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T_0 functional Alexandroff topologies are partial metrizable

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Abstract

If $f : X \to X$ is a function, the associated functional Alexandroff topology on X is the topology whose closed sets are $\{A \subseteq X : f(A) \subseteq A\}$. We prove that every functional Alexandroff topology is pseudopartial metrizable and every T_0 functional Alexandroff topology is partial metrizable.

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1. Introduction

Alexandroff spaces (also called principal spaces) first discussed in [1] are topological spaces in which arbitrary intersections of open sets are open. Obviously, every topology on a finite set is an Alexandroff topology. Alexandroff spaces play an important role in domain theory. In fact, in [16] Steve Matthews discusses constructing each semantic domain as an Alexandroff topology. The key is that the set of upper sets is an Alexandroff topology. These spaces are also applied in digital topology as they are determined uniquely by the family of all finite subspaces and are considered a generalization of finite topological spaces; to learn more about this application one can see [10] and [11] .

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In [5], functional Alexandroff topologies, a new subclass of Alexandroff topologies on a set were introduced. There they defined the functional Alexandroff topology on a set X induced by the mapping $f : X \to X$, to be the topology whose closed sets are $\{A \subseteq X : f(A) \subseteq A\}$. One can see that arbitrary unions of closed sets are closed. They denoted this topology by τ_f. However, other notations are used. For instance in [15], it is denoted by $\mathcal{P}(f)$. In [7] Echi calls these spaces, primal spaces. A topology τ on a set X is functional Alexandroff topology if it is realized as τ_f for some function $f : X \to X$.

The functional Alexandroff topologies, have been applied and studied extensively. For example in [8] they show that Ulam-Kakutani-Collatz conjecture is true if and only if $\mathbb{N}$ is supercompact with respect to a specific functional Alexandroff topology. To learn more about properties of these spaces one can see [3], [4], [5], [9], and [14].

In the second section we investigate topological properties of connected functional Alexandroff topologies and will give a characterization of these topologies based on the specialization order.

In the third and last section we use the results of the second section to prove that T_0 functional Alexandroff topologies are partial metrizable and functional Alexandroff topologies in general are pseudopartial metrizable. This is helpful because metrics and in general distance functions allow us to talk about distance between the points and sets, closeness of sets and points, and more importantly one can talk about Cauchy sequences and convergence. This can be useful in the theory of functional Alexandroff topologies.

2. Connected functional Alexandroff topologies

In this section we find conditions that are equivalent to connectedness of functional Alexandroff topologies.

Lemma 2.1. *Let $f : X \to X$ be a mapping. Then the following are equivalent:*

(1) *τ_f is T_2,*
(2) *τ_f is T_1,*
(3) *f is the identity map,*
(4) *τ_f is the discrete topology*

Proof. Note that (a), (b), and (d) are equivalent for every Alexandroff topology. We prove (b)⇒(c)⇒(d). Suppose τ_f is T_1. Then for every $x \in X$ the singleton set $\{x\}$ is closed. Thus, $f(\{x\}) \subseteq \{x\}$ or in other words $f(x) = x$. The implication (c)⇒(d) is trivial as every set is a closed set. □

Recall that $\mathbb{N}_0 = \{x \in \mathbb{Z} : x \geq 0\}$. In the following lemma $f^0(a) = a$ and $f^n(a) = f(f^{n-1}(a))$ for every $n \in \mathbb{N}$. Also, in a topological space X we denote the closure of a subset A of X by $cl(A)$.

Lemma 2.2 (Proposition 1.2-(2) of [7])**.** *Let $f : X \to X$ be a mapping and $A \subseteq X$. Then $cl(A) = \bigcup_{i \in \mathbb{N}_0} f^i(A)$.*

Lemma 2.3. *Let $f : X \to X$ be a mapping. If τ_f is T_0 and $f(t) \neq t$, then $t \notin cl(f(t))$.*

Proof. Suppose τ_f is T_0 and $f(t) \neq t$. Since τ_f is T_0, either $t \notin cl(f(t))$ or $f(t) \notin cl(t)$. Since $f(t) \in cl(t)$, we have $t \notin cl(f(t))$. □

Recall that for every topology τ, its associated preorder (that is, a reflexive and transitive relation) $\leq_\tau$ is called the specialization preorder and defined by $a \leq b$ if and only if $a \in cl(\{b\})$. Then, one can see that this preorder is a partial order if and only if τ is T_0 which in this case we call it specialization order.

If $(X, \leq)$ is a preordered set, $a \in X$, and $A \subseteq X$, then $\uparrow a = \{x \in X : a \leq x\}$, $\downarrow a = \{x \in X : x \leq a\}$, $\uparrow A = \bigcup_{a\in A} \uparrow a$, and $\downarrow A = \bigcup_{a\in A} \downarrow a$.

Let $f : X \to X$ be a mapping. By using Lemma 2.2, we have the following lemma.

Lemma 2.4. *Let $f : X \to X$ be a mapping and $\leq$ be the specialization preorder of τ_f. If a is an arbitrary element of X, then*

(1) $\downarrow a = cl(\{a\}) = \{a, f(a), f^2(a), \cdots\}$.
(2) $\cdots \leq f^3(a) \leq f^2(a) \leq f(a) \leq a$.

Let $f : X \to X$ be a mapping, $a \in X$, and $n \in \mathbb{N}_0$. Then we define $f^{-n}(a) = \{x : f^n(x) = a\}$.

Lemma 2.5. *Let $f : X \to X$ be a mapping and $(X, \leq_{\tau_f})$ be a partially ordered set. If C is a chain in $(X, \leq_{\tau_f})$, then there is an embedding from C to $\mathbb{Z}$, in particular C is countable.*

Proof. Let a be an arbitrary element of C. Note that by Lemma 2.4, if $x \leq a$ then $x \in \{a, f(a), f^2(a), \cdots\}$ and if $x \geq a$ then $x \in f^{-n}(a)$ for some non-negative integer n. Thus, $C = (\downarrow a \cap C) \cup \bigcup_{n\in\mathbb{N}} H_n$, where

$H_n = \{x \in C : f^n(x) = a \text{ and } f^m(x) \neq a \text{ for } 0 \leq m < n \} \setminus \downarrow a$.

