first prove that (10) holds. For this purpose, fix $k \in \mathbb{N}$. We may assume that $\|e_n\|_k \neq 0$ for all $n \in \mathbb{N}$, for otherwise the desired equality would hold trivially. By the Banach-Steinhaus theorem, there exists $\delta > 0$ such that

$$x := (x_n)_{n\in\mathbb{Z}} \in X \text{ and } d(x, 0) < \delta \implies \|x_n e_n\|_k < 1 \text{ for all } n \in \mathbb{Z}. \tag{12}$$

Fix $r > 0$. By hypothesis, there is a δ-chain $(z_j)_{j=0}^m$ for B from re_1 to itself. For each $j \in \{1, \dots, m\}$, let $y_j := z_j - Bz_{j-1}$ and write $y_j = (y_{j,n})_{n\in\mathbb{Z}}$. By (12),

$$|y_{j,m-j+1}| < \frac{1}{\|e_{m-j+1}\|_k} \quad \text{for all } j \in \{1, \dots, m\}. \tag{13}$$

Since

$$z_m = B^m z_0 + B^{m-1} y_1 + B^{m-2} y_2 + \cdots + By_{m-1} + y_m, \tag{14}$$

we obtain

$$\begin{aligned} r &= y_{1,m} + y_{2,m-1} + \cdots + y_{m-1,2} + y_{m,1} \\ &\leq |y_{1,m}| + |y_{2,m-1}| + \cdots + |y_{m-1,2}| + |y_{m,1}| \\ &< \frac{1}{\|e_m\|_k} + \frac{1}{\|e_{m-1}\|_k} + \cdots + \frac{1}{\|e_2\|_k} + \frac{1}{\|e_1\|_k}, \end{aligned}$$

where in the last inequality we used (13). Since $r > 0$ is arbitrary, we are done.

Let us now establish (11). Assume $k \in \mathbb{N}$, $\|e_{-n}\|_k \neq 0$ for all $n \in \mathbb{N}$, $\delta > 0$ such that (12) holds, and $r > 0$. Let $(z_j)_{j=0}^m$ be a δ-chain for B from re_0 to itself. For each $j \in \{1, \dots, m\}$, let $y_j := z_j - Bz_{j-1}$ and write $y_j = (y_{j,n})_{n\in\mathbb{Z}}$. By (12),

$$|y_{j,-j}| < \frac{1}{\|e_{-j}\|_k} \quad \text{for all } j \in \{1, \dots, m\}. \tag{15}$$

Since (14) holds, we obtain $0 = r + y_{1,-1} + y_{2,-2} + \cdots + y_{m,-m}$. Thus, by (15),

$$r \leq |y_{1,-1}| + |y_{2,-2}| + \cdots + |y_{m,-m}| < \frac{1}{\|e_{-1}\|_k} + \frac{1}{\|e_{-2}\|_k} + \cdots + \frac{1}{\|e_{-m}\|_k}.$$

Since $r > 0$ is arbitrary, the proof is complete. □

The previous theorem can be generalized to bilateral weighted backward shifts as follows.

Theorem 14. *Suppose that X is a Fréchet sequence space over $\mathbb{Z}$ in which the sequence $(e_n)_{n\in\mathbb{Z}}$ of canonical vectors is a basis, $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is an increasing sequence of seminorms that induces the topology of X, $w := (w_n)_{n\in\mathbb{Z}}$ is a sequence of nonzero scalars, and the bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X$$

is a well-defined operator. Then B_w is chain recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{1}{|w_{-n+1}\cdots w_0|\|e_{-n}\|_k} = \sum_{n=1}^{\infty} \frac{|w_1\cdots w_n|}{\|e_n\|_k} = \infty \quad \textit{for all } k \in \mathbb{N}. \tag{16}$$

The above theorem follows from the previous one by means of a suitable conjugacy. The method can be found in [27, Section 4.1], but we will recall it here briefly for the sake of completeness. Consider the weights

$$v_0 := 1, \qquad v_{-n} := w_{-n+1}\cdots w_0 \quad \text{and} \quad v_n := \frac{1}{w_1\cdots w_n} \quad \text{for } n \geq 1,$$

the vector space

$$X_v := \{(x_n)_{n\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}} : (v_n x_n)_{n\in\mathbb{Z}} \in X\},$$

and the vector space isomorphism

$$\phi_v : (x_n)_{n\in\mathbb{Z}} \in X_v \mapsto (v_n x_n)_{n\in\mathbb{Z}} \in X.$$

Use ϕ_v to transfer the topology of X to X_v: $U \subset X_v$ is declared to be open in X_v if and only if $\phi_v(U)$ is open in X. Then X_v is a Fréchet space whose topology is induced by the sequence of seminorms given by

$$\|(x_n)_{n\in\mathbb{Z}}\|'_k := \|\phi_v((x_n)_{n\in\mathbb{Z}})\|_k \quad \text{for } (x_n)_{n\in\mathbb{Z}} \in X_v.$$

Moreover, $(e_n)_{n\in\mathbb{Z}}$ is a basis of X_v and $B_w \circ \phi_v = \phi_v \circ B$, that is, ϕ_v establishes a conjugacy between B_w and B. Hence, B_w is chain recurrent if and only if so is B. Thus, Theorem 14 follows from Theorem 13 applied to X_v endowed with the seminorms $\|\cdot\|'_k$.

Let us now consider the case of unilateral (unweighted) backward shifts.

Theorem 15. *Suppose that X is a Fréchet sequence space in which the sequence $(e_n)_{n\in\mathbb{N}}$ of canonical vectors is a basis, $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is an increasing sequence of seminorms that induces the topology of X, and the unilateral backward shift*

$$B : (x_1, x_2, x_3, \ldots) \in X \mapsto (x_2, x_3, x_4, \ldots) \in X$$

is a well-defined operator. Then B is chain recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} = \infty \quad \textit{for all } k \in \mathbb{N}. \tag{17}$$

Proof. If (17) holds, then the proof of Claim 1 in Theorem 13 shows that $e_i \in O_0(B)$ for all $i \in \mathbb{N}$. Since B is a unilateral backward shift, this implies that B is chain recurrent. The converse is proved as in the penultimate paragraph of the proof of Theorem 13. □

The above theorem can be generalized to unilateral weighted backward shifts as follows.

Theorem 16. *Suppose that X is a Fréchet sequence space in which the sequence $(e_n)_{n\in\mathbb{N}}$ of canonical vectors is a basis, $(\|\cdot\|_k)_{k\in\mathbb{N}}$ is an increasing sequence of seminorms that induces the topology of X, $w := (w_n)_{n\in\mathbb{N}}$ is a sequence of nonzero scalars, and the unilateral weighted backward shift*

$$B_w : (x_1, x_2, x_3, \ldots) \in X \mapsto (w_2x_2, w_3x_3, w_4x_4, \ldots) \in X$$

is a well-defined operator. Then B_w is chain recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{|w_1 \cdots w_n|}{\|e_n\|_k} = \infty \quad \textit{for all } k \in \mathbb{N}.$$

As in the case of bilateral shifts, Theorem 16 can be easily deduced from Theorem 15 by means of a suitable conjugacy.

The characterization of transitivity for weighted shifts on Fréchet sequence spaces was obtained in [26] (see also 4.1 in [27]). Actually, under the above notation, the unilateral weighted shift B_w is transitive on X if and only if there exists an increasing sequence $(n_k)_k$ in $\mathbb{N}$ tending to infinity such that

$$\frac{1}{|w_1 \cdots w_{n_k}|}\, e_{n_k} \to 0 \ \text{ in } X.$$

In the case of a bilateral weighted shift, transitivity is equivalent to the existence of an increasing sequence $(n_k)_k$ in $\mathbb{N}$ tending to infinity such that

$$\frac{1}{|w_{j+1} \cdots w_{j+n_k}|}\, e_{j+n_k} \to 0 \ \text{ and } \ |w_{j+1-n_k} \cdots w_j|\, e_{j-n_k} \to 0,$$

in X, for any $j \in \mathbb{Z}$.

Since transitivity obviously implies chain recurrence, the main interesting examples of chain recurrent weighted shifts are those that are not transitive. We will provide some natural examples, and to do so we need to recall the concept of Köthe sequence spaces (see [31,34]), which can be viewed as an intersection of a decreasing sequence of weighted ℓ^p-spaces, for a fixed p, or weighted c_0-spaces, when the matrix below consists of non-zero weights:

An infinite matrix $A := (a_{j,k})_{j,k\in\mathbb{N}}$ of non-negative weights is a *Köthe matrix* if, for each $j \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{j,k} > 0$, and $0 \le a_{j,k} \le a_{j,k+1}$ for all $j, k \in \mathbb{N}$. Given $1 \le p < \infty$, we consider the Fréchet sequence spaces

$$\lambda_p(A) := \Big\{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k := \left(\sum_{j=1}^{\infty} |x_j|^p a_{j,k}^p\right)^{1/p} < \infty,\ \forall\ k \in \mathbb{N}\Big\},$$

and for $p = 0$,

$$\lambda_0(A) := \Big\{x \in \mathbb{K}^{\mathbb{N}} : \lim_{j\to\infty} x_j a_{j,k} = 0, \|x\|_k := \sup_{j\in\mathbb{N}} |x_j| a_{j,k},\ \forall\ k \in \mathbb{N}\Big\},$$

which are the associated Köthe sequence spaces.

Köthe spaces are certainly a natural class of Fréchet sequence spaces. Obviously, if the entries $a_{j,k} = 1$ for all $j, k \in \mathbb{N}$, then we have $\lambda_p(A) = \ell^p$ and $\lambda_0(A) = c_0$.

For instance, the derivative operator D of many function spaces X can be represented as a weighted backward shift if the Taylor representation around 0 of functions $f \in X$ allows an isomorphism of X with a Köthe space.

