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ABSTRACT

In this article, we analyze the chaotic behavior of finite difference operators associated with certain differential equations. Our examples range from numerical schemes for a birth-and-death model with proliferation to a class of second-order partial differential equations that includes the hyperbolic heat transfer equation, the telegraph equation, and the wave equation. We provide sufficient conditions on the spatial and time steps of the scheme that guarantee chaos for the corresponding operators, and we compare them with the conditions needed to ensure chaotic analytical solutions.

The analysis of chaos for partial differential equations is a fashionable area of study nowadays. Most natural phenomena of interest like cell proliferation, electrostatics, electrodynamics, elasticity, fluid flow, and heat conduction are described by partial differential equations (PDEs). It is well-known that solutions to linear PDEs can be represented in terms of C_0-semigroups whose chaotic dynamics have been widely analyzed. With this motivation, we investigate sufficient conditions to ensure chaos for numerical discretizations of these equations. This comparison between chaos for the numerical schemes and the exact solutions is completely new. Our findings need interaction between different areas of mathematics, such as numerical analysis, complex analysis, and operator theory.

I. INTRODUCTION

The theory of chaos has been extensively studied in finite-dimensional dynamical systems, which include discrete maps and ordinary differential equations. This field has resulted in significant applications in physics, chemistry, biology, and engineering. The analysis of chaos for partial differential equations (PDEs) is much more complicated. Most natural phenomena of interest like cell proliferation, electrostatics, electrodynamics, elasticity, fluid flow, heat conduction, sound propagation, and traffic modeling are described by PDEs.

The study of C_0-semigroups has commonly been associated with the study of linear partial differential equations of parabolic and hyperbolic types. It is now well-established that solutions to these equations can be represented in terms of C_0-semigroups.[1] The study of chaotic dynamics of C_0-semigroups solution of PDEs was initiated by Desch *et al.*[2] Herzog[3] analyzed chaos for C_0-semigroups on certain spaces of analytic functions with controlled growth. Since then, the dynamics on this kind of phase spaces has been intensively investigated.[4–9,25–27]

Finite difference methods are among the most significant numerical methods for solving differential equations. These methods involve a system of difference equations where the derivatives are estimated by divided finite differences, and the solution of this system provides an approximation of the differential equation solution in a discrete set of points. Many of these numerical schemes for differential equations can be modeled as dynamical systems. Therefore, it is pertinent to study the conditions that ensure the chaotic behavior of these systems, as was recently shown in Ref. 10.

There exist a wide range of finite divided differences methods[11,12] for approximating the derivatives of a real function $f:\mathbb{R}\to\mathbb{R}$. However, one of the most popular corresponds to the standard finite differences approximations for the first and second derivatives, namely, the forward, backward, and centered discretizations. When applying these approximations in an ordinary or partial differential equation, in many cases, this leads to a finite difference equation that can be described as a linear dynamical system, that is, in terms of the iterates of an operator T in an appropriate space of sequences. In some cases, this operator is a diagonal-constant operator, also called a Toeplitz operator, whose dynamics have been extensively studied in the literature.[13–15]

In this paper, we will consider finite difference schemes for a birth-and-death model with constant coefficients given by an infinite system of first-order ordinary differential equations, and also for a class of second-order partial differential equations. The chaotic dynamics of the C_0-semigroups solution of these models have been characterized in Ref. 16 and Refs. 7 and 8, respectively. In our work, we will go a step further, and we will provide sufficient conditions for these numerical discretizations being chaotic, and we will compare them with the ones obtained in Ref. 16 and Refs. 7 and 8, respectively. The analysis of the dynamics of finite difference operators is inspired in Ref. 10 but, as far as we know, the comparison between chaos for the numerical schemes and the exact solutions is completely new. Moreover, chaos in the sense of Devaney implies that the operator is not contractive. Hence, by definition of stability[17] for a numerical algorithm, chaotic finite difference operators are unstable. Thus, our results provide sufficient conditions on the spatial and time steps that provide unstable numerical algorithms. Also, to obtain stable numerical algorithms, the spatial and time steps cannot satisfy our conditions. Due to the nature of the subject, it needs the interaction between different areas of mathematics, such as numerical analysis, complex analysis, and operator theory, among others.

This paper is organized as follows. In Sec. II, we collect some basic notions and results about the dynamics of linear operators. We will recall some criteria from the literature that provide sufficient conditions to ensure chaos for Toeplitz operators, sequences of operators, and C_0-semigroups. In Sec. III, we will concentrate on the existence of chaos for finite difference schemes applied to a birth-and-death model with constant coefficients given by an infinite system of first-order ordinary differential equations. It will be crucial in our results to represent this model in terms of a Toeplitz operator. We will show the same conditions on the parameters of the model are needed to guarantee chaos of the numerical and analytical solution. In Sec. IV, we will apply different numerical schemes to a class of second-order partial differential equations with respect to time and space. By using the eigenvector field criterion for the sequence of operators, we will obtain sufficient conditions on the time and spatial step to ensure chaos for the numerical solutions. Finally, in Sec. V, we will show, as an application of our results, sufficient conditions for obtaining chaotic numerical solutions in the case of the hyperbolic heat transfer equation, the telegraph equation, and the wave equation.

II. PRELIMINARIES

In this section, we recall some basic notions and results about dynamics of linear operators, paying special attention to Toeplitz operators and C_0-semigroups.

A. Chaos for Toeplitz operators

In what follows, we will consider the following notation. If $M = \mathbb{N}_0$ or $\mathbb{Z}$, let us denote the space of sequences $\ell^p(M) := \{(a_n)_{n\in M} : \sum_{n\in M} |a_n|^p < \infty\}$. Let us denote $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$L^p(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} : \int_{\mathbb{T}} |f(z)|^p dz < \infty \right\},$$

$$L^\infty(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} : \sup |f(z)| < \infty \right\},$$

and

$$\mathcal{H}^2(\mathbb{D}) := \{ f : \mathbb{D} \to \mathbb{C} \quad \text{is holomorphic:} \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty \}.$$

Recall that a Toeplitz operator $T_\phi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$, with symbol $\phi \in L^\infty(\mathbb{T})$ is defined as $T_\phi(f) = P(M_\phi(f))$, $f \in \mathcal{H}^2(\mathbb{D})$, where M_ϕ is the multiplication operator by ϕ and $P : L^2(\mathbb{T}) \to \mathcal{H}^2(\mathbb{D})$ is the Riesz projection. Given $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in L^\infty(\mathbb{T})$, and $f(z) = \sum_{n=0}^\infty b_n z^n \in \mathcal{H}^2(\mathbb{D})$, we can write $(T_\phi f)(z) = \sum_{n=0}^\infty c_n z^n$, where the sequence $c = (c_n)_n$ is obtained as the convolution of $a = (a_n)_n$, with $b = (b_n)_n$, i.e.,

$$c_n = (a * b)_n = \sum_{j=-\infty}^{n} a_j b_{n-j}, \quad n \in \mathbb{N}_0,$$

and the Toeplitz operator is then considered as an infinite matrix operator $T_\phi : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$. In case $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is so that $a = (a_n)_n \in \ell^1(\mathbb{Z})$, then T_ϕ is a well-defined bounded operator in $\ell^p(\mathbb{N}_0)$, $1 \le p < \infty$.