We show that $|H_n| \leq 1$ for every $n \in \mathbb{Z}^+$. By way of contradiction suppose there is an integer n such that H_n has more than one element. Thus, there are two different elements $b, c \in C$ such that $b, c \in H_n$. Either $b \leq c$ or $c \leq b$. Without lost of generality suppose that $b \leq c$. Thus, there is a positive integer m such that $b = f^m(c)$. Either $m < n$ or $n \leq m$. If $m < n$ then $a = f^n(c) = f^{n-m}(f^m(c)) = f^{n-m}(b)$ and $n - m < n$ which is in contradiction with $b \in H_n$. If $m \geq n$ then $b = f^m(c) = f^{m-n}(f^n(c)) = f^{m-n}(a) \in \downarrow a$ which is a contradiction. Thus, $|H_n| \leq 1$. Now define $\theta : C \to \mathbb{Z}$ by

$$\theta(x) = \begin{cases} n, & \text{if } x \in H_n; \\ -n & \text{if } x = f^n(a) \text{ and } x \neq f^m(a) \text{ for } m < n. \end{cases}$$

One can easily see that θ is one-to-one and order preserving. Consequently, θ is an embedding.

□

Consider the mapping $f : X \to X$ and define the relation R_f on X as follows:

aR_fb if and only if there are $m, n \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$.

This relation has been defined in [5], except that there the authors assume $m, n \in \mathbb{N}$. So, some of the results related to connectedness of τ_f that are already in [5] are stated here due to their role later on in the next section.

One can easily verify that R_f is reflexive and symmetric. For transitivity suppose $f^m(a) = f^n(b)$ and $f^r(b) = f^s(c)$, where $m, n, r, s \in \{0, 1, 2, \cdots,\}$. Then $f^{m+r}(a) = f^{n+r}(b) = f^{n+s}(c)$ and so, aR_fb and bR_fc implies aR_fc.

One can see that $cl(\{a\}) = \{a, f(a), f^2(a), f^3(a), \cdots,\} \subseteq [a]$ for $a \in X$ where $[a]$ is the equivalence class of a. In fact, $[a] = cl(\{a\}) \cup f^{-1}(cl(\{a\})) \cup f^{-2}(cl(\{a\})) \cup f^{-3}(cl(\{a\})) \cdots$.

Lemma 2.6. *If $f : X \to X$ is a mapping, then $[a]$ is a clopen set of τ_f for every $a \in X$.*

Proof. Suppose $a \in X$. We show that $[a]$ is clopen. Since $m \in [a]$ if and only if $f(m) \in [a]$, $f([a]) \subseteq [a]$ and $f(X \setminus [a]) \subseteq X \setminus [a]$. Consequently, both $[a]$ and $X \setminus [a]$ are closed. So, $[a]$ is a clopen set. □

Corollary 2.7. *If $f : X \to X$ is a mapping, then τ_f is connected if and only if for every $a, b \in X$ there are $m, n \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$.*

Corollary 2.8. *If $f : X \to X$ is a mapping and $a \in X$, then τ_f is connected if and only if $X = \bigcup_{m \in \mathbb{N}_0} \bigcup_{n \in \mathbb{N}_0} f^{-n}(f^m(\{a\}))$*

Recall that a preordered set $(X, \leq)$ is called connected if for any $a, b \in X$, there exists a finite sequence $(a = x_1, \cdots, x_n = b)$ such that x_i and x_{i+1} are comparable for every $i = 1, \cdots, n-1$.

One can easily verify the proof of the following lemma.

Lemma 2.9. *A preordered set $(X, \leq)$ is connected if and only if $X = \uparrow\downarrow \{a\} \cup \uparrow\downarrow\uparrow\downarrow \{a\} \cup \cdots$. for every $a \in X$*

Theorem 2.10. *Let $f : X \to X$ be a mapping. Then the following are equivalent:*

1. *τ_f is connected,*
2. *$(X, \leq_{\tau_f})$ is a connected preordered set,*
3. *$X = \uparrow\downarrow \{a\}$ for every $a \in X$.*

Proof. We prove the theorem by showing that both (a) and (b) are equivalent to (c).

For (a) ⇒ (c) suppose τ_f is connected. We prove that $X = \uparrow\downarrow \{a\}$ for every $a \in X$. If $b \in X$, then by Corollary 2.7 there are $m, n \in \{0, 1, 2, \cdots,\}$ such that $f^m(a) = f^n(b)$. Since $f^m(a) = f^n(b) \leq_{\tau_f} a$, we have $f^n(b) \in \downarrow \{a\}$. On the other hand, $f^n(b) \leq_{\tau_f} b$ or $b \in \uparrow f^n(b)$. Thus, $b \in \uparrow f^n(b) \subseteq \uparrow\downarrow \{a\}$. Consequently, $X = \uparrow\downarrow \{a\}$.

For (c) ⇒ (a) , suppose $X = \uparrow\downarrow \{a\}$ for every $a \in X$. We prove that τ_f is connected. It is enough to show that for every $x \in X$ there are $m, n \in \{0, 1, 2, \cdots,\}$ such that $f^m(x) = f^n(a)$. Note that every $x \in X$ belongs to the set $\uparrow\downarrow \{a\} = \uparrow \{a, f(a), f^2(a), \cdots\}$. Thus, there is an $n \in \{0, 1, 2, \cdots,\}$ such that $f^n(a) \leq_{\tau_f} x$ or $f^n(a) \in\downarrow \{x\}$ and by Lemma 2.4, $f^n(a) = f^m(x)$ for some non-negative integer m. Hence, by Corollary 2.8, τ_f is connected.

For (b) $\Rightarrow$ (c) it is enough to prove $\uparrow\downarrow\uparrow\downarrow \{a\} \subseteq\uparrow\downarrow \{a\}$ for every $a \in X$. Assume $x \in\uparrow\downarrow\uparrow\downarrow \{a\}$. Thus, there is a $y \in\downarrow\uparrow\downarrow \{a\}$ such that $y \leq_{\tau_f} x$. Since $y \in\downarrow\uparrow\downarrow \{a\}$, there is a $b \in\uparrow\downarrow \{a\}$ such that $y \leq_{\tau_f} b$. By Lemma 2.4, $b \in\uparrow \{a, f(a), f^2(a), \cdots\}$. There is a non-negative integer m such that $f^m(a) \leq_{\tau_f} b$. Thus, $f^m(a) \in\downarrow \{b\}$ and therefore, by Lemma 2.4, there is a non-negative integer n such that $f^m(a) = f^n(b)$. On the other hand, since $y \leq_{\tau_f} b$, there is a non-negative integer r such that $y = f^r(b)$. Either $r \geq n$ or $r < n$. If $r \geq n$, then $f^{m+(r-n)}(a) = f^{n+(r-n)}(b) = f^r(b) = y \leq_{\tau_f} x$. Thus, $x \in\uparrow\downarrow \{a\}$. In the case of $r < n$ we have $f^m(a) = f^n(b) \leq_{\tau_f} f^r(b) = y \leq_{\tau_f} x$. So, $x \in\uparrow\downarrow \{a\}$. Consequently, $\uparrow\downarrow\uparrow\downarrow \{a\} =\uparrow\downarrow \{a\}$ and therefore, (b) implies (c). Thus, parts (b) and (c) are equivalent as (c) implies (b). □