In order to have that a unilateral weighted backward shift B_w is well-defined (equivalently, continuous) on a Köthe sequence space, we need to consider some conditions that relate the weight sequence w with the Köthe matrix A. It is well known (see, e.g., [32]) that B_w is continuous if and only if

$$\forall\ n \in \mathbb{N},\ \exists\ m > n\ :\ \sup_{i\in\mathbb{N}} w_{i+1} \frac{a_{i,n}}{a_{i+1,m}} < \infty. \tag{18}$$

Examples 17. (A) We consider three different Hilbert spaces of holomorphic functions on the unit disc. Namely, the *Bergman space* A^2 of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^2 := \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 d\lambda(z) < \infty,$$

the *Dirichlet space* $\mathcal{D}$ of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^2 := |f(0)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 d\lambda(z) < \infty,$$

where in both cases λ denotes the two-dimensional Lebesgue measure, and the *Hardy space* H^2 of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^2 := \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty.$$

We have that $\mathcal{D} \subset H^2 \subset A^2$. Moreover, via the identification with a sequence space by $f(z) = \sum_{n\geq 0} a_n z^n \mapsto (a_n)_n$, we know that

$$\mathcal{D} = \ell^2(v) \text{ for } v := (1,1,2,3,\dots),\ H^2 = \ell^2, \text{ and } A^2 = \ell^2(v) \text{ for } v := (1, \frac{1}{2}, \frac{1}{3}, \dots),$$

where

$$\ell^2(v) := \{a = (a_n)_n \in \mathbb{C}^{\mathbb{N}_0} : \|a\|^2 := \sum_{n=0}^{\infty} |a_n|^2 v_n < \infty\}.$$

A natural operator on these spaces is the (unweighted) backward shift that corresponds to $(Bf)(z) := (f(z)-f(0))/z$, $z \neq 0$, $(Bf)(0) := f'(0)$. The behavior concerning transitivity of B is different on these spaces, since B is transitive on the Bergman space A^2 by the above characterization, but it is not transitive on $\mathcal{D}$ or H^2 since $\|B\| = 1$ in both spaces. On the other hand, we have that

$$\sum_{n=0}^{\infty} \frac{1}{\|e_n\|} = \sum_{n=0}^{\infty} 1 = \infty \text{ on } H^2, \text{ and } \sum_{n=0}^{\infty} \frac{1}{\|e_n\|} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ on } \mathcal{D},$$

that is, B is chain recurrent on these spaces.
(B) Now we will consider non-normable Köthe spaces. For $A := (k^j)_{j,k\in\mathbb{N}}$, we have that $\lambda_p(A) = \lambda_2(A) = \mathcal{H}(\mathbb{C})$. If $A := (j^k)_{j,k\in\mathbb{N}}$, we have that $\lambda_p(A) = \lambda_2(A) =: s$, the space of rapidly decreasing sequences, and for the matrix $A := (e^{-j/k})_{j,k\in\mathbb{N}}$, we have that $\lambda_p(A) = \lambda_2(A) = \mathcal{H}(\mathbb{D})$, the space of the holomorphic functions on the unit disc. We obviously have that $\mathcal{H}(\mathbb{C}) \subset s \subset \mathcal{H}(\mathbb{D})$. The derivative operator D corresponds to the weighted shift B_w given by $w := (1,2,3,\dots)$, and D is transitive (thus, chain recurrent) on the three spaces. If, as in (A), we consider the unweighted shift B, then

$$\sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} < \infty \quad \text{for} \quad k \geq 2,$$

in $\mathcal{H}(\mathbb{C})$ or s, so B is not chain recurrent on these spaces. For the space $\mathcal{H}(\mathbb{D})$ we have

$$\sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} = \sum_{n=1}^{\infty} e^{n/k} = \infty \quad \text{for every } k \in \mathbb{N},$$

and B is chain recurrent on $\mathcal{H}(\mathbb{D})$. Actually, the transitivity condition is also fulfilled. If we set $X := \lambda_2(A)$ for $A := ((\log(j+1))^k)_{j,k\in\mathbb{N}}$, then $s \subset X \subset \mathcal{H}(\mathbb{D})$ and

$$\sum_{n=1}^{\infty} \frac{1}{\|e_n\|_k} = \sum_{n=1}^{\infty} \frac{1}{(\log(n+1))^k} = \infty \quad \text{for every } k \in \mathbb{N},$$

so B is chain recurrent on X too, but the transitivity condition is not satisfied in this case.

5. Periodic shadowing for operators on Banach spaces

Our goal in this section is to investigate the notion of periodic shadowing for continuous linear operators on Banach spaces.

Given a metric space X, recall that $f : X \to X$ has the *positive periodic shadowing property* [30,38] if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any periodic δ-pseudotrajectory $(x_n)_{n\in\mathbb{N}_0}$ of f is ε-shadowed by a periodic trajectory of f (a sequence $(y_n)_{n\in\mathbb{N}_0}$ is *periodic* if there exists $p \in \mathbb{N}$ such that $y_{n+p} = y_n$ for all $n \in \mathbb{N}_0$; such a p is called a *period* for sequence $(y_n)_{n\in\mathbb{N}_0}$). If f is bijective, then the *periodic shadowing property* is defined by replacing the set $\mathbb{N}_0$ by the set $\mathbb{Z}$ in the above definition.

Let us say that a continuous linear operator T on a Banach space X is *generalized hyperbolic* if there is a direct sum decomposition

$$X = M \oplus N,$$

where M and N are closed subspaces of X with the following properties ($r(T)$ denotes the spectral radius of T):

(a) $T(M) \subset M$ and $r(T|_M) < 1$;
(b) $T|_N : N \to T(N)$ is bijective, $T(N)$ is closed, $T(N) \supset N$ and $r((T|_N)^{-1}|_N) < 1$.

If T is invertible, then condition (b) can be rewritten as follows:

(b') $T^{-1}(N) \subset N$ and $r(T^{-1}|_N) < 1$.

In the case of invertible operators, this class appeared in [9], where it was proved that these operators have the shadowing property, enabling the construction of the first examples of operators that have the shadowing property but are not hyperbolic. The terminology "generalized hyperbolic" was introduced in [16], where many additional dynamical properties of these operators were investigated. The fact that every invertible generalized hyperbolic operator is structurally stable was established in [12]. For not necessarily invertible operators, this class appeared in [10].

It is known that generalized hyperbolic operators exhibit several types of shadowing properties (see [1,9,10,28]). We shall now prove that they also have the periodic shadowing property.

Theorem 18. *Every generalized hyperbolic operator T on a Banach space X has the positive periodic shadowing property.*

Proof. Let S denote the operator $(T|_N)^{-1}|_N$ on N. For each $x \in X$, write $x = x^{(1)} + x^{(2)}$ with $x^{(1)} \in M$ and $x^{(2)} \in N$. Let $\alpha > 0$ be such that

$$\|x^{(1)}\| \leq \alpha \, \|x\| \quad \text{and} \quad \|x^{(2)}\| \leq \alpha \, \|x\| \quad \text{for all } x \in X. \tag{19}$$

Since $r(T|_M) < 1$ and $r(S) < 1$, there exist $0 < t < 1$ and $\beta \geq 1$ such that

$$\|T^n y\| \leq \beta \, t^n \|y\| \quad \text{and} \quad \|S^n z\| \leq \beta \, t^n \|z\| \qquad (n \in \mathbb{N}_0, y \in M, z \in N). \tag{20}$$

Given $\varepsilon > 0$, put $\delta := \frac{(1-t)\varepsilon}{3\alpha\beta}$. Let $(x_n)_{n\in\mathbb{N}_0}$ be a periodic δ-pseudotrajectory of T with period p, say. For each $n \in \mathbb{N}_0$, let $y_n := x_{n+1} - Tx_n$. Note that the sequence $(y_n)_{n\in\mathbb{N}_0}$ is also periodic with period p. We claim that

$$x := x_0 + \sum_{j=1}^{\infty} S^j y_{j-1}^{(2)} - \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} T^{kp+j} y_{p-j-1}^{(1)}$$

is a periodic vector whose trajectory ε-shadows $(x_n)_{n\in\mathbb{N}_0}$. Indeed,

$$\begin{aligned}
T^p x &= T^p x_0 + \sum_{j=1}^{p} T^{p-j} y_{j-1}^{(2)} + \sum_{j=p+1}^{\infty} S^{j-p} y_{j-1}^{(2)} - \sum_{j=0}^{p-1} \sum_{k=1}^{\infty} T^{kp+j} y_{p-j-1}^{(1)} \\
&= T^p x_0 + \sum_{j=0}^{p-1} T^j y_{p-j-1}^{(2)} + \sum_{j=1}^{\infty} S^j y_{j-1}^{(2)} - \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} T^{kp+j} y_{p-j-1}^{(1)} + \sum_{j=0}^{p-1} T^j y_{p-j-1}^{(1)} \\
&= T^p x_0 + \sum_{j=0}^{p-1} T^j y_{p-j-1} + \sum_{j=1}^{\infty} S^j y_{j-1}^{(2)} - \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} T^{kp+j} y_{p-j-1}^{(1)} \\
&= x,
\end{aligned}$$

because

$$\begin{aligned}
T^p x_0 + \sum_{j=0}^{p-1} T^j y_{p-j-1} &= T^p x_0 + \sum_{j=0}^{p-1} T^j (x_{p-j} - T x_{p-j-1}) \\
&= \sum_{j=0}^{p} T^j x_{p-j} - \sum_{j=1}^{p} T^j x_{p-j} \\
&= x_0.
\end{aligned}$$

On the other hand,

$$x_n - T^n x = \sum_{j=0}^{n-1} T^j y^{(1)}_{n-j-1} - \sum_{j=1}^{\infty} S^j y^{(2)}_{n+j-1} + \sum_{j=0}^{p-1}\sum_{k=0}^{\infty} T^{kp+j+n} y^{(1)}_{p-j-1}, \tag{21}$$

for all $n \in \mathbb{N}_0$. In fact, the case $n = 0$ follows immediately from the definition of x. Assume that (21) holds for a certain $n \geq 0$. Then,

$$\begin{aligned}
x_{n+1} - T^{n+1}x &= y_n + T(x_n - T^n x) \\
&= y_n + \sum_{j=0}^{n-1} T^{j+1} y^{(1)}_{n-j-1} - \sum_{j=1}^{\infty} S^{j-1} y^{(2)}_{n+j-1} + \sum_{j=0}^{p-1}\sum_{k=0}^{\infty} T^{kp+j+n+1} y^{(1)}_{p-j-1} \\
&= y^{(1)}_n + y^{(2)}_n + \sum_{j=1}^{n} T^j y^{(1)}_{n-j} - y^{(2)}_n - \sum_{j=1}^{\infty} S^j y^{(2)}_{n+j} + \sum_{j=0}^{p-1}\sum_{k=0}^{\infty} T^{kp+j+n+1} y^{(1)}_{p-j-1} \\
&= \sum_{j=0}^{n} T^j y^{(1)}_{(n+1)-j-1} - \sum_{j=1}^{\infty} S^j y^{(2)}_{(n+1)+j-1} + \sum_{j=0}^{p-1}\sum_{k=0}^{\infty} T^{kp+j+(n+1)} y^{(1)}_{p-j-1},
\end{aligned}$$

proving that (21) also holds with $n+1$ in place of n. Now, by (19), (20) and (21),

$$\|x_n - T^n x\| < \frac{3\alpha\beta\delta}{1-t} = \varepsilon \quad \text{for all } n \in \mathbb{N}_0,$$

which completes the proof. □

Corollary 19. *Every invertible generalized hyperbolic operator T on a Banach space X has the periodic shadowing property.*

Let us recall the following basic fact (see [11, Lemma 19], for instance).

