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k-spaces of non-domain-valued geometric points

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Abstract

The aim of this paper is to study the topological properties of algebraic sets with zero divisors. We impose a subbasic topology on the set of proper ideals of a k-algebra and this new "k-space" becomes a generalization of the corresponding Zariski space. We prove that a k-space is T_0, quasi-compact, spectral, and connected. Moreover, we study continuous maps between such k-spaces. We conclude with a question about construction of a sheaf of k-spaces similar to affine schemes.

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1. Introduction

In the introduction of [2], Grothendieck described the process of getting the spectrum of prime ideals (also called geometric points) starting from a system of polynomial equations. In brief, it is as follows.

Suppose k is a commutative ring with identity. Let $P_I = k[(x_i)_{i\in I}]$ be a ring of polynomials in the indeterminates x_i with coefficients in k, and I be an index set (not necessarily finite). Let $S = \{p_j\}_{j\in J}$ be a system of polynomials of P_I, where the index set J is also not necessarily finite. An element $a = (a_i)_{i\in I}$ of a k-algebra A is called a *solution* of the system S if $p_j(a) = 0$ for all $j \in J$.

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If $\mathbb{A}\mathrm{lg}_k$ and $\mathbb{S}\mathrm{ets}$ respectively denote the categories of k-algebras and sets, then a functor

$$\mathcal{V}_S \colon \mathbb{A}\mathrm{lg}_k \to \mathbb{S}\mathrm{ets}$$

represents the solutions of some system S of polynomial equations with coefficients in k if and only if $\mathcal{V}_S$ is representable, *i.e.*, $\mathcal{V}_S$ is isomorphic to $\mathcal{R}_A$ for some object A in $\mathbb{A}\mathrm{lg}_k$. Conversely, for every object A in $\mathbb{A}\mathrm{lg}_k$ the representable functor $\mathcal{R}_A$ is isomorphic to some $\mathcal{V}_S$. The functor $\mathcal{R}_A$ is called the *affine algebraic space over* k represented by A. The category $\mathbb{A}\mathrm{ff}_k$ of affine algebraic spaces has representable functors $\mathcal{R}_A$ (A is an object in $\mathbb{A}\mathrm{lg}_k$) as objects and morphisms are defined as natural transformations, *i.e.*, they are the induced maps

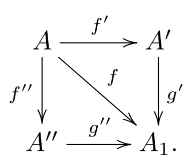
$$\mu(f) \colon \mathrm{Hom}_{\mathbb{A}\mathrm{lg}_k}(A, B) \to \mathrm{Hom}_{\mathbb{A}\mathrm{lg}_k}(A', B)$$

obtained from morphisms $f \colon A' \to A$ in $\mathbb{A}\mathrm{lg}_k$. An A'-*valued point* is a k-algebra homomorphism $f \colon A \to A'$, *i.e.*, f is an element of the set $\mathrm{Hom}_{\mathbb{A}\mathrm{lg}_k}(A, A')$. If we restrict A' to be an object of the full subcategory $\mathbb{F}\mathrm{ield}_k$ of $\mathbb{A}\mathrm{lg}_k$ then the elements of $\mathrm{Hom}_{\mathbb{A}\mathrm{lg}_k}(A, A')$ are called *geometric points*.

We define an equivalence relation between geometric points as follows. We say two geometric points $f' \colon A \to A'$ and $f'' \colon A \to A''$ are equivalent if there exists a third geometric point $f \colon A \to A_1$ and k-algebra morphisms $g' \colon A' \to A_1$ and $g'' \colon A'' \to A_1$ such that

$$f = g'' \circ f'' = g' \circ f' \tag{1.1}$$

i.e., the following diagram commutes:



Since g' and g'' are monomorphisms, we observe that the condition (1.1) is equivalent to $\mathrm{ker} f' = \mathrm{ker} f''$. Therefore, the equivalence classes of the above relation are in bijection with the prime ideals of A. Now there is a bijection between geometric points and prime ideals of a k-algebra A. The *loci* of a k-algebra A is the equivalence classes of geometric points.