Lemma 2.11. *Let $f : X \to X$ be a mapping and τ_f be T_0. If X is connected, then X with respect to the inequality $\leq_{\tau_f}$ is a $\wedge$-semilattice.*

Proof. Suppose $a, b \in X$. We prove that $a\wedge b$ exists. Since X is connected, there are $m, n \in \mathbb{N}_0$ such that $f^m(a) = f^n(b)$. Thus, $t = f^m(a) = f^n(b) \leq_{\tau_f} a, b$. If $s \leq_{\tau_f} a, b$ with $s \geq_{\tau_f} t$ then $s \in \{a, f(a), \cdots, f^m(a)\} \cap \{b, f(b), \cdots, f^n(b)\}$. Since both sets $\{a, f(a), \cdots, f^m(a)\}$ and $\{b, f(b), \cdots, f^n(b)\}$ are finite and form a chain, we can find the biggest element s in their intersection which will be $a \wedge b$. □

Remark 2.12. Let $f : X \to X$ be a mapping and τ_f be connected and T_0. By Lemma 2.11 $(X, \leq_{\tau_f})$ is a $\wedge$-semilattice. We claim that for every $x, y \in X$ there are $l, k \in \mathbb{N}_0$ such that $f^l(x) = f^k(y)$ and whenever, $f^i(x) = f^j(y)$ for $i, j \in \mathbb{N}_0$ we have $l \leq i$ and $k \leq j$. Suppose $k = min\{d \in \mathbb{N}_0 : \exists c \in \mathbb{N}_0\ f^d(y) = f^c(x)\}$. Thus, there is an $c_0 \in \mathbb{N}_0$ such that $f^k(y) = f^{c_0}(x)$. Let $l = min\{c \in \mathbb{N}_0 :\ f^k(y) = f^c(x)\}$. We prove that if $f^i(x) = f^j(y)$ for $i, j \in \mathbb{N}_0$ we have $l \leq i$ and $k \leq j$. By definition of k we have $k \leq j$. We prove that $l \leq i$. If $l > i$ then, $f^k(y) = f^l(x) = f^{i+(l-i)}(x) = f^{j+(l-i)}(y) \leq_{\tau_f} f^j(y) \leq_{\tau_f} f^k(y)$. Thus, $f^k(y) = f^l(x) = f^j(y) = f^i(x)$. By definition of l, we have $l \leq i$ which contradicts, $l > i$. Thus, $l \leq i$. By definition of infimum, $x \wedge y = f^l(x) = f^k(y)$.

We call the pair (l, k) the (x, y)-connector. It is clear by the procedure that if (l, k) is the (x, y)-connector, then (k, l) is the (y, x)-connector. Thus, by definition, if (l, k) is the (x, y)-connector, then $x \wedge y = f^l(x) = f^k(y)$.

Lemma 2.13. *Let $f : X \to X$ be a mapping and τ_f be T_0 and connected. If $x, z \in X$, then for every $y \in X$ either $x \wedge z \geq_{\tau_f} x \wedge y$ or $x \wedge z \geq_{\tau_f} y \wedge z$.*

Proof. Suppose (θ, η) is the (x, y)-connector and (ξ, φ) is the (y, z)-connector. Thus, $x \wedge y = f^\theta(x) = f^\eta(y)$ and $y \wedge z = f^\xi(y) = f^\varphi(z)$. Either $\xi \leq \eta$ or $\eta \geq \xi$. If $\xi \leq \eta$, then $f^\theta(x) = f^\eta(y) = f^{\xi+\eta-\xi}(y) = f^{\varphi+\eta-\xi}(z)$. Thus, $x \wedge z \geq_{\tau_f} f^\theta(x) = x \wedge y$. Similarly, for the case $\eta \geq \xi$ one can show that $x \wedge z \geq_{\tau_f} y \wedge z$. □

Recall that a nonempty subset C of the preordered set $(X, \leq)$ is called cyclic if $x \leq y$ for every $x, y \in C$. We call a cyclic subset nontrivial, if it has more than one element.

We call a preordered set $(X, \leq)$ cyclic free if it does not contain any nontrivial cyclic subset.

Lemma 2.14. *Let $f : X \to X$ be a mapping. If τ_f is connected, then every cyclic subset of $(X, \leq_{\tau_f})$ is finite.*

Proof. Suppose C is a cyclic subset of X. Assume that a is a fixed element of C. Then for $x \in C$ we have $x \leq_{\tau_f} a$ and $a \leq_{\tau_f} x$. Thus, there are non-negative integers m, n such that $x = f^n(a)$ and $a = f^m(x)$. Thus, $a = f^{m+n}(a)$. Therefore, $C \subseteq \{a, f(a), \cdots, f^{m+n-1}(a)\}$. Consequently, C is finite. □

Remark 2.15. Note that in Lemma 2.14, the cyclic $\{a, f(a), \cdots, f^{m+n-1}(a)\}$ is a maximal cyclic. Because if $\{a, f(a), \cdots, f^{m+n-1}(a)\} \subseteq D$ then $d \leq a$ for every $d \in D$. Thus, $d = f^q(a)$ for some non-negative integer q and therefore, $d \in \{a, f(a), \cdots, f^{m+n-1}(a)\}$.

Definition 2.16. Let $(X, \leq)$ be a partially ordered set. We call X a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) if X is $\wedge$-semilattice and every chain of X can be embedded in $\mathbb{Z}$ ($\mathbb{N}$).

Definition 2.17. Let $(X, \leq)$ be a preordered set. We call X cyclic primal if $X = H \cup C$, where C is a finite nontrivial cyclic subset of X and H is either empty set or an $\mathbb{N}$-primal such that every element of C is less than or equal to every element of X.

Example 2.18. For the following functions we will classify the specialization preorder of the functional Alexandroff topology induced on the domain by the function.

(1) For the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = max\{x - 1, 1\}$, the specialization preorder $\leq_{\tau_f}$ is the usual order on $\mathbb{N}$. Thus, $\mathbb{N}$ with respect to $\leq_{\tau_f}$ is a connected $\mathbb{N}$-primal partially ordered set.

(2) For the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = x - 1$, the specialization preorder $\leq_{\tau_g}$ is the usual order on $\mathbb{Z}$. Thus, $\mathbb{Z}$ with respect to $\leq_{\tau_g}$ is a connected $\mathbb{Z}$-primal partially ordered set.