Lemma 20. *If $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence of scalars, then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdots w_{k+n}|^{\frac{1}{n}} < 1$;

(ii) $\sup_{k\in\mathbb{N}} \sum_{n=0}^{\infty} |w_k w_{k+1} \cdots w_{k+n}| < \infty$;

(iii) $\sup_{k\in\mathbb{N}} \sum_{n=0}^{k-1} |w_k w_{k-1} \cdots w_{k-n}| < \infty$.

We shall now prove that positive shadowing and positive periodic shadowing coincide for unilateral weighted backward shifts on the classical Banach sequence spaces $\ell_p(\mathbb{N})$ $(1 \leq p < \infty)$ and $c_0(\mathbb{N})$.

Theorem 21. *Let* $X := \ell_p(\mathbb{N})$ $(1 \le p < \infty)$ *or* $X := c_0(\mathbb{N})$. *Let* $w := (w_n)_{n\in\mathbb{N}}$ *be a bounded sequence of nonzero scalars and consider the unilateral weighted backward shift*

$$B_w : (x_1, x_2, x_3, \dots) \in X \mapsto (w_2x_2, w_3x_3, w_4x_4, \dots) \in X.$$

The following assertions are equivalent:

(i) B_w *has the positive shadowing property;*
(ii) B_w *has the positive periodic shadowing property;*
(iii) B_w *is generalized hyperbolic;*
(iv) *One of the following conditions holds:*
  (a) $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k w_{k+1} \cdots w_{k+n}|^{\frac{1}{n}} < 1$;
  (b) $\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_k w_{k+1} \cdots w_{k+n}|^{\frac{1}{n}} > 1$.

The equivalences (i) ⇔ (iii) ⇔ (iv) can be found in [10]. Our goal here is to include (ii) among these equivalences. For this purpose, we will adapt an argument used in the proof of [11, Theorem 18], but taking care to construct a δ-pseudotrajectory which is periodic.

Proof. By Theorem 18, (iii) ⇒ (ii). Suppose that (ii) holds and let us prove that (iv) must be true. We assume that (a) is false and prove that (b) holds. Let $\delta > 0$ be associated to $\varepsilon := 1$ in the definition of positive periodic shadowing. By Lemma 20, there are integers $k \ge 2$ and $m_0 \ge 1$ such that

$$\sum_{n=0}^{m_0} |w_k w_{k+1} \cdots w_{k+n}| \ge \frac{1+\delta}{\delta^2}. \tag{22}$$

Fix $m > m_0$ and let $\theta_j \in \mathbb{R}$ satisfy

$$e^{i\theta_j} w_k w_{k+1} \cdots w_{k+m-j} = |w_k w_{k+1} \cdots w_{k+m-j}| \quad (0 \le j \le m).$$

Define

$$\begin{aligned}
x_0 &:= \delta e^{i\theta_0} e_{k+m},\\
x_1 &:= B_w(x_0) + \delta e^{i\theta_1} e_{k+m-1},\\
x_2 &:= B_w(x_1) + \delta e^{i\theta_2} e_{k+m-2},\\
&\ \vdots\\
x_m &:= B_w(x_{m-1}) + \delta e^{i\theta_m} e_k,\\
x_{m+1} &:= B_w(x_m),
\end{aligned}$$

$$x_{m+2} := B_w(x_{m+1}),$$
$$\vdots$$
$$x_{m+k} := B_w(x_{m+k-1}) = 0.$$

Since

$$(x_n)_{n\in\mathbb{N}_0} := (x_0, \ldots, x_{m+k}, x_0, \ldots, x_{m+k}, x_0, \ldots, x_{m+k}, \ldots)$$

is a periodic δ-pseudotrajectory of T, there exists $a := (a_n)_{n\in\mathbb{N}} \in X$ such that

$$\|x_n - B_w^n(a)\| < 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{23}$$

Write $a_{k+m} = \delta e^{i\theta_0} + \gamma$ with $|\gamma| < 1$. Since the $(k-1)^{\text{th}}$ coordinate of x_{m+1} is equal to

$$\big(|w_k w_{k+1} \cdots w_{k+m}| + |w_k w_{k+1} \cdots w_{k+m-1}| + \cdots + |w_k w_{k+1}| + |w_k|\big)\delta$$

and the $(k-1)^{\text{th}}$ coordinate of $B_w^{m+1}(a)$ is $w_k w_{k+1} \cdots w_{k+m}(\delta e^{i\theta_0} + \gamma)$, (23) gives

$$\big|\big(|w_k w_{k+1} \cdots w_{k+m-1}| + \cdots + |w_k w_{k+1}| + |w_k|\big)\delta - w_k w_{k+1} \cdots w_{k+m}\gamma\big| < 1. \tag{24}$$

By (22) and (24), $|w_k w_{k+1} \cdots w_{k+m}| > 1/\delta$. Hence, by dividing both sides of (24) by $|w_k w_{k+1} \cdots w_{k+m}|\delta$, we get

$$\frac{1}{|w_{k+m}|} + \frac{1}{|w_{k+m} w_{k+m-1}|} + \cdots + \frac{1}{|w_{k+m} w_{k+m-1} \cdots w_{k+1}|} < 1 + \frac{1}{\delta}\cdot \tag{25}$$

Since this holds for every $m > m_0$, we have that $\inf_{n\in\mathbb{N}} |w_n| > 0$. Let

$$v_n := w_n^{-1} \ (n \in \mathbb{N}), \quad t := k + m \quad \text{and} \quad C := \sum_{n=0}^{k-1} |v_k v_{k-1} \cdots v_{k-n}|.$$

By (25),

$$\begin{aligned}
\sum_{n=0}^{t-1} |v_t v_{t-1} \cdots v_{t-n}| &= \sum_{n=0}^{m-1} |v_t v_{t-1} \cdots v_{t-n}| + \sum_{n=m}^{t-1} |v_t v_{t-1} \cdots v_{t-n}| \\
&< \Big(1 + \frac{1}{\delta}\Big) + |v_t v_{t-1} \cdots v_{t-m+1}| \sum_{n=0}^{k-1} |v_k v_{k-1} \cdots v_{k-n}| \\
&< \Big(1 + \frac{1}{\delta}\Big)\Big(1 + C\Big).
\end{aligned}$$

Since this holds for all $t > k + m_0$, Lemma 20 gives

$$\lim_{n\to\infty} \sup_{t\in\mathbb{N}} |v_t v_{t+1} \cdots v_{t+n}|^{\frac{1}{n}} < 1,$$

which is equivalent to (b). □

On the other hand, we will see below that the notions of shadowing and periodic shadowing do not coincide in general for invertible operators on Banach spaces. For this purpose, we will need the result below, which gives us another class of operators that have the periodic shadowing property. We denote by S_X the unit sphere of the Banach space X.

Theorem 22. *Suppose that T be an invertible continuous linear operator on a Banach space X for which there is a direct sum decomposition*

$$X = M \oplus N,$$

where M and N are closed subspaces of X with $T(M) \subset M$ and $T^{-1}(N) \subset N$ such that both $T|_M$ and $T^{-1}|_N$ are uniformly positively expansive, i.e., there are $n, m \in \mathbb{N}$ such that

$$\|T^n y\| \geq 2 \quad \textit{and} \quad \|T^{-m} z\| \geq 2 \quad \textit{for all } y \in S_M \textit{ and } z \in S_N. \tag{26}$$

For each $x \in X$ and each $k \in \mathbb{Z}$, let $x = x^{1,k} + x^{2,k}$ be the unique decomposition of x with $x^{1,k} \in T^k(M)$ and $x^{2,k} \in T^k(N)$. Suppose also that there is a constant $c > 0$ such that

$$\|x^{1,k}\| \leq c\|x\| \quad \textit{and} \quad \|x^{2,k}\| \leq c\|x\| \quad \textit{for all } x \in X \textit{ and } k \in \mathbb{Z}. \tag{27}$$

Then T has the periodic shadowing property.