The *spectrum* of A (denoted by $\mathrm{Spec}\, A$) is defined as the set of prime ideals of A, *i.e.*, $X = \mathrm{Spec}\, A = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } A\}$. For $S = \{p_j\}_{j \in J}$ be a system of polynomials of P_I, let $\mathcal{V}(S)$ be the subset of $\mathrm{Spec}\, A$ defined by

$$\mathcal{V}(S) = \{\text{set of loci of } u \in \mathcal{R}_A(A') \mid f_i(u) = 0, \forall_{i \in I}\, f_i \in S\},$$

where $f_i(u)$ is defined by $f_i(u) = u(f_i)$. From the above definition of $\mathcal{V}(S)$, we immediately see

$$\mathcal{V}(S) = \{\mathfrak{p} \in X \mid S \subseteq \mathfrak{p}\}.$$

From the above, we observe that in order to obtain the $\mathrm{Spec}\, A$, we worked with the full subcategory $\mathbb{F}\mathrm{ield}_k$ of $\mathbb{A}\mathrm{lg}_k$.

If we remove this 'restriction' on the k-algebra, what we obtain is a spectrum $\operatorname{Idl} A$ of all ideals (instead of prime ideals) of A. This will allow us to study polynomial equations having solutions in any k-algebra (not necessarily a field). Since $A \notin \operatorname{Spec} A$, we also consider the set $\operatorname{Spi} A$ of all proper ideals of A as our 'generalized' spectrum on which we endow a topology and call it a k-space. Our choice of the notation $\operatorname{Spi} A$ is to have an 'alignment' with the notation $\operatorname{Spm} A$ of maximal ideals of A as in Grothendieck [2]. A k-space is a generalization of a Zariski space (*i.e.*, $\operatorname{Spec} A$ endowed with a Zariski topology). The purpose of this paper is to study topological properties of k-spaces and simultaneously compare them with Zariski spaces.

2. k-spaces

To construct a k-space, we use two maps defined in Proposition 2.2. Similar maps also appear when we take values of polynomials over an integral domain (to impose a Zariski topology on a $\operatorname{Spec} A$). Before we discuss properties of these maps, let us see an example in our context.

Example 2.1. We consider a k-algebra with zero divisors and its algebraic sets. The polynomials listed in the Table 1 are of minimal degrees.

Subsets of $\mathbb{Z}_4$	**Polynomials**	**Algebraic sets of $\mathbb{Z}_4$**	**Polynomials**
$\varnothing$	$\mathbb{Z}_4 \setminus \{0\}$	$\varnothing$	1
$\{0\}$	$\{ax\}$	$\{0\}$	x
$\{1\}$	$\{ax+b \mid a+b=0\}$	$\{1\}$	$x+3$
$\{2\}$	$\{ax+b \mid 2a+b=0\}$	$\{2\}$	$(x+2)$
$\{3\}$	$\{ax+b \mid 3a+b=0\}$	$\{3\}$	$x+1$
$\{0,1\}$	$\{ax^2+bx+c \mid a+b=0, c=0\}$	$\{0,1\}$	$x(x+3)$
$\{0,2\}$	$\{ax^2+bx+c \mid 2b=0, c=0\}$	$\{0,2\}$	$x(x+2)$
$\{0,3\}$	$\{ax^2+bx+c \mid a+3b=0, c=0\}$	$\{0,3\}$	$x(x+1)$
$\{1,2\}$	$\{ax^2+bx+c \mid a-b=0, 2a+c=0\}$	$\{1,2\}$	$(x+3)(x+2)$
$\{1,3\}$	$\{ax^2+bx+c \mid 2b=0, 2a+2c=0\}$	$\{1,3\}$	$(x+3)(x+1)$
$\{2,3\}$	$\{ax^2+bx+c \mid a+b=0, 2b+c=0\}$	$\{2,3\}$	$(x+2)(x+1)$
$\{0,1,2\}$	$\{ax^2+bx+c \mid a+b=0, 2b=0\}$	$\{0,1,2\}$	$x(x+3)(x+2)$
$\{0,1,3\}$	$\{ax^2+bx+c \mid a+b=0, 2b=0\}$	$\{0,1,3\}$	$x(x+3)(x+1)$
$\{0,2,3\}$	$\{ax^2+bx+c \mid a+b=0, 2b=0\}$	$\{0,2,3\}$	$x(x+2)(x+1)$
$\{1,2,3\}$	$\{ax^2+bx+c \mid a+b=0, 2b=0\}$	$\{1,2,3\}$	$(x+3)(x+2)(x+1)$
$\mathbb{Z}_4$	$\{ax+b \mid a=0, b=0\}$	$\mathbb{Z}_4$	$x(x+3)(x+2)(x+1)$