(3) Consider the function $h : \mathbb{Z} \to \mathbb{Z}$ defined by

$$h(x) = \begin{cases} x - 2, & \text{if } x \text{ is even;} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$$

The specialization preorder $\leq_{\tau_h}$ on the set of even numbers is the usual order of $\mathbb{Z}$. However, $1 \leq_{\tau_h} x$ for every odd number x. One can see that every even number is incomparable with every odd number. Therefore, $\mathbb{Z}$ with respect to $\leq_{\tau_h}$ is not connected and consists of two components, $2\mathbb{Z}$ which is a $\mathbb{Z}$-primal with respect to $\leq_{\tau_h}$ and the set of odd integers which is an $\mathbb{N}$-primal with respect to $\leq_{\tau_h}$.

(4) Consider the function $r : \mathbb{N} \to \mathbb{N}$ defined by

$$r(x) = \begin{cases} x-1, & \text{if } x > 1; \\ 4 & \text{if } x = 1. \end{cases}$$

Then $\mathbb{N}$ with respect to $\leq_{\tau_r}$ is connected and cyclic primal such that $\mathbb{N} = \{5, 6, 7, \cdots\} \cup \{1, 2, 3, 4\}$ with $\{5, 6, 7, \cdots\}$ being $\mathbb{N}$-primal and $\{1, 2, 3, 4\}$ being cyclic and we have $1, 2, 3, 4 \leq_{\tau_r} x$ for every $x \in \mathbb{N}$.

Lemma 2.19. *Let $f : X \to X$ be a mapping and τ_f be connected. Then $(X, \leq_{\tau_f})$ is either a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) or a cyclic primal.*

Proof. If $(X, \leq_{\tau_f})$ is a partially ordered set then by Lemma 2.5 and Lemma 2.11, $(X, \leq_{\tau_f})$ is a $\mathbb{Z}$-primal ($\mathbb{N}$-primal). If $(X, \leq_{\tau_f})$ is not a partially ordered set, then there are at least two different elements a and b such that $a \leq_{\tau_f} b$ and $b \leq_{\tau_f} a$. Then by the proof of Lemma 2.14, there is a positive integer m such that $a = f^m(a)$ which results in a maximal finite cyclic $C = \{a, f(a), \cdots, f^{m-1}(a)\}$ that contains both a and b. Next we prove that $z \leq x$ for every $x \in X$ and $z \in C$. Since X is connected, there are $r, s \in \mathbb{N}_0$ such that $f^r(x) = f^s(a)$. Assume $z = f^q(a)$ for some $q \in \mathbb{N}_0$. There is a $j \in \mathbb{N}_0$ such that $q + jm \geq s$. Thus, $z = f^q(a) = f^{q+jm}(a) = f^{(q+jm-s)+s}(a) = f^{(q+jm-s)+r}(x) \leq x$. It remains to show that $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is either the empty set or an $\mathbb{N}$-primal. If $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is the empty set, then $X = \{a, f(a), \cdots, f^{m-1}(a)\}$ and therefore, X is cyclic. So, suppose $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is non-empty.

We show the set $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is cyclic free; meaning that any cyclic of $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ must be singleton. By the way of contradiction suppose J is a cyclic of $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ with at least two elements u and v. Using $u \leq v$ and $v \leq u$ will result in the existence of a positive integer k with $f^k(u) \neq u$ and $f^{k+1}(u) = u$. Since τ_f is connected, there are non-negative integers r, s such that $f^r(a) = f^s(u)$. Now, $u = f^{k+1}(u) = f^{s(k+1)}(u) = f^{sk+s}(u) = f^{sk+r}(a) \in \{a, f(a), \cdots, f^{m-1}(a)\}$ which is a contradiction. Thus, $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is cyclic free.

Now we prove that $X \setminus \{a, f(a), \cdots, f^{n-1}(a)\}$ is a partially ordered set. By the way of contradiction suppose there are two different elements $e, k \in X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ such that $e \leq k$ and $k \leq e$. Thus, $\{e, k\}$ will be a nontrivial cyclic of $X \setminus \{a, f(a), \cdots, f^{n-1}(a)\}$ which by the previous paragraph is impossible. Thus, $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is a partially ordered set.

Next we prove that $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ is an $\mathbb{N}$-primal.

Define $g : X \setminus \{a, f(a), \cdots, f^{m-1}(a)\} \to X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$ by

$$g(x) = \begin{cases} x, & \text{if } x \in f^{-1}(a); \\ f(x) & \text{otherwise} \end{cases}$$

We prove that for every positive integer n, $g^n(x)$ is in the form of $f^i(x)$ where $i \in \{0, \cdots, n\}$. Note that by definition it is true for $k = 1$. Next suppose it is true for $k-1$. Then $g^k(x) = g(g^{k-1}(x)) = \begin{cases} g^{k-1}(x), & \text{if } g^{k-1}(x) \in f^{-1}(a); \\ f(g^{k-1}(x)) & \text{otherwise} \end{cases}$

But by hypothesis, $g^{k-1}(x)$ is in the form of $f^i(x)$, where $i \in \{0, \cdots, k-1\}$. Thus, for every positive integer n we have $g^n(x) = f^i(x)$ where $i \in \{0, \cdots, n\}$.

Next we show that $\leq_{\tau_g} = \leq_{\tau_f}$ on $X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}$. If $x \leq_{\tau_g} y$ then by Lemma 2.2 $x \in \{y, g(y), g^2(y), \cdots\}$. Since for every positive integer n, $g^n(y)$ is in the form of $f^i(y)$ where $i \in \{0, \cdots, n\}$, we have $x \leq_{\tau_f} y$. Thus, $\leq_{\tau_g} \subseteq \leq_{\tau_f}$. On the other hand if $x \leq_{\tau_f} y$, then $x = f^l(y)$ for some non-negative integer l. Note that $g^l(y)$ is in the form of $f^i(y)$ where $i \in \{0, \cdots, l\}$ and so, $g^{l+l-i}(y) = f^{i+l-i}(y) = f^l(y) = x$. Consequently, $x \leq_{\tau_g} y$. Hence, $\leq_{\tau_g} = \leq_{\tau_f}$. Thus, $(X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}, \leq_{\tau_g})$ is a partial order set and therefore, by Lemma 2.11 $(X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}, \leq_{\tau_g})$ is an $\wedge$-semilattice. We show that every chain C of $(X \setminus \{a, f(a), \cdots, f^{m-1}(a)\}, \leq_{\tau_g})$ can be embedded in $\mathbb{N}$. By Lemma 2.5 every chain C can be embedded in $\mathbb{Z}$. So, it is enough to show that C has an smallest element. If t is a fixed element of C then we show that $\downarrow t = \{t, f(t), f^2(t), \cdots\}$ is a finite set and therefore, C has a minimum element. Since X is connected, there are $k_1, k_2 \in \mathbb{N}_0$ such that $f^{k_1}(t) = f^{k_2}(a)$ and so, $f^{mk_1}(t) = f^{mk_2}(a) = a \notin C$. Thus, $\{x \in C : x \leq_{\tau_g} t\} = \downarrow t \cap C \subseteq \{t, f(t), f^2(t), \cdots, f^{mk_1-1}(t)\}$ and therefore, $\downarrow t \cap C$ is finite. Consequently, C can be embedded in $\mathbb{N}$ and the proof is complete. $\square$

