



Banach spaces with small weakly open subsets of the unit ball and massive sets of Daugavet and Δ -points

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Abstract

We prove that there exists an equivalent norm $|||\cdot|||$ on $L_\infty[0, 1]$ with the following properties:

- (1) The unit ball of $(L_\infty[0, 1], |||\cdot|||)$ contains non-empty relatively weakly open subsets of arbitrarily small diameter;
- (2) The set of Daugavet points of the unit ball of $(L_\infty[0, 1], |||\cdot|||)$ is weakly dense;
- (3) The set of ccw Δ -points of the unit ball of $(L_\infty[0, 1], |||\cdot|||)$ is norming.

We also show that there are points of the unit ball of $(L_\infty[0, 1], |||\cdot|||)$ which are not Δ -points, meaning that the space $(L_\infty[0, 1], |||\cdot|||)$ fails the diametral local diameter 2 property. Finally, we observe that the space $(L_\infty[0, 1], |||\cdot|||)$ provides both alternative and new examples that illustrate the differences between the various diametral notions for points of the unit ball of Banach spaces.

Keywords Daugavet points · Δ -Points · Points of continuity · Renormings · Space of essentially bounded measurable functions

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1 Introduction

Recall that a Banach space X is said to have the *Daugavet property* if every rank one bounded operator $T : X \rightarrow X$ satisfies the *Daugavet equation*

$$\|I + T\| = 1 + \|T\|, \quad (\text{DE})$$

where $I : X \rightarrow X$ stands for the identity operator. Furthermore, if X has the Daugavet property, then every weakly compact operator $T : X \rightarrow X$ satisfies (DE). Since the Daugavet equation is a stress of the operator norm's triangle inequality, it is natural to expect that it will impose severe restrictions on the underlying operator. As a matter of fact, if $\|T\|$ is an

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eigenvalue of T , then T satisfies the Daugavet equation, and the converse holds true if the space X is uniformly convex [7, Lemma 11.3 and Theorem 11.10].

Actually, the Daugavet property puts very strong constraints on the structure of the underlying Banach space. An old result in this line is that a Banach space with the Daugavet property cannot be linearly embedded into any Banach space with an unconditional basis (see e.g. [24, Theorem 3.2]). Further restrictions follow from the celebrated geometric characterisation of the Daugavet property exhibited in [18, Lemma 2.1] stated as follows: a Banach space X has the Daugavet property if and only if every point $x \in S_X$ satisfies the following condition: given any slice S of B_X and any $\varepsilon > 0$, there exists $y \in S$ such that

$$\|x - y\| > 2 - \varepsilon.$$

The latter characterisation, which still holds with respect to non-empty relatively weakly open subsets (resp. convex combinations of slices) [21, Lemma 3], shows that spaces with the Daugavet property live in the universe of Banach spaces far away from Asplundness and Radon–Nikodym property. Indeed, the above characterisation allows to prove that if X has the Daugavet property, then X contains an isomorphic copy of ℓ_1 , and every slice, relatively weakly open subset and convex combination of slices of B_X has diameter two.

Very recently, local versions of the Daugavet property have been considered in the following sense.

Definition 1.1 Let X be a Banach space and let $x \in S_X$. We say that x is

- (1) a *Daugavet point* if, for every slice S of B_X and every $\varepsilon > 0$, there exists $y \in S$ such that $\|y - x\| > 2 - \varepsilon$,
- (2) a *super Daugavet point* if, for every non-empty relatively weakly open subset W of B_X and every $\varepsilon > 0$, there exists $y \in W$ such that $\|y - x\| > 2 - \varepsilon$,
- (3) a *ccs Daugavet point* if, for every convex combination of slices C of B_X and every $\varepsilon > 0$, there exists $y \in C$ such that $\|y - x\| > 2 - \varepsilon$.

A classical result, often known as Bourgain’s lemma, establishes that every non-empty relatively weakly open subset of B_X contains a convex combination of slices of B_X (see e.g. [11, Lemma II.1]). As an immediate consequence we infer that every ccs Daugavet point is a “ccw Daugavet point”, meaning that the property of the definition actually holds for every convex combination of non-empty relatively weakly open subsets of B_X . In particular, every ccs Daugavet point is a super Daugavet point. Furthermore, it is known that the mere existence of a ccs Daugavet point implies that every convex combination of slices (and of weak open subsets) of the unit ball of the underlying space has diameter 2 [20, Proposition 3.12]. Apart from finite dimensional considerations, this is surprisingly the only known isomorphic obstruction to the existence of diametral points, see below for more details.

Variants of the above notions restricting to slices, weakly open subsets, and convex combinations of slices and weakly open subsets containing a given point, were also considered.

Definition 1.2 Let X be a Banach space and let $x \in S_X$. We say that x is

- (1) a Δ -point if, for every slice S of B_X with $x \in S$ and every $\varepsilon > 0$, there exists $y \in S$ such that $\|y - x\| > 2 - \varepsilon$,
- (2) a *super Δ -point* if, for every non-empty relatively weakly open subset W of B_X with $x \in W$ and every $\varepsilon > 0$, there exists $y \in W$ such that $\|y - x\| > 2 - \varepsilon$,
- (3) a *ccs Δ -point* if, for every convex combination of slices C of B_X with $x \in C$ and every $\varepsilon > 0$, there exists $y \in C$ such that $\|y - x\| > 2 - \varepsilon$,

- (4) a *ccw* Δ -point if, for every convex combination of non-empty relatively weakly open subsets D of B_X with $x \in D$ and every $\varepsilon > 0$, there exists $y \in D$ such that $\|y-x\| > 2-\varepsilon$.

The notions of Daugavet and Δ -points were introduced in [4, Section 1], whereas the rest of notions go back to [20, Definitions 2.4 and 2.5]. See [1, 2, 16, 19, 20, 23] for further research on these notions. In particular, note that it is still unknown whether every *ccs* Δ -point has to be a super Δ -point, and whether the notions of *ccs* and *ccw* Δ -points are different. This is due to the subtle failure of a localization of Bourgain's lemma (see e.g. [20, Remark 2.3]). However, all the other notions are known to be different and can even present extreme differences, see [20] for more details.

In view of the fact that the Daugavet property imposes strong restrictions on the geometric structure of the given space, a natural question is how the mere presence of Daugavet or Δ -points affect the geometric structure of the underlying Banach space. Although it was proved in [5] that finite dimensional spaces contain no Δ -points and that the notion strongly negates some isometric properties of Banach spaces (asymptotic uniform smoothness and weak* asymptotic uniform convexity [2, 5], or existence of subsymmetric bases [6] as well as unconditional bases with small constants of unconditionality which are either shrinking or boundedly complete [2]), surprising examples have recently shown the purely isometric nature of these local notions. To name a few, there exists a space with a 1-unconditional basis and a weakly dense subset of Daugavet point [6], there exists a Lipschitz-free space with the RNP and a Daugavet point which is both isomorphic to ℓ_1 and isometric to a dual space [2, 23], and there exists an equivalent norm on ℓ_2 for which the unit vector basis e_1 is simultaneously a super Daugavet point and a *ccw* Δ -point [15]. Actually, every infinite dimensional Banach space can be renormed with a Δ -point [2], and every Banach space with a weakly-null unconditional Schauder basis can be renormed with a super Daugavet point [15].

