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Some results on weaker forms of star^n-CCC, weakly Lindelöf and star^n-DCCC spaces

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Abstract

In this paper we provide some general results about topological spaces X satisfying any of the following properties: star^n-$\mathcal{P}$, weakly star^n-$\mathcal{P}$ or almost star^n-$\mathcal{P}$, for $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$, where κ is an infinite cardinal number. The particular cases when $\kappa = \omega$, $\mathcal{P} \in \{\text{CCC}, \text{weakly Lindelöf}, \text{DCCC}\}$ are obtained. Furthermore, for the same classes of spaces defined by such $\mathcal{P}$, by applying Erdős-Radó's theorem and using the rank l-diagonal notion, we establish some cardinal inequalities.

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1. Introduction

Let $A \subseteq X$ and $\mathcal{U}$ be a family of subsets of X, the *star of A* with respect to the family $\mathcal{U}$, is defined by $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \varnothing\}$. Recursively, $\mathrm{St}^0(A,\mathcal{U}) = A$ and for each $n \in \omega$, with $0 < n$, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$. As usual, we write $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$.

Given a topological space X and a topological property $\mathcal{P}$, the generic notion of star-$\mathcal{P}$ property was introduced in [15] by van Mill, Tkachuk and Wilson. A space X is called star-$\mathcal{P}$, if for every open cover $\mathcal{U}$ of X, there exists a subspace Y of X such that Y satisfies the property $\mathcal{P}$ and $\mathrm{St}(Y,\mathcal{U}) = X$. Furthermore, some star-type properties have been studied during the last decades (see [7, 13]) and more recently in [2, 24, 26]. Establishing these types of properties generically allows a better understanding of the subject and opens the possibility to consider variants of these notions. For instance, if $\mathcal{P}$ is a topological property, we say that X is weakly star-$\mathcal{P}$ (respectively, almost star-$\mathcal{P}$) if for every open cover $\mathcal{U}$ of X, there exist a subspace Y of X such that $\mathrm{cl}(\mathrm{St}(Y,\mathcal{U})) = X$ (respectively, $\bigcup\{\mathrm{cl}(\mathrm{St}(y,\mathcal{U}) : y \in Y\}) = X$) (see [17, 18]).

In this paper, following some ideas from [25], in Section 2, we provide some general results about star-$\mathcal{P}$, weakly star-$\mathcal{P}$ and almost star-$\mathcal{P}$ spaces, for $\mathcal{P} \in \{\kappa\mathrm{CC}, W\kappa L, \mathrm{D}\kappa\mathrm{CC}\}$. Furthermore, in Section 3 we obtain some cardinal inequalities, using Erdős-Radó's theorem and the rank l-diagonal notion.

2. Some general results

In this section, we obtain some general results about topological spaces X which satisfy any of the properties: star^n-$\mathcal{P}$, weakly star^n-$\mathcal{P}$ or almost star^n-$\mathcal{P}$. Before introducing these notions, we present some basic facts. In what follows, unless otherwise stated, we consider a Hausdorff topological space X and an infinite cardinal number κ. All notations and terminology not explained in this paper can be found in [8, 12]. The next propositions follow immediately from definitions.

Proposition 2.1. *Let X be a topological space X and $A \subseteq X$. For every open cover $\mathcal{U}$ of X and $n \in \omega$, we have that $cl(\mathrm{St}^n(A,\mathcal{U})) \subseteq \mathrm{St}^{n+1}(A,\mathcal{U})$.*

Proposition 2.2. *Let X be a topological space and $\mathcal{U}$ an open cover of X. For every $x, y \in X$, with $x \neq y$, and $n \in \omega$, the next conditions are equivalent:*

(1) $y \notin \mathrm{St}^n(x,\mathcal{U})$;
(2) $x \notin \mathrm{St}^n(y,\mathcal{U})$,
(3) $\mathrm{St}^i(x,\mathcal{U}) \cap \mathrm{St}^j(y,\mathcal{U}) = \varnothing$, *for* $i + j = n$.

Let X be a topological space and $l \in \omega$. A *diagonal sequence of rank l on X* (see [3]), is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^l(x,\mathcal{U}_n) : n \in \omega\}$, for every $x \in X$. A space X has a *rank l-diagonal*, where $l \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank l.

The proof of the next result follows immediately from Erdős-Radó's theorem (see [21, Lemma 3.2]).

Lemma 2.3. *Let X be a space with rank l-diagonal and $Y \subseteq X$. If $|Y| > 2^\kappa$, then for every diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank l, there are $n_0 \in \omega$ and $S \subseteq Y$, such that:*

(1) $|S| > \kappa$;
(2) *S is closed and discrete;*
(3) *For any $x, y \in S$ with $x \neq y$, $y \notin \mathrm{St}^l(x, \mathcal{U}_{n_0})$.*

Note that in Lemma 2.3, the third condition means:

(a) If l is even, the family $\{\mathrm{St}^j(s, \mathcal{U}_{n_0}) : s \in S\}$ is cellular, where $l = 2j$.
(b) If l is odd, the family $\{\mathrm{St}^j(s, \mathcal{U}_{n_0}) : s \in S\}$ is discrete, where $l = 2j+1$.

Recall that for a Hausdorff space X the *Hausdorff pseudo-character of X*, denoted by $H\psi(X)$, is the smallest infinite cardinal κ such that for every $x \in X$, there is a collection $\mathcal{B}_x$ of open neighbourhoods of x with $|\mathcal{B}_x| \leq \kappa$ such that if $x \neq y$, there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ with $U \cap V = \varnothing$ (see [11]). In addition, $d(X)$ denotes the density of X.