We recall that if X is a partially ordered set and $x, y \in X$, $x \prec y$ means $x < y$ and whenever $z < y$ we have $z \leq x$.

Theorem 2.20. *Let τ be a topology on a set X such that every component of $(X, \leq_\tau)$ is either a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) or a cyclic primal. Then there is a function $f : X \to X$ such that τ_f is finer than τ and $\leq_{\tau_f} = \leq_\tau$.*

Proof. First we assume X is connected. If X is $\mathbb{Z}$-primal or $\mathbb{N}$-primal, define $f : X \to X$ by

$$f(x) = \begin{cases} \bigvee\{t : t <_\tau x\}, & \text{if } \{t : t <_\tau x\} \neq \varnothing; \\ x, & \text{if } \{t : t <_\tau x\} = \varnothing \end{cases}.$$

One can see that if $\{t : t <_\tau x\} \neq \varnothing$, $f(x) \prec x$. Also since X is connected and $\mathbb{Z}$-primal ($\mathbb{N}$-primal), one can see that for every $x, y \in X$ with $x <_\tau y$, there are $y_0, \cdots, y_n \in X$ with $x = y_0 \prec \cdots \prec y_n = y$. Thus, by definition of f one can easily verify that $x = f^n(y)$.

We prove that τ_f is finer than τ. Suppose $V \in \tau$. We prove $V \in \tau_f$ or equivalently, $f(X \setminus V) \subseteq X \setminus V$. Note that if $a \in X \setminus V$ and $t \leq_\tau a$, then $t \in cl(\{a\}) \subseteq cl(X \setminus V) = X \setminus V$. In particular, $f(a) \in X \setminus V$ as $f(a) \leq_\tau a$. Thus, $f(X \setminus V) \subseteq X \setminus V$. Consequently, $\tau \subseteq \tau_f$.

The inclusion $\tau \subseteq \tau_f$ implies $\leq_{\tau_f} \subseteq \leq_\tau$. For the converse, let $x <_\tau y$. Then, $x = f^n(y)$ for some positive integer n. On the other hand by Lemma 2.2 closure of $\{y\}$ with respect to τ_f is $\{y, f(y), f^2(y), \cdots\}$. Hence, $x <_{\tau_f} y$. Consequently, $\leq_{\tau_f} = \leq_\tau$.

For the case that X is a cyclic primal, assume $X = H \cup C$, where $C = \{c_1, \cdots, c_m\}$ is a finite cyclic subset of X and H is either the empty set or an $\mathbb{N}$-primal such that every element of C is less than or equal to every element of X.

If $H = \varnothing$, then $X = \{c_1, \cdots, c_m\}$. Define $f : C \to C$ by $f(x_i) = x_{i+1}$ for $i = 1, \cdots, m-1$ and $f(x_m) = x_1$. It is straightforward to verify that $\tau_f = \tau = \{\varnothing, X\}$ and therefore, $\leq_{\tau_f} = \leq_\tau$ and we leave it to the reader.

In the case that H is an $\mathbb{N}$-primal, define $f : X \to X$ by

$$f(x) = \begin{cases} \bigvee\{t \in H \setminus C : t < x\}, & \text{if } x \in H \setminus C \text{ and } \{t \in H \setminus C : t < x\} \neq \varnothing; \\ x, & \text{if } x \in H \setminus C \text{ and } \{t \in H \setminus C : t < x\} = \varnothing \\ x_{i+1}, & \text{if } x = x_i,\ i = 1, \cdots, m-1 \\ x_1, & \text{if } x = x_m \end{cases}.$$

Similar to the other case one can prove that τ_f is finer than τ and $\leq_{\tau_f} = \leq_\tau$ and we leave it to the reader.

For the case that X is not connected, assume $X = \bigcup_{i \in I} X_i$ with $|I| > 1$ where X_i's are the components of X. Then define $f : X \to X$ such that $f(x) = f_i(x)$ for every $x \in X_i$ where f_i is defined on the connected components X_i exactly the way we defined f on a connected case in the first part. Since X_i's form a partition for X, f is well-defined. It is straightforward to verify that $\tau \subseteq \tau_f$ and $\leq_{\tau_f} = \leq_\tau$ and we leave it to the reader. □

Theorem 2.21. *Let $f : X \to X$ be a mapping. If τ_f is compact, then for every $a \in X$ there is a positive integer n such that $f^n(a) = f^{n-1}(a)$. The converse is true if and only if τ_f has finitely many component.*

Proof. By the way of contradiction suppose that there is a $z \in X$ such that $f^n(z) \neq f^{n-1}(z)$ for every positive integer n. Note $\{z, f(z), f^2(z), f^3(z), \cdots,\} \supseteq \{f(z), f^2(z), f^3(z), \cdots,\} \supseteq \{f^2(z), f^3(z), \cdots,\} \supseteq \cdots$ is a descending chain of closed sets with $\bigcap_{n \in \mathbb{N}_0} \{f^{n-1}(z), f^n(z), \cdots,\} = \varnothing$. Thus, $X = \bigcup_{n \in \mathbb{N}_0} \{f^{n-1}(z), f^n(z), \cdots,\}^c$. Since X is compact, and $\{f^{n-1}(z), f^n(z), \cdots,\}^c$'s form a chain, $X = \{f^{m-1}(z), f^m(z), \cdots,\}^c$ for some $m \in \mathbb{N}_0$. So, $\{f^{m-1}(z), f^m(z), \cdots,\} = \varnothing$ which is a contradiction.