The various Δ -notions can be seen as extreme opposites to the classical notions of denting points, points of continuity and points of strong regularity (also see [9] for precise quantitative formulations of this statement). They are localized versions of the so-called "diametral diameter 2 properties" (*DLD2P*, *DD2P* and *DSD2P*) that have previously appeared in the literature under various names, but were formally introduced in [14]. With this terminology, the *DLD2P* (resp. *DD2P*) asks all the elements of the unit sphere of a Banach space to be Δ -points (resp. super Δ -points). The *DSD2P* was originally defined by asking all the points inside of the unit ball of a Banach space to be *ccs* Δ -points, but it turned out to be equivalent to the Daugavet property [17]. On the other hand, its restricted version (the *restricted DSD2P* [20]), as well as the *DLD2P* and the *DD2P*, are known to be strictly weaker properties. Although the Daugavet property can be characterized by Daugavet, super Daugavet or *ccs* Daugavet points, it is currently unknown whether the three remaining diametral properties are equivalent. Furthermore, it is unknown whether the *DLD2P* forces all the weakly open subsets of the unit ball to have diameter 2 (but note that there exists a space with the *DD2P*, the *restricted DSD2P* and convex combinations of slices of arbitrarily small diameter in its unit ball [3]).

The example from [6] provides an interesting insight to this question. Indeed, the space that was constructed there with a weakly dense subset of Daugavet points and a 1-unconditional basis admits non-empty relatively weakly open subsets of arbitrarily small diameter in its unit ball. In fact, each of the Daugavet points in the considered weakly dense set is a point of continuity for the identity mapping $I : (B_X, w) \rightarrow (B_X, \|\cdot\|)$ (in other words, it has relative weak open neighborhoods of arbitrarily small diameter). However, this space cannot contain any point satisfying a stronger diametral condition, as it was proved in [6] (resp. [20]) that

spaces with a 1-unconditional basis contain neither super Δ -points nor ccs Δ -points. Thus, at this point, a natural question is how big the set of stronger notions than Daugavet and Δ -points can be in a Banach space where there are non-empty relatively weakly open subsets of arbitrarily small diameter.

In view of this fact, during the last week of June 2023, in the framework of the 2023 ICMAT-IMAG Doc-Course in Functional Analysis, a supervised research program was celebrated at IMAG (Granada), where we considered the following question: How massive can the sets of Daugavet, super Δ , super Daugavet and ccs/ccw Δ -points be in a Banach space having non-empty relatively weakly open subsets of arbitrary small diameter in its unit ball? The main goal of the project was to study the renorming techniques from [12], where it is proved that every Banach space containing c_0 can be renormed in such a way that all the slices of the new unit ball have diameter 2, whereas it admits weakly open subsets of arbitrarily small diameter, and to try to build a similar renorming in a more suitable context for our study, namely in the space $L_\infty[0, 1]$. The idea is also inspired by the construction from [20, Section 4.6], where similar techniques were used in order to produce an example of a super Daugavet point that is not a ccs Δ -point.

The main aim of the present paper is to present the results obtained in this workshop. We prove that the space $L_\infty[0, 1]$ admits an equivalent renorming such that the new unit ball contains non-empty relatively weakly open subsets of arbitrarily small diameter and such that the sets of Daugavet points and super Δ -points are as big as they can be taking into account that its unit ball contains non-empty weakly open subsets of small diameter. This is a big difference with the above-mentioned example of [6], where the set of super Δ -points is empty. Furthermore, we show that this space also contains points which are simultaneously super Daugavet and ccw Δ , which is the strongest diametral notion we can get in this context. We collect the results in the following theorem.

Theorem 1.3 *For every $\varepsilon \in (0, 1)$, there exists an equivalent norm $\|\cdot\|_\varepsilon$ on $L_\infty[0, 1]$ with the following properties:*

- (1) *For every $f \in L_\infty[0, 1]$, $\|f\|_\infty \leq \|f\|_\varepsilon \leq \frac{1}{1-\varepsilon} \|f\|_\infty$;*
- (2) *The unit ball of $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ contains non-empty relatively weakly open subsets of arbitrarily small diameter;*
- (3) *The set of Daugavet points of the unit ball of $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ is weakly dense;*
- (4) *The set of ccw Δ -points of the unit ball of $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ is norming (in other words, every slice of the unit ball contains a ccw Δ -point);*
- (5) *There are points of the unit ball of $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ which are:*
 - (a) *Simultaneously super Daugavet points and ccw Δ -points;*
 - (b) *Simultaneously Daugavet points and preserved extreme points (hence also ccw Δ -points), but not super Daugavet points;*
 - (c) *Simultaneously Daugavet points and points of continuity.*

Furthermore, if ε is smaller than $1/7$, then there are points of the unit ball of $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ which are not Δ -points (in other words, $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$ fails the DLD2P).

In particular, in the above renorming there are Daugavet points which are not super Δ -points and there are ccw Δ -points which are not super Daugavet points. Even though it was already known that these notions are not equivalent (see [20] for references), the various counterexamples from the literature were obtained with different techniques. Theorem 1.3 shows that such counterexamples may live in the same Banach space. Furthermore, it is, to our knowledge, the first example of a Banach space which contains points that are both Daugavet and ccw Δ , but not super Daugavet.

2 Notation and preliminary results

Given a Banach space X , B_X (resp. S_X) stands for the closed unit ball (resp. the unit sphere) of X . We denote by X^* the topological dual of X . By a slice of B_X , we mean any non-empty subset of B_X given as the intersection of B_X with an open half-space. Every slice S of B_X can be written as $S = S(B_X, f, \delta)$, where f is a norm one functional on X , $\delta > 0$ and

$$S(B_X, f, \delta) := \{x \in B_X : f(x) > 1 - \delta\}.$$

If A is a subset of a Banach space X , we denote by $\text{co } A$ (resp. $\overline{\text{co}} A$) the convex hull (resp. the closure of the convex hull) of A . Recall that a subset A in the unit ball of Banach space X is said to be *norming* if $\|x^*\| = \sup_{x \in A} |x^*(x)|$ for every $x^* \in X^*$. In particular, if A is a symmetric subset of B_X , then this property is equivalent to A satisfying $B_X = \overline{\text{co}} A$ (in other words, to every slice of B_X containing an element of A). We deal with real Banach spaces only.

Let μ be the Lebesgue measure on the segment $[0, 1]$. Recall that two measurable subsets A and B of $[0, 1]$ are said to be *essentially disjoint* if $\mu(A \cap B) = 0$. The space $L_\infty[0, 1]$ stands for the classical Banach space of all equivalent classes of μ -essentially bounded functions on $[0, 1]$ equipped with the norm given by the essential supremum. Recall that the following criteria provides a practical way of testing whether a given sequence in $L_\infty[0, 1]$ is weakly-null (see e.g. [22, Theorem 8.7]).

Theorem 2.1 *A bounded sequence (u_n) in $L_\infty[0, 1]$ converges weakly to 0 if and only we can find, for every $\delta > 0$ and (k_j) increasing sequence of natural numbers, some $J \in \mathbb{N}$ such that*

$$\mu \left(\bigcap_{j=1}^J \{t \in [0, 1] : |u_{k_j}(t)| > \delta\} \right) = 0.$$

In particular, every bounded sequence of functions with pairwise essentially disjoint supports in $L_\infty[0, 1]$ is weakly-null.

We now recall some classical definitions from Banach space geometry. Given a convex set A in a vector space X , a point $x_0 \in A$ is said to be *extreme* if the condition $x_0 = \frac{y+z}{2}$ for $y, z \in A$ forces $y = z = x_0$. Given a bounded, closed, and convex subset C of a Banach space X , a point $x_0 \in C$ is a *preserved extreme point* if x_0 is an extreme point in \overline{C}^{w^*} , where the closure is taken in the w^* -topology of X^{**} . For easy reference, let us point out the following characterisation of preserved extreme points (which proof can be found, for instance, in [10, Proposition 0.1.3]).