Lemma 2.4. *Let X be an infinite topological space, then $|X| \leq d(X)^{H\psi(X)}$.*

Proof. Let $\kappa = H\psi(X)$ and for any $x \in X$, let $\{U^x_\alpha : \alpha \in \kappa\}$ be a family of open neighbourhoods of x, which satisfies the condition of Hausdorff pseudo-character of X. We put $V^x_{\alpha,\beta} = U^x_\alpha \cap U^x_\beta$ and consider a dense subset D of X with $|D| = d(X)$. Now we define $f : X \to D^{\kappa\times\kappa}$ as follows: for each $x \in X$, let $f(x) = f_x$, where $f_x : \kappa \times \kappa \to D$ is given by $f_x(\alpha, \beta) = d^x_{\alpha,\beta}$, where $d^x_{\alpha,\beta}$ is an arbitrary point in $D \cap V^x_{\alpha,\beta}$. We see that f is injective. Indeed, let $x, y \in X$ with $x \neq y$, then there are U^x_α and U^y_β such that $U^x_\alpha \cap U^y_\beta = \varnothing$, so, $V^x_{\alpha,\beta} \cap V^y_{\alpha,\beta} = \varnothing$. Hence, $d^x_{\alpha,\beta} \neq d^y_{\alpha,\beta}$, which implies that $f_x(\alpha, \beta) \neq f_y(\alpha, \beta)$, thus $f(x) \neq f(y)$. ☐

Corollary 2.5. *Let X be an infinite topological space with rank l-diagonal, for $l \geq 2$. Then $|X| \leq d(X)^\omega$.*

In the next definition we present some notions investigated previously; for instance, the star^n-$\mathcal{P}$ property, with different terminology, was introduced in [14, Definition 20] and the almost and weakly star^1-$\mathcal{P}$ properties were defined in [2].

Definition 2.6. Let $\mathcal{P}$ be a topological property and $n \in \omega$. We say that a topological space X is:

- *star^n-$\mathcal{P}$* if for any open cover $\mathcal{U}$ of X, there is $Y \subseteq X$ such that Y satisfies $\mathcal{P}$ and $\mathrm{St}^n(Y, \mathcal{U}) = X$.
- *almost star^n-$\mathcal{P}$* if given any open cover $\mathcal{U}$ of X, there is a subspace $Y \subseteq X$ with property $\mathcal{P}$ and such that $\bigcup\{\mathrm{cl}(\mathrm{St}^n(x, \mathcal{U})) : x \in Y\} = X$.
- *weakly star^n-$\mathcal{P}$* if for every open cover $\mathcal{U}$ of X, there exists $Y \subseteq X$ such that Y satisfies $\mathcal{P}$ and $\mathrm{cl}(\mathrm{St}^n(Y, \mathcal{U})) = X$.

When $n = 1$, we write *star-$\mathcal{P}$*, *almost star-$\mathcal{P}$* and *weakly star-$\mathcal{P}$*, respectively.

Clearly star^n-$\mathcal{P}$ implies almost star^n-$\mathcal{P}$ and almost star^n-$\mathcal{P}$ implies weakly star^n-$\mathcal{P}$. Furthermore, star^n-$\mathcal{P}$, almost star^n-$\mathcal{P}$ and weakly star^n-$\mathcal{P}$ are preserved under continuous functions, when $\mathcal{P}$ does it. Even more, all of these notions can be established in terms of basic open covers. Moreover, it is clear that star^n-$\mathcal{P}$ implies star^{n+1}-$\mathcal{P}$, however the reciprocal does not hold (see Examples 2.10 and 2.11). In addition, we get the following.

Proposition 2.7. *Let X be a topological space.*

(1) *If X contains a dense subspace which is star^n-$\mathcal{P}$, then X is star^{n+1}-$\mathcal{P}$.*
(2) *If X contains a dense subspace which is weakly star^n-$\mathcal{P}$, then X is weakly star^n-$\mathcal{P}$.*

Proof. We will prove just (1). Let Y be a dense subset of X which is star^n-$\mathcal{P}$. Let $\mathcal{U}$ be an open cover of X, then there is $Z \subseteq Y$ which satisfies $\mathcal{P}$ and $Y \subseteq \mathrm{St}^n(Z,\mathcal{U})$. Hence, $\mathrm{St}^{n+1}(Z,\mathcal{U}) = X$. □

Recall that a space X is *Menger* if for each countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a family $\{\mathcal{V}_n : n \in \omega\}$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, for each $n \in \omega$, and $\bigcup_{n\in\omega} \mathcal{V}_n$ is an open cover of X. It follows immediately from the definitions that star^n-countable implies star^n-Menger and star^n-Menger implies star^n-Lindelöf.

Recall that a *cellular family* is a collection of nonempty pairwise disjoint open sets.

Definition 2.8. A space X satisfies the κ *chain condition* (denoted by κCC) if every cellular family in X has cardinality at most κ.

When $\kappa = \omega$, it is said that X satisfies the *countable chain condition* (denoted by CCC) [23, Definition 1.1]. Furthermore, if Y is a dense subset of a space X which satisfies κCC, then X is κCC. The proof of the next result follows immediately.