For the second part, suppose that for every $a \in X$ there is a positive integer n such that $f^n(a) = f^{n-1}(a)$. We prove that τ_f is compact if and only if τ_f has finitely many components. If τ_f has infinity many components, then obviously, τ_f is not compact. For the converse, assume that there are $a_1, \cdots, a_m$ such that $X = [a_1] \cup \cdots \cup [a_m]$. To prove that τ_f is compact suppose $X = \bigcup_{\alpha \in I} V_\alpha$. By the assumption, for every a_i there is an integer r_i such that $f^{r_i}(a_i) = f^{r_i - 1}(a_i)$, $i = 1, \cdots, m$. For every a_i there is an $\alpha_i \in I$ such that $f^{r_i}(a_i) \in V_{\alpha_i}$. We show that $[a_i] \subseteq V_{\alpha_i}$. Suppose $x \in [a_i]$. Thus, there are non-negative integers m and n with $n \geq r_i$ such that $f^m(x) = f^n(a_i)$ and so, $f^m(x) = f^n(a_i) = f^{r_i}(a_i) \in V_{\alpha_i}$. Since V_{α_i} is open and $f^m(x) \in V_{\alpha_i}$, we have $x \in V_{\alpha_i}$. Therefore, $[a_i] \subseteq V_{\alpha_i}$. Consequently, $X = \bigcup_{i=1}^m V_{\alpha_i}$. □

3. Partial metrics and functional Alexandroff topologies

In this section we are investigating when a functional Alexandroff topology comes from a generalized metric. Note that by Lemma 2.1, if a functional Alexandroff topology is T_1, then it is the discrete topology and therefore, is metrizable. So, it seems that the more interesting functional Alexandroff topologies are those that are not T_1. So, we first consider T_0 functional Alexandroff topologies and later on any functional Alexandroff topology. Therefore, we are after distance functions that satisfy weaker axioms than metrics. Partial metrics are one of the distance functions that have a T_0 induced topology and are a good candidate here.

Partial metrics were introduced in the early 1990's in [17], by a computer scientist, Steve Matthews, to deal with the reality of having only partial information and to model the gaining of partial knowledge about ideal objects through a computer program.

Partial metrics are less well known than metrics, but are useful in the study of the asymmetric topologies that arise in domain theory which is a part of computer logic. The structures of domain theory are continuous posets, and each continuous poset has a partial metric on it that gives rise to all three topologies studied there: Scott topology, lower topology, and the join of these two, Lawson topology; see [12]. Each partial metric also has a natural completion which turns out to be the order theoretic round ideal completion, that yields a continuous poset, see [13]. To learn more about partial metrics and their development see [2], [6], [12], [16], [17], [19], and [20].

As you see below in the definition of partial metrics it is not required that the distance of a point from itself be zero.

Definition 3.1. A *partial metric* is a function $p : X \times X \to [0, \infty)$ satisfying the following axioms:

- For every $x, y \in X$, $p(x, y) \geq p(x, x)$,
- For every $x, y \in X$, $p(x, y) = p(y, x)$,
- For every $x, y, z \in X$, $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$,
- For every $x, y \in X$, $x = y$ if $p(x, y) = p(x, x) = p(y, y)$.

One can see that a partial metric is a metric if and only if $p(x, x) = 0$ for every $x \in X$.

A *pseudopartial metric* is a function $p\colon X \times X \to [0, \infty)$ satisfying all of the axioms of partial metrics except the last one.

For a pseudopartial metric p on the set X, $a \in X$, and $r > 0$, define

$N_r(a) = \{x \in X : p(x, a) - p(x, x) \leq r\}$ and $N_r^*(a) = \{x \in X : p(x, a) - p(a, a) \leq r\}$. Then define,

$\tau_p = \{U \subseteq X : a \in U \Rightarrow \exists r > 0 \ N_r(a) \subseteq U\}$ and

$\tau_p^* = \{U \subseteq X : a \in U \Rightarrow \exists r > 0 \ N_r^*(a) \subseteq U\}$.

Then both τ_p and τ_p^* form topologies on the set X and τ_p is called the induced topology by p. Both topologies τ_p and τ_p^* are T_0 if and only if p is a partial metric.

In the following theorem we will prove that every T_0 functional Alexandroff topology is partial metrizable.

Theorem 3.2. *Let $f : X \to X$ be a mapping. If τ_f is T_0 then there is a partial metric $p : X \times X \to \mathbb{R}$ such that $\tau_f = \tau_p$.*

Proof. Suppose $f : X \to X$ is a function such that τ_f is T_0. First assume τ_f is connected. Then, $X = [a]$ for every $a \in X$. Since τ_f is T_0, by then by Lemma 2.5 and Lemma 2.11, $(X, \leq_{\tau_f})$ is either $\mathbb{N}$-primal or $\mathbb{Z}$-primal. We prove the theorem separately for each case.

Case 1 ($\mathbb{Z}$-primal case): Here we assume that X is $\mathbb{Z}$-primal and not $\mathbb{N}$-primal as we prove the case of $\mathbb{N}$-primal later. Since $(X, \leq_{\tau_f})$ is $\mathbb{Z}$-primal and not $\mathbb{N}$-primal, there exists a $b \in X$ such that $b > f(b) > f^2(b) > \cdots$. Since X is connected by Lemma 2.8, $X = \bigcup_{m,n\in\mathbb{N}_0} f^{-m}(f^n(b))$. Now define $\alpha : X \to \mathbb{R}$ such that for $x \in f^{-m}(f^n(b))$,

$$\alpha(x) = \begin{cases} \frac{1}{-m+n+2}, & \text{if } -m+n \geq 0; \\ \frac{-m+n-1}{-m+n-2}, & \text{if } -m+n < 0. \end{cases}$$

First we show that α is well defined. It is enough to prove that if $x \in f^{-r}(f^s(b)) \cap f^{-m}(f^n(b))$ then $-r+s = -m+n$. If $x \in f^{-r}(f^s(b)) \cap f^{-m}(f^n(b))$ then $f^s(b) = f^r(x)$ and $f^n(b) = f^m(x)$. Either $r \leq m$ or $m \leq r$. Without lost of generality assume $r \leq m$. Thus, $f^n(b) = f^m(x) = f^{(m-r)+r}(x) = f^{(m-r)}(f^s(b))$ and so, $n = m - r + s$ or $-r + s = -m + n$.

Next we show that $x < y$ implies $\alpha(x) < \alpha(y)$. Since $x < y$ there is a positive integer l such that $x = f^l(y)$. On the other hand, there are non-negative integers r, s, m and n such that $f^r(x) = f^s(b)$ and $f^m(y) = f^n(b)$. Either $l \leq m$ or $l > m$.