In Ref. 14, the authors characterized Devaney chaos for a Toeplitz operator T_ϕ when the symbol ϕ has the form $\phi(z) = \frac{a_{-1}}{z} + a_0 + a_1 z$ in terms of the coefficients. The equivalence was based on the characterization given in Ref. 13, where sufficient conditions for chaos were also provided for more general symbols. In the context of linear dynamics, we recall that an operator T on a Banach space X is Devaney chaotic if it is hypercyclic, that is, there is a vector $x \in X$ such that its orbit $\mathrm{Orb}(x, T) = \{x, Tx, T^2x, \ldots\}$ is dense in X and the set of periodic points $\mathrm{Per}(T)$ is dense in X, a concept that comes from Refs. 18 and 19.

For the applications to numerical schemes for first-order PDEs, the tridiagonal case when the coefficients are real was especially useful, as obtained in Ref. 10. Here, we will need it for the birth-and-death model.

Proposition 1 Ref. 10**.** *Let $T_\phi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$ be a Toeplitz operator with symbol $\phi(z) = \frac{a_{-1}}{z} + a_0 + a_1 z$, where $a_0, a_{-1}, a_1 \in \mathbb{R}$, $a_{-1}, a_1 \in \mathbb{R}\setminus\{0\}$. Then, T_ϕ is chaotic if and only if $|a_{-1}| > |a_1|$ and*

$$||a_0| - 1| < |a_{-1} + a_1|, \tag{1}$$

except when $a_1 \cdot a_{-1} < 0$, $|a_1| + |a_{-1}| < 1$, and $\alpha = (|a_1| + |a_{-1}|)^2 \ge |a_{-1} + a_1|$, in which case (1) has to be changed to

$$2\sqrt{\frac{|a_{-1}||a_1|(1-\alpha)}{\alpha}}|a_0| < 1 + |a_{-1} + a_1|. \tag{2}$$

B. Universality

We now recall some basic notions about dynamics for sequences of operators. We refer the reader to Ref. 20 for more information about this topic.

Definition 1. *A sequence of operators $T_n : Z \to X$, $n \in \mathbb{N}_0$, is hypercyclic (or universal) if there exists $z \in Z$ such that $Orb(z, (T_n)) := \{T_n z : n \in \mathbb{N}_0\}$ is dense in X.*

Let us now introduce a useful sequence of operators that will be needed in order to prove our results for the finite differences applied to the second-order PDEs.

Definition 2. *Let $\mathcal{T} : X \times X \to X \times X$ be an operator, then we define the projection sequence of operators $T_n : X \times X \to X$ given by*

$$T_n(x_1, x_2) := \pi(\mathcal{T}^n(x_1, x_2)) \quad \text{for } (x_1, x_2) \in X \times X,$$

where $\pi : X \times X \to X$ is the first coordinate projection operator.

Within this framework, $(T_n)_n$ is said to be Devaney chaotic if it satisfies the following two conditions:

(i) There exists a $\mathcal{T}$-invariant subspace Y of $X \times X$ with $\overline{\pi(Y)} = X$.
(ii) The operator $\mathcal{T}|_Y$ is Devaney chaotic.

Remark 1. *Observe that if a projection sequence $(T_n)_n$ as considered in Definition 2 is Devaney chaotic, then the sequence of operators $T_n : X \times X \to X$, $n \in \mathbb{N}_0$, is hypercyclic, and the set $\{x \in X : \exists z \in X \text{ with } (x, z) \in Per(\mathcal{T})\}$ is dense in X.*

Given an operator $T : X \to X$ on a complex Banach space X, a function $E : A \to X$ for certain $A \subset \mathbb{C}$ is an eigenvector field if $E(\lambda) \in \ker(\lambda I - T)$ for any $\lambda \in A$. The following criterion adapts the classical eigenvector field criterion for operators[21] to a projection sequence of operators.

Theorem 1. *Given a projection sequence of operators $T_n : X \times X \to X$, $n \in \mathbb{N}_0$, induced by the operator $\mathcal{T} : X \times X \to X \times X$, if $U \subset \mathbb{C}$ is a nonempty connected open set such that $U \cap \mathbb{T} \neq \emptyset$ and $G : U \to X \times X$ is a weakly holomorphic eigenvector field of $\mathcal{T}$ such that $\overline{\pi(Y)} = X$, where*

$$Y := span\{G(\lambda) : \lambda \in U\},$$

then $(T_n)_n$ is Devaney chaotic.

C. Chaos for C_0-semigroups

We now recall the corresponding notions in linear dynamics for C_0-semigroups.

Definition 3. *Let X be a Banach space. A one-parameter family $\{T_t\}_{t\geq 0} \subset \mathcal{B}(X)$ is a C_0-semigroup if $T_0 = I, T_{t+s} = T_t \circ T_s$ and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$. The operator*

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x)$$

exists on a dense subspace of X denoted by $D(A)$; the so-called domain of A and $(A, D(A))$ is called the infinitesimal generator of the semigroup.

As a consequence of the Hille–Yosida theorem (Theorem 7.4 of Ref. 22), the solution of the abstract Cauchy problem on X given by

$$\begin{cases} \partial_t u(t) = Au(t), \\ u(0) = \varphi \end{cases} \tag{3}$$

can be obtained in terms of a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X whose infinitesimal generator is A. If $A \in \mathcal{B}(X)$, that is, A is a bounded linear operator, then the semigroup is uniformly continuous and can be represented as $T_t = e^{tA} = \sum_{k=0}^{\infty} (tA)^n/n!$ for all $t \geq 0$ (see Chap. I, Prop. 3.5 of Ref. 1). We recall the definitions of Devaney chaos and sub-chaos for a C_0-semigroup, the second one introduced in Ref. 23.