Proof. Without loss of generality, we may assume $m = n$ in (26). By arguing as in the proof of Proposition 32 in the Appendix, we see that T has the periodic shadowing property if and only if so does T^n. Therefore, we may assume $n = 1$. Fix $\varepsilon > 0$ and let $\delta := \frac{\varepsilon}{12c} > 0$. Let $(x_j)_{j\in\mathbb{Z}}$ be a periodic δ-pseudotrajectory of T. We claim that

$$\|x_j^{2,0}\| < \frac{\varepsilon}{2} \quad \text{for all } j \in \mathbb{Z}. \tag{28}$$

Indeed, suppose that this is false and choose $\ell \in \mathbb{Z}$ such that

$$\|x_\ell^{2,0}\| \geq \frac{\varepsilon}{2}\cdot \tag{29}$$

We shall prove by induction that

$$\|x_{\ell-k}^{2,-k}\| \geq \frac{2^k\varepsilon}{3} + \frac{\varepsilon}{6} \quad \text{for all } k \in \mathbb{N}_0. \tag{30}$$

The case $k = 0$ is exactly (29). Assume that (30) holds for a certain $k \in \mathbb{N}_0$. Since

$$\|Tx^{2,-k-1}_{\ell-k-1} - x^{2,-k}_{\ell-k}\| = \|(Tx_{\ell-k-1})^{2,-k} - x^{2,-k}_{\ell-k}\| \leq c\,\|Tx_{\ell-k-1} - x_{\ell-k}\| \leq c\,\delta,$$

we obtain

$$\frac{2^k\varepsilon}{3} + \frac{\varepsilon}{6} - c\,\delta \leq \|x^{2,-k}_{\ell-k}\| - c\,\delta \leq \|Tx^{2,-k-1}_{\ell-k-1}\| \leq \frac{1}{2}\|x^{2,-k-1}_{\ell-k-1}\|,$$

and so

$$\|x^{2,-k-1}_{\ell-k-1}\| \geq \frac{2^{k+1}\varepsilon}{3} + \frac{\varepsilon}{3} - 2c\,\delta = \frac{2^{k+1}\varepsilon}{3} + \frac{\varepsilon}{6}\cdot$$

Hence, (30) holds with $k+1$ instead of k. By (27) and (30),

$$\|x_{\ell-k}\| \geq \frac{1}{c}\,\|x^{2,-k}_{\ell-k}\| \geq \frac{1}{c}\Big(\frac{2^k\varepsilon}{3} + \frac{\varepsilon}{6}\Big) \quad \text{for all } k \in \mathbb{N}_0.$$

Since the sequence $(x_j)_{j\in\mathbb{Z}}$ is periodic, we have a contradiction. This proves that (28) holds. A similar argument shows that

$$\|x^{1,0}_j\| < \frac{\varepsilon}{2} \quad \text{for all } j \in \mathbb{Z}. \tag{31}$$

By (28) and (31), $\|x_j\| < \varepsilon$ for all $j \in \mathbb{Z}$. Hence, the periodic δ-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ is ε-shadowed by the trajectory of the zero vector, proving that T has the periodic shadowing property. □

Let $X := \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X := c_0(\mathbb{Z})$. Let $w := (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of scalars with $\inf_{n\in\mathbb{Z}} |w_n| > 0$ and consider the bilateral weighted backward shift

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X.$$

It was proved in [11, Theorem 18] that B_w has the shadowing property if and only if one of the following conditions holds:

(A) $\lim\limits_{n\to\infty} \sup\limits_{k\in\mathbb{Z}} |w_k \cdots w_{k+n}|^{\frac{1}{n}} < 1.$

(B) $\lim\limits_{n\to\infty} \inf\limits_{k\in\mathbb{Z}} |w_k \cdots w_{k+n}|^{\frac{1}{n}} > 1.$

(C) $\lim\limits_{n\to\infty} \sup\limits_{k\in\mathbb{N}} |w_{-k-n} \cdots w_{-k}|^{\frac{1}{n}} < 1$ and $\lim\limits_{n\to\infty} \inf\limits_{k\in\mathbb{N}} |w_k \cdots w_{k+n}|^{\frac{1}{n}} > 1.$

Note that (A) and (B) are exactly the cases in which B_w is hyperbolic. In case (C), B_w is not hyperbolic but it is generalized hyperbolic. It follows immediately from the above result that:

B_w has the shadowing property if and only if it is generalized hyperbolic.

In view of Corollary 19, we conclude that:

If B_w has the shadowing property, then it has the periodic shadowing property.

We shall now see that the converse of this fact is false.

Corollary 23. *Let $X := \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X := c_0(\mathbb{Z})$. Let $w := (w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of scalars with $\inf_{n\in\mathbb{Z}} |w_n| > 0$ and consider the bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in X \mapsto (w_{n+1}x_{n+1})_{n\in\mathbb{Z}} \in X.$$

If

$$\lim_{n\to\infty} \inf_{k\in\mathbb{N}} |w_{-k-n+1} \cdots w_{-k}| = \infty \quad \textit{and} \quad \lim_{n\to\infty} \sup_{k\in\mathbb{N}} |w_k \cdots w_{k+n-1}| = 0, \tag{32}$$

then B_w has the periodic shadowing property but does not have the shadowing property.

As a concrete example, we can take a weight sequence of the form

$$w := (\ldots, a, a, a, a^{-1}, a^{-1}, a^{-1}, \ldots), \quad \text{where } a > 1.$$

Proof. By the above-mentioned result from [11], B_w does not have the shadowing property. On the other hand, let

$$M := \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0 \text{ for all } n \geq 0\}$$

and

$$N := \{(x_n)_{n\in\mathbb{Z}} \in X : x_n = 0 \text{ for all } n < 0\},$$

which are closed subspaces of X such that $X = M \oplus N$, $B_w(M) \subset M$ and $B_w^{-1}(N) \subset N$. The conditions in (32) imply that both $B_w|_M$ and $B_w^{-1}|_N$ are uniformly positively expansive. Since condition (27) holds with $c := 1$, Theorem 22 guarantees that B_w has the periodic shadowing property. □

6. Appendix: generalities on shadowing and chain recurrence for operators

Throughout this appendix, X denotes an arbitrary topological vector space over $\mathbb{K}$, unless otherwise specified. We emphasize that X is not assumed to be a Hausdorff space. Recall that a set $A \subset X$ is *balanced* if $\lambda A \subset A$ whenever $|\lambda| \leq 1$. We denote by $\mathcal{V}_X$ the set of all balanced neighborhoods of 0 in X. It is well known that every neighborhood of 0 in X contains an element of $\mathcal{V}_X$. We denote by $L(X)$ the set of all continuous linear operators on X and by $GL(X)$ be the set of those operators that have a continuous inverse.

Since X has a canonical underlying uniform structure, the notion of pseudotrajectory in the uniform space setting given in Section 3 can be rewritten as follows in the present context: Given a neighborhood V of 0 in X, a V*-pseudotrajectory* of a map $f : X \to X$ is a finite or infinite sequence $(x_j)_{i<j<k}$ in X such that

$$f(x_j) - x_{j+1} \in V \quad \text{for all } i < j < k-1.$$

Finite V-pseudotrajectories are also called V*-chains*. With these notions at hand, the concepts of positive shadowing, shadowing, finite shadowing, chain recurrence, chain transitivity and chain mixing are defined in the obvious way. We observe that if X is metrizable and we endow X with a compatible invariant metric, then these notions coincide with the corresponding ones in the metric space setting. Clearly, it is always true that

$$\text{chain mixing} \quad \Rightarrow \quad \text{chain transitivity} \quad \Rightarrow \quad \text{chain recurrence.}$$

The fact that these notions always coincide in the linear setting was observed in [1] (see also [3]):

Proposition 24. *For any linear operator $T : X \to X$ (not necessarily continuous), the following assertions are equivalent:*

(i) *T is chain recurrent;*
(ii) *T is chain transitive;*
(iii) *T is chain mixing.*

Given $T : X \to X$ and $x, y \in X$, we write $x\mathcal{R}y$ if for every $V \in \mathcal{V}_X$, there exist V-chains for T from x to y and from y to x. With this notation, the chain recurrent set of T can be written as $CR(T) = \{x \in X : x\mathcal{R}x\}$. Restricted to $CR(T)$, the relation $\mathcal{R}$ is an equivalence relation and its equivalence classes are called the *chain recurrent classes* of T.

Chain recurrence is closely connected to the notion of recurrence, a property that is deserving special attention for linear dynamics in recent years (see, e.g., [18,44,2,13,17,23]). A continuous linear operator $T : X \to X$ is said to be *recurrent* if, for every non-empty open set $U \subset X$, there exists $k \in \mathbb{N}$ such that $T^k(U) \cap U \neq \emptyset$. By a *recurrent vector* x for T we mean that, for any neighborhood U of x, there exists $k \in \mathbb{N}$ with $T^k x \in U$, and the set of recurrent vectors of T is denoted by $\mathrm{Rec}(T)$. We easily have that any recurrent operator is chain recurrent. Indeed, if $x \in X$ and $V \in \mathcal{V}_X$, there exists

$W \in \mathcal{V}_X$ open with $W \subset V \cap T^{-1}(V)$. We set $U := x + W$ and find $k \in \mathbb{N}$ and $y \in U$ such that $T^k y \in U$. Therefore, $(x, Ty, \dots, T^{k-1}y, x)$ is a V-chain for T from x to itself. Since x and V were arbitrary, we conclude that $CR(T) = X$.

Proposition 25. *For any linear operator $T : X \to X$ (not necessarily continuous), the set $CR(T)$ is a subspace of X and is the unique chain recurrent class of T.*

Proof. Take $x, y \in CR(T)$ and $V \in \mathcal{V}_X$. Choose $U \in \mathcal{V}_X$ with $U+U+U+U \subset V$. There are U-chains $(x_j)_{j=0}^r$ and $(y_j)_{j=0}^s$ for T with $x_0 = x_r = x$ and $y_0 = y_s = y$. Since the set $F := \{x_1, \dots, x_r, y_1, \dots, y_s\}$ is bounded, there is $t \in \mathbb{N}$ such that $F \subset \lambda U$ whenever $|\lambda| \geq t$. Choose an integer $k \geq t$ which is a multiple of both r and s. Let

$$(x'_j)_{j=0}^k := (x_0, x_1, \dots, x_r, x_1, \dots, x_r, \dots, x_1, \dots, x_r),$$

which is also a U-chain for T from x to itself. Similarly,

$$(y'_j)_{j=0}^k := (y_0, y_1, \dots, y_s, y_1, \dots, y_s, \dots, y_1, \dots, y_s)$$

is also a U-chain for T from y to itself. For each $0 \leq j \leq k$, let $z_j := \left(1 - \frac{j}{k}\right)x'_j + \frac{j}{k}\, y'_j$. Then $z_0 = x$, $z_k = y$ and

$$Tz_j - z_{j+1} = \left(1 - \frac{j}{k}\right)(Tx'_j - x'_{j+1}) + \frac{j}{k}(Ty'_j - y'_{j+1}) + \frac{1}{k}\, x'_{j+1} - \frac{1}{k}\, y'_{j+1} \in V,$$

for all $0 \leq j < k$. Thus, $(z_j)_{j=0}^k$ is a V-chain for T from x to y. This proves that $CR(T)$ is a chain recurrent class.