Proposition 2.9. *If X is a CCC space, then X is star^2-countable.*

Example 2.10. There exists a star^2-countable space which is not star-countable. Indeed, in [18, Example 2.2], it was shown that the space $X = (\beta D \times (\omega_1+1))\backslash((\beta D\backslash D) \times \{\omega_1\})$, as a subspace of the product of $\beta D \times (\omega_1+1)$, where D is a discrete space of cardinality ω_1 and βD denote the Čech-Stone compactification of D, is a Tychonoff almost star-countable space X which is not star-countable. On the other hand, given that $\beta D \times \omega_1$ is dense in X and countably compact, X is star^2-countable.

Example 2.11. Assuming CH, there exists a star^2-Menger space which is not star-Menger. Indeed, let $\mathcal{K}[\mathbb{P}]$ be the space of all compact nowhere dense subsets of $\mathbb{P}$ endowed with the Pixley-Roy topology, where $\mathbb{P}$ is the subspace of irrational numbers. In [6, Example 2.5], it was shown that $\mathcal{K}[\mathbb{P}]$ is a CCC space but is not star-Menger. Note that, by Proposition 2.9, $\mathcal{K}[\mathbb{P}]$ is star^2-Menger.

Question 2.12. *Is there an almost star^n-$\mathcal{P}$ (weakly star^n-$\mathcal{P}$) space which is not almost star-$\mathcal{P}$ (weakly star-$\mathcal{P}$), for some $n \geq 2$ and some property $\mathcal{P}$?*

A collection $\mathcal{U}$ of subsets of X is *discrete* if every point of X has an open neighbourhood which intersects at most one element of $\mathcal{U}$. The next notion can be found in [9, Definition 2.3].

Definition 2.13. Let X be a topological space. We say that X satisfies the *discrete κ chain condition* (denoted by DκCC) if every discrete family of nonempty open subsets of X has cardinality at most κ.

If $\kappa = \omega$ in the previous definition, we obtain the known property *discrete countable chain condition* (denoted by DCCC) [21, Definition 2.1]. Note that κCC implies DκCC.

Recall that a space X is *weakly Lindelöf* if every open cover $\mathcal{U}$ of X contains a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathrm{cl}(\bigcup \mathcal{V}) = X$.

Definition 2.14. We say that a space X is *weakly κ-Lindelöf* (denoted by $W\kappa L$) if every open cover $\mathcal{U}$ of X contains a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \le \kappa$ and $\mathrm{cl}(\bigcup \mathcal{V}) = X$.

Note that a space X is weakly κ-Lindelöf if and only if its weakly Lindelöf number $wL(X) \le \kappa$ (see [4]). It is not difficult to show that, if X is a topological space with $d(X) \le \kappa$, then X is weakly star-$\mathcal{P}$, for each $\mathcal{P} \in \{\kappa\mathrm{CC}, W\kappa L, \mathrm{D}\kappa\mathrm{CC}\}$. Furthermore, from definitions, we immediately obtain that the properties $\kappa\mathrm{CC}, W\kappa L, \mathrm{D}\kappa\mathrm{CC}$ are closed under unions of at most κ many subspaces. It is not difficult to show the next result.

Proposition 2.15. *The following hold:*

(1) *A weakly star-κCC space is weakly star-weakly κ-Lindelöf.*
(2) *A weakly star-weakly κ-Lindelöf space is weakly star-DκCC.*
(3) *A star^2-weakly κ-Lindelöf space is star^2-DκCC.*

The authors do not know if the reciprocal implications of Proposition 2.15 hold.

Remark 2.16. The space X in Example 2.10 satisfies the properties weakly star-CCC and weakly star-weakly Lindelöf, however X is not star-CCC nor star-weakly Lindelöf. Indeed, it can been shown, by using the same ideas as in [19, Example 3.1, for the space S_2], that X is not a star-weakly Lindelöf space. Furthermore, given that every CCC space is weakly Lindelöf, the space X can not be star-CCC. Moreover, given that $\beta D \times \omega_1$ is a dense countably compact subspace of X, we have that $\beta D \times \omega_1$ is star-finite (see [7, Theorem 2.1.4], where the star-finite property is called strongly 1-starcompact). Hence, X is weakly star-finite. Thus, X is weakly star-CCC and so, it is weakly star-weakly Lindelöf.

Lemma 2.17. *Let X and Y be spaces such that $d(X) \le \kappa$. If Y satisfies the property $\mathcal{P}$, then $X \times Y$ satisfies the property $\mathcal{P}$, for $\mathcal{P} \in \{\kappa\mathrm{CC}, W\kappa L, \mathrm{D}\kappa\mathrm{CC}\}$.*

Proof. When $\mathcal{P} \in \{\kappa\mathrm{CC}, \mathrm{D}\kappa\mathrm{CC}\}$, the proof follows the same ideas as [20, Theorem 3.3]. Now suppose that $\mathcal{P} = W\kappa L$. Take a dense subset $D = \{d_\delta : \delta < \kappa\}$ of X. Each $\{d_\delta\} \times Y$ is $W\kappa L$. Then, $D \times Y = \bigcup_{\delta<\kappa}\{d_\delta\} \times Y$ is $W\kappa L$. Now, $D \times Y$ is a dense subset of $X \times Y$. It follows that $X \times Y$ is $W\kappa L$. □

Theorem 2.18. *Let X and Y be spaces such that $d(X) \leq \kappa$. If Y satisfies the property weakly star-$\mathcal{P}$, then $X \times Y$ satisfies the property weakly star-$\mathcal{P}$, for $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$.*