Table 1: Algebraic sets of $\mathbb{Z}_4$

Proposition 2.2. *Define the maps* $\operatorname{Spi} A \underset{\mathcal{V}}{\overset{\mathcal{I}}{\rightleftarrows}} \mathfrak{P}(A)$ *as follows:*

$$\mathcal{V}(\{x\}) = \{\mathfrak{a} \in \operatorname{Spi} A \mid x \in \mathfrak{a}\}, \quad \mathcal{I}(S) = \cap\{\mathfrak{s} \mid \mathfrak{s} \in S\},$$

with $S \subseteq \operatorname{Spi} A$. *Here* $\mathfrak{P}(A)$ *denotes the power set of* A. *Then* $\mathcal{V}$ *and* $\mathcal{I}$ *satisfy the following properties.*

(1) $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle)$, *where* $\langle S \rangle$ *is the ideal of* A *generated by the subset* S.
(2) *The map* $\mathcal{V}$ *is order reversing and surjective.*
(3) *If* $\mathfrak{a}$ *is a non-radical ideal of* A, *then* $\mathcal{V}(\mathfrak{a}) \subsetneq \mathcal{V}(\sqrt{\mathfrak{a}})$ *if and only if* A *has non-zero zero divisors.*
(4) *For any two ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of* A, *we have*

$$\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a}\mathfrak{b}).$$

(5) *For a family of sets* $\{\mathcal{V}(\mathfrak{a}_\alpha)\}_{\alpha \in \Gamma}$, *we have* $\bigcap_{\alpha \in \Gamma} \mathcal{V}(\mathfrak{a}_\alpha) = \mathcal{V}\left(\sum_{\alpha \in \Gamma} \mathfrak{a}_\alpha\right)$.
(6) $\mathcal{V}(\mathfrak{a}) = \operatorname{Spi} A$ *if and only if* $\mathfrak{a} = \mathfrak{o}$, *where* $\mathfrak{o}$ *is the zero ideal of* A. *If* $\mathcal{V}(\mathfrak{a}) = \varnothing$, *then* $\mathfrak{a} = A$.
(7) *For any two ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of* A *and* $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$ *implies* $\mathcal{V}(\sqrt{\mathfrak{a}}) \subseteq \mathcal{V}(\mathfrak{b})$.
(8) *The map* $\mathcal{I}$ *is order reversing and surjective.*
(9) $\mathcal{I}(\varnothing) = A$ *and* $\mathcal{I}\left(\bigcup_{\lambda \in \Lambda} T_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathcal{I}(T_\lambda)$.
(10) *If* T *is a subset of* $\operatorname{Spi} A$ *and* $\mathfrak{a}$ *is an ideal of* A, *then* $\mathcal{I}\mathcal{V}(\mathfrak{a}) \supseteq \mathfrak{a}$, *and* $\mathcal{V}\mathcal{I}(T) = T$.
(11) *the collections* $\mathcal{C}_\mathcal{V} = \{\mathcal{V}(\mathfrak{a}) \mid \mathfrak{a} \in \operatorname{Idl}(A)\}$ *and* $\mathcal{C}_{\mathcal{V}\mathcal{I}} = \{\mathcal{V}\mathcal{I}(S) \mid S \in \mathcal{P}(\operatorname{Spi} A)\}$ *of sets are identical, where* $\operatorname{Idl} A$ *denotes the poset (under inclusion) of all ideals of* A.