Let $x, y \in CR(T)$ and $a, b \in \mathbb{K}$. Given $V \in \mathcal{V}_X$, choose $U \in \mathcal{V}_X$ with $aU + bU \subset V$. There are U-chains $(x_j)_{j=0}^k$ and $(y_j)_{j=0}^t$ for T from x to itself and from y to itself, respectively, and we may assume $k = t$. Hence, $(ax_j + by_j)_{j=0}^k$ is a V-chain for T from $ax + by$ to itself, proving that $ax + by \in CR(T)$. This shows that $CR(T)$ is a subspace of X. □

It is worth to note that, in contrast with the above situation, one cannot ensure that the set $\mathrm{Rec}(T)$ is a subspace of X, in general. This is related to the problem of the recurrence of n-direct sum $T \oplus \cdots \oplus T$ for every $n \in \mathbb{N}$ (see [18,24]).

Proposition 26. *For any $T \in L(X)$, the set $CR(T)$ is a T-invariant closed subspace of X. Moreover, if $T \in GL(X)$, then $CR(T^{-1}) = CR(T)$ and $T(CR(T)) = CR(T)$; in particular, T^{-1} is chain recurrent if and only if so is T.*

Proof. $CR(T)$ is T-invariant: Let $x \in CR(T)$ and $V \in \mathcal{V}_X$. By the continuity of T, there exists $U \in \mathcal{V}_X$ with $T(U) \subset V$. Since $x \in CR(T)$, there is a U-chain $(x_j)_{j=0}^k$ for T from x to itself. Hence, $(Tx_j)_{j=0}^k$ is a V-chain for T from Tx to itself, proving that $Tx \in CR(T)$.

$CR(T)$ is closed: Let $x \in \overline{CR(T)}$ and $V \in \mathcal{V}_X$. Choose $U \in \mathcal{V}_X$ with $U + U \subset V$ and $W \in \mathcal{V}_X$ with $W \subset U$ and $T(W) \subset U$. Take an $y \in (x + W) \cap CR(T)$ and let $(y_j)_{j=0}^k$ be a W-chain for T from y to itself. Then $(x, y_1, \dots, y_{k-1}, x)$ is a V-chain for T from x to itself, proving that $x \in CR(T)$.

$CR(T^{-1}) = CR(T)$: Let $x \in CR(T)$ and $V \in \mathcal{V}_X$. Choose $U \in \mathcal{V}_X$ with $T^{-1}(U) \subset V$ and let $(x_j)_{j=0}^k$ be a U-chain for T from x to itself. Since

$$T^{-1}x_{j+1} - x_j = T^{-1}(-(Tx_j - x_{j+1})) \in T^{-1}(U) \subset V \quad \text{for all } 0 \le j < k,$$

we have that $(x_k, x_{k-1}, \dots, x_1, x_0)$ is a V-chain for T^{-1} from x to itself. Thus, $x \in CR(T^{-1})$.

$T(CR(T)) = CR(T)$: By what we have seen above,

$$T(CR(T)) \subset CR(T) \quad \text{and} \quad T^{-1}(CR(T)) = T^{-1}(CR(T^{-1})) \subset CR(T^{-1}) = CR(T),$$

which implies the desired equality. □

The simplest example of a chain recurrent operator is the identity operator. The next result gives classes of operators that are not chain recurrent.

Proposition 27. *If either*

(a) $T \in L(X)$, $V \in \mathcal{V}_X$ *is convex,* $V \neq X$, $\lambda \in (0, 1)$ *and* $T(V) \subset \lambda V$, *or*
(b) $T \in GL(X)$, $V \in \mathcal{V}_X$ *is convex,* $V \neq X$, $\lambda \in (1, \infty)$ *and* $T(V) \supset \lambda V$,

then

$$CR(T) \subset \bigcap_{n=1}^{\infty} \frac{1}{n} V.$$

In particular, T *is not chain recurrent. Moreover,* $CR(T) = \{0\}$ *if* $\bigcap_{n=1}^{\infty} \frac{1}{n} V = \{0\}$.

Proof. Without loss of generality, we may assume that V is closed. If (a) holds, choose $\delta \in (0, 1)$ such that $\lambda + \delta < 1$. Given $x \in X \setminus \bigcap_{n=1}^{\infty} \frac{1}{n} V$, define

$$a := \inf\{b > 0 : x \in bV\}.$$

We have that $a > 0$, $x \in aV$ and $x \notin bV$ for every $b \in (0, a)$. Let $U := a\delta V \in \mathcal{V}_X$ and let $(x_j)_{j=0}^k$ be any U-chain for T starting at $x_0 = x$. If $j \in \{0, \dots, k-1\}$ and $x_j \in aV$, then

$$x_{j+1} = Tx_j - (Tx_j - x_{j+1}) \in a\lambda V + a\delta V = a(\lambda + \delta)V \subset aV.$$

Hence, by induction, $x_j \in a(\lambda + \delta)V$ for all $j \in \{1, \dots, k\}$. In particular, $x_k \neq x$, proving that $x \notin CR(T)$. Case (b) follows from case (a) and the fact that $CR(T^{-1}) = CR(T)$. □

Corollary 28. *Suppose that X is a locally convex space whose topology is defined by a family $(q_i)_{i\in I}$ of semi-norms, where no q_i is identically zero. If either*

(a) $T \in L(X)$, $i \in I$, $\lambda \in (0,1)$ *and* $q_i(Tx) \leq \lambda\, q_i(x)$ *for all* $x \in X$, *or*
(b) $T \in GL(X)$, $i \in I$, $\lambda \in (1,\infty)$ *and* $q_i(Tx) \geq \lambda\, q_i(x)$ *for all* $x \in X$,

then

$$CR(T) \subset \{x \in X : q_i(x) = 0\}.$$

In particular, T is not chain recurrent. Moreover, $CR(T) = \{0\}$ if q_i is a norm.

Proposition 29. *If $T \in L(X)$, $\lambda \in \mathbb{K}$ and $|\lambda| = 1$, then:*

(a) $CR(\lambda T) = CR(T)$.
(b) *λT is chain recurrent if and only if so is T.*
(c) *λT has the positive shadowing property if and only if so does T.*

Proof. (a): Let $x \in CR(T)$ and $V \in \mathcal{V}_X$. We may assume that V is open in X. Let $(x_j)_{j=0}^k$ be a V-chain for T from x to itself. Given any integer $n \geq 1$, the sequence

$$(x_0, \lambda x_1, \ldots, \lambda^k x_k, \lambda^{k+1} x_1, \ldots, \lambda^{2k} x_k, \ldots, \lambda^{(n-1)k+1} x_1, \ldots, \lambda^{nk} x_k) \tag{33}$$

is a V-chain for λT from x to $\lambda^{nk} x$. If λ corresponds to a rational rotation on the unit circle, then we choose n such that $\lambda^n = 1$, and so (33) is a V-chain for λT from x to itself. In the case of an irrational rotation, we choose n such that λ^{nk} is so close to 1 that we can replace the last term in (33) by x and so obtain a V-chain for λT from x to itself. This proves that $CR(T) \subset CR(\lambda T)$. Hence, $CR(\lambda T) \subset CR(\lambda^{-1}\lambda T) = CR(T)$.
(b): It follows immediately from (a).
(c): Suppose that T has the positive shadowing property. Given $V \in \mathcal{V}_X$, let $U \in \mathcal{V}_X$ be associated to V according to positive shadowing. If $(x_j)_{j\in\mathbb{N}_0}$ is a U-pseudotrajectory of λT, then $(\lambda^{-j} x_j)_{j\in\mathbb{N}_0}$ is a U-pseudotrajectory of T, and so it is V-shadowed by the trajectory of a certain $x \in X$ under T. It follows that $(x_j)_{j\in\mathbb{N}_0}$ is V-shadowed by the trajectory of x under λT, proving that λT has the positive shadowing property. □

Proposition 30. *For any $T \in L(X)$, $CR(T^n) = CR(T)$ for all $n \in \mathbb{N}$.*

Proof. Fix $n \geq 2$, $x \in X$ and $V \in \mathcal{V}_X$. If $(x_j)_{j=0}^k$ is a V-chain for T^n from x to itself, then

$$(x_0, Tx_0, \ldots, T^{n-1}x_0, x_1, Tx_1, \ldots, T^{n-1}x_1, \ldots, x_{k-1}, Tx_{k-1}, \ldots, T^{n-1}x_{k-1}, x_k)$$

is a V-chain for T from x to itself. Conversely, if $U \in \mathcal{V}_X$ satisfies

$$U + T(U) + T^2(U) + \cdots + T^{n-1}(U) \subset V,$$

$(x_j)_{j=0}^k$ is a U-chain for T from x to itself and

$$(y_j)_{j=0}^{kn} := (x_0, x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k),$$

then $(y_0, y_n, y_{2n}, \ldots, y_{kn})$ is a V-chain for T^n from x to itself. □

Corollary 31. *For any $T \in L(X)$, the following assertions are equivalent:*

(i) *T is chain recurrent;*
(ii) *T^n is chain recurrent for some $n \in \mathbb{N}$.*
(iii) *T^n is chain recurrent for every $n \in \mathbb{N}$.*

Proposition 32. *For any $T \in L(X)$, the following assertions are equivalent:*

(i) *T has the positive shadowing property;*
(ii) *T^n has the positive shadowing property for some $n \in \mathbb{N}$.*
(iii) *T^n has the positive shadowing property for every $n \in \mathbb{N}$.*

Proof. (i) ⇒ (iii): It is enough to note that if $(x_j)_{j\in\mathbb{N}_0}$ is a U-pseudotrajectory of T^n, then

$$(x_0, Tx_0, \ldots, T^{n-1}x_0, x_1, Tx_1, \ldots, T^{n-1}x_1, x_2, Tx_2, \ldots, T^{n-1}x_2, \ldots)$$

is a U-pseudotrajectory of T.
(iii) ⇒ (ii): Obvious.
(ii) ⇒ (i): Given $V \in \mathcal{V}_X$, let $V' \in \mathcal{V}_X$ be such that

$$V' + T(V') + T^2(V') + \cdots + T^{n-1}(V') \subset V.$$

Let $U' \in \mathcal{V}_X$ be associated to V' according to the hypothesis that T^n has the positive shadowing property. We may assume that $U' \subset V'$. Let $U \in \mathcal{V}_X$ be such that

$$U + T(U) + T^2(U) + \cdots + T^{n-1}(U) \subset U'.$$

If $(x_j)_{j\in\mathbb{N}_0}$ is a U-pseudotrajectory of T, then $(x_{jn})_{j\in\mathbb{N}_0}$ is a U'-pseudotrajectory of T^n, and so it is V'-shadowed by the trajectory of a certain $x \in X$ under T^n. It follows that $(x_j)_{j\in\mathbb{N}_0}$ is V-shadowed by the trajectory of x under T. □