Proof. Let $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$ and suppose that $\mathcal{U}$ is an open cover of basic open sets of $X \times Y$. So, $\{V : U \times V \in \mathcal{U}\}$ is an open cover of Y. Hence, there is $Z \subseteq Y$ which witnesses that Y is weakly star-$\mathcal{P}$. From Lemma 2.17, $X \times Z$ satisfies the property $\mathcal{P}$. Furthermore, $\text{St}(X \times Z, \mathcal{U})$ is dense in $X \times Y$ and the proof is complete. □

Theorem 2.19. *Let X and Y be spaces such that $d(X) \leq \kappa$. If Y satisfies the property almost star-$\mathcal{P}$, then $X \times Y$ satisfies the property star^2-$\mathcal{P}$, for $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$.*

Proof. Let $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$ and suppose that $\mathcal{U}$ is an open cover of basic open sets of $X \times Y$ and $D = \{d_\alpha : \alpha \in \kappa\}$ is a dense subset of X. Then $\mathcal{V}_\alpha = \{\{d_\alpha\} \times V : d_\alpha \in U, U \times V \in \mathcal{U}\}$ is an open cover of $\{d_\alpha\} \times Y$. Hence, there is $\{d_\alpha\} \times W_\alpha \subseteq \{d_\alpha\} \times Y$ which witnesses that $\{d_\alpha\} \times Y$ is almost star-$\mathcal{P}$. Let $Z_\alpha = X \times W_\alpha$. By Lemma 2.17, Z_α satisfies $\mathcal{P}$. So, $Z = \bigcup\{Z_\alpha : \alpha \in \kappa\}$ satisfies $\mathcal{P}$. We claim that $\text{St}^2(Z, \mathcal{U}) = X \times Y$. Indeed, let $(a, b) \in X \times Y$ and consider $U \times V \in \mathcal{U}$ with $(a, b) \in U \times V$. Fix $\alpha \in \kappa$ such that $d_\alpha \in U$. Then $(d_\alpha, b) \in \text{cl}_{\{d_\alpha\} \times Y}(\text{St}((d_\alpha, y), \mathcal{V}_\alpha))$, for some $(d_\alpha, y) \in Z_\alpha$. Hence, there is $\{d_\alpha\} \times B \in \mathcal{V}_\alpha$ such that $y \in B$ and $(\{d_\alpha\} \times V) \cap (\{d_\alpha\} \times B) \neq \varnothing$. Furthermore, there is $A \subseteq X$ with $A \times B \in \mathcal{U}$ and $d_\alpha \in A$. We have that $(A \times B) \cap (U \times V) \neq \varnothing$. Thus, $(a, b) \in \text{St}^2((d_\alpha, y), \mathcal{U})$ and the proof is complete. □

Question 2.20. *Let X and Y be spaces such that $d(X) \leq \kappa$ and suppose that Y satisfies the property almost star-$\mathcal{P}$. Is it true that $X \times Y$ satisfies the property almost star-$\mathcal{P}$, for $\mathcal{P} \in \{\kappa\text{CC}, W\kappa L, \text{D}\kappa\text{CC}\}$?*

The next result generalizes [25, Theorem 3.1] (see Corollary 2.22 below). Recall that a topological space X is *quasi-regular* ([16, p. 2]) if for every nonempty open set U, there is a nonempty open set V such that $\text{cl}(V) \subseteq U$.

Theorem 2.21. *Let X be a quasi-regular topological space. Then X is weakly star-*DκCC *if and only if X is* DκCC.

Proof. Suppose that X is not DκCC and let $\mathcal{W} = \{W_\alpha : \alpha \in \kappa^+\}$ be a discrete family of nonempty open sets in X. Given that X is quasi-regular, we have that for each $\alpha \in \kappa^+$, there is an open subset $V_\alpha \neq \varnothing$ which satisfies $\text{cl}(V_\alpha) \subseteq W_\alpha$.

Let $\mathcal{U} = \mathcal{W} \cup \{X \setminus \text{cl}(\bigcup\{V_\alpha : \alpha \in \kappa^+\})\}$. Since the family $\{V_\alpha : \alpha \in \kappa^+\}$ is discrete, we have that $\bigcup\{\text{cl}(V_\alpha) : \alpha \in \kappa^+\} = \text{cl}\left(\bigcup\{V_\alpha : \alpha \in \kappa^+\}\right)$. So, $\mathcal{U}$ is an open cover of X. Then, there exists $Y \subseteq X$ such that Y is DκCC and $\text{cl}(\text{St}(Y, \mathcal{U})) = X$.

We claim that for each $\alpha \in \kappa^+$, $W_\alpha \cap Y \neq \varnothing$. Indeed, fix $\alpha \in \kappa^+$. Given that $\text{cl}(\text{St}(Y, \mathcal{U})) = X$, we have that $V_\alpha \cap \text{St}(Y, \mathcal{U}) \neq \varnothing$; so, there exist $z \in V_\alpha$, $y_z \in Y$ and $W \in \mathcal{U}$ such that $\{z, y_z\} \subseteq W$. We assert that $W = W_\beta$, for some $\beta \in \kappa^+$. In fact, the case $W = X \setminus \text{cl}(\bigcup\{V_\gamma : \gamma \in \kappa^+\})$ is not possible, otherwise, $z \in V_\alpha \cap (X \setminus \mathrm{cl}(\bigcup\{V_\gamma : \gamma \in \kappa^+\})) \subseteq V_\alpha \cap (X \setminus \bigcup\{V_\gamma : \gamma \in \kappa^+\})$, but $V_\alpha \cap (X \setminus \bigcup\{V_\gamma : \gamma \in \kappa^+\}) = \varnothing$. As $\mathcal{U}$ is a discrete family, necessarily we have $\alpha = \beta$; hence, $y_z \in W_\alpha \cap Y$.