Remark 2.3. Notice that for $\operatorname{Spec} A$ and for any ideal $\mathfrak{a}$ of A, we always have equalities in (3) and (4). Note that for $\operatorname{Spec} A$, we always have: $\mathcal{I}\mathcal{V}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$ (*cf.* Proposition 2.2 (10))

2.1. ***k*-topologies.** In case of $\operatorname{Spec} A$, the sets $\{\mathcal{V}(\mathfrak{a})\}_{\mathfrak{a} \in \operatorname{Idl} A}$ are closed under finite unions and we obtain the usual Zariski topology on $\operatorname{Spec} A$. But that closure property fails to hold for $\operatorname{Spi} A$ (see Theorem 2.2(4)). However, as a sub-base, $\mathcal{C}_\mathcal{V}$ or equivalently by $\mathcal{C}_{\mathcal{V}\mathcal{I}}$ (see Proposition 2.2 (11)) induces a unique topology on $\operatorname{Spi} A$, which we call the *k-topology*. We denote the corresponding topological space by $(\operatorname{Spi} A, \mathcal{C}_\mathcal{V})$, and in short, call it a *k-space*. With the abuse of notation we shall also denote the space by $\operatorname{Spi} A$. A k-topology coincides with the Zariski topology whenever we restrict $\operatorname{Spi} A$ to $\operatorname{Spec} A$. Note that a study of a similar topology on various classes of ideals of a ring has been done in Dube and Goswami [1].

It is well-known that a *Zariski space* is quasi-compact. The same holds for a k-space. In the proof we shall use the Alexander Subbase Theorem.

Proposition 2.4. *A* k*-space is quasi-compact.*

Proof. Let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a family of subbasic closed sets of an k-space $\operatorname{Spi} A$ such that $\bigcap_{\alpha \in \Lambda} K_\alpha = \varnothing$. Let $\{\mathfrak{s}_\alpha\}_{\alpha \in \Lambda}$ be a family of ideals of A such that $\forall \alpha \in \Lambda$, $K_\alpha = \mathcal{V}(\mathfrak{s}_\alpha)$. Since

$$\bigcap_{\alpha \in \Lambda} \mathcal{V}(\mathfrak{s}_\alpha) = \mathcal{V}\left(\sum_{\alpha \in \Lambda} \mathfrak{s}_\alpha\right),$$

we get $\mathcal{V}\left(\sum_{\alpha\in\Lambda}\mathfrak{s}_\alpha\right)=\varnothing$, and that by Proposition 2.2 (6) implies $\sum_{\alpha\in\Lambda}\mathfrak{s}_\alpha = A$. Then, in particular, we obtain $1=\sum_{\alpha_i\in\Lambda} s_{\alpha_i}$, where $s_{\alpha_i}\in\mathfrak{s}_{\alpha_i}$ and $s_{\alpha_i}\neq 0$ for $i=1,\dots,n$. This implies $A=\sum_{i=1}^{n}\mathfrak{s}_{\alpha_i}$. Therefore, $\bigcap_{i=1}^{n}K_{\alpha_i}=\varnothing$, and hence by Alexander subbase theorem, $\operatorname{Spi}A$ is quasi-compact. □

Since $\mathcal{V}(\mathfrak{a})\neq\mathcal{V}(\mathfrak{a}')$ for any two distinct elements $\mathfrak{a}$ and $\mathfrak{a}'$ of $\operatorname{Idl}A$, we immediately have

Proposition 2.5. *Every k-space is T_0.*

It is known that $\{\mathcal{V}(\mathfrak{p})\mid\mathfrak{p}\in\operatorname{Spec}A\}$ are exactly the irreducible closed subsets of a Zariski space. For a k-space, the situation is more intriguing.