Proposition 33. *If $T \in GL(X)$, then T^{-1} has the (finite) shadowing property if and only if so does T.*

Proof. Suppose that T has the shadowing property. Given $V \in \mathcal{V}_X$, let $U \in \mathcal{V}_X$ be associated to V according to the definition of shadowing. Choose $W \in \mathcal{V}_X$ with $T(W) \subset U$. If $(x_j)_{j\in\mathbb{Z}}$ is a W-pseudotrajectory of T^{-1}, then $(x_{-j+1})_{j\in\mathbb{Z}}$ is a U-pseudotrajectory of T, and so it is V-shadowed by the trajectory of some $x \in X$ under T. Hence, $(x_j)_{j\in\mathbb{Z}}$ is V-shadowed by the trajectory of Tx under T^{-1}, proving that T^{-1} has the shadowing property. The case of finite shadowing is analogous. □

Recall that X is said to be the *topological direct sum* of the subspaces $M_1, \ldots, M_n$ if X is the algebraic direct sum of $M_1, \ldots, M_n$ and the canonical algebraic isomorphism

$$(y_1, \ldots, y_n) \mapsto y_1 + \cdots + y_n$$

is a homeomorphism from the product space $M_1 \times \cdots \times M_n$ onto X.

Proposition 34. *Let $T \in L(X)$. If*

$$X = M_1 \oplus \cdots \oplus M_n$$

is a topological direct sum of T-invariant subspaces $M_1, \ldots, M_n$, then:

(a) $CR(T) = CR(T|_{M_1}) \oplus \cdots \oplus CR(T|_{M_n})$.
(b) $CR(T|_{M_i}) = CR(T) \cap M_i$ *for all* $i \in \{1, \ldots, n\}$.
(c) T *is chain recurrent if and only if so are* $T|_{M_1}, \ldots, T|_{M_n}$.
(d) T *has the positive shadowing property if and only if so do* $T|_{M_1}, \ldots, T|_{M_n}$.

Proof. (a): Since $CR(T|_{M_i}) \subset CR(T)$ for all $i \in \{1, \ldots, n\}$, we have that

$$CR(T|_{M_1}) + \cdots + CR(T|_{M_n}) \subset CR(T).$$

Conversely, let $x \in CR(T)$ and write $x = y_1 + \cdots + y_n$, where $y_1 \in M_1, \ldots, y_n \in M_n$. We fix $i \in \{1, \ldots, n\}$ and prove that $y_i \in CR(T|_{M_i})$. For this purpose, let $P_i : X \to M_i$ be the canonical projection. Given $U \in \mathcal{V}_{M_i}$, we have that $V := P_i^{-1}(U) \in \mathcal{V}_X$, because P_i is continuous. Let $(x_j)_{j=0}^k$ be a V-chain for T from x to itself. Since

$$(T|_{M_i})(P_i x_j) - P_i x_{j+1} = P_i(Tx_j - x_{j+1}) \in P_i(V) \subset U \quad (0 \leq j < k),$$

we have that $(P_i x_j)_{j=0}^k$ is a U-chain for $T|_{M_i}$ from y_i to itself. Thus, $y_i \in CR(T|_{M_i})$.
(b) and (c): They follow immediately from (a).
(d): Suppose that T has the positive shadowing property. Fix $i \in \{1, \ldots, n\}$ and $U \in \mathcal{V}_{M_i}$. Define $V := P_i^{-1}(U) \in \mathcal{V}_X$ and let $V' \in \mathcal{V}_X$ be associated to V according to the definition of positive shadowing. Let $U' := V' \cap M_i \in \mathcal{V}_{M_i}$. If a sequence $(y_j)_{j\in\mathbb{N}_0}$ is a U'-pseudotrajectory of $T|_{M_i}$, then it is also a V'-pseudotrajectory of T, and so it is V-shadowed by the trajectory of a certain $x \in X$ under T. Since

$$y_j - (T|_{M_i})^j(P_i x) = P_i(y_j - T^j x) \in P_i(V) \subset U \quad (j \in \mathbb{N}_0),$$

we have that $(y_j)_{j\in\mathbb{N}_0}$ is U-shadowed by the trajectory of $P_i x$ under $T|_{M_i}$, proving that $T|_{M_i}$ has the positive shadowing property.

Conversely, suppose that each $T|_{M_i}$ has the positive shadowing property. Given $V \in \mathcal{V}_X$, choose $U_1 \in \mathcal{V}_{M_1}, \ldots, U_n \in \mathcal{V}_{M_n}$ with $U_1 + \cdots + U_n \subset V$. Let $U_i' \in \mathcal{V}_{M_i}$ be associated to U_i according to the definition of positive shadowing. Let $V' := P_1^{-1}(U_1') \cap \ldots \cap P_n^{-1}(U_n') \in \mathcal{V}_X$. Then every V'-pseudotrajectory of T is V-shadowed by a real trajectory of T, proving that T has the positive shadowing property. □

Recall that a *topological supplement* of a subspace M of X is a subspace N of X such that X is the topological direct sum of M and N.

Corollary 35. *Let $T \in L(X)$. If $CR(T)$ admits a T-invariant topological supplement, then $T|_{CR(T)}$ is chain recurrent.*

Proof. Let $M := CR(T)$. By hypothesis, there is a T-invariant subspace N of X such that $X = M \oplus N$ as a topological direct sum. Since M is also T-invariant, Proposition 34(b) gives $CR(T|_{CR(T)}) = CR(T|_M) = CR(T) \cap M = CR(T)$, and so $T|_{CR(T)}$ is chain recurrent. □

Proposition 36. *Let $T \in L(X)$ and let M be a T-invariant subspace of X. Suppose that M admits a T-invariant topological supplement.*

(a) *If T is chain recurrent, then so is $T|_M$.*
(b) *If T has the positive shadowing property, then so does $T|_M$.*

Proof. It follows immediately from Proposition 34(c,d). □

Remark 37. In Proposition 36, it is not enough to assume that M has a topological supplement, the hypothesis of T-invariance is essential. In order to give a counterexample, assume that X is a separable Banach space and that $T \in GL(X)$ is generalized hyperbolic but not hyperbolic (*shifted hyperbolic* in the terminology of [16]). Let $X = M \oplus N$ be the direct sum decomposition given by the definition of generalized hyperbolicity. By the spectral radius formula, there exist constants $t \in (0,1)$ and $c \geq 1$ such that

$$\|T^n y\| \leq c\,t^n \|y\| \text{ and } \|T^{-n} z\| \leq c\,t^n \|z\| \text{ for all } n \in \mathbb{N}_0, y \in M, z \in N.$$

Choose $\lambda \in (t,1)$ and a nonzero $w \in M \cap T(N)$, and define

$$u := \sum_{n=-\infty}^{\infty} T^n w \quad \text{and} \quad v := \sum_{n=-\infty}^{\infty} \lambda^n T^n w.$$

We have that u is a nontrivial fixed point of T and v is an eigenvector of T associated to the eigenvalue λ^{-1}. Therefore,

$$F := \text{span}\{u\}, \quad G := \text{span}\{v\} \quad \text{and} \quad H := \text{span}\{u, v\}$$

are T-invariant subspaces of X, which admit topological supplements since they are finite-dimensional. Moreover:

- $T|_F$ is chain recurrent but does not have the positive shadowing property,
- $T|_G$ has the shadowing property but is not chain recurrent,
- $T|_H$ neither is chain recurrent nor has the positive shadowing property.

Although T has the shadowing property, it may fail to be chain recurrent. However, by [16, Corollary 2], the restriction of T to the smallest closed T-invariant subspace Y of X containing $M \cap T(N)$ satisfies the so-called *frequent hypercyclicity criterion* [27, Section 9.2], and so it exhibits several types of chaotic behaviors, including *frequent hypercyclicity*, *mixing*, *Devaney chaos*, *dense distributional chaos* and *dense mean Li-Yorke chaos* [7,8,27]. In particular, $T|_Y$ is chain recurrent. Concrete examples of shifted hyperbolic operators on the Banach spaces $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ were obtained in [11, Theorem 9].

For invertible operators on Banach spaces, the next proposition shows that the closed T-invariant subspace $CR(T)$ has the property that the restricted operator $T|_{CR(T)}$ has the shadowing property whenever so does T.

Proposition 38. *Let X be a Banach space. If $T \in GL(X)$ has the shadowing property, then $T|_{CR(T)}$ is chain recurrent and has the shadowing property.*

Proof. We claim that

$$\{x \in X : (T^j x)_{j \in \mathbb{N}} \text{ is bounded}\} \subset I_0(T). \tag{34}$$

Indeed, assume $C := \sup_{j \in \mathbb{N}} \|T^j x\| < \infty$. Given $\delta > 0$, there exists a decreasing sequence $1 = t_0 > t_1 > \cdots > t_{k-1} > t_k = 0$ of real numbers such that $t_j - t_{j+1} < \delta/C$ for all $j \in \{0, \ldots, k-1\}$. Then, $(t_j T^j x)_{j=0}^k$ is a δ-chain for T from x to 0. Similarly,

$$\{x \in X : (T^{-j} x)_{j \in \mathbb{N}} \text{ is bounded}\} \subset O_0(T). \tag{35}$$

By combining (34) and (35), we obtain

$$\{x \in X : (T^j x)_{j \in \mathbb{Z}} \text{ is bounded}\} \subset CR(T). \tag{36}$$

Let us prove that $T|_{CR(T)}$ has the shadowing property. By Theorem 1, it is enough to show that $T|_{CR(T)}$ has the finite shadowing property. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by the shadowing property of T. Let $(x_j)_{j=0}^k$ be a δ-chain for $T|_{CR(T)}$. Since $x_0, x_k \in CR(T)$, there are δ-chains for T of the forms $(0, x_{-i}, \dots, x_0)$ and $(x_k, \dots, x_\ell, 0)$. Hence,

$$(x_j)_{j\in\mathbb{Z}} := (\dots, 0, 0, x_{-i}, \dots, x_0, \dots, x_k, \dots, x_\ell, 0, 0, \dots)$$

is a δ-pseudotrajectory of T. Thus, there exists $x \in X$ with $\|T^j x - x_j\| < \varepsilon$ for all $j \in \mathbb{Z}$. By (36), $x \in CR(T)$ and we are done.