If $l \leq m$, then $f^n(b) = f^m(y) = f^{m-l}(f^l(y)) = f^{m-l}(x)$. Therefore, $x \in f^{-(m-l)}(f^n(b)) \cap f^{-r}(f^s(b))$ and so, $-r + s = -m + l + n > -m + n$. If $-m + n \geq 0$, then $\alpha(y) = \frac{1}{-m+n+2} > \frac{1}{-r+s+2} = \alpha(x)$. If $-m + n < 0$ then, $\alpha(y) = \frac{-m+n-1}{-m+n-2}$. On the other hand, if $-r + s \geq 0$ we have $\alpha(x) = \frac{1}{-r+s+2} \leq \frac{1}{2} < \frac{-m+n-1}{-m+n-2} = \alpha(y)$ and if $-r + s < 0$ we have $\alpha(x) = \frac{-r+s-1}{-r+s-2} < \frac{-m+n-1}{-m+n-2} = \alpha(y)$ as $-r + s = -m + l + n > -m + n$.

If $l > m$, then $x = f^l(y) = f^{l-m}(f^m(y)) = f^{l-m}(f^n(b))$. So, $x \in f^{l-m}(f^n(b))$ with $l - m + n > 0$. Thus, $\alpha(x) = \frac{1}{l-m+n+2}$. On the other hand, either $\alpha(y) = \frac{1}{-m+n+2}$ if $-m + n \geq 0$ or $\alpha(y) = \frac{-m+n-1}{-m+n-2}$ if $-m + n < 0$ and in either case we have $\alpha(y) > \alpha(x)$.

Now define $p_a : X \times X \to \mathbb{R}$ by $p_a(x, y) = \alpha(x) + \alpha(y) - \alpha(x \wedge y)$. We prove that p_a is a partial metric and $\tau_{p_a} = \tau_f$. One can easily verify that since α is order preserving, for every $x, y \in X$, $p_a(x, y) \geq p_a(x, x)$. It is obvious that $p_a(x, y) = p_a(y, x)$. For the triangularity, we have to show for every x, y, z we have $p_a(x, z) + p_a(y, y) \leq p_a(x, y) + p_a(y, z)$, or $\alpha(x) + \alpha(z) - \alpha(x \wedge z) + \alpha(y) \leq \alpha(x) + \alpha(y) - \alpha(x \wedge y) + \alpha(z) + \alpha(y) - \alpha(z \wedge y)$ or $\alpha(x \wedge y) + \alpha(z \wedge y) \leq \alpha(x \wedge z) + \alpha(y)$. By Lemma 2.13 either $x \wedge y \leq x \wedge z$ or $y \wedge z \leq x \wedge z$. If $x \wedge y \leq x \wedge z$, since α is order preserving, $\alpha(x \wedge y) \leq \alpha(x \wedge z)$. On the other hand, α being order preserving implies $\alpha(y \wedge z) \leq \alpha(y)$ and therefore, $\alpha(x \wedge y) + \alpha(z \wedge y) \leq \alpha(x \wedge z) + \alpha(y)$. For the case that $y \wedge z \leq x \wedge z$ similarly, can be proved $\alpha(x \wedge y) + \alpha(z \wedge y) \leq \alpha(x \wedge z) + \alpha(y)$.

For the last property note that $p_a(x, y) = p_a(x, x) = p_a(y, y)$ implies $\alpha(y) = \alpha(x \wedge y)$ and $\alpha(x) = \alpha(x \wedge y)$. Thus, $x \wedge y = x$ as $x \wedge y < x$ implies $\alpha(x \wedge y) < \alpha(x)$. Similarly, $x \wedge y = y$. Therefore, $x \wedge y = x = y$.

We now show that $\tau_{p_a} = \tau_f$. Suppose $U \in \tau_f$. We show that $U \in \tau_{p_a}$. If U is the empty set, we are done. So, let $x \in U$. We show that there is an $r > 0$ such that $N_r(x) \subseteq U$. Either $\alpha(x) = \frac{1}{n+2}$ or $\alpha(x) = \frac{n+1}{n+2}$ for some non-negative integer number n. If $\alpha(x) = \frac{1}{n+2}$, let $r = \frac{1}{2}(\frac{1}{n+2} - \frac{1}{n+3})$ and if $\alpha(x) = \frac{n+1}{n+2}$ let $r = \frac{1}{2}(\frac{n+1}{n+2} - \frac{n}{n+1})$. We prove that $N_r(x) \subseteq U$. Note that $N_r(x) = \{y : p_a(x, y) - p_a(y, y) \leq r\} = \{y : \alpha(x) - \alpha(x \wedge y) \leq r\}$. For every $y \in X$, either $\alpha(x) - \alpha(x \wedge y) = 0$ or $\alpha(x) - \alpha(x \wedge y) > 0$. If $\alpha(x) = \alpha(x \wedge y)$, since x and $x \wedge y$ are comparable, $x = x \wedge y$ or $x \leq y$. In this case $y \in U$, since U is open in τ_f. If $\alpha(x) - \alpha(x \wedge y) > 0$ then by the way α is defined, $\alpha(x) - \alpha(x \wedge y) \geq \alpha(x) - \alpha(f(x)) > r$ and therefore, $y \notin N_r(x)$. Thus, $N_r(x) \subseteq \{y : \alpha(x) - \alpha(x \wedge y) = 0\} = \{y : x \leq y\} \subseteq U$.

Next assume $U \in \tau_{p_a}$. We show that $U \in \tau_f$ or equivalently, $f(X \setminus U) \subseteq (X \setminus U)$. By the way of contradiction suppose $t \in X \setminus U$ but $f(t) \in U$. Since U is open in τ_{p_a}, there is an $r > 0$ such that $N_r(f(t)) \subseteq U$. By definition, $N_r(f(t)) = \{y : \alpha(f(t)) - \alpha(f(t) \wedge y) \leq r\}$. So, $t \in N_r(f(t))$ as $t \geq f(t)$. This is a contradiction with $t \in X \setminus U$. Thus, $f(X \setminus U) \subseteq (X \setminus U)$. Consequently, $\tau_{p_a} \subseteq \tau_f$.

Case 2 ($\mathbb{N}$-primal case): If $(X, \leq_{\tau_f})$ is $\mathbb{N}$-primal, then every maximal chain containing $\{a, f(a), f^2(a), \cdots\}$ has an smallest element. Therefore, there is a non-negative integer number such that $f^m(a) = f^{m+1}(a)$. Let $b = f^t(a)$ where $t = min\{i \in \mathbb{N}_0 : f^i(a) = f^{i+1}(a)\}$. Let $B_0 = \{b\}$, $B_1 = f^{-1}(b) \setminus B_0$, $B_2 = f^{-2}(b) \setminus (B_0 \cup B_1)$,..., $B_n = f^{-n}(b) \setminus (\bigcup_{i=0}^{n-1} B_i)$. Note that for every $x \in X$ there are non-negative integers r and s such that $f^r(a) = f^s(x)$ and so, $f^{s+t}(x) = f^{r+t}(a) = f^r(b) = b$. Thus, $x \in f^{-(s+t)}(b)$. Consequently, $x \in \bigcup_{i \in \mathbb{N}_0} B_i$. Thus, $X = \bigcup_{i \in \mathbb{N}_0} B_i$.