Definition 4. *An element $x \in X$ is called a periodic point for $\{T_t\}_{t\geq 0}$ if there exists some $t > 0$ such that $T_t x = x$. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is called Devaney chaotic if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X and the set of periodic points is dense in X. It is said to be sub-chaotic if there exists a closed subspace $Y \neq \{0\}$ invariant under $\{T_t\}_{t\geq 0}$, such that $\{T_t|_Y\}_{t\geq 0}$ is Devaney chaotic as a C_0-semigroup on Y.*

For linear second-order PDEs, it is usual to convert them into first-order PDEs, to which one can apply the theory of solution C_0-semigroups to the Cauchy problem by setting $u_1 = u$ and $u_2 = \partial_t u$. The solution depends on the initial conditions for u and its derivative with respect to t, belonging to a common Banach space X. If we want to study the chaotic behavior for the analytic solutions, it makes sense to consider a projection (uniparametric) family of operators, as we did in the discrete case.

Definition 5. *Let $\{\mathcal{T}_t : X \times X \to X \times X\}_{t\geq 0}$ be a C_0-semigroup. Then we define the projection family of operators $T_t : X \times X \to X$, $t \geq 0$, given by*

$$T_t(x_1, x_2) := \pi(\mathcal{T}_t(x_1, x_2)), \quad \text{for } (x_1, x_2) \in X \times X,$$

where $\pi : X \times X \to X$ is the first coordinate projection operator. $\{T_t\}_{t\geq 0}$ is said to be Devaney chaotic if $\{\mathcal{T}_t\}_{t\geq 0}$ is sub-chaotic with respect to a closed subspace $Y \neq \{0\}$, invariant for the C_0-semigroup, with $\overline{\pi(Y)} = X$.

As we will see when dealing with second-order PDEs, once they are converted into a first-order PDE, we will be interested in the chaotic behavior of the associated projection family. This means the existence of initial conditions $\varphi_i \in X$, $i = 1, 2$, such that the solution $u(t)_{t\geq 0}$ with $u(0) = \varphi_1$ and $\partial_t u(0) = \varphi_2$ is such that the set $\{u(t) : t \geq 0\}$ is dense in X (hypercyclic behavior) and, at the same time, we can find a sequence $(\varphi_{n,1})_n \in X^{\mathbb{N}}$, dense in X, and another sequence $(\varphi_{n,2})_n \in X^{\mathbb{N}}$, such that the solutions $u_n(t)_{t\geq 0}$ with $u_n(0) = \varphi_{n,1}$ and $\partial_t u_n(0) = \varphi_{n,2}$, are periodic.

The following criterion was stated in Ref. 23 and provides sufficient conditions to ensure sub-chaos for C_0-semigroups. We refer the reader to Ref. 2 (see also Th. 7.30 of Ref. 20) for the so-called Desch–Schappacher–Webb (DSW) criterion of chaos.

Proposition 2 Ref. 23. *Let X be a complex separable infinite-dimensional Banach space and let $(A, D(A))$ be the generator of a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X. Assume that there exists an open connected subset U and a weakly holomorphic function $f : U \to X$ such that*

(i) $U \cap i\mathbb{R} \neq \emptyset$,
(ii) $f(\lambda) \in ker(\lambda I - A)$ *for every* $\lambda \in U$.

Then, the restriction of the C_0-semigroup $\{T_t\}_{t\geq 0}$ to the invariant space

$$X_U := \overline{span\{f(\lambda) : \lambda \in U\}}$$

is Devaney chaotic. In particular, the C_0-semigroup $\{T_t\}_{t\geq 0}$ is subchaotic. Also, if $X_U = X$, then $\{T_t\}_{t\geq 0}$ is chaotic.

Finally, we recall the definition of the space of analytic functions of Herzog type.[3] Given $\rho > 0$, let

$$X_\rho = \left\{ f : \mathbb{R} \to \mathbb{C} : f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\}$$

endowed with the norm $\|f\| = \sup_{n\geq 0} |a_n|$. This space is isometrically isomorphic to $c_0 = \{a_n : \mathbb{N}_0 \to \mathbb{C} : \lim_{n\to\infty} |a_n| = 0\}$.

III. CHAOS FOR THE DISCRETIZATION OF BIRTH-AND-DEATH MODELS WITH PROLIFERATION

In Ref. 24, the authors studied a model of growth population for drug-resistant cancer cells, i.e., tumor cells that are not affected by chemotherapy. The model considers a population of copies of drug-resistant genes, which is divided into subpopulations. They assume that a gene in a subpopulation j can generate only gens in the neighboring subpopulations $j+1$ and $j-1$, at a rate b and d, respectively, or in the same subpopulation j at a rate a. This leads to the following birth-and-death model with constant coefficients:

$$\begin{aligned} f_1{}' &= af_1 + df_2, \\ f_n{}' &= af_n + bf_{n-1} + df_{n+1}, \quad n \geq 2. \end{aligned} \tag{4}$$

Let us define L as the operator in $\ell^p(\mathbb{N}_0)$, $1 \leq p < \infty$, and c_0 given by the infinite matrix

$$L := \begin{pmatrix} a & d & 0 & 0 & \dots \\ b & a & d & 0 & \dots \\ 0 & b & a & d & \dots \\ 0 & 0 & b & a & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}.$$

In Ref. 16, the authors obtained the following result, which states sufficient conditions to ensure Devaney chaos for the analytic solution of (4) given by the C_0-semigroup $(e^{tL})_{t\geq 0}$ when the initial conditions belong to the spaces $\ell^p(\mathbb{N}_0)$ or c_0.

Theorem 2. *If $a, b, d \in \mathbb{R}$ are such that*

$$|b| < |d|, \quad |a| < |b+d|,$$

then the semigroup solution of (4) given by $(e^{tL})_{t\geq 0}$ is Devaney chaotic in $\ell^p(\mathbb{N}_0)$ or c_0 for $1 \leq p < \infty$.

We will now show that under the same assumptions on the coefficients a, b, d, the solution of the forward discretization of model (4) is also chaotic. To do so, let us recall that the forward discretization of the first derivative is given by the following approximation:

$$f_k'(t) = \frac{f_k(t+h) - f_k(t)}{h} = \frac{f_k^{n+1} - f_k^n}{h},$$

where h denotes the time step and $t = nh, n \in \mathbb{N}_0$. Applying the forward discretization to (4) we get the following numerical scheme:

$$\begin{aligned} f_1^{k+1} &= f_1^k(ha+1) + hdf_2^k, \\ f_n^{k+1} &= hbf_{n-1}^k + f_n^k(ha+1) + hdf_{n+1}^k, \quad n \geq 2. \end{aligned} \tag{5}$$

If $f^k \in \ell^2(\mathbb{N}_0)$, we can express system (5) as $f^{k+1} = T_\phi f^k$ where T_ϕ is the tridiagonal Toeplitz operator with symbol $\phi(z) = hd\frac{1}{z} + (ha+1) + hb \cdot z$. We can now state our theorem.