Let us now prove that $T|_{CR(T)}$ is chain recurrent. Fix $x \in CR(T)$ and $\delta > 0$. Let $\eta > 0$ be associated to $\delta/(1 + \|T\|)$ according to the shadowing property of T. Since $x \in CR(T)$, there exists an η-chain $(x_j)_{j=0}^k$ for T from x to itself. As in the previous paragraph, we can extend this η-chain to a bounded η-pseudotrajectory $(x_j)_{j\in\mathbb{Z}}$ of T. By our choice of η, there exists $y \in X$ such that

$$\|T^j y - x_j\| < \frac{\delta}{1 + \|T\|} \quad \text{for all } j \in \mathbb{Z}.$$

By (36), $y \in CR(T)$. Hence, $(x, Ty, T^2 y, \dots, T^{k-1} y, x)$ is a δ-chain for $T|_{CR(T)}$ from x to itself, proving that $T|_{CR(T)}$ is chain recurrent. □

In general, properties associated with chaotic behavior are not possible for compact operators. Certainly, although chain recurrence is equivalent to chain transitivity in linear dynamics, and the second property can be thought as a chaotic behavior, the fact that, e.g., the identity of a finite dimensional space is an operator which is chain recurrent and compact spoils this impossibility for chain recurrence. The natural question is whether we can have anything else than the “finite dimensional” case for chain recurrent compact operators. We will show that, as a consequence of the previous results, there is nothing else. We recall that a linear operator $T : X \to X$ on a topological vector space X is *compact* if there is $V \in \mathcal{V}_X$ such that $T(V)$ is relatively compact.

Proposition 39. *Let $T \in L(X)$ be a compact operator on a locally convex space X with continuous norm. Then $CR(T)$ is a finite dimensional subspace.*

Proof. We first suppose that the scalar field is $\mathbb{C}$ and X is a Banach space. In this case, the spectrum of T consists, at most, of 0 and a sequence of eigenvalues of T whose limit is 0, and each eigenvalue has a finite dimensional associated eigenspace. The Riesz decomposition theorem yields that we can write $X = M_1 \oplus M_2$ with M_i closed and T-invariant subspace, $i = 1, 2$, M_1 finite dimensional and consisting of sums of eigenvectors, and M_2 so that, for $T_2 := T|_{M_2}$, we have that the spectral radius of T_2 is strictly less than 1. In particular, there is $n \in \mathbb{N}$ such that $\|T_2^n\| < 1$. By Propositions 27 and 30, we have that

$$CR(T_2) = CR(T_2^n) = \{0\}.$$

Thus, by applying Proposition 34, we get $CR(T) = CR(T_1) \oplus CR(T_2) \subset M_1$, and $CR(T)$ is finite dimensional.

Still in the complex case, but now allowing X to be an arbitrary locally convex space with continuous norm, we take $V \in \mathcal{V}_X$ absolutely convex such that $K := \overline{T(V)}$ is compact and the gauge p of V is a norm. In particular, T naturally induces an operator T_V on the local Banach space X_V (which is the completion of the normed space (X, p)), which is also compact. We easily have that $CR(T) \subset CR(T_V)$, with a natural inclusion, and $CR(T)$ is finite dimensional.

Finally, when $\mathbb{K} = \mathbb{R}$, we consider the complexification $\tilde{X} = X + iX = X \oplus X$ with the (compact) operator $\tilde{T} = T + iT = T \oplus T$. Proposition 34 yields $CR(\tilde{T}) = CR(T) + i\,CR(T)$, and $CR(T)$ is finite dimensional. □

Remark 40. When X is a complex Banach space, there is another typical property satisfied by the spectrum of T when it has certain chaotic behavior: Namely, every connected component of $\sigma(T)$ intersects $\mathbb{T}$. This is also the case for chain recurrence. Indeed, if K is a connected component of $\sigma(T)$, again Riesz decomposition theorem brings the existence of a T-invariant closed subspace M, with a complement which is also T-invariant, such that $\sigma(T|_M) = K$. Then, by Proposition 36, $T|_M$ is also chain recurrent. If K does not intersect $\sigma(T)$, either it is contained in $\mathbb{D}$, which is impossible since its spectral radius would be strictly less than 1, or it is contained in the complementary of $\overline{\mathbb{D}}$, also impossible since this would mean that $T|_M$ is invertible with a spectral radius of its inverse strictly less than 1.

Proposition 41. *Suppose that X is the product of a family $(X_i)_{i\in I}$ of topological vector spaces over $\mathbb{K}$, $T_i \in L(X_i)$ for each $i \in I$, and $T \in L(X)$ is the product operator given by*

$$T((x_i)_{i\in I}) := (T_i x_i)_{i\in I}.$$

The following properties hold:

(a) $CR(T) = \prod_{i\in I} CR(T_i)$.
(b) *T is chain recurrent if and only if so is each T_i.*
(c) *T has the positive shadowing property if and only if so does each T_i.*

Proof. For each $i \in I$, let $\pi_i : X \to X_i$ denote the canonical projection.
(a): Let $x := (x_i)_{i\in I} \in \prod_{i\in I} CR(T_i)$ and $V \in \mathcal{V}_X$. We may assume that $V = \prod_{i\in I} V_i$, where $V_i \in \mathcal{V}_{X_i}$ for each $i \in I$ and $V_i = X_i$ except for i in a finite subset J of I. For each $i \in J$, there is a V_i-chain $(x_i^{(j)})_{j=0}^{k_i}$ for T_i from x_i to itself, and we may assume all the k_i's equal to the same k. For each $j \in \{0, \ldots, k\}$, let $x_i^{(j)} := x_i$ for all $i \in I \backslash J$, and let $x^{(j)} := (x_i^{(j)})_{i\in I} \in X$. Then $(x^{(j)})_{j=0}^{k}$ is a V-chain for T from x to itself. For the converse, simply note that if $y \in CR(T)$, $i \in I$, $U_i \in \mathcal{V}_{X_i}$ and $(y^{(j)})_{j=0}^{\ell}$ is a $\pi_i^{-1}(U_i)$-chain for T from y to itself, then $(\pi_i(y^{(j)}))_{j=0}^{\ell}$ is a U_i-chain for T_i from $\pi_i(y)$ to itself.

(b): It follows immediately from (a).
(c): Suppose that each T_i has the positive shadowing property. Let $V := \prod_{i\in I} V_i$, where $V_i \in \mathcal{V}_{X_i}$ for each $i \in I$ and $V_i = X_i$ except for i in a finite subset J of I. For each $i \in J$, let $U_i \in \mathcal{V}_{X_i}$ be associated to V_i according to the definition of positive shadowing. Let $U := \bigcap_{i\in J} \pi_i^{-1}(U_i) \in \mathcal{V}_X$. Then every U-pseudotrajectory of T is V-shadowed by some real trajectory of T, proving that T has the positive shadowing property. Conversely, if T has the positive shadowing property and $i \in I$, we regard the product space X as the topological direct sum of the T-invariant "subspaces" X_i and $\prod_{\ell\neq i} X_\ell$ in a canonical way and apply Proposition 34(d) to conclude that T_i has the positive shadowing property. □

Remark 42. Consider the notations of the previous proposition. Let $Y := \bigoplus_{i\in I} X_i$ be the external direct sum of the family $(X_i)_{i\in I}$, that is, the set of all $(x_i)_{i\in I} \in X$ such that $x_i = 0$ except for a finite number of indices. If we consider Y as a subspace of the product topological vector space X and $S \in L(Y)$ is the operator obtained by restricting T to Y, then the following properties hold:

(a) $CR(S) = \bigoplus_{i\in I} CR(T_i)$.
(b) S is chain recurrent if and only if so is each T_i.
(c) S has the positive shadowing property if and only if so does each T_i.

The proof is similar to the previous one and so we leave it to the reader.

Proposition 43. *Suppose that X is the locally convex direct sum of a family $(X_i)_{i\in I}$ of locally convex spaces over $\mathbb{K}$, $T_i \in L(X_i)$ for each $i \in I$, and $T \in L(X)$ is given by*

$$T((x_i)_{i\in I}) := (T_i x_i)_{i\in I}.$$

The following properties hold:

(a) *$CR(T) = \bigoplus_{i\in I} CR(T_i)$.*
(b) *T is chain recurrent if and only if so is each T_i.*
(c) *If T has the positive shadowing property, then so does each T_i.*

Proof. (a): Let $x := (x_i)_{i\in I} \in \bigoplus_{i\in I} CR(T_i)$ and $V \in \mathcal{V}_X$. There is a finite subset J of I such that $x_i = 0$ for all $i \in I\backslash J$. We may regard X as the topological direct sum of the T-invariant "subspaces" $Y := \bigoplus_{i\in J} X_i$ and $Z := \bigoplus_{i\in I\backslash J} X_i$ in a canonical way. Let $U \in \mathcal{V}_Y$ and $W \in \mathcal{V}_Z$ be such that $U + W \subset V$. Since J is finite, Y coincides with the product space $\prod_{i\in J} X_i$. Hence, by Proposition 41(a), $(x_i)_{i\in J} \in \prod_{i\in J} CR(T_i) = CR(T|_Y)$. Thus, there is a U-chain for $T|_Y$ from $(x_i)_{i\in J}$ to itself. Each element of this U-chain can be regarded as an element of X by completing the remaining coordinates with 0's. In this way we obtain a V-chain for T from x to itself, proving that $x \in CR(T)$. For the converse, note that each X_i can be regarded as a T-invariant subspace of X that admits a T-invariant topological supplement, and so we can apply Proposition 34(a).