Proposition 2.2 *Let X be a Banach space and let $C \subseteq X$ be a bounded, closed, and convex set. Let $x_0 \in C$. The following are equivalent:*

- (1) x_0 is a preserved extreme point of C ;
- (2) The slices of C containing x_0 form a neighbourhood basis of x_0 in C for the weak topology;
- (3) For every pair of nets (y_s) and (z_s) in C such that $\frac{y_s+z_s}{2} \rightarrow x_0$ weakly, we have $y_s \rightarrow x_0$ weakly.

Given a Banach space X and a subset $A \subseteq X$, a point $x_0 \in A$ is said to be a *point of continuity* if, for every $\varepsilon > 0$, there exists a weakly open subset $W \subseteq A$ with $x_0 \in W$ and $\text{diam}(W) < \varepsilon$. Observe that this means that the identity mapping $I : (A, w) \rightarrow (A, \|\cdot\|)$

is continuous at x_0 . In turn, this is equivalent to the fact that if a net (x_s) of elements of A satisfies that $x_s \xrightarrow{w} x_0$, then $\|x_s - x_0\| \rightarrow 0$. A closed and bounded set B (resp. a closed convex and bounded set C) in a Banach space X is said to have the *point of continuity property* (resp. *convex point of continuity property (CPCP)*) if every closed subset A of B (resp. every closed and convex subset A of C) contains a point of continuity.

We finally recall the definition of the ‘‘Summing Tree Simplex’’ from [8] that was constructed in order to distinguish between the CPCP and the PCP for subsets of Banach spaces. This set will be the stepping stone for our renorming of $L_\infty[0, 1]$. Let $\mathbb{N}^{<\omega}$ be the set of all ordered finite sequences of positive integers including the empty sequence denoted by \emptyset . If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^{<\omega}$, the length of α is $|\alpha| = n$ and $|\emptyset| = 0$. For simplicity, we will sometimes identify \mathbb{N}^1 with \mathbb{N} and denote by i the sequence (i) with one element $i \in \mathbb{N}$. We use the natural order in $\mathbb{N}^{<\omega}$ given by:

$$\alpha \preceq \beta \text{ if } |\alpha| \leq |\beta| \text{ and } \alpha_i = \beta_i \text{ for all } i \in \{1, \dots, |\alpha|\},$$

and $\emptyset \preceq \alpha$ for any $\alpha \in \mathbb{N}^{<\omega}$. We denote by $\alpha \# i$ the finite sequence resulting from the concatenation of an element $\alpha \in \mathbb{N}^{<\omega}$ with the sequence $i = (i)$ with only one element $i \in \mathbb{N}$.

Let $c_0(\mathbb{N}^{<\omega})$ be the completion of the space $c_{00}(\mathbb{N}^{<\omega})$ of all finitely supported families of real numbers indexed by $\mathbb{N}^{<\omega}$ with the supremum norm. Then $c_0(\mathbb{N}^{<\omega})$ is isometric to the usual space c_0 . From now on, in order to distinguish with the norm from the space $L_\infty[0, 1]$, we will denote by $\|\cdot\|$ the supremum norm on c_0 and $c_0(\mathbb{N}^{<\omega})$, and by $\|\cdot\|_\infty$ the essential supremum norm on $L_\infty[0, 1]$. Let $(e_\alpha)_{\alpha \in \mathbb{N}^{<\omega}}$ be the unit vector basis of $c_{00}(\mathbb{N}^{<\omega})$ and $(e_\alpha^*)_{\alpha \in \mathbb{N}^{<\omega}}$ be the sequence of biorthogonal functionals. For a given $\alpha \in \mathbb{N}^{<\omega}$, let

$$x_\alpha := \sum_{\beta \preceq \alpha} e_\beta.$$

We consider the set

$$K := \overline{\text{co}}\{x_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}} \subset S_{c_0}^+.$$

Some properties of the set K are given in [8, Theorem 1.1]. In particular, it is proved there that K has the CPCP but fails the PCP. We end the present section by providing a few more properties for K .

Lemma 2.3 *For every $x \in K$ and for every slice S of K , $\sup_{y \in S} \|x - y\| = 1$.*

Proof Observe that for every $z \in \text{co}\{x_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}}$ and for every $\alpha \in \mathbb{N}^{<\omega}$, we have $\lim \|z - x_{\alpha \# n}\| = 1$. Thus, since every slice of K contains some x_α , and since $x_{\alpha \# n} \rightarrow_n x_\alpha$ weakly, the conclusion follows from an easy density argument. \square

From the fact that K has the CPCP it is immediate to infer that K contains non-empty relatively weakly open subsets of arbitrarily small diameter. However, we will describe a particular family of non-empty relatively weakly open subsets of small diameter because they will be useful in order to localise ccw Δ -points which are not super Daugavet points in the final renorming of $L_\infty[0, 1]$ (see Remark 3.8).

Lemma 2.4 *For $n \in \mathbb{N}$ and $\rho \in (0, 1/n)$, let*

$$V_{n,\rho} := \bigcap_{i=1}^n \{z \in K : e_i^*(z) > 1/n - \rho\}.$$

Then $V_{n,\rho}$ is a non-empty relatively weakly open subset of K with diameter smaller than $2/n + 2n\rho$.

Proof For $i \in \{1, \dots, n\}$, let $x_i := x_{(i)}$. Then $x_0 := \frac{1}{n} \sum_{i=1}^n x_i \in V_{n,\rho}$. Clearly, it is enough to prove that for every $z \in \text{co}\{x_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}} \cap V_{n,\rho}$, $\|x_0 - z\| \leq 1/n + n\rho$. Fix such a z , and write $z = \sum_{l=1}^L \lambda_l x_{\alpha_l}$ with $\lambda_l > 0$, $\sum_{l=1}^L \lambda_l = 1$, and $\alpha_l \in \mathbb{N}^{<\omega}$. For every $i \in \mathbb{N}$, let

$$A_i := \{l : (i) \leq \alpha_l\}.$$

Since $z \in V_{n,\rho}$, we have $e_i^*(z) = \sum_{l \in A_i} \lambda_l > 1/n - \rho$ for every $i \leq n$. So observe that for any given $j \in \{1, \dots, n\}$, we have

$$\sum_{l \in A_j} \lambda_l = \sum_{i=1}^n \sum_{l \in A_i} \lambda_l - \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{l \in A_i} \lambda_l \leq 1 - (n-1)(1/n - \rho) = 1/n + (n-1)\rho.$$

In the same way,

$$\sum_{i > n} \sum_{l \in A_i} \lambda_l \leq \sum_{i \in \mathbb{N}} \sum_{l \in A_i} \lambda_l - \sum_{i=1}^n \sum_{l \in A_i} \lambda_l \leq 1 - n(1/n - \rho) = n\rho.$$

Now let us define $v = x_0 - z = \frac{1}{n} \sum_{i=1}^n x_i - \sum_{l=1}^L \lambda_l x_{\alpha_l}$ and let us fix $\beta \in \mathbb{N}^{<\omega}$. We want to evaluate $|v(\beta)|$. There are three cases to consider.

Case 1. If $\beta = \emptyset$, then $v(\beta) = 0$, so there is nothing to do.

Case 2. If $|\beta| > 1$, take $j_\beta \in \mathbb{N}$ such that $(j_\beta) \leq \beta$. Then, either there is no $l \in \{1, \dots, L\}$ such that $(j_\beta) \leq \alpha_l$ in which case $v(\beta) = 0$, or there is an $l \in \{1, \dots, L\}$ such that $(j_\beta) \leq \alpha_l$, and $|v(\beta)| = \left| \sum_{l=1}^L \lambda_l x_{\alpha_l}(\beta) \right| \leq \sum_{l \in A_{j_\beta}} \lambda_l$. Hence $|v(\beta)| \leq \max\{1/n + (n-1)\rho, n\rho\} \leq 1/n + n\rho$.