Theorem 3. *Given the birth-and-death model (4), if the coefficients $a, b, d \in \mathbb{R}$ satisfy*

$$|b| < |d|, \quad |a| < |b+d|,$$

then for every time step h such that $h < \frac{|b+d|}{(|b|+|d|)^2}$, the operator T_ϕ, which defines the forward derivative discretization of (4) is Devaney chaotic.

Proof. In order to prove our result, we will use Theorem 1 identifying $a_{-1} = hd, a_0 = ha+1$ and $a_1 = hb$. Since $|b| < |d|$, we immediately have $|a_1| < |a_{-1}|$. We now divide the proof into two cases.

- *Case 1.* If $b \cdot d > 0$, since $h < \frac{|b+d|}{(|b|+|d|)^2} \leq \frac{1}{|b|+|d|}$, we have that $|hb| + |hd| = h(|b|+|d|) < 1$. Also, since $|a| < |b+d|$, we get for $a \neq 0$, $h < \frac{|b+d|}{(|b|+|d|)^2} < \frac{1}{|a|}$. Thus, $||ha+1|-1| = |ha+1-1| = |ha| < h|b+d|$ and the coefficients a, b, d satisfy (1) of Proposition 1. This implies that for $b \cdot d > 0$ and $h < \frac{|b+d|}{(|b|+|d|)^2}$ the operator is chaotic if, and only if, $|b| < |d|$ and $|a| < |b+d|$.
- *Case 2.* If $b \cdot d < 0$, then $|hb| + |hd| = h(|b|+|d|) < 1$, and since $h < \frac{|b+d|}{(|b|+|d|)^2}$, we have $h(|b|+|d|)^2 < |b+d|$, which is equivalent to $(|hb|+|hd|)^2 < |hd+hb|$. It is also verified that $||ha+1|-1| = |ha+1-1| = |ha| < h|b+d|$, and, therefore, the coefficients satisfy (1) of Proposition 1, which implies that for $b \cdot d < 0$ and $h < \frac{|b+d|}{(|b|+|d|)^2}$ the operator is chaotic if and only if $|b| < |d|$ and $|a| < |b+d|$.

□ □

As an immediate consequence, we obtain the following corollary that provides a sufficient condition that ensure both the analytical solution and the forward derivative discretization solution of the birth-and-death model (4) are chaotic.

Corollary 1. *Given the birth-and-death model (4), if the coefficients a, b, d satisfy*

$$|b| < |d|, \quad |a| < |b+d|,$$

then for $h < \frac{|b+d|}{(|b|+|d|)^2}$ both the semigroup $(e^{tL})_{t\geq 0}$ and the operator T_ϕ associated to the forward discretization with step h are chaotic.

IV. CHAOS FOR THE DISCRETIZATION OF A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

We will now consider the following second-order partial differential equation with respect to the time and space given by

$$\frac{\partial^2 u}{\partial t^2}(t,x)+\gamma\frac{\partial u}{\partial t}(t,x)+\theta u(t,x)=\alpha\frac{\partial^2 u}{\partial x^2}(t,x),\quad t\geq 0,\quad x\in\mathbb{R},\tag{6}$$

where γ,θ and $\alpha\in\mathbb{R}$. This equation can be reduced into a first-order system on the phase space that is the product of a certain space of Herzog type with itself. Setting $u_1=u$ and $u_2=\frac{\partial u}{\partial t}$, we have

$$\begin{cases}\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0 & I\\ \alpha\frac{\partial^2}{\partial x^2}-\theta I & -\gamma I\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix},\\ \begin{pmatrix}u_1(0,x)\\u_2(0,x)\end{pmatrix}=\begin{pmatrix}\varphi_1(x)\\\varphi_2(x)\end{pmatrix},\quad x\in\mathbb{R}.\end{cases}\tag{7}$$

Since the second-order differential operator $\frac{\partial^2}{\partial x^2}$ turns out to be a bounded operator on X_ρ, then the operator-valued matrix

$$A:=\begin{pmatrix}0 & I\\ \alpha\frac{\partial^2}{\partial x^2}-\theta I & -\gamma I\end{pmatrix}$$

defines a bounded operator on $X:=X_\rho\bigoplus X_\rho$ for every $\rho>0$ and, consequently, we have that $(e^{tA})_{t\geq 0}$ is the uniformly continuous solution semigroup. The following result stated in Ref. 8 showed sufficient conditions to guarantee the solution of Eq. (6) is chaotic.

Theorem 4. *Let γ,θ,α be real positive numbers. Suppose that*

$$\gamma^2=4\theta.$$

Then, A generates a uniformly continuous semigroup, which is Devaney chaotic on $X_\rho\bigoplus X_\rho$ for each $\rho>\frac{\gamma}{2\sqrt{\alpha}}$.

Remark 2. *From the proof of Theorem 4 in Ref. 8, it is easy to show that no restriction on the positive parameters γ,θ,α are needed if we just want to ensure Devaney chaos, in the sense of Section II, for the analytical solution of (6) in X_ρ applying Theorem 2.*

We will now obtain sufficient conditions in terms of the space and time steps that ensure Devaney chaos for the numerical solution of Eq. (6) when applying different discretizations. More concretely, we will consider forward discretization in time and space, forward discretization in time and centered in space and forward discretization in time and backward in space. Let $f:\mathbb{R}\to\mathbb{R}$, then fixed $x_0\in\mathbb{R}$ and with the notation $h>0$, $x_n=x_0+kh$, $k\in\mathbb{Z}$, $f^k=f(x_k)$ and $(f^k)'=f'(x_k)$, the first and second derivatives can be approximated by the following schemes:

$$(f^k)'\approx\frac{f^{k+1}-f^k}{h},\qquad (f^k)''\approx\frac{f^{k+2}-2f^{k+1}+f^k}{h^2},\qquad \text{Forward},$$

$$(f^k)'\approx\frac{f^{k}-f^{k-1}}{h},\qquad (f^k)''\approx\frac{f^{k}-2f^{k-1}+f^{k-2}}{h^2},\qquad \text{Backward},$$

$$(f^k)'\approx\frac{f^{k+1/2}-f^{k-1/2}}{h},\qquad (f^k)''\approx\frac{f^{k+1}-2f^{k}+f^{k-1}}{h^2},\qquad \text{Centered}.$$

When applying a numerical scheme to system (7), it can be defined as an operator $\mathcal{T}$ acting on the complex Banach space $X:=c_0\times c_0$, where $c_0:=\{(a_n)_n:\lim_n|a_n|=0\}$. This operator $\mathcal{T}$ defines the projection sequence of operators $(T_n)_n:c_0\times c_0\to c_0$ that gives the numerical solution of Eq. (6) when taking finite difference derivatives. We will denote these operators as $(T_n^f)_n$, $(T_n^c)_n$, and $(T_n^b)_n$ when applying forward discretization in time and forward in space, centered in space, and backward in space, respectively. In the following result, we analyze the chaotic behavior of such sequences of operators.