From the claim above, we have that $\{W_\alpha \cap Y : \alpha \in \kappa^+\}$ is a discrete family of nonempty open subsets in Y, which contradicts that Y is DκCC, and the proof is complete. □

Recall that a *neighbourhood assignment* for a topological space (X, τ) is a function $\varphi\colon X \to \tau$ with $x \in \varphi(x)$, for every $x \in X$. The space X is *weakly dually* DCCC [1] if for any neighbourhood assignment φ on X, there exists a subspace DCCC $Y \subseteq X$ such that $\bigcup\{\varphi(y) : y \in Y\}$ is dense in X.

Corollary 2.22. *Let X be a quasi-regular topological space. The following are equivalent:*

(1) *X is* DCCC*;*
(2) *X is weakly dually* DCCC*;*
(3) *X is weakly star-*DCCC.

Proof. (1) ⇒ (2): Follows from definitions.

(2) ⇒ (3): Let $\mathcal{U}$ be an open cover of X. Clearly we can choose a neighbourhood assignment φ such that $\varphi(x) \in \mathcal{U}$, for each $x \in X$. Hence, there exists $Y \subseteq X$ such that Y is DCCC and $\bigcup\{\varphi(y) : y \in Y\}$ is dense in X. Note that for any $y \in Y$, $\varphi(y) \subseteq \mathrm{St}(Y, \mathcal{U})$. Thus, $X = \mathrm{cl}(\bigcup\{\phi(y) : y \in Y\}) \subseteq \mathrm{cl}(\mathrm{St}(Y, \mathcal{U}))$.

(3) ⇒ (1): Follows from Theorem 2.21. □

Question 2.23. *Is there a Hausdorff star^2-*DCCC *space which is not* DCCC*?*

We finish this section with the next result (compare with [25, Theorem 3.2]), before we recall that a space X is *developable* [24, Definition 2.7] if there is a collection of open covers $\{\mathcal{U}_n : n \in \omega\}$ of X such that for every $x \in X$, the family $\{\mathrm{St}(x, \mathcal{U}_n) : n \in \omega\}$ is a local base at x. Moreover, a topological space X is *Baire*, if the countable intersection of open dense sets is dense.

Theorem 2.24. *Let X be a topological space and let $\mathcal{P}$ be a topological property such that the following conditions hold:*

(i) *$\mathcal{P}$ is closed under countable unions;*
(ii) *If $Y \in \mathcal{P}$ and Y is a dense subset in X, then $X \in \mathcal{P}$.*

Then, when X is Baire and a developable space, the properties weakly star-$\mathcal{P}$ and $\mathcal{P}$ coincide.

Proof. Let X be a weakly star-$\mathcal{P}$ space. In order to prove that X satisfies $\mathcal{P}$, we show that X has a dense subset which satisfies $\mathcal{P}$. Consider a collection $\{\mathcal{U}_n : n \in \omega\}$ of open covers which witnesses that X is developable. Now, given that for each $n \in \omega$, $\mathcal{U}_n$ is an open cover of X, there exists $Y_n \subseteq X$ such that Y_n satisfies $\mathcal{P}$ and $\mathrm{cl}(\mathrm{St}(Y_n, \mathcal{U}_n)) = X$. Let $Y = \bigcup\{Y_n : n \in \omega\}$. From (i), we obtain that Y satisfies $\mathcal{P}$.

We claim that Y is dense in X. Indeed, let $D = \bigcap_{n\in\omega} \mathrm{St}(Y_n, \mathcal{U}_n)$. As X is a Baire space, we have that D is dense in X. Hence, for each nonempty open set W, there is $d \in D \cap W$. Thus, from the fact that X is developable, we get $n_d \in \omega$, such that $d \in \mathrm{St}(d, \mathcal{U}_{n_d}) \subseteq W$. Given that $D \subseteq \mathrm{St}(Y_{n_d}, \mathcal{U}_{n_d})$, there are $y_d \in Y_{n_d}$ and $V \in \mathcal{U}_{n_d}$ which satisfy $d, y_d \in V$; this implies that $y_d \in \mathrm{St}(d, \mathcal{U}_{n_d}) \subseteq W$. So, $y_d \in W \cap Y$. We conclude that Y is dense in X and by (ii), X satisfies $\mathcal{P}$. □

We say that a topological space X is *κ-separable* if its density $d(X) \leq \kappa$.

Corollary 2.25. *Consider $\mathcal{P} \in \{$weakly κ-Lindelöf, κCC, κ-separable$\}$. In the class of Baire and developable spaces, the property weakly star-$\mathcal{P}$ coincides with the property $\mathcal{P}$.*

3. Some cardinal inequalities

In this section we show some cardinal inequalities for topological spaces with rank l-diagonal which satisfies star^n-$\mathcal{P}$, weakly star^n-$\mathcal{P}$ or almost star^n-$\mathcal{P}$, for $\mathcal{P} \in \{\kappa\mathrm{CC}, \mathrm{D}\kappa\mathrm{CC}\}$ and some $l, n \in \omega$. The proofs of Theorems 3.1, 3.3 and 3.6 generalize and follow, respectively, the same structure as [25, Theorems 3.8, 3.9 and 3.10].