Now define $\alpha : X \to \mathbb{R}$ such that $\alpha(x) = \frac{n+1}{n+2}$ if $x \in B_n$. We prove that $P_a : X \times X \to \mathbb{R}$ such that $p_a(x, y) = \alpha(x) + \alpha(y) - \alpha(x \wedge y)$ is a partial metric such that $\tau_f = \tau_{p_a}$.

First we show that $x < y$ implies $\alpha(x) < \alpha(y)$. Suppose $y \in B_j$. Thus, $\alpha(y) = \frac{j+1}{j+2}$. Since $x < y$, $x = f^l(y)$ for some positive integer l. Either $l \leq j$ or $j < l$. If $l \leq j$, then $b = f^j(y) = f^{j-l}(f^l(y)) = f^{j-l}(x)$. Thus, $x \in B_k$ with $k \leq j - l < j$. Thus, $\alpha(x) = \frac{k+1}{k+2} < \frac{j+1}{j+2} = \alpha(y)$. In case that $j < l$ we have, $x = f^l(y) = f^{l-j+j}(y) = f^{l-j}(f^j(y)) = f^{l-j}(b) = b$. Thus, $B_0 = \{x\}$ and therefore, $\alpha(x) = \frac{1}{2}$. Since $x \neq y$, we have $y \notin B_0$. Thus, $j > 0$. Hence, $\alpha(y) = \frac{j+1}{j+2} > \frac{0+1}{0+2} = \alpha(x)$.

In this case similar to Case 1 we use α being order preserving to show that p_a is a partial metric. Also similarly it can be shown that $\tau_{p_a} = \tau_f$.

For the case that X is not connected, $X = \bigcup_{i\in I}[a_i]$ such that $|I| > 1$. Define $p : X \times X \to \mathbb{R}$ by

$$p(x,y) = \begin{cases} \frac{p_{a_k}(x,y)}{1+p_{a_k}(x,y)}, & \text{if } \exists k \text{ such that } x, y \in P_{a_k}; \\ 1 & \text{otherwise} \end{cases}$$

Note that since $[a_i]$'s are disjoint, p is well defined. The proof that p is a partial metric can be found in [18]. Since for every partial metric the induced topology by the partial metric $\frac{p}{1+p}$ is the same as the induced metric by p, we will have $\tau_f = \tau_p$ in this case. □

Since the partial metric in Theorem 3.2 is defined based on the specialization order, Theorem 2.20 and Theorem 3.2 yield to the following corollary.

Corollary 3.3. *Let τ be a topology on a set X such that every component of X is a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) with respect to $\leq_\tau$. Then τ is partial metrizable.*

Theorem 3.4. *Every functional Alexandroff topology is pseudopartial metrizable.*

Proof. Suppose $f : X \to X$ is a mapping. We prove that there is a pseudopartial metric $p : X \times X \to \mathbb{R}$ such that $\tau_f = \tau_p$. If τ_f is T_0, then by Theorem 3.2 τ_f is partial metrizable and therefore, pseudopartial metrizable. So, assume τ_f is not T_0. We first assume τ_f is connected. So, suppose $X = [a]$ for a fixed $a \in X$. By Lemma 2.19 $(X, \leq_{\tau_f})$ is cyclic primal as in the case of $\mathbb{Z}$-primal ($\mathbb{N}$-primal) τ_f is T_0. Thus, $X = H \cup C$, where C is a finite nontrivial cyclic subset of X and H is either empty set or an $\mathbb{N}$-primal such that every element of C is less than or equal to every element of X. If $H = \varnothing$ then $X = C$. In this case define $e_a : X \times X \to \mathbb{R}$ by $e_a(x,y) = 0$ for every $x, y \in X$. Then one can easily show that e_a is a pseudopartial metric and $\tau_{e_a} = \tau_f = \{\varnothing, X\}$. In the case that $H \neq \varnothing$, define $e_a : X \times X \to \mathbb{R}$ by using the partial metric p_a that was defined in Theorem 3.2 as following

$$e_a(x,y) = \begin{cases} p_a(x,y), & \text{if } x, y \in X \setminus C \\ & \text{and } x \in (\uparrow\downarrow \{y\} \cup \uparrow\downarrow\uparrow\downarrow \{y\} \cup \cdots) \cap (X \setminus C); \\ p_a(x,x) + p_a(y,y), & \text{if } x, y \in X \setminus C \\ & \text{and } x \notin (\uparrow\downarrow \{y\} \cup \uparrow\downarrow\uparrow\downarrow \{y\} \cup \cdots) \cap (X \setminus C); \\ 0, & \text{if } x, y \in C; \\ p_a(x,x) & \text{if } x \in H \setminus C \text{ and } y \in C, \\ p_a(y,y) & \text{if } x \in C \text{ and } y \in H \setminus C. \end{cases}$$

It is straightforward to prove that e_a is a pseudopartial metric and $\tau_{e_a} = \tau_f$ and we will leave it to the reader.

In the case that τ_f is not connected, we define a pseudopartial metric on X based on the pseudopartial metrics on the components as we did in Theorem 3.2 □

Corollary 3.5. *Let τ be a topology on a set X such that every component of $(X, \leq_\tau)$ is either a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) or a cyclic primal. Then there is a pseudopartial metric p such that $\leq_\tau = \leq_p$.*

Proof. Suppose τ is a topology on a set X such that every component of $(X, \leq_\tau)$ is either a $\mathbb{Z}$-primal ($\mathbb{N}$-primal) or a cyclic primal. Then by Theorem 2.20 there is a function $f : X \to X$ such that $\leq_\tau = \leq_{\tau_f}$. Then, by Theorem 3.4 there is a pseudopartial metric p such that $\leq_{\tau_f} = \leq_p$ and so, $\leq_\tau = \leq_p$. □

4. Conclusion

The focus of this article is mainly about general metrizability of functional Alexandroff topologies which enables us to talk about distance. In other words, it connects these topologies with analysis. These topologies might have interesting properties in the context of topological groups when the domain of the function is a group rather than a set and the mapping is a group homomorphism.

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