Theorem 5. *Let γ,θ,α be real positive numbers and let h and m denote the time step and spatial step of a numerical scheme applied to Eq. (6). The following assertions hold:*

(1) If $h<\frac{2}{\gamma}$ and $m<2h\sqrt{\frac{\alpha}{2(2-h\gamma)+h^2\theta}}$, then the sequence of operators $(T_n^f)_n$ is Devaney chaotic in c_0.
(2) If $h<\frac{2}{\gamma}$, then the sequence of operators $(T_n^c)_n$ is Devaney chaotic in c_0.
(3) If $h<\frac{2}{\gamma}$ and $m<h\sqrt{\frac{\alpha}{2(2-h\gamma)+h^2\theta}}$, then the sequence of operators $(T_n^b)_n$ is Devaney chaotic in c_0.

Proof. (1) Applying the forward discretization in time and space to system (7), we obtain the following system:

$$\begin{cases}\frac{(u_1)_n^{k+1}-(u_1)_n^k}{h}=(u_2)_n^k\\ \frac{(u_2)_n^{k+1}-(u_2)_n^k}{h}=\alpha\frac{(u_1)_{n+2}^k-2(u_1)_{n+1}^k+(u_1)_n^k}{m^2}-\theta(u_1)_n^k-\gamma(u_2)_n^k.\end{cases}\tag{8}$$

After a simple computation system (8) reduces to

$$\begin{cases}(u_1)_n^{k+1}=h(u_2)_n^k+(u_1)_n^k,\\ (u_2)_n^{k+1}=\frac{h\alpha}{m^2}\left[(u_1)_{n+2}^k-2(u_1)_{n+1}^k+(u_1)_n^k\right]-h\theta(u_1)_n^k\\ \quad+(1-h\gamma)(u_2)_n^k.\end{cases}\tag{9}$$

If we consider $(u_1)^k,(u_2)^k\in c_0$ the previous system can be expressed as a matrix system,

$$\begin{pmatrix}(u_1)^{k+1}\\(u_2)^{k+1}\end{pmatrix}=\begin{pmatrix}I & hI\\ \frac{h\alpha}{m^2}(B^2-2B+I)-h\theta & (1-h\gamma)I\end{pmatrix}\begin{pmatrix}(u_1)^k\\(u_2)^k\end{pmatrix},$$

where $I:c_0\to c_0$ is the identity operator and $B:c_0\to c_0$ is the backward shift operator. The operator $\mathcal{T}_f:c_0\times c_0\to c_0\times c_0$ is defined by the matrix

$$\begin{pmatrix}I & hI\\ \frac{h\alpha}{m^2}(B^2-2B+I)-h\theta & (1-h\gamma)I\end{pmatrix}$$

and the projection sequence of operators $(T_n^f)_n:c_0\times c_0\to c_0$ that gives the numerical solution of (6) is defined by $T_n^f:=\pi((\mathcal{T}^f)^n)$, where $\pi:c_0\times c_0\to c_0$ is the first coordinate projection. Let now $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathcal{T}^f$. Solving system (9) by the substitution, we arrive to

$$\begin{cases}(u_2)_n^k=\frac{\lambda-1}{h}(u_1)_n^k\\ (u_1)_n^k\left[\frac{\lambda^2-\lambda}{h}-\frac{h\alpha}{m^2}+h\theta-\frac{\lambda-1}{h}(1-h\gamma)\right]\\ \quad=\frac{h\alpha}{m^2}\left[(u_1)_{n+2}^k-2(u_1)_{n+1}^k\right].\end{cases}\tag{10}$$

Now, let us define

$$\begin{aligned}\varphi(\lambda) &:= \frac{(\lambda^2-\lambda)m^2}{h^2\alpha} - 1 + \frac{m^2\theta}{\alpha} - \frac{(\lambda-1)m^2}{h^2\alpha}(1-h\gamma)\\ &= \frac{m^2}{h^2}\frac{\lambda-1}{\alpha}(\lambda-1+h\gamma) - 1 + \frac{m^2\theta}{\alpha}.\end{aligned}$$

With this notation, the second equation in (10) can be reformulated as follows:

$$(u_1)_n^k\varphi(\lambda) = \left[(u_1)_{n+2}^k - 2(u_1)_{n+1}^k\right].$$

Let now define $g(\lambda) := 1 + \varphi(\lambda)$ and $f(\lambda) := 1 - e^{\frac{1}{2}\log(g(\lambda))}$, where $\log(z)$ is a branch of the logarithm defined on $\mathbb{C}\backslash\{z : \mathrm{Re}(z) \le 0\}$. Let us now consider the vector sequence

$$e_{f(\lambda)} = \left(f(\lambda), f(\lambda)^2, \ldots\right).$$

It is easy to prove that $e_{f(\lambda)}$ verifies the second equation in (10), and, thus, $\left(e_{f(\lambda)}, \frac{\lambda-1}{h}e_{f(\lambda)}\right)$ is a candidate for an eigenvector of the operator $\mathcal{T}^f$. Nevertheless, it is necessary to define the domain of g that makes f a holomorphic function. Indeed, notice that $g(-1) = \frac{m^2}{h^2}\frac{2}{\alpha}(2-2h\gamma) + \frac{m^2\theta}{\alpha} > 0$. Since g is an entire function there exists an open and connected neighborhood V of -1 such that $g(V)\bigcap\{z : \mathrm{Re}(z) \le 0\} = \emptyset$ and f is holomorphic in V. Since by hypothesis $m < 2h\sqrt{\frac{\alpha}{2(2-h\gamma)+h^2\theta}}$ it follows that

$$f(-1) = 1 - \frac{m}{h}\sqrt{\frac{2(2-h\gamma)+h^2\theta}{\alpha}} \in \mathbb{D}.$$

Since f is holomorphic in V, there exists $U \subset V$ an open and connected neighborhood of -1 such that $f(U) \subset \mathbb{D}$, and, therefore, $\left(e_{f(\lambda)}, \frac{\lambda-1}{h}e_{f(\lambda)}\right)$ is an eigenvector of $\mathcal{T}^f$ for all $\lambda \in U$. Let us now define $G : U \to c_0 \times c_0$ as

$$G(\lambda) := \left(e_{f(\lambda)}, \frac{\lambda-1}{h}e_{f(\lambda)}\right).$$

It is clear that $G(\lambda)$ is a weakly holomorphic map. Now, by the open mapping theorem for holomorphic functions, we have that $f(U)$ is an open and connected subset of $\mathbb{D}$, and, therefore, it has an accumulation point on the disk. Finally, from Example 3.2 of Ref. 20, we get that $\mathrm{span}\{e_{f(\lambda)}, \lambda \in U\}$ is dense in c_0 and the conclusion holds from the eigenvalue field criterion 1 for the projection sequence of operators $(T_n^f)_n$.