Case 3. If $\beta = (j_\beta)$ for some $j_\beta \in \mathbb{N}$, then either $j_\beta > n$, and $|v(\beta)| \leq \sum_{l \in A_{j_\beta}} \lambda_l \leq n\rho$, or $j_\beta \leq n$, and $|v(\beta)| = \left| 1/n - \sum_{l \in A_{j_\beta}} \lambda_l \right| \leq n\rho$ because $1/n - \rho < \sum_{l \in A_{j_\beta}} \lambda_l \leq 1/n + (n-1)\rho$.

It follows that $\|x_0 - z\| = \sup_{\beta \in \mathbb{N}^{<\omega}} |v(\beta)| \leq 1/n + n\rho$, as we wanted. \square

For easy future reference, we end the section with the following lemma, whose proof follows from [8, P.82].

Lemma 2.5 Every x_α is a preserved extreme point of K .

3 Main result

The aim of the section is to prove Theorem 1.3. Let $\{A_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}}$ be a family of pairwise disjoint non-empty open subsets of $[1/2, 1]$. Then, for every $\alpha \in \mathbb{N}^{<\omega}$, let $f_\alpha := \mathbb{1}_{A_\alpha}$ and let E_α be the norm one functional on $L_\infty[0, 1]$ given by

$$E_\alpha(f) := 1/\mu(A_\alpha) \cdot \int_{A_\alpha} f d\mu$$

for every $f \in L_\infty[0, 1]$. We will use the f_α to construct a positive isometric embedding of $c_0(\mathbb{N}^{<\omega})$ into $L_\infty[0, 1]$. For every $z := (z_\alpha)_{\alpha \in \mathbb{N}^{<\omega}} \in c_{00}(\mathbb{N}^{<\omega})$, we define

$$\Phi(z) := \sum_{\alpha \in \mathbb{N}^{<\omega}} z_\alpha f_\alpha.$$

By the disjointness of the support of the functions f_α , this map is clearly isometric and sends positive sequences to positive functions. Thus it can be extended by density to a positive isometric embedding $\Phi: c_0(\mathbb{N}^{<\omega}) \rightarrow L_\infty[0, 1]$. Furthermore, observe that, by construction, Φ satisfies the equation

$$E_\alpha \circ \Phi = e_\alpha^*, \tag{3.1}$$

that is $E_\alpha(\Phi(z)) = z_\alpha$ for every $z := (z_\alpha)_{\alpha \in \mathbb{N}^{<\omega}} \in c_0(\mathbb{N}^{<\omega})$.

Let $K_0 := \Phi(K)$ be the image of the subset K of $c_0(\mathbb{N}^{<\omega})$ from the preliminary section, and fix $\varepsilon \in (0, 1)$. We consider the equivalent norm $|||\cdot|||_\varepsilon$ on $L_\infty[0, 1]$ given by the Minkowski functional of the set

$$B_\varepsilon := \overline{\text{co}}\left((2K_0 - \mathbb{1}) \cup (-2K_0 + \mathbb{1}) \cup ((1 - \varepsilon)B_{L_\infty[0,1]} + \varepsilon B_{\ker(E_0)})\right),$$

where $\mathbb{1} := \mathbb{1}_{[0,1]}$ and E_0 is the norm one functional on $L_\infty[0, 1]$ given by $E_0(f) := 4 \cdot \int_0^{1/4} f d\mu$ for every $f \in L_\infty[0, 1]$. Observe that $(1 - \varepsilon)B_{L_\infty[0,1]} \subset B_\varepsilon \subset B_{L_\infty[0,1]}$, which means

$$(1 - \varepsilon) |||\cdot|||_\varepsilon \leq \|\cdot\|_\infty \leq |||\cdot|||_\varepsilon.$$

We will prove that this renorming of $L_\infty[0, 1]$ satisfies all the properties of Theorem 1.3.

Let $A := (2K_0 - \mathbb{1})$, $B := (1 - \varepsilon)B_{L_\infty[0,1]} + \varepsilon B_{\ker(E_0)}$ and $A_\varepsilon := A \cup -A \cup B$. For every $\alpha \in \mathbb{N}^{<\omega}$, let $h_\alpha := \sum_{\beta \leq \alpha} f_\beta$ and $u_\alpha := 2h_\alpha - \mathbb{1}$. Observe that $A = \overline{\text{co}}\{u_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}}$, and that $E_0(\psi) = -1$ for every $\psi \in A$ and $E_0(\varphi) \in [-1 + \varepsilon, 1 - \varepsilon]$ for every $\varphi \in B$. We will start by proving that our renorming produces weakly open subsets of arbitrarily small diameter in the new unit ball.

Proposition 3.1 *The set B_ε admits non-empty relatively weakly open subsets of arbitrarily small diameter.*

Proof For $n \in \mathbb{N}$ and $\rho > 0$, consider

$$\tilde{V}_{n,\rho} := \left\{ f \in B_\varepsilon : E_0(f) < -1 + \rho \text{ and } E_i(f) > 2\left(\frac{1}{n} - \rho\right) - 1 \text{ for every } i \in \{1, \dots, n\} \right\},$$

where i stands the sequence (i) . Note that for $x_0 := \frac{1}{n} \sum_{i=1}^n x_i$, we have that $f_0 := 2\Phi(x_0) - \mathbb{1} \in \tilde{V}_{n,\rho}$. By density, it is enough to find an upper bound for the distance of f to f_0 for every $f \in \tilde{V}_{n,\rho} \cap \text{co } A_\varepsilon$. So pick such an f , and write

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$$

with $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$, $\sum_{i=1}^3 \lambda_i = 1$, $f_1 \in A$, $f_2 \in -A$ and $f_3 \in B$. Then, evaluating against the functional E_0 , we get

$$-1 + \rho > E_0(f) \geq -\lambda_1 + \lambda_2 - (1 - \varepsilon)\lambda_3 \geq -\lambda_1 - (1 - \varepsilon)\lambda_3,$$

hence

$$1 - \rho < \lambda_1 + (1 - \varepsilon)\lambda_3.$$

In particular,

$$1 - \rho < \lambda_1 + \lambda_3 = 1 - \lambda_2,$$

and we get $\lambda_2 < \rho$. Furthermore, since

$$\lambda_1 = 1 - \lambda_2 - \lambda_3 \leq 1 - \lambda_3,$$

we have

$$1 - \rho < 1 - \lambda_3 + (1 - \varepsilon)\lambda_3,$$

and thus $\lambda_3 < \rho/\varepsilon$. Finally

$$\lambda_1 = 1 - \lambda_2 - \lambda_3 > 1 - (1 + 1/\varepsilon)\rho.$$

It follows that

$$\|f - f_1\|_\varepsilon \leq (1 - \lambda_1) \|f_1\|_\varepsilon + \lambda_2 \|f_2\|_\varepsilon + \lambda_3 \|f_3\|_\varepsilon < 2(1 + 1/\varepsilon)\rho.$$

Write $f_1 = 2\Phi(z) - \mathbb{1}$ with $z \in K$. For every $i \in \{1, \dots, n\}$, we have

$$2(1/n - \rho) - 1 < E_i(f) = \lambda_1 E_i(f_1) + \lambda_2 E_i(f_2) + \lambda_3 E_i(f_3) < E_i(f_1) + 2(1 + 1/\varepsilon)\rho,$$

thus

$$z_i = E_i(\Phi(z)) = \frac{E_i(f_1) + 1}{2} > \frac{1}{n} - (2 + 1/\varepsilon)\rho,$$

which means $z \in V_{n, \tilde{\rho}}$ for $\tilde{\rho} = (2 + 1/\varepsilon)\rho$. So by Lemma 2.4, we get

$$\begin{aligned} \|f_1 - f_0\|_\varepsilon &= 2 \|\Phi(x_0) - \Phi(z)\|_\varepsilon \leq \frac{2}{1 - \varepsilon} \|\Phi(x_0) - \Phi(z)\|_\infty \\ &= \frac{2}{1 - \varepsilon} \|x_0 - z\| \leq \frac{4/n + 4n\tilde{\rho}}{1 - \varepsilon}. \end{aligned}$$

The conclusion follows. □

Actually, we can say a bit more in that regard: the set B_ε admits points of continuity. Indeed, the latter claim immediately follows from the fact that the set K itself admits points of continuity (it has the CPCP, see [8, Theorem 1.1(c)]) together with the following result.