(b): It follows immediately from (a).
(c): It is enough to apply Proposition 36(b), since each X_i can be regarded as a T-invariant subspace of X that admits a T-invariant topological supplement. □

Remark 44. The converse of Proposition 43(c) is false in general. For instance, consider the locally convex direct sum $\mathbb{K}^{(\mathbb{N})}$, where $\mathbb{K}$ is endowed with its usual topology, and

$$T((x_n)_{n\in\mathbb{N}}) := (2x_n)_{n\in\mathbb{N}} \quad \text{for all } (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{(\mathbb{N})}.$$

We know that the operator $x \in \mathbb{K} \mapsto 2x \in \mathbb{K}$ has the shadowing property, but we will show that the operator T does not have the positive shadowing property. For this purpose, let $j_n : \mathbb{K} \to \mathbb{K}^{(\mathbb{N})}$ denote the n^{th} canonical injection and let $\overline{\Delta}(0;\delta) := \{\lambda \in \mathbb{K} : |\lambda| \leq \delta\}$ for $\delta > 0$. Consider the following neighborhood of 0 in $\mathbb{K}^{(\mathbb{N})}$:

$$V := \mathrm{co}\Big(\bigcup_{n=1}^{\infty} j_n(\overline{\Delta}(0;1))\Big),$$

where $\mathrm{co}(A)$ denotes the convex hull of the set $A \subset \mathbb{K}^{(\mathbb{N})}$. Given any neighborhood U of 0 in $\mathbb{K}^{(\mathbb{N})}$ of the form

$$U := \mathrm{co}\Big(\bigcup_{n=1}^{\infty} j_n(\overline{\Delta}(0;\delta_n))\Big),$$

with $\delta_n > 0$ for all $n \in \mathbb{N}$, define $x^{(0)} := 0$ and $x^{(j)} := Tx^{(j-1)} + \delta_j e_j$ for $j \geq 1$, where e_j is the sequence whose j^{th} coordinate is 1 and the others are 0. Then $(x^{(j)})_{j\in\mathbb{N}_0}$ is a U-pseudotrajectory of T, but it cannot be V-shadowed by a trajectory of T, because each $x \in \mathbb{K}^{(\mathbb{N})}$ has finite support.

Remark 45. Propositions 29(c), 32, 34(d), 36(b), 41(c) and 43(c), as well as Remark 42(c), remain true if we replace positive shadowing by finite shadowing. Moreover, all these results have analogous formulations with shadowing instead of positive shadowing in the case of invertible operators.

We close the paper by proposing the following open problems:

Problem A. To characterize the Fréchet spaces in which shadowing and finite shadowing coincide for operators or at least find sufficient (resp. necessary) conditions for the validity of this equivalence in the case of non-normable Fréchet spaces.

Problem B. Does Theorem 12 hold for every Fréchet space? If not, for which Fréchet spaces does the property described in Theorem 12 hold?

Problem C. To characterize the periodic shadowing property for bilateral weighted shifts on Banach sequence spaces.

Problem D. If $T \in L(X)$ is an (invertible) operator on a Banach space X, is it true that $T|_{CR(T)}$ is always chain recurrent?

Note. We were informed that Antoni López-Martínez and Dimitris Papathanasiou have recently solved Problem D in the negative.

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References

[1] F.F. Alves, N.C. Bernardes Jr., A. Messaoudi, Chain recurrence and average shadowing in dynamics, Monatshefte Math. 196 (4) (2021) 665–697.
[2] M. Amouch, O. Benchiheb, N. Karim, Recurrence of multiples of composition operators on weighted Dirichlet spaces, Adv. Oper. Theory 7 (2) (2022) 23.
[3] M.B. Antunes, G.E. Mantovani, R. Varão, Chain recurrence and positive shadowing in linear dynamics, J. Math. Anal. Appl. 506 (1) (2022) 125622.
[4] N. Aoki, K. Hiraide, Topological Theory of Dynamical Systems – Recent Advances, North-Holland, Amsterdam, 1994.
[5] F. Bayart, É. Matheron, Dynamics of Linear Operators, Cambridge University Press, Cambridge, 2009.
[6] F. Bayart, I.Z. Ruzsa, Difference sets and frequently hypercyclic weighted shifts, Ergod. Theory Dyn. Syst. 35 (3) (2015) 691–709.
[7] N.C. Bernardes Jr., A. Bonilla, V. Müller, A. Peris, Distributional chaos for linear operators, J. Funct. Anal. 265 (9) (2013) 2143–2163.
[8] N.C. Bernardes Jr., A. Bonilla, A. Peris, Mean Li-Yorke chaos in Banach spaces, J. Funct. Anal. 278 (3) (2020) 108343.
[9] N.C. Bernardes Jr., P.R. Cirilo, U.B. Darji, A. Messaoudi, E.R. Pujals, Expansivity and shadowing in linear dynamics, J. Math. Anal. Appl. 461 (1) (2018) 796–816.
[10] N.C. Bernardes Jr., A. Messaoudi, Shadowing and structural stability in linear dynamical systems, arXiv:1902.04386v1, 2019, 20 pp.
[11] N.C. Bernardes Jr., A. Messaoudi, Shadowing and structural stability for operators, Ergod. Theory Dyn. Syst. 41 (4) (2021) 961–980.
[12] N.C. Bernardes Jr., A. Messaoudi, A generalized Grobman-Hartman theorem, Proc. Am. Math. Soc. 148 (10) (2020) 4351–4360.
[13] A. Bonilla, K.-G. Grosse-Erdmann, A. López-Martínez, A. Peris, Frequently recurrent operators, J. Funct. Anal. 283 (12) (2022) 109713.
[14] N. Bourbaki, General Topology, Chapters 1-4, Springer-Verlag, Berlin, 1989.
[15] R. Bowen, ω-limit sets for axiom A diffeomorphisms, J. Differ. Equ. 18 (2) (1975) 333–339.

[16] P.R. Cirilo, B. Gollobit, E.R. Pujals, Dynamics of generalized hyperbolic linear operators, Adv. Math. 387 (2021) 107830.
[17] R. Cardeccia, S. Muro, Multiple recurrence and hypercyclicity, Math. Scand. 128 (3) (2022) 589–610.
[18] G. Costakis, A. Manoussos, I. Parissis, Recurrent linear operators, Complex Anal. Oper. Theory 8 (8) (2014) 1601–1643.
[19] G. Costakis, M. Sambarino, Topologically mixing hypercyclic operators, Proc. Am. Math. Soc. 132 (2) (2004) 385–389.
[20] C. Conley, The gradient structure of a flow, I, IBM Research, RC 3932 (#17806), July 17, 1972; reprinted in Ergod. Theory Dyn. Syst. 8 (1988) 11–26.
[21] U.B. Darji, D. Gonçalves, M. Sobottka, Shadowing, finite order shifts and ultrametric spaces, Adv. Math. 385 (2021) 107760.
[22] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, second edition, Addison-Wesley Publishing Company, Inc., Redwood City, 1989.
[23] S. Grivaux, A. López-Martínez, Recurrence properties for linear dynamical systems: an approach via invariant measures, J. Math. Pures Appl. 169 (9) (2023) 155–188.
[24] S. Grivaux, A. López-Martínez, A. Peris, Questions in linear recurrence: from the $T \oplus T$-problem to lineability, https://arxiv.org/abs/2212.03652.
[25] S. Grivaux, É. Matheron, Q. Menet, Linear dynamical systems on Hilbert spaces: typical properties and explicit examples, Mem. Am. Math. Soc. 269 (1315) (2021), v+147 pp.
[26] K.-G. Grosse-Erdmann, Hypercyclic and chaotic weighted shifts, Stud. Math. 139 (1) (2000) 47–68.
[27] K.-G. Grosse-Erdmann, A. Peris Manguillot, Linear Chaos, Springer-Verlag, London, 2011.
[28] L. Jiao, L. Wang, F. Li, Average shadowing property and asymptotic average shadowing property of linear dynamical systems, Int. J. Bifurc. Chaos 29 (12) (2019) 1950170.
[29] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995.
[30] P. Kościelniak, On genericity of shadowing and periodic shadowing property, J. Math. Anal. Appl. 310 (1) (2005) 188–196.
[31] G. Köthe, Topological Vector Spaces. I, Translated from the German by D.J.H. Garling, Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag, New York Inc., New York, 1969.
[32] F. Martínez-Giménez, A. Peris, Chaos for backward shift operators, Int. J. Bifurc. Chaos Appl. Sci. Eng. 12 (8) (2002) 1703–1715.
[33] M. Mazur, Hyperbolicity, expansivity and shadowing for the class of normal operators, Funct. Differ. Equ. 7 (1–2) (2000) 147–156.
[34] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press, Oxford University Press, New York, 1997.
[35] Q. Menet, Linear chaos and frequent hypercyclicity, Trans. Am. Math. Soc. 369 (7) (2017) 4977–4994.
[36] A. Morimoto, Some stabilities of group automorphisms, in: Manifolds and Lie Groups, in: Progr. Math., vol. 14, Birkhäuser, 1981, pp. 283–299.
[37] J. Ombach, The shadowing lemma in the linear case, Univ. Iagel. Acta Math. 31 (1994) 69–74.
[38] A.V. Osipov, S.Yu. Pilyugin, S.B. Tikhomirov, Periodic shadowing and Ω-stability, Regul. Chaotic Dyn. 15 (2–3) (2010) 404–417.
[39] K. Palmer, Shadowing in Dynamical Systems - Theory and Applications, Mathematics and Its Applications, vol. 501, Kluwer Academic Publishers, Dordrecht, 2000.
[40] S.Yu. Pilyugin, Shadowing in Dynamical Systems, Lecture Notes in Mathematics, vol. 1706, Springer-Verlag, Berlin, 1999.
[41] H.N. Salas, Hypercyclic weighted shifts, Trans. Am. Math. Soc. 347 (3) (1995) 993–1004.
[42] M. Shub, Global Stability of Dynamical Systems (with the collaboration of A. Fathi and R. Langevin), Springer-Verlag, New York, 1987.
[43] Ja.G. Sinaĭ, Gibbs measures in ergodic theory (in Russian), Usp. Mat. Nauk 27 (4(166)) (1972) 21–64.
[44] Z. Yin, Y. Wei, Recurrence and topological entropy of translation operators, J. Math. Anal. Appl. 460 (2018) 203–215.