Theorem 2.6. *Every non-empty subbasic closed subset of a k-space is irreducible.*

Proof. Since for every non-empty subbasic closed subset $\mathcal{V}(\mathfrak{a})$ of a k-space $\operatorname{Spi}A$, the ideal $\mathfrak{a}$ is also in $\operatorname{Spi}A$, it is sufficient to show that $\mathcal{V}(\mathfrak{a})=\mathcal{C}\ell(\mathfrak{a})$ for every $\mathfrak{a}\in\operatorname{Spi}A$. Observe that $\mathcal{C}\ell(\mathfrak{a})$ is the smallest closed set containing $\mathfrak{a}$ and $\mathcal{V}(\mathfrak{a})$ is a closed set such that $\mathfrak{a}\in\operatorname{Spi}A$.Therefore, $\mathcal{C}\ell(\mathfrak{a})\subseteq\mathcal{V}(\mathfrak{a})$.To obtain the reverse inclusion, first consider the case: $\mathcal{C}\ell(\mathfrak{a})=\operatorname{Spi}A$. Since

$$\operatorname{Spi}A=\mathcal{C}\ell(\mathfrak{a})\subseteq\mathcal{V}(\mathfrak{a})\subseteq\operatorname{Spi}A,$$

we obtain $\mathcal{V}(\mathfrak{a})=\mathcal{C}\ell(\mathfrak{a})$. Now, let $\mathcal{C}\ell(\mathfrak{a})\neq\operatorname{Spi}A$. For $\mathcal{C}\ell(\mathfrak{a})$, there exists an index set, Ω, such that for each $\alpha\in\Omega$, there is a positive integer n_α and $\mathfrak{a}_{\alpha 1},\dots,\mathfrak{a}_{\alpha n_\alpha}\in\operatorname{Idl}A$ such that

$$\mathcal{C}\ell(\mathfrak{a})=\bigcap_{\alpha\in\Omega}\left(\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{\alpha i})\right).$$

Since by hypothesis, $\mathcal{C}\ell(\mathfrak{a})\neq\operatorname{Spi}A$, without loss of generality, assume that $\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{\alpha i})\neq\varnothing$, for each α. Therefore, $\mathfrak{a}\in\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{\alpha i})$, for each α, and from that we have

$$\mathcal{V}(\mathfrak{a})\subseteq\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{\alpha i}),$$

i.e., $\mathcal{V}(\mathfrak{a})\subseteq\mathcal{C}\ell(\mathfrak{a})$, and this completes the proof. □

A Zariski space $\operatorname{Spec}A$ is connected if and only if the k-algebra A does not have any non-trivial idempotent elements. For a k-space the situation is much simpler.

Theorem 2.7. *Every k-space* $\operatorname{Spi}A$ *is connected.*

Proof. Since by Proposition 2.2 (6), $\operatorname{Spi}A=\mathcal{V}(\mathfrak{o})$ and since irreducibility implies connectedness, the desired claim immediately follows from Theorem 2.6. □

It is known that every Noetherian space can be represented as a finite union of non-empty irreducible closed subsets. For a k-space $\operatorname{Spi}A$, this representation is always possible irrespective of A being Noetherian and hence $\operatorname{Spi} A$ being Noetherian. This follows from the fact that $\mathcal{V}(\mathfrak{o})$ is irreducible in $\operatorname{Spi} A$.

Next, we wish to prove that every non-empty irreducible closed subset of a k-space has a unique generic point. To this end, notice that if K is an irreducible closed subset of a topological space X and $\mathcal{S}$ is a closed subbase of X, then it is known (see Harris [3, §7.2]) that K is the intersection of members of $\mathcal{S}$. For a k-space we get more. In other words, the converse of Theorem 2.6 is also true.

Lemma 2.8. *If K is a non-empty irreducible closed subset of a k-space* $\operatorname{Spi} A$*, then $K = \mathcal{V}(\mathfrak{a})$ for some $\mathfrak{a} \in \operatorname{Spi} A$.*

Proposition 2.9. *Every k-space is sober.*

Proof. It follows from Lemma 2.8 that every non-empty irreducible closed subset of $\operatorname{Spi} A$ is of the form $\mathcal{V}(\mathfrak{a})$, where $\mathfrak{a} \in \operatorname{Spi} A$. Let $\mathcal{V}(\mathfrak{a})$ be a non-empty irreducible closed subset of $\operatorname{Spi} A$. Since $\mathfrak{a} \in \mathcal{V}(\mathfrak{a})$, we have $\mathcal{C}\ell(\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{a})$. Therefore, to show $\mathcal{V}(\mathfrak{a})$ has a generic point, it is now sufficient to show that $\mathcal{C}\ell(\mathfrak{a}) \supseteq \mathcal{V}(\mathfrak{a})$. Since $\mathcal{C}_{\mathcal{V}}$ is a closed subbase of $\operatorname{Spi} A$, the required containment follows from Lemma 2.6. Moreover, by Proposition 2.5, every k-space is T_0. So, we have the uniqueness of a generic point. □