Proposition 3.2 *Let z be a point of continuity of K . Then $2\Phi(z) - \mathbb{1}$ is a point of continuity of B_ε .*

Proof Let $z \in K$ be a point of continuity. To show that $f := 2\Phi(z) - \mathbb{1}$ is a point of continuity of B_ε , it is enough by density to prove that for every net (f_s) in $\text{co } A_\varepsilon$, we have that $f_s \rightarrow f$ weakly if and only if $\|f - f_s\|_\varepsilon \rightarrow 0$. So consider such a net, and for every s , write

$$f_s = \lambda_s^1 f_s^1 + \lambda_s^2 f_s^2 + \lambda_s^3 f_s^3$$

with $\lambda_s^1, \lambda_s^2, \lambda_s^3 \in [0, 1]$, $\sum_{i=1}^3 \lambda_s^i = 1$, $f_s^1 \in A$, $f_s^2 \in -A$ and $f_s^3 \in B$. If $f_s \rightarrow f$ weakly, then

$$E_0(f_s) \rightarrow E_0(f) = -1.$$

Now since $E_0(f_s^1) = -1$, $E_0(f_s^2) = 1$ and $E_0(f_s^3) \in [-1 + \varepsilon, 1 - \varepsilon]$ for every s , it immediately follows that

$$\lambda_s^1 \rightarrow 1 \quad \text{and} \quad \lambda_s^2, \lambda_s^3 \rightarrow 0.$$

From the above we conclude that $f_s^1 \rightarrow f$ weakly. For every s , pick $z_s \in K$ such that $f_s^1 = 2\Phi(z_s) - \mathbb{1}$. Then $\Phi(z_s) \rightarrow \Phi(z)$ weakly, and since Φ is a linear isometry, we get that $z_s \rightarrow z$ weakly. But z is a point of continuity of K , so it follows that $\|z - z_s\| \rightarrow 0$. Going back to $L_\infty[0, 1]$, we get that $\|\Phi(z) - \Phi(z_s)\|_\varepsilon \rightarrow 0$, and thus $\|f - f_s\|_\varepsilon \rightarrow 0$, as we wanted. □

We will now prove that the set of Daugavet points of B_ε is weakly dense. The following approximation lemma will be useful throughout the rest of the article.

Lemma 3.3 *Let $\varphi \in B$ and let $(W_i)_{i \in I}$ be a collection of pairwise essentially disjoint measurable subsets of $[1/2, 1]$ with non-zero measures. Then there exists a sequence $(\varphi_n) \subset B$ such that $\varphi_n \rightarrow \varphi$ weakly and such that the sets*

$$\{t \in W_i : \varphi_n(t) = 1\} \quad \text{and} \quad \{t \in W_i : \varphi_n(t) = -1\}$$

have positive measure for every $i \in I$ and every $n \in \mathbb{N}$.

Proof For every $i \in I$, let (B_i^n) and (C_i^n) be sequences of measurable subsets of W_i with positive measure such that any two distinct sets in these families are essentially disjoint (the existence of these subsets immediately follows from the fact that μ is σ -finite and purely non-atomic). Then set $b_i^n := \mathbb{1}_{B_i^n}$ and $c_i^n := \mathbb{1}_{C_i^n}$, and write $\varphi = (1 - \varepsilon)f + \varepsilon g$ with $f \in B_{L_\infty[0,1]}$ and $g \in B_{\ker(E_0)}$. We define

$$f_n := f \cdot \left(\mathbb{1} - \sum_{i \in I} (b_i^n + c_i^n) \right) + \sum_{i \in I} (b_i^n - c_i^n)$$

and

$$g_n := g \cdot \left(\mathbb{1} - \sum_{i \in I} (b_i^n + c_i^n) \right) + \sum_{i \in I} (b_i^n - c_i^n).$$

Note that these functions are well defined because the B_i^n and C_i^n are pairwise essentially disjoint. Furthermore, we have that $f_n = f$ and $g_n = g$ a.e. on $[0, 1] \setminus (\bigcup_{i \in I} (B_i^n \cup C_i^n))$, $\|f_n\|_\infty, \|g_n\|_\infty \leq 1$, and f_n and g_n are equal to 1 a.e. on B_i^n and to -1 a.e. on C_i^n . So $f_n \in B_{L_\infty[0,1]}$ and $g_n \in B_{\ker(E_0)}$, and calling to the weak convergence criteria (Theorem 2.1), we get that $f_n \rightarrow f$ weakly and $g_n \rightarrow g$ weakly. Thus $\varphi_n = (1 - \varepsilon)f_n + \varepsilon g_n$ does the job. □

Proposition 3.4 *Let E be the set of all functions u in B_ε of the form $u = \theta\lambda\psi + (1 - \lambda)\varphi$, where $\theta, \lambda, \psi, \varphi$ satisfy the following conditions:*

- (1) $\theta \in \{-1, 1\}$ and $\lambda \in [0, 1]$;
- (2) $\psi = 2h - 1$, where $h \in K_0$ is the image of a finitely supported element of K ;
- (3) $\varphi \in B$ and, for every $\alpha \in \mathbb{N}^{<\omega}$, the sets $\{t \in U_\alpha : \varphi(t) = 1\}$ and $\{t \in U_\alpha : \varphi(t) = -1\}$ have positive measure.

Then E is weakly dense in B_ε . Furthermore, every $u \in E$ is a Daugavet point in $(L_\infty([0, 1]), \|\cdot\|_\varepsilon)$.

Proof Let us first prove that the set E is weakly dense in B_ε . Since the sets A and B are convex, and since $\frac{A-A}{2} \subset B_{\ker(E_0)} \subset B$, it follows from [13, Lemma 2.4] that

$$\text{co}(A \cup -A \cup B) = \text{co}(A \cup B) \cup \text{co}(-A \cup B).$$

Hence, by symmetry, it is sufficient to prove that every element of $\text{co}(A \cup B)$ is the weak limit of a sequence in E . Let $u = \lambda\psi + (1 - \lambda)\varphi$ with $\lambda \in [0, 1]$, $\psi \in A$ and $\varphi \in B$. First, write $\psi = 2h - 1$ with $h \in K_0$. Since the set of all finitely supported elements of K is dense in K , and since K is mapped isometrically onto K_0 , we can find a sequence (h_n) in K_0 which converges in norm to h and such that every h_n is the image of a finitely supported element of K . We define $\psi_n := 2h_n - 1$. Next, for every $\alpha \in \mathbb{N}^{<\omega}$, pick a sequence $(W_\alpha^n)_{n \in \mathbb{N}}$ of pairwise disjoint non-empty open subsets of A_α . By Lemma 3.3, we can find a sequence (φ_n) in B which converges weakly to φ and such that the sets $\{t \in W_\alpha^n : \varphi_n(t) = 1\}$ and

$\{t \in W_\alpha^n : \varphi_n(t) = -1\}$ have positive measure for every α, n . Clearly, $u_n = \lambda\psi_n + (1 - \lambda)\varphi_n$ belongs to E and converges weakly to u , so we are done.