(2) Applying the forward discretization in time and centered in space to system (7), we obtain the following scheme:

$$\begin{cases}(u_1)_n^{k+1} = h(u_2)_n^k + (u_1)_n^k,\\ (u_2)_n^{k+1} = \frac{h\alpha}{m^2}\left[(u_1)_{n+1}^k - 2(u_1)_n^k + (u_1)_{n-1}^k\right]\\ \qquad -h\theta(u_1)_n^k + (1-h\gamma)(u_2)_n^k.\end{cases} \tag{11}$$

As in (9), the previous system (11) defines an operator $\mathcal{T}^c$ acting on $c_0 \times c_0$ given by the following matrix:

$$\begin{pmatrix} I & hI \\ \frac{h\alpha}{m^2}(B - 2I + F) - h\theta & (1-h\gamma)I\end{pmatrix},$$

where $F : c_0 \to c_0$ is the forward shift operator. Furthermore, this operator defines a projection sequence of operators $T_n^c : c_0 \times c_0 \to c_0$, such that $T_n^c := \pi((\mathcal{T}^c)^n)$ and it gives the numerical solution of (6) taking forward derivatives in time and centered in space. If $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{T}^c$, then we obtain

$$\begin{cases}(u_2)_n^k = \frac{\lambda-1}{h}(u_1)_n^k,\\ (u_1)_n^k\varphi(\lambda) = (u_1)_{n+1}^k + (u_1)_{n-1}^k,\end{cases} \tag{12}$$

where

$$\varphi(\lambda) := \frac{m^2}{h^2}\frac{\lambda-1}{\alpha}(\lambda-1+h\gamma) + 2 + \frac{\theta m^2}{\alpha}.$$

Let us now define $f(\lambda)$

$$f(\lambda) := \frac{\varphi(\lambda) - e^{\frac{1}{2}\log(\varphi(\lambda)^2-4)}}{2},$$

where $\log(z)$ is a branch of the logarithm defined on $\mathbb{C}\backslash\{z : \mathrm{Re}(z) \le 0\}$. It is clear that

$$e_{f(\lambda)} := \left(f(\lambda), f(\lambda)^2, \ldots\right)$$

verifies the second equation in (12). Also, $\varphi(-1)^2 - 4 = \frac{m^2}{h^2}\frac{2}{\alpha}(2-h\gamma) + 2 + \frac{\theta m^2}{\alpha} > 0$, and since $(\varphi^2 - 4)$ is an entire function there exists V an open and connected neighborhood of -1 such that $(\varphi(V)^2 - 4)\bigcap\{z : \mathrm{Re}(z) \le 0\} = \emptyset$ and, therefore, f is holomorphic in V. Since $\varphi(-1) > 2$, it follows that

$$f(-1) = \frac{\varphi(-1) - \sqrt{\varphi(-1)^2 - 4}}{2} \in \mathbb{D},$$

so there exists $U \subset V$ an open and connected neighborhood of -1 such that $f(U) \subset \mathbb{D}$. Hence, $\left(e_{f(\lambda)}, \frac{\lambda-1}{h}e_{f(\lambda)}\right)$ is an eigenvector of $\mathcal{T}^c$ for all $\lambda \in U$. As before, $G : U \to c_0 \times c_0$ given by

$$G(\lambda) := \left(e_{f(\lambda)}, \frac{\lambda-1}{h}e_{f(\lambda)}\right)$$

is a weakly holomorphic map and the conclusion holds as in part (1).

(3) We now apply the forward discretization in time and backward in space to system (7), which leads to the difference system,

$$\begin{cases}(u_1)_n^{k+1} = h(u_2)_n^k + (u_1)_n^k,\\ (u_2)_n^{k+1} = \frac{h\alpha}{m^2}\left[(u_1)_n^k - 2(u_1)_{n-1}^k + (u_1)_{n-2}^k\right]\\ \qquad -h\theta(u_1)_n^k + (1-h\gamma)(u_2)_n^k,\end{cases} \tag{13}$$

which can be defined in terms of the operator $\mathcal{T}^b : c_0 \times c_0 \to c_0 \times c_0$ given by the matrix

$$\begin{pmatrix} I & hI \\ \frac{h\alpha}{m^2}(I - 2F + F^2) - h\theta & (1-h\gamma)I\end{pmatrix}.$$

The operator $\mathcal{T}^b$ defines the projection sequence of operators $T_n^b : c_0 \times c_0 \to c_0$ given by $T_n^b := \pi((\mathcal{T}^b)^n)$. Proceeding as before, if $\lambda \in \mathbb{C}$ is an eigenvalue, then

$$\begin{cases} (u_2)_n^k = \frac{\lambda-1}{h}(u_1)_n^k, \\ (u_1)_n^k \varphi(\lambda) = -2(u_1)_{n-1}^k + (u_1)_{n-2}^k, \end{cases} \tag{14}$$

with

$$\varphi(\lambda) := \frac{m^2}{h^2}\frac{\lambda-1}{\alpha}(\lambda - 1 + h\gamma) - 1 + \frac{\theta m^2}{\alpha}.$$

We define

$$f(\lambda) := \frac{-1 + e^{\frac{1}{2}\log(1+\varphi(\lambda))}}{\varphi(\lambda)},$$

where $\log(z)$ is a principal branch of the logarithm defined on $\mathbb{C}\backslash\{z : \mathrm{Re}(z) \le 0\}$. Notice that $\varphi(-1) := \frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) - 1 + \frac{\theta m^2}{\alpha}$, and since φ is an entire function, there exists V_1 an open and connected neighborhood of -1 such that $0 \notin \varphi(V_1)$. Let us also observe that

$$1 + \varphi(-1) = \frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha} > 0$$

and by the continuity of φ, there exists V_2 an open and connected neighborhood of -1 such that $(1 + \varphi(V_2)) \bigcap \{z : \mathrm{Re}(z) \le 0\} = \emptyset$. Let us define $U = V_1 \cap V_2$. It is clear that f is holomorphic in U. Furthermore, due to the fact that $m < h\sqrt{\frac{\alpha}{2(2-h\gamma)+h^2\theta}}$ we have $\frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha} < 1$ and, therefore,

$$|f(-1)| = \left| \frac{-1 + \sqrt{\frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha}}}{-1 + \frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha}} \right| = \frac{1 - \sqrt{\frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha}}}{1 - \frac{m^2}{h^2}\frac{2}{\alpha}(2 - h\gamma) + \frac{\theta m^2}{\alpha}} < 1.$$

Since f is continuous in U, we can find $V \subset U$ an open and connected neighborhood of -1 such that $f(\lambda) \subset \mathbb{D}$ for all $\lambda \in V$ and, therefore, $\left(e_{f(\lambda)}, \frac{\lambda-1}{h} e_{f(\lambda)}\right)$ is an eigenvector of $\mathcal{T}^b$ for all $\lambda \in V$, where $e_{f(\lambda)} := \left(f(\lambda), f(\lambda)^2, \ldots\right)$. The conclusion now follows exactly as in parts (1) and (2).