Theorem 3.1. *Let X be a weakly star^n-κCC and Baire topological space with rank $2n$-diagonal for some $n \in \omega$. Then $|X| \leq 2^\kappa$.*

Proof. Given that X has rank $2n$-diagonal, there is a countable family $\{\mathcal{U}_l : l \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^{2n}(x, \mathcal{U}_l) : l \in \omega\}$. Hence, if $x, y \in X$ with $x \neq y$, then there exists $l \in \omega$, such that $y \notin \mathrm{St}^{2n}(x, \mathcal{U}_l)$.

Note that for any $l \in \omega$, there is $Y_l \subseteq X$ such that Y_l satisfies κCC and $\mathrm{cl}(\mathrm{St}^n(Y_l, \mathcal{U}_l)) = X$. Let $D = \bigcap_{l \in \omega} \mathrm{St}^n(Y_l, \mathcal{U}_l)$. As X is a Baire space, we have that D is dense in X.

We claim that $|D| \leq 2^\kappa$. Indeed, suppose that $|D| > 2^\kappa$. Then, by Lemma 2.3, there exist $m \in \omega$ and $S \subseteq D$, such that: $|S| > \kappa$, S is closed and discrete and for any $x, y \in S$, with $x \neq y$, $y \notin \mathrm{St}^{2n}(x, \mathcal{U}_m)$. Let $s \in S$, since $S \subseteq D \subseteq \mathrm{St}^n(Y_m, \mathcal{U}_m)$, there is $y_s \in Y_m$ such that $s \in \mathrm{St}^n(y_s, \mathcal{U}_m)$, so $y_s \in \mathrm{St}^n(s, \mathcal{U}_m)$. That is, for any $s \in S$, $\mathrm{St}^n(s, \mathcal{U}_m) \cap Y_m \neq \varnothing$. Hence, $\{\mathrm{St}^n(x, \mathcal{U}_m) \cap Y_m : x \in S\}$ is a cellular family in Y_m with $|\{\mathrm{St}^n(x, \mathcal{U}_m) \cap Y_m : x \in S\}| > \kappa$, which contradicts that Y_m is κCC. So, the claim holds.

From Corollary 2.5, $|X| \leq |D|^\omega \leq 2^\kappa$ and the proof is complete. □

Corollary 3.2. *Let X be a weakly star-κCC and Baire topological space with rank 2-diagonal. Then $|X| \leq 2^\kappa$.*

Theorem 3.3. *Let X be a star^2-κCC topological space with rank 4-diagonal. Then $|X| \leq 2^\kappa$.*

Proof. Suppose that $|X| > 2^\kappa$. Since X has rank 4-diagonal, there is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^4(x, \mathcal{U}_n) : n \in \omega\}$. Then, by Lemma 2.3, there exist $m \in \omega$ and $S \subseteq X$, such that: $|S| > \kappa$, S is closed and discrete and for any $x, y \in S$, with

$x \neq y$, $y \notin \mathrm{St}^4(x,\mathcal{U}_m)$. That is, the collection $\{\mathrm{St}^2(s,\mathcal{U}_m) : s \in S\}$ is a cellular family.

We claim that the collection $\{\mathrm{St}(s,\mathcal{U}_m) : s \in S\}$ is discrete. Otherwise, there exist $x_0 \in X$ and $U \in \mathcal{U}_m$ such that $x_0 \in U$ and $|\{s \in S : \mathrm{St}(s,\mathcal{U}_m) \cap U \neq \varnothing\}| > 1$. Let $s_1, s_2 \in S$, with $s_1 \neq s_2$ such that $U \cap \mathrm{St}(s_i,\mathcal{U}_m) \neq \varnothing$, for $i \in \{1,2\}$. Thus $U \subseteq \mathrm{St}^2(s_1,\mathcal{U}_m)$, hence $\mathrm{St}^2(s_1,\mathcal{U}_m) \cap \mathrm{St}(s_2,\mathcal{U}_m) \neq \varnothing$. So, $s_2 \in \mathrm{St}^3(s_1,\mathcal{U}_m)$, which is a contradiction.

From the previous claim, $\bigcup\{\mathrm{cl}(\mathrm{St}(s,\mathcal{U}_m)) : s \in S\} = \mathrm{cl}(\bigcup\{\mathrm{St}(s,\mathcal{U}_m) : s \in S\}) = \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))$. For any $x \in \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))$, there is $s_x \in S$ such that $x \in \mathrm{cl}(\mathrm{St}(s_x,\mathcal{U}_m)) \subseteq \mathrm{St}^2(s_x,\mathcal{U}_m)$ (see Proposition 2.1). Let $U_x \in \mathcal{U}_m$ such that $x \in U_x$ and $U_x \cap \mathrm{St}(s_x,\mathcal{U}_m) \neq \varnothing$. We define $\mathcal{U} = \{U_x : x \in \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))\} \cup \{X \setminus \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))\}$. As $\mathcal{U}$ is an open cover of X, there exists $Y \subseteq X$ such that $\mathrm{St}^2(Y,\mathcal{U}) = X$ and Y is κCC.