Now let us prove that every element of E is a Daugavet point. Since E is symmetric, it is sufficient to show that every element of the form $u = \lambda\psi + (1 - \lambda)\varphi$, where ψ and φ are as in the statement of the lemma, is a Daugavet point. So take such a function u and, for every $\alpha \in \mathbb{N}^{<\omega}$, call $P_\alpha := \{t \in A_\alpha : \varphi(t) = 1\}$ and $N_\alpha := \{t \in A_\alpha : \varphi(t) = -1\}$, which have positive measure. Note that by assumption, $\psi = -1$ a.e. on all but finitely many A_β , which means $u = -1$ a.e. on all but finitely many N_β . Let S be a slice of B_ε . Then S has non-empty intersection with $A, -A$ or B , so there are three cases to consider.

Case 1. If $A \cap S$ is non-empty, then, since $A = \overline{\text{co}}\{u_\alpha\}_{\alpha \in \mathbb{N}^{<\omega}}$, we have that S must contain one of the functions u_α . As previously observed, $u = -1$ a.e. on all but finitely many N_β , so since $u_{\alpha:n} \rightarrow u_\alpha$ weakly, we can find $n_0 \in \mathbb{N}$ such that $u_{\alpha:n_0} \in S$ and u is equal to -1 a.e. on $N_{\alpha:n_0}$. But by definition, $u_{\alpha:n_0} = 1$ a.e. on $N_{\alpha:n_0} \subseteq A_{\alpha:n_0}$, so this implies that

$$\{t \in [0, 1] : |u_{\alpha:n_0}(t) - u(t)| \geq 2\}$$

has positive measure. Hence

$$|||u - u_{\alpha:n_0}|||_\varepsilon \geq \|u - u_{\alpha:n_0}\|_\infty = 2.$$

Case 2. If $-A \cap S$ is non-empty, then S must contain one of the $-u_\alpha$. Since those take the constant value 1 on all except finitely many A_β , it suffices to compute the value of $u + u_\alpha$ on N_β , where β is such that u is equal to -1 a.e. on U_β and $-u_\alpha$ is equal 1 a.e. on $A_\beta \supset N_\beta$.

Case 3. Assume that $B \cap S$ is non-empty, and pick $\tilde{\varphi}$ in this set. By assumption, there exists $\alpha \in \mathbb{N}^{<\omega}$ such that $u = -1$ a.e. on the open set N_α . Calling to Lemma 3.3, we can then approximate $\tilde{\varphi}$ by a sequence of elements of B which are a.e. equal to 1 on some subsets of positive measure of N_α . Hence, without loss of generality, $\tilde{\varphi}$ satisfies this latter property, and it immediately follows that $|||u - \tilde{\varphi}|||_\varepsilon \geq \|u - \tilde{\varphi}\|_\infty = 2$. □

As a direct consequence, we get:

Corollary 3.5 *Every function u in the set A is a Daugavet point in $(L_\infty[0, 1], |||\cdot|||_\varepsilon)$.*

Proof It immediately follows from Proposition 3.4 that $2\Phi(z) - 1$ is a Daugavet point in $(L_\infty[0, 1], |||\cdot|||_\varepsilon)$ for every $z \in K$ with finite support. Since these elements are dense in $A = 2K_0 - 1$, and since the set of all Daugavet points is closed, we immediately get the desired conclusion. □

Remark 3.6 Since B_ε admits non-empty relatively weakly open subsets of arbitrarily small diameter, it follows from Proposition 3.4 that some of the Daugavet points from the set E are not super Δ -points. Furthermore, it follows from Proposition 3.2 and Corollary 3.5 that B_ε contains points which are simultaneously Daugavet points and points of continuity, so we get new examples in the line of [6].

In particular, we get that the u_α are Daugavet points in $(L_\infty[0, 1], |||\cdot|||_\varepsilon)$. But we can say a bit more about these functions.

Proposition 3.7 *Every u_α is a preserved extreme point of B_ε . In particular, every u_α is simultaneously a Daugavet point and a ccw Δ -point (in particular a super Δ -point) in $(L_\infty[0, 1], |||\cdot|||_\varepsilon)$.*

Proof In order to prove that u_α is preserved extreme, it is sufficient to prove in virtue of Proposition 2.2 that for any nets (u_s) and (v_s) in $\text{co}(A \cup -A \cup B)$ such that $\frac{u_s+v_s}{2} \rightarrow u_\alpha$ weakly, we have that $u_s \rightarrow u_\alpha$ weakly. So pick such nets, and for every s , let us write

$$u_s = \lambda_s^1 f_s^1 - \lambda_s^2 f_s^2 + \lambda_s^3 f_s^3 \quad \text{and} \quad v_s = \mu_s^1 g_s^1 - \mu_s^2 g_s^2 + \mu_s^3 g_s^3$$

with $\lambda_s^i, \mu_s^i \in [0, 1]$, $\sum_{i=1}^3 \lambda_s^i = \sum_{i=1}^3 \mu_s^i = 1$, $f_s^1, f_s^2, g_s^1, g_s^2 \in A$ and $f_s^3, g_s^3 \in B$. Testing against E_0 , we get:

$$E_0\left(\frac{u_s + v_s}{2}\right) = -\frac{\lambda_s^1 + \mu_s^1}{2} + \frac{\lambda_s^2 + \mu_s^2}{2} + \frac{\lambda_s^3}{2} E_0(f_s^3) + \frac{\mu_s^3}{2} E_0(g_s^3) \rightarrow E_0(u_\alpha) = -1.$$

But since $E_0(f_s^3), E_0(g_s^3) > -1 + \varepsilon$, this implies that $\frac{\lambda_s^1 + \mu_s^1}{2} \rightarrow 1$ and $\frac{\lambda_s^2 + \mu_s^2}{2}, \frac{\lambda_s^3}{2}, \frac{\mu_s^3}{2} \rightarrow 0$. Hence, $\lambda_s^1, \mu_s^1 \rightarrow 1, \lambda_s^2, \mu_s^2, \lambda_s^3, \mu_s^3 \rightarrow 0$ and $\frac{f_s^1 + g_s^1}{2} \rightarrow u_\alpha = 2\Phi(x_\alpha) - \mathbb{1}$ weakly. Write $f_s^1 = 2\Phi(y_s) - \mathbb{1}$ and $g_s^1 = 2\Phi(z_s) - \mathbb{1}$ with $y_s, z_s \in K$. Since Φ is isometric, we get, by the weak-to-weak continuity of bounded operators, that $\frac{y_s+z_s}{2} \rightarrow x_\alpha$ weakly in K . Since x_α is preserved extreme in K , it follows that $y_s \rightarrow x_\alpha$ weakly in K , and going back through Φ , we conclude that $f_s^1 \rightarrow u_\alpha$ weakly. Hence $u_s \rightarrow u_\alpha$ weakly, and we are done. The final part follows since it was observed in [20, Proposition 3.13] that every preserved extreme Δ -point is a ccw Δ -point. \square

Remark 3.8 None of the u_α is super Daugavet.