□ □

Corollary 2. *Given $h < 2\tau$ and $m < h\sqrt{\frac{\alpha_d}{2(2\tau-h)}}$, the numerical solution of Eq. (6), when considering forward discretization in time and forward in space, forward discretization in time and centered in space and forward discretization in time, and backward in space are always Devaney chaotic.*

Remark 3. *Observe that when $\gamma = 0$, the sequence of operators $(T_n^c)_n$ is chaotic in c_0 for any time and spatial steps.*

Figure 1(a) represents three consecutive iterations of the sequence of operators T_n^f with $\alpha = 4$, $\theta = 1$ and $\gamma = 1$, $h = 2/300$, $m = 0.004$ and the initial conditions are of order $\frac{1}{n}$ with some random multiplication factors. In particular, in yellow, red, and blue colors we see the representation of the solution for the iterations 149, 150, and 151, respectively. Figures 1(b) and 1(c) also represent three consecutive iterations of T_n^f for the same α, θ and γ but taking $h = 2.5 \times 10^{-6}$ and $m = 0.2$. It is easy to verify that in Fig. 1(a), h and m satisfy the condition of Corollary 2 while the other case does not. We can observe that Fig. 1(a) represents, as predicted, an unstable numerical solution of Eq. (6) since for only 150 iterations the error is of order 10^{174}. We can also notice how the error rapidly grows between the three consecutive iterations. On the other hand, Figs. 1(b) and 1(c) represent a stable solution and even taking 10 000 iterations error is small and the graphics of the three consecutive iterations overlap.

V. APPLICATIONS

A. The hyperbolic heat transfer equation

The hyperbolic heat transfer equation in the absence of internal heat sources can be seen as a particular case of the class of PDEs (6) and it is given by

$$\begin{cases} \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \alpha_d \frac{\partial^2 u}{\partial x^2}, \\ u(0, x) = \varphi_1(x), \quad x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0, x) = \varphi_2(x), \quad x \in \mathbb{R}, \end{cases} \tag{15}$$

where φ_1 and φ_2 represent the initial variation of temperature, respectively, $\alpha_d > 0$ is the thermal diffusivity, and $\tau > 0$ is the thermal relaxation time. In Ref. 7 (see also Chap. 7 in Ref. 20), the authors stated that the solution semigroup of the hyperbolic heat equation is chaotic on $X_\rho \times X_\rho$ under some conditions on ρ, which can be relaxed if we just care about the chaotic behavior of the solution.

We get that the heat equation (15) is a particular case of model (6) when $\theta = 0$, $\gamma = \frac{1}{\tau}$, and $\alpha = \frac{\alpha_d}{\tau}$. As an immediate consequence of Theorem 5, we obtain sufficient conditions in terms of the space and time steps that ensure Devaney chaos for the numerical solution of model (15) when considering forward discretization in time and space, forward discretization in time and centered in space, and forward discretization in time and backward in space. Following a similar notation as in Sec. IV, we will denote the corresponding projection sequence of operators $(T_n)_n : c_0 \times c_0 \to c_0$ that gives the numerical solution of the hyperbolic heat transfer equation (15) as $(T_n^{HF})_n$, $(T_n^{HC})_n$, and $(T_n^{HB})_n$, respectively.

Corollary 3. *Let τ, α_d be real positive numbers and let h and m denote the time step and spatial step of a numerical scheme applied to Eq. (15). The following assertions hold:*

(1) If $h < 2\tau$ and $m < 2h\sqrt{\frac{\alpha_d}{2(2\tau-h)}}$, then the sequence of operators $(T_n^{HF})_n$ is Devaney chaotic in c_0.
(2) If $h < 2\tau$, then the sequence of operators $(T_n^{HC})_n$ is Devaney chaotic.
(3) If $h < 2\tau$ and $m < h\sqrt{\frac{\alpha_d}{2(2\tau-h)}}$, then the sequence of operators $(T_n^{HB})_n$ is Devaney chaotic in c_0.

In particular, if $h < 2\tau$ and $m < h\sqrt{\frac{\alpha_d}{2(2\tau-h)}}$, the projection sequences of operators $(T_n^{HF})_n$, $(T_n^{HC})_n$, and $(T_n^{HB})_n$ are all Devaney chaotic in c_0.

B. The telegraph equation

The telegraph equation models a telegraph wire as an electrical circuit consisting of a resistor R and an inductance L. The function $u(t, x)$ gives the voltage on the wire in time t and at position x. The