According to Hochster [4], a topological space is called *spectral* if it is quasi-compact, sober, admitting a basis of quasi-compact open subspaces that is closed under finite intersections. It has also been shown in [4] that a Zariski space is spectral. We wish to show that a k-space is also spectral and our proof is constructible topology-independent and avoids the checking of the existence of a basis of quasi-compact open subspaces that is closed under finite intersections. The key to our proof is the following

Lemma 2.10. *A quasi-compact, sober, open subspace of a spectral space is spectral.*

Proof. Suppose S is a quasi-compact, sober, open subspace of a spectral space X. Since S is quasi-compact and sober, it is sufficient to prove that the set $\mathcal{O}_S$ of compact open subsets of S forms a basis of a topology that is closed under finite intersections. It is obvious that a subset T of S is open in S if and only if T is open in X, and hence a subset T of S belongs to $\mathcal{O}_S$ if and only if T belongs to $\mathcal{O}_X$. Now using these facts, we argue as follows.

Let U be an open subset of S. Since U is also open in X, we have $U = \cup\mathcal{U}$, for some subset $\mathcal{U}$ of $\mathcal{O}_X$. But each element of $\mathcal{U}$ being a subset of U is a subset of S, and it belongs to $\mathcal{O}_S$. Therefore, every open subset of S can be presented as a union of compact open subsets of S. Now it remains to prove that $\mathcal{O}_S$ is closed under finite intersections, but this immediately follows from the fact that $\mathcal{O}_X$ is closed under finite intersections. □

Theorem 2.11. *Every k-space is spectral.*

Proof. It is well known (see Priestley [5, Theorem 4.2]) that the set $\operatorname{Idl} A$ endowed with a k-topology is spectral. Now, if we extend the domain of $\mathcal{V}$ to

Idl A, then it is easy to see that with some routine changes of notation in the proof, Theorem 2.6 still holds. Moreover, we have $\{A\} = \mathcal{V}(A) = \mathcal{C}\ell(A)$, and therefore Idl $A\backslash$Spi A is closed, and that implies Spi A is open. The desired claim now follows from Lemma 2.10, Proposition 2.4, and Proposition 2.9. $\square$

Once we have k-spaces, it is natural to consider the continuous maps between such spaces. Using subbasic-closed-set formulation of continuity, we obtain the following properties.

Proposition 2.12. *Let $\phi\colon A \to A'$ be a k-algebra homomorphism and $\mathfrak{b} \in \operatorname{Spi} A'$. Then*

(1) *the map $\phi^*\colon \operatorname{Spi} A' \to \operatorname{Spi} A$ defined by $\phi^*(\mathfrak{b}) = f^{-1}(\mathfrak{b})$ is continuous;*
(2) *if ϕ is surjective, then the k-space $\operatorname{Spi} A'$ is homeomorphic to the closed subspace $\mathcal{V}(\ker\phi)$ of the k-space $\operatorname{Spi} A$;*
(3) *the image $\phi^*(\operatorname{Spi} A')$ is dense in $\operatorname{Spi} A$ if and only if*
$$\ker\phi \subseteq \bigcap_{\mathfrak{s}\in \operatorname{Spi} A} \mathfrak{s};$$
(4) *if A_S is the localization of a k-algebra A at a multiplicative closed subset S, then there is a closed, continuous, and injective map from the k-space $\operatorname{Spi}(R_S)$ to the k-space*
$$(\operatorname{Spi} A)_S := \{\mathfrak{s} \in \operatorname{Spi} A \mid \mathfrak{s} \cap S = \varnothing\}.$$