Proof For every $n \in \mathbb{N}$, let $u_\alpha^n := 2(h_\alpha + \frac{1}{n} \sum_{i=1}^n f_{\alpha ii}) - \mathbb{1}$. We have

$$\| |u_\alpha - u_\alpha^n| |_\varepsilon \leq \frac{1}{1 - \varepsilon} \|u_\alpha - u_\alpha^n\|_\infty = \frac{1}{1 - \varepsilon} \left\| \Phi\left(\frac{1}{n} \sum_{i=1}^n x_{\alpha ii}\right) \right\|_\infty = \frac{1}{n(1 - \varepsilon)}.$$

Consider

$$\tilde{W}_{n,\rho} := \left\{ f \in B_\varepsilon : E_0(f) < -1 + \rho, \text{ and } E_{\alpha ii}(f) > 2\left(\frac{1}{n} - \rho\right) - 1 \text{ for every } i \in \{1, \dots, n\} \right\}.$$

Then we can show as in the proof of Proposition 3.1 that the diameter of $\tilde{W}_{n,1/n^2}$ goes to 0 as n goes to infinity. Since $u_\alpha^n \in \tilde{W}_{n,1/n^2}$ and since the distance from u_α to u_α^n goes to 0, we get that u_α is not a super Daugavet point. \square

We will now show that B_ε contains points satisfying stronger diametral notions. We start by the following easy observation.

Lemma 3.9 Let $\varphi \in B_\varepsilon$ be a function such that the sets

$$P_\alpha := \{t \in U_\alpha : \varphi(t) = 1\} \text{ and } N_\alpha := \{t \in U_\alpha : \varphi(t) = -1\}$$

have positive measure for every $\alpha \in \mathbb{N}^{<\omega}$. Then φ belongs to B .

Proof Fix $\delta \in (0, 1)$. We can find $h_1, h_2 \in K_0, \psi \in B$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\|\varphi - (\lambda_1(2h_1 - \mathbb{1}) + \lambda_2(-2h_2 + \mathbb{1}) + \lambda_3\psi)\|_\infty < \delta$. Also, we may assume without loss of generality that h_1 and h_2 are the images of finitely supported elements of K . So choose $\alpha \in \mathbb{N}^{<\omega}$ belonging in neither of these supports. Then $2h_1 - \mathbb{1} = -1$ and $-2h_2 + \mathbb{1} = 1$ a.e. on U_α . Taking into account that

$$\text{ess sup}_{t \in P_\alpha} (\lambda_1(2h_1 - \mathbb{1}) + \lambda_2(-2h_2 + \mathbb{1}) + \lambda_3\psi)(t) > 1 - \delta$$

we infer

$$1 - \delta < -\lambda_1 + \lambda_2 + \lambda_3 \operatorname{ess\,sup}_{t \in P_\alpha} \psi(t) < -\lambda_1 + \lambda_2 + \lambda_3 = 1 - 2\lambda_1,$$

which implies $\lambda_1 < \delta/2$. Similarly we get

$$-1 + \delta > -\lambda_1 + \lambda_2 + \lambda_3 \operatorname{ess\,sup}_{t \in N_\alpha} \psi(t) > -\lambda_1 + \lambda_2 - \lambda_3 = -1 + 2\lambda_2,$$

which implies $\lambda_2 < \delta/2$. Hence $\lambda_3 > 1 - \delta$, and it follows that

$$\begin{aligned} \|\varphi - \psi\|_\infty &\leq \lambda_1 \|2h_1 - \mathbb{1}\|_\infty + \lambda_2 \|2h_2 - \mathbb{1}\|_\infty + (1 - \lambda_3) \|\psi\|_\infty \\ &\quad + \|\varphi - (\lambda_1(2h_1 - \mathbb{1}) + \lambda_2(-2h_2 + \mathbb{1}) + \lambda_3\psi)\|_\infty < 3\delta. \end{aligned}$$

Since B is closed, the conclusion follows. □

Proposition 3.10 *Let $\varphi \in B_\varepsilon$ be a function such that the sets*

$$P_\alpha := \{t \in U_\alpha : \varphi(t) = 1\} \quad \text{and} \quad N_\alpha := \{t \in U_\alpha : \varphi(t) = -1\}$$

have positive measure for every $\alpha \in \mathbb{N}^{<\omega}$. Then φ is simultaneously a super Daugavet point and a ccw Δ -point in $(L_\infty[0, 1], \|\cdot\|_\varepsilon)$.

Proof We first prove that φ is a super Daugavet point. Let W be a non-empty relatively weakly open subset of B_ε . By Proposition 3.4 there exists a function $u \in E$ which belongs to the weakly open set W . Write $u = \theta\lambda u_1 + (1 - \lambda)u_2$ with $\theta \in \{-1, 1\}$, $\lambda \in [0, 1]$, $u_1 \in A$ and $u_2 \in B$. We will assume that $\theta = 1$, because the other case can be shown by the analogous method. Up to approximating u_2 as in Lemma 3.3, we may assume without loss of generality that u_2 is equal to -1 a.e. on a subset of non-zero measure of P_α for every $\alpha \in \mathbb{N}^{<\omega}$. Hence u is equal to -1 a.e. on subsets of positive measure of all but finitely many P_α , and it follows that $\|\varphi - u\|_\varepsilon \geq \|\varphi - u\|_\infty = 2$, as we wanted.

Next, let us prove that φ is a ccw Δ -point. Assume that $\varphi \in \sum_{i=1}^n \lambda_i W_i$, where the W_i are non-empty relatively weakly open subset of B_ε , $\lambda_i > 0$, and $\sum_{i=1}^n \lambda_i = 1$. Then write $\varphi = \sum_{i=1}^n \lambda_i \varphi_i$ with $\varphi_i \in W_i$. Since $\|\varphi_i\|_\infty \leq 1$ for every i , it follows that

$$\{t \in [0, 1] : \varphi(t) = 1\} = \bigcap_{i=1}^n \{t \in [0, 1] : \varphi_i(t) = 1\}$$

and

$$\{t \in [0, 1] : \varphi(t) = -1\} = \bigcap_{i=1}^n \{t \in [0, 1] : \varphi_i(t) = -1\}.$$

Hence, every φ_i is equal to 1 a.e. on the P_α and to -1 a.e. on the N_α . In particular, every φ_i belongs to B by Lemma 3.9.

By Lemma 3.3, we can find for every i a sequence of functions (φ_i^k) in B_ε which converges weakly to φ_i and such that $\varphi_i^k = -1$ a.e. on some subsets $A_\alpha \subseteq P_\alpha$ of positive measure for every $\alpha \in \mathbb{N}^{<\omega}$ and every i, k . Then we get that for large enough k 's, each φ_i^k belongs to the corresponding W_i . It follows that $\sum_{i=1}^n \lambda_i \varphi_i^k \in \sum_{i=1}^n \lambda_i W_i$. Finally,

$$\left\| \left\| \varphi - \sum_{i=1}^n \lambda_i \varphi_i^k \right\|_\varepsilon \right\|_\infty \geq \left\| \varphi - \sum_{i=1}^n \lambda_i \varphi_i^k \right\|_\infty \geq \operatorname{ess\,sup}_{t \in A_\alpha} \left| \varphi(t) - \sum_{i=1}^n \lambda_i \varphi_i^k(t) \right| = 2,$$

so the conclusion follows. □

Corollary 3.11 *The set of all points of B_ε which are simultaneously Daugavet points and ccw Δ -points is norming.*

Proof Every slice S of B_ε intersects either A , $-A$ or B . But since every slice of A must contain one of the u_α , and since functions taking value 1 and -1 on some subsets of positive measure of every U_α are weakly dense in B , the result immediately follows from Proposition 3.7 and 3.10. \square

Finally, let us prove that, for small enough ε 's, there are points in B_ε which are not Δ -points.