We assert that for every $s \in S$, it holds that $\mathrm{St}^2(s,\mathcal{U}_m) \cap Y \neq \varnothing$. Indeed, let $s \in S$. Given that $s \in \mathrm{St}^2(Y,\mathcal{U})$, there are $U, V \in \mathcal{U}$ and $y \in Y$ such that $y \in V$, $s \in U$ and $U \cap V \neq \varnothing$. As $s \in U$, $U \subseteq \mathrm{St}(S,\mathcal{U}_m)$. Hence, $V \neq X \setminus \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))$. Then, $V = U_z$, for some $z \in \mathrm{cl}(\mathrm{St}(S,\mathcal{U}_m))$. From which, $V \subseteq \mathrm{St}^2(s,\mathcal{U}_m)$ and thus, $\mathrm{St}^2(s,\mathcal{U}_m) \cap Y \neq \varnothing$. Finally, we conclude that the collection $\{\mathrm{St}^2(s,\mathcal{U}_m) \cap Y : s \in S\}$ is a cellular family in Y with cardinality $> \kappa$, which is a contradiction. □

Corollary 3.4. *Let X be a star^2-CCC topological space with rank 4-diagonal. Then $|X| \leq 2^\omega$.*

Question 3.5. *Suppose that X is a star^n-κCC topological space with rank $2n$-diagonal, for $n > 2$. Does $|X| \leq 2^\kappa$ hold?*

Theorem 3.6. *Let X be a weakly star^n-DκCC Baire topological space with rank $(2n+1)$-diagonal for some $n \in \omega$. Then $|X| \leq 2^\kappa$.*

Proof. Let $\{\mathcal{U}_l : l \in \omega\}$ be a countable family of open covers which witnesses that X has rank $(2n+1)$-diagonal. Now, for any $l \in \omega$, there is $Y_l \subseteq X$ such that Y_l satisfies DκCC and $\mathrm{cl}(\mathrm{St}^n(Y_l,\mathcal{U}_l)) = X$. Let $D = \bigcap_{l\in\omega} \mathrm{St}^n(Y_l,\mathcal{U}_l)$. As X is a Baire space, we have that D is dense in X.

We claim that $|D| \leq 2^\kappa$. Indeed, suppose that $|D| > 2^\kappa$. Then, by Lemma 2.3, there exist $m \in \omega$ and $S \subseteq D$, such that: $|S| > \kappa$, S is closed and discrete and for any $x, y \in S$, with $x \neq y$, $y \notin \mathrm{St}^{2n+1}(x,\mathcal{U}_m)$. Note that the collection $\{\mathrm{St}^n(s,\mathcal{U}_m) : s \in S\}$ is discrete. Otherwise, there exists $x_0 \in X$ such that for any open set V with $x_0 \in V$, $|\{s \in S : V \cap \mathrm{St}^n(s,\mathcal{U}_m) \neq \varnothing\}| > 1$. Let $U \in \mathcal{U}_m$ such that $x_0 \in U$. By assumption, there exist $s_1, s_2 \in S$, with $s_1 \neq s_2$ such that $U \cap \mathrm{St}^n(s_1,\mathcal{U}_m) \neq \varnothing$ and $U \cap \mathrm{St}^n(s_2,\mathcal{U}_m) \neq \varnothing$. This implies that $s_1 \in \mathrm{St}^{2n+1}(s_2,\mathcal{U}_m)$, which contradicts the property of S. Given that $S \subseteq D \subseteq \mathrm{St}^n(Y_m,\mathcal{U}_m)$, for any $s \in S$, there is $y_s \in Y_m$ such that $s \in \mathrm{St}^n(y_s,\mathcal{U}_m)$. That is, $y_s \in \mathrm{St}^n(s,\mathcal{U}_m)$, so $\mathrm{St}^n(s,\mathcal{U}_m) \cap Y_m \neq \varnothing$. We conclude that $\{\mathrm{St}^n(s,\mathcal{U}_m) \cap Y_m : s \in S\}$ is a discrete family of open sets in Y_m with cardinality greater than κ, which contradicts that Y_m is DκCC. Therefore, $|D| \leq 2^\kappa$. From Corollary 2.5, $|X| \leq |D|^\omega \leq 2^\kappa$ and the proof is complete. □

Corollary 3.7. *Let X be a weakly star-*DκCC*, Baire topological space with rank 3-diagonal. Then $|X| \leq 2^\kappa$.*

Corollary 3.8. *Let X be a weakly star-weakly κ-Lindelöf and Baire topological space and suppose that it has rank 3-diagonal. Then $|X| \leq 2^\kappa$. In particular, when X is a weakly Lindelöf and Baire space, $|X| \leq 2^\omega$.*

Question 3.9. *Let X be a weakly star-*DκCC *(or weakly star-weakly κ-Lindelöf), Baire topological space and suppose that it has rank 2-diagonal. Can it be proven that $|X| \leq 2^\kappa$?*

Remark 3.10. The results 2.15–2.22, 2.24, 2.25, 3.1, 3.2, 3.6–3.8 hold when we put almost star $\mathcal{P}$, instead of weakly star $\mathcal{P}$.

Theorem 3.11. *If X is a star^n-*DκCC *topological space with rank $(2n+1)$-diagonal. Then $|X| \leq 2^\kappa$.*

Proof. Suppose that $|X| > 2^\kappa$. Consider a countable family $\{\mathcal{U}_m : m \in \omega\}$ of open covers of X such that for every $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^{2n+1}(x, \mathcal{U}_m) : m \in \omega\}$. By Lemma 2.3, there exist $S \subseteq X$ and $m_0 \in \omega$, which satisfy $|S| > \kappa$, S is closed and discrete and for any $x, y \in S$, with $x \neq y$, $y \notin \mathrm{St}^{2n+1}(x, \mathcal{U}_{m_0})$. Note that $\mathcal{U} = \{\mathrm{St}^n(s, \mathcal{U}_{m_0}) : s \in S\}$ is a discrete family in X, as proven in Theorem 3.6.