Proof. To show (1), let $\mathcal{V}(\mathfrak{a})$ be a subbasic closed set of the ideal space Spi A. Observe that
$$(\phi^*)^{-1}(\mathcal{V}(\mathfrak{a})) = \{\mathfrak{b} \in \operatorname{Spi} A' \mid \phi(\mathfrak{a}) \subseteq \mathfrak{b}\} = \mathcal{V}(\langle \phi(\mathfrak{a})\rangle),$$
and hence the map ϕ^* continuous. For the homeomorphism in (2), observe that $\ker\phi \subseteq \phi^{-1}(\mathfrak{b})$, in other words, $\phi^*(\mathfrak{b}) \in \mathcal{V}(\ker\phi)$. This implies that $\operatorname{im}\phi^* = \mathcal{V}(\ker\phi)$. Since for all $\mathfrak{b} \in \operatorname{Spi} A'$,
$$\phi(\phi^*(\mathfrak{b})) = \mathfrak{b} \cap \operatorname{im}\phi = \mathfrak{b},$$
the map ϕ^* is injective. To show that ϕ^* is a closed map, first we observe that for any subbasic closed subset $\mathcal{V}(\mathfrak{a})$ of Spi A', we have
$$\phi^*(\mathcal{V}(\mathfrak{a})) = \phi^{-1}\{\mathfrak{i}' \in \operatorname{Spi} A' \mid \mathfrak{a} \subseteq \mathfrak{i}'\} = \mathcal{V}(\phi^{-1}(\mathfrak{a})).$$
Now if K is a closed subset of Spi A' and if
$$K = \bigcap_{\alpha\in\Omega}\left(\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{i\alpha})\right),$$
then
$$\phi^*(K) = \phi^{-1}\left(\bigcap_{\alpha\in\Omega}\left(\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\mathfrak{a}_{i\alpha})\right)\right) = \bigcap_{\alpha\in\Omega}\bigcup_{i=1}^{n_\alpha}\mathcal{V}(\phi^{-1}(\mathfrak{a}_{i\alpha})),$$
a closed subset of Spi A. Since ϕ^* is continuous, we have the proof. To prove (3), we first show that $\mathcal{C}\ell(\phi^*(\mathcal{V}(\mathfrak{b}))) = \mathcal{V}(\phi^{-1}(\mathfrak{b}))$, for all ideals $\mathfrak{b} \in R'$. To this end, let $\mathfrak{s} \in \phi^*(\mathcal{V}(\mathfrak{b}))$. This implies $\phi(\mathfrak{s}) \in \mathcal{V}(\mathfrak{b})$, which means $\mathfrak{b} \subseteq \phi(\mathfrak{s})$.

In other words, $\mathfrak{s} \in \mathcal{V}(\phi^{-1}(\mathfrak{b}))$. The other inclusion follows from the fact that $\phi^{-1}(\mathcal{V}(\mathfrak{b})) = \mathcal{V}(\phi^{-1}(\mathfrak{b}))$. Since

$$\mathcal{C}\ell(\phi^*(\mathrm{Spi}\, A')) = \mathcal{V}(\phi^{-1}(\mathfrak{o})) = \mathcal{V}(\ker \phi),$$

the closed subspace $\mathcal{V}(\ker \phi)$ is equal to $\mathrm{Spi}\, A$ if and only if $\ker \phi \subseteq \cap_{\mathfrak{s} \in \mathrm{Spi}\, A} \mathfrak{s}$. Finally, to have (4), it is easy to see that the ring homomorphism $\phi\colon A \to A_S$ defined by $\phi(r) = r/1$ induces a map $\phi^*\colon \mathrm{Spi}\, A_S \to \mathrm{Spi}\, A$ defined by $\phi^*(\mathfrak{a}) = \phi^{-1}(\mathfrak{a})$. We claim that $\phi^*(\mathfrak{a}) \cap S = \varnothing$. If not, let $s \in \phi^*(\mathfrak{a}) \cap S$. Then

$$\phi(s) \in \phi(\phi^{-1}(\mathfrak{a}) \cap S) = \phi(\phi^{-1}(\mathfrak{a})) \cap \phi(S) = \mathfrak{a} \cap \phi(S),$$