Proposition 3.12 *If $\varepsilon < \frac{1}{7}$, then the space $(L_\infty([0, 1]), \|\cdot\|_\varepsilon)$ fails the DLD2P.*

Proof Consider the function $f := (1 - \varepsilon) \left(\mathbb{1}_{[0, \frac{1}{4}]} - \mathbb{1}_{[\frac{1}{4}, \frac{1}{2}]} \right)$, which belongs to $(1 - \varepsilon)B_{L_\infty([0,1])}$, and thus to B_ε . We will show that $\tilde{f} = \frac{f}{\|f\|_\varepsilon}$ is not a Δ -point. Consider the functional F on $L_\infty[0, 1]$ given by

$$F(\varphi) = 4 \cdot \left(\int_0^{\frac{1}{4}} \varphi(t)dt - \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(t)dt \right)$$

for every $\varphi \in L_\infty[0, 1]$. Since $F(\varphi) = 0$ for every $\varphi \in A$, we obtain the following estimate on the

norm of F :

$$\|F\|_\varepsilon = \sup\{F(\varphi) : \varphi \in A \cup (-A) \cup B\} \leq 2 - \varepsilon.$$

This in turn allows us to provide a lower bound on the norm of f :

$$\|f\|_\varepsilon \geq \frac{F(f)}{2 - \varepsilon} = \frac{2(1 - \varepsilon)}{2 - \varepsilon}.$$

We now define the slice of B_ε which will witness that \tilde{f} is not a Δ -point. Let $\eta > 0$ be such that $3\eta < 1 - 4\varepsilon$ and $\frac{5}{2}\varepsilon + \eta < \frac{1}{2}$, and consider

$$S := \{\varphi \in B_\varepsilon : F(\varphi) > 2(1 - \varepsilon) - \eta\}.$$

The set S is non-empty, since f , and consequently \tilde{f} , belong to S , so it does indeed define a slice of B_ε . To finish the proof, it suffices to uniformly bound away from 2 the distance of any function in S to \tilde{f} . In fact, since B_ε is the closed convex hull of $A \cup -A \cup B$, and since no function in A or $-A$ belongs to S , it is enough to uniformly bound away from 2 the distance of any function in $S \cap B$ to \tilde{f} (calling to [16, Lemma 2.1]). So fix $\varphi \in B$ such that $F(\varphi) > 2(1 - \varepsilon) - \eta$. Since $\|\varphi\|_\infty \leq 1$, we obtain that

$$\begin{aligned} 2(1 - \varepsilon) - \eta < F(\varphi) &= E_0(\varphi) - 4 \cdot \int_{1/4}^{1/2} \varphi(t)dt \\ &\leq E_0(\varphi) + 1. \end{aligned}$$

With the previous equation and with the fact that $E_0(\varphi) \leq (1 - \varepsilon)$, we get that $E_0(\varphi)$ belongs to $[1 - 2\varepsilon - \eta, 1 - \varepsilon]$. On the other hand, we have that

$$1 - \varepsilon \leq E_0(\tilde{f}) = \frac{1 - \varepsilon}{\|f\|_\varepsilon} \leq \frac{2 - \varepsilon}{2}.$$

Defining $\tau = E_0(\tilde{f} - \varphi)$, we obtain, combining both estimates, that:

$$0 \leq \tau \leq \frac{2 - \varepsilon}{2} - (1 - 2\varepsilon - \eta) = \frac{3}{2}\varepsilon + \eta. \tag{3.2}$$

Now, consider the function $h = \tilde{f} - \varphi - \tau \cdot \left(\mathbb{1}_{[0, \frac{1}{4}]} - \mathbb{1}_{[\frac{1}{4}, \frac{1}{2}]} \right)$, which clearly satisfies $E_0(h) = 0$. Since $\|\varphi\|_\infty \leq 1$ and since $1 - \varepsilon \leq \tilde{f} \leq 1 - \frac{\varepsilon}{2}$ almost everywhere on $[0, 1/4]$, we have that $-\varepsilon \leq \tilde{f} - \varphi \leq 2 - \frac{\varepsilon}{2}$ almost everywhere on $[0, \frac{1}{4}]$. Similarly, $-2 + \frac{\varepsilon}{2} \leq \tilde{f} - \varphi \leq \varepsilon$ almost everywhere on $[\frac{1}{4}, \frac{1}{2}]$. By the choice of η and equation (3.2), we have that $\varepsilon + \tau \leq \frac{5}{2}\varepsilon + \eta < 1/2$, and thus we immediately get from the previous inequalities that $\|h\|_\infty \leq 2 - \frac{\varepsilon}{2} - \tau$. Next, observe that $\frac{h}{\|h\|_\infty}$ belongs to B , since $\frac{h}{\|h\|_\infty}$ is contained in both $B_{L^\infty[0,1]}$ and $B_{\ker(E_0)}$, and can be trivially written as $\frac{h}{\|h\|_\infty} = (1 - \varepsilon) \frac{h}{\|h\|_\infty} + \varepsilon \frac{h}{\|h\|_\infty}$. In other words, we have that $\|h\|_\varepsilon = \|h\|_\infty$. On the one hand, this immediately yields the estimate

$$\|h\|_\varepsilon = \|h\|_\infty \leq 2 - \frac{\varepsilon}{2} - \tau.$$

On the other hand, using the triangle inequality, and the fact that $\| \cdot \|_\varepsilon \leq \frac{1}{1-\varepsilon} \| \cdot \|_\infty$, we obtain

$$\|h\|_\varepsilon \geq \left\| \tilde{f} - \varphi \right\|_\varepsilon - \left\| \tau \cdot \left(\mathbb{1}_{[0, \frac{1}{4}]} - \mathbb{1}_{[\frac{1}{4}, \frac{1}{2}]} \right) \right\|_\varepsilon \geq \left\| \tilde{f} - \varphi \right\|_\varepsilon - \frac{\tau}{1 - \varepsilon}.$$

Combining the two previous bounds, doing a few more simple computations, and using equation (3.2), we get that

$$\left\| \tilde{f} - \varphi \right\|_\varepsilon \leq 2 - \frac{\varepsilon}{2} - \tau + \frac{\tau}{1 - \varepsilon} = 2 - \frac{\varepsilon}{2} + \frac{\tau\varepsilon}{1 - \varepsilon} \leq 2 - \frac{\varepsilon}{2} + \frac{\varepsilon(\frac{3}{2}\varepsilon + \eta)}{1 - \varepsilon} = 2 - \varepsilon \left(\frac{1}{2} - \frac{\frac{3}{2}\varepsilon + \eta}{1 - \varepsilon} \right),$$

and the last term is smaller than or equal to $2 - \frac{\varepsilon}{4}$. Indeed

$$\begin{aligned} 2 - \varepsilon \left(\frac{1}{2} - \frac{\frac{3}{2}\varepsilon + \eta}{1 - \varepsilon} \right) &\leq 2 - \frac{\varepsilon}{4} \Leftrightarrow \frac{1}{4} \leq \frac{1}{2} - \frac{\frac{3}{2}\varepsilon + \eta}{1 - \varepsilon} \Leftrightarrow \frac{3}{2}\varepsilon + \eta \leq \frac{1}{4}(1 - \varepsilon) \\ &\Leftrightarrow 6\varepsilon + 4\eta \leq 1 - \varepsilon \Leftrightarrow 7\varepsilon + 4\eta \leq 1, \end{aligned}$$

and the last inequality holds by the choice of η . □

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