Since $\mathcal{U}_{m_0}$ is an open cover of X, by hypothesis, there exists $Y \subseteq X$ such that Y is DκCC and $\mathrm{St}^n(Y, \mathcal{U}_{m_0}) = X$. We claim that for every $s \in S$, $Y \cap \mathrm{St}^n(s, \mathcal{U}_{m_0}) \neq \varnothing$. Indeed, let $s_0 \in S$. Given that $\mathrm{St}^n(Y, \mathcal{U}_{m_0}) = X$, there is $U_i \in \mathcal{U}_{m_0}$ such that $s_0 \in U_1$, $U_i \cap U_{i+1} \neq \varnothing$ and $U_n \cap Y \neq \varnothing$. So, $Y \cap \mathrm{St}^n(s_0, \mathcal{U}_{m_0}) \neq \varnothing$. Hence, we conclude that $\{Y \cap \mathrm{St}^n(s, \mathcal{U}_{m_0}) : s \in S\}$ is a discrete family in Y with cardinality greater than κ, which contradicts that Y is DκCC. So $|X| \leq 2^\kappa$. □

Corollary 3.12. *If X is a star-*DCCC *topological space with rank 3-diagonal. Then $|X| \leq 2^\omega$.*

Corollary 3.13. *Let X be a star^2-*DCCC *topological space and suppose that it has rank 5-diagonal. Then $|X| \leq 2^\omega$.*

From Proposition 2.15, we obtain the following.

Corollary 3.14. *Let X be a star^2-weakly κ-Lindelöf topological space and suppose that it has rank 5-diagonal. Then $|X| \leq 2^\kappa$. In particular, when X is weakly Lindelöf with rank 5-diagonal, $|X| \leq 2^\omega$.*

Question 3.15. *Let X be a weakly star-*DκCC *(or weakly star-weakly κ-Lindelöf) topological space. Can it be proven that $|X| \leq 2^\kappa$, if X has rank l-diagonal, for some $l < 5$?*

The next result was proved in [5] for $\kappa = \omega$ and its proof follows the same ideas. Furthermore, from Theorem 2.21 it follows that, in the class of normal spaces, the properties star-DCCC and DCCC are equivalent. Hence, Theorem 3.16 coincides with [21, Theorem 3.4].

Theorem 3.16. *If X is a star-DκCC and normal space with rank 2-diagonal. Then $|X| \leq 2^\kappa$.*

Question 3.17. *Suppose that X is a star^n-DκCC and normal space with rank $2n$-diagonal, for $n > 1$. Does $|X| \leq 2^\kappa$ hold?*

We finish this paper with the next result (compare with [22, Theorem 5.4]). Recall that the *extent* of a space X is defined by $e(X) = \sup\{|A| : A \text{ is a closed subset of } X \text{ and } A = A \setminus A^d\} + \omega$, where A^d denotes the set of all limit points of A. Moreover, if X is a space and $A \subseteq X$, a family $\mathcal{U}$ is an *open expansion of* A if $\mathcal{U} = \{U_a : a \in A\}$ and U_a is an open neighbourhood of a for any $a \in A$.

Theorem 3.18. *If X is a star-DCCC and normal space with $H\psi(X) = \omega$. Then $e(X) \leq 2^\omega$.*

Proof. Suppose that $e(X) > 2^\omega$. That is, there exists a closed and discrete subset E of X such that $|E| > 2^\omega$. We take, for every $x \in X$, the collection $\mathcal{B}_x = \{B_m^x : m \in \omega\}$ of open neighbourhoods of x satisfying the condition for Hausdorff pseudo-character. Suppose that for each $m \in \omega$, $B_{m+1}^x \subseteq B_m^x$. Denote by $\mathcal{P}_m = \{\{x, y\} \in [E]^2 : B_m^x \cap B_m^y = \varnothing\}$, for every $m \in \omega$. It can be shown that $[E]^2 = \bigcup\{\mathcal{P}_m : m \in \omega\}$. So, by Erdős-Radó's theorem (see [10, Theorem 2.3]), there exist $m_0 \in \omega$ and a subset S of E, with $|S| > \omega$ such that $[S]^2 \subseteq \mathcal{P}_{m_0}$. Note that S is a closed and discrete subset of X. Even more, $\{B_{m_0}^s : s \in S\}$ is a disjoint open expansion of S. By [21, Lemma 3.3], there exists a discrete open expansion $\mathcal{V} = \{V_s : s \in S\}$ of S such that $s \in V_s \subseteq B_{m_0}^s$, for each $s \in S$.

Now, we consider the open cover of X, $\mathcal{U} = \{U_x : x \in X\}$, where $U_x = V_x$, if $x \in S$ and when $x \notin S$, U_x is an open neighbourhood of x which witnesses that $\mathcal{V}$ is a discrete family and $U_x \cap S = \varnothing$. Hence, there exists $Y \subseteq X$ such that Y is DCCC and $\mathrm{St}(Y, \mathcal{U}) = X$. We claim that for every $s \in S$, $V_s \cap Y \neq \varnothing$. Indeed, let $s \in S$. Since $\mathrm{St}(Y, \mathcal{U}) = X$, there is $U \in \mathcal{U}$ such that $s \in U$ and $U \cap Y \neq \varnothing$. Note that $U = V_s$ and the claim follows. Thus, $\{V_s \cap Y : s \in S\}$ is a discrete family in Y, which is a contradiction. Therefore, $e(X) \leq 2^\omega$. □

Question 3.19. *Suppose that X is a star^n-DκCC and normal space such that $H\psi(X) = \kappa$, for $n > 1$ and $\kappa > \omega$. Does $e(X) \leq 2^\kappa$ hold?*

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