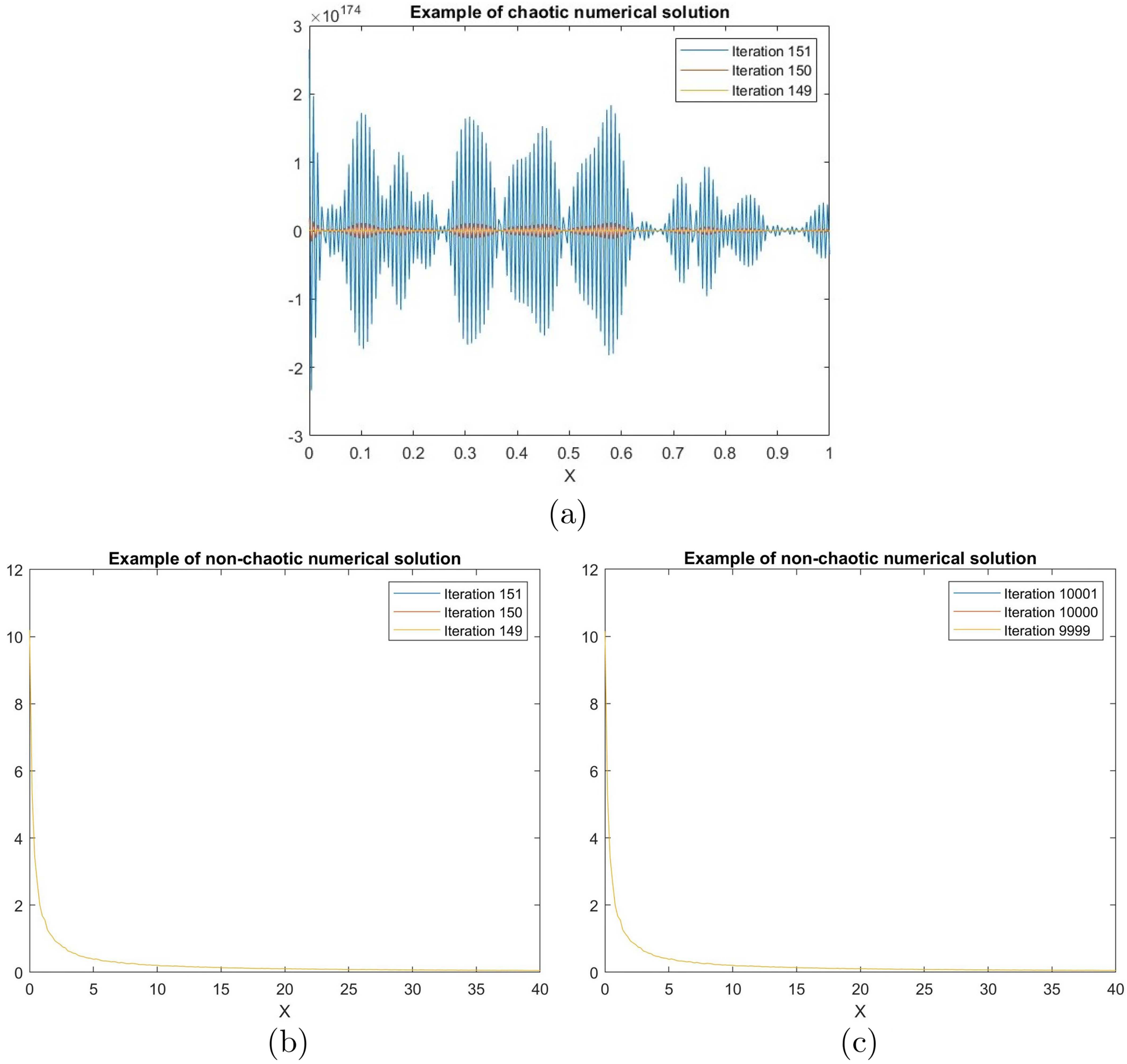

FIG. 1. Iterations of the operator $(T_n^f)_n$ for $\alpha = 4, \theta = 1$, and $\gamma = 1$. (a) Chaotic discretization taking iterations 151, 150, and 149. (b) Non-chaotic discretization taking iterations 151, 150, and 149. (c) Non-chaotic discretization taking iterations 10 001,10 000, and 9999.

equation also considers the possibility of a current leakage to the ground either through a resistor G or through a capacitance C. It is given by

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + (a+b)\frac{\partial u}{\partial t} + abu = 0, \tag{16}$$

where $a = \frac{G}{L}$, $b = \frac{R}{L}$, and $c^2 = \frac{1}{LC}$. In Ref. 8, the authors proved that the solution semigroup of the telegraph equation is chaotic on $X_\rho \times X_\rho$ for $a = b$ and $\rho > \frac{a}{c}$. It is easy to prove that applying Proposition 2, no restrictions on the parameters a, b, c are needed to ensure Devaney chaos for the analytic solution in X_ρ.

The telegraph equation (16) labels into model (6) when $\alpha = c^2$, $\gamma = (a+b)$, and $\theta = ab$. As in the hyperbolic heat equation, we can obtain the projection sequences of operators $(T_n^{TF})_n$, $(T_n^{TC})_n$, and $(T_n^{TB})_n$ that gives the numerical solution of (16) when taking forward derivatives in time and space, forward derivatives in time and centered in space, and forward derivatives in time and backward in space, respectively. Applying Theorem 5, we obtain the following result.

Corollary 4. *Let a, b, c be real positive numbers and let h and m be the time step and spatial step of a numerical scheme applied to Eq. (16). The following assertions hold:*

(1) If $h < \frac{2}{a+b}$ and $m < 2hc\sqrt{\frac{1}{2(2-h(a+b))+h^2ab}}$, then the sequence of operators $(T_n^{TF})_n$ is Devaney chaotic in c_0.
(2) If $h < \frac{2}{a+b}$, then the sequence of operators $(T_n^{TC})_n$ is Devaney chaotic.
(3) If $h < \frac{2}{a+b}$ and $m < hc\sqrt{\frac{1}{2(2-h(a+b))+h^2ab}}$, then the sequence of operators $(T_n^{TB})_n$ is Devaney chaotic in c_0.

Also, if $h < \frac{2}{a+b}$ and $m < hc\sqrt{\frac{1}{2(2-h(a+b))+h^2ab}}$ the numerical solutions of Eq. (16), when considering forward discretization in time and forward in space, forward discretization in time and centered in space, and forward discretization in time and backward in space are always Devaney chaotic.

C. The wave equation

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{17}$$

is clearly a particular case of Eq. (6) when $\gamma = \theta = 0$ and $\alpha = c^2$. In Ref. 7, the authors stated that the semigroup solution of the wave equation is always chaotic in $X_\rho \times X_\rho$.

As in models (15) and (16), we denote by $(T_n^{WF})_n$, $(T_n^{WC})_n$, and $(T_n^{WB})_n$ the projection sequences of operators that give the numerical solution of the wave equation taking forward derivatives in time and space, forward derivatives in time and centered in space, and forward derivatives in time and backward in space, respectively. We immediately get the following result that states sufficient conditions on the spatial and time steps to ensure chaotic numerical discretizations.

Corollary 5. *Let c be a real number and let h and m be the time step and spatial step of a numerical scheme applied to Eq. (17). The following assertions hold:*

(1) If $m < h|c|$, then the sequence of operators $(T_n^{WF})_n$ is Devaney chaotic in c_0.
(2) The sequence of operators $(T_n^{WC})_n$ is Devaney chaotic for every h and m.
(3) If $m < h\frac{|c|}{2}$, then the sequence of operators $(T_n^{WB})_n$ is Devaney chaotic in c_0.

Given $m < h\frac{|c|}{2}$, the numerical solution of Eq. (17), when considering forward discretization in time and forward in space, forward discretization in time and centered in space, and forward discretization in time and backward in space are all Devaney chaotic.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

All authors have contributed equally to this work. All authors read and approved the final manuscript.

Alfred Peris: Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Alfred Peris:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Writing – review & editing (equal). **Álvaro Vargas:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Validation (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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