and hence $\phi(s) \in \mathfrak{a}$. Since $\phi(s)$ is a unit in A_S, this implies $\mathfrak{a} = A_S$, a contradiction. Therefore, ϕ^* is indeed a map from $\mathrm{Spi}\, A_S$ to $(\mathrm{Spi}\, A)_S$. If $\phi^*(\mathfrak{a}) = \phi^*(\mathfrak{b})$ for some $\mathfrak{a}$, $\mathfrak{b} \in \mathrm{Spi}\, A_S$, then

$$\mathfrak{a} = \phi(\phi^{-1}(\mathfrak{a})) = \phi(\phi^{-1}(\mathfrak{b})) = \mathfrak{b}$$

shows that ϕ^* is injective. The map $\phi^*\colon \mathrm{Spi}\, A_S \to \mathrm{Spi}\, A \backslash S$ is continuous follows from (1). Since $\phi^*(\mathcal{V}(\mathfrak{a})) = \mathcal{V}(\phi^{-1}(\mathfrak{a}))$, the map ϕ^* is also closed. Therefore, ϕ^* has the desired properties. □

Corollary 2.13. *The k-space* $\mathrm{Spi}\,(A/\mathfrak{a})$ *is homeomorphic to the closed subspace* $\mathcal{V}(\mathfrak{a})$ *of* $\mathrm{Spi}\, A$.

Remark 2.14. From Proposition 2.12, we get the well-known result that the Zariski spaces $\mathrm{Spec}\, A$ and $\mathrm{Spec}(A/\sqrt{\mathfrak{o}})$ are canonically homeomorphic, and $\phi^*(\mathrm{Spec}\, A')$ is dense in $\mathrm{Spec}\, A$ if and only if $\ker \phi \subseteq \mathcal{V}(\mathfrak{o})$.

3. Conclusion

The generalizations like schemes, algebraic spaces of algebraic varieties still do not answer how to do algebraic geometry when polynomial equations have solutions over a k-algebra which is not an integral domain. Inclusion of zero divisors immediately brings the following two problems in Grothendieck's scheme theory:

(1) the algebraic sets $\{\mathcal{V}(\mathfrak{a})\}_{\mathfrak{a} \in \mathrm{Idl}\, A}$ no longer form a Zariski topology; and
(2) we do not get a sheaf of local rings.

The alternative topology (k-topology) proposed here handles the problem (1). But we do not know how to resolve problem (2). Considering $\mathrm{Spi}\, A$ is not good enough to obtain a sheaf of local rings. For, if $S = A \backslash \mathfrak{s}$, for some $\mathfrak{s} \in \mathrm{Spec}\, A$, then S is a multiplicatively closed set and hence we can have a local algebra A_S at S. But if $\mathfrak{s} \in \mathrm{Spi}\, A$, then the set S is no longer necessarily multiplicatively closed. Therefore, inclusion of zero divisors in varieties might require a completely different treatment to study local properties of geometric objects.

Table 2 summarizes major results in this paper and compared them with Zariski spaces.

EGA I	**Proposed idea**
Field-valued polynomial equations	k-algebra-valued polynomial equations
The set $\operatorname{Spec} A$ of prime ideals of A	The set $\operatorname{Spi} A$ of proper ideals of A
$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$	$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{s} \in \operatorname{Spi} A \mid \mathfrak{a} \subseteq \mathfrak{s}\}$
$\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) = \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{ab})$.	$\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{ab})$.
$\mathcal{V}(\mathfrak{a}) = \mathcal{V}(\sqrt{\mathfrak{a}})$	$\mathcal{V}(\mathfrak{a}) \supseteq \mathcal{V}(\sqrt{\mathfrak{a}})$
$\mathcal{I}\mathcal{V}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$	$\mathcal{I}\mathcal{V}(\mathfrak{a}) \supseteq \mathfrak{a}$,
Zariski topology	k-topology
Compact and T_0	Compact and T_0
$\{\mathcal{V}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} A\}$ are irreducible	Non-empty subbasic closed sets are irreducible
Sober	Sober
Spectral	Spectral
$\operatorname{Spec} A$ is connected iff A does not have any non-trivial idempotent elements	$\operatorname{Spi} A$ is always connected

Table 2: Zariski spaces vs. k-spaces

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