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On the hyperspaces of meager and regular continua

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Abstract

Given a metric continuum X, we consider the collection of all regular subcontinua of X and the collection of all meager subcontinua of X, these hyperspaces are denoted by $D(X)$ and $M(X)$, respectively. It is known that $D(X)$ is compact if and only if $D(X)$ is finite. In this way, we find some conditions related about the cardinality of $D(X)$ and we reduce the fact to count the elements of $D(X)$ to a Graph Theory problem, as an application of this, we prove in particular that $|D(X)| \notin \{2,3,4,5,8,9\}$ for any continuum X. Also, we prove that $D(X)$ is never homeomorphic to $\mathbb{N}$. On the other hand, given a point $p \in X$, we consider the meager composant and the filament composant of p in X, denoted by M_p^X and $Fcs_X(p)$, respectively, and we study some relations between M_p^X and $Fcs_X(p)$ such as the equality of them as a subset of X. Also, we construct examples showing that the collection $Fcs(X) = \{Fcs_X(p) : p \in X\}$ can be homeomorphic to: any finite discrete space, the harmonic sequence, the closure of the harmonic sequence and the Cantor set. Finally, we study the contractibility of $M(X)$; we prove the arc of pseudo-arcs, which is a no contractible continuum, satisfies that its hyperspace of meager subcontinua is contractible, given a solution to Problem 3 of [10]. Most of the results shown in this paper are focus to answer problems and questions posed in [6], [9] and [10]. Also, we rise open problems.

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1. Introduction

A *continuum* is a nonempty compact connected metric space. A closed subset A of a continuum X is said to be *regular* provided that the closure of its interior is equal to A, and A is said to be *meager* if the interior of A is empty. Given a continuum X, by a *hyperspace* of X we mean a specified collection of subsets of X endowed with the *Hausdorff metric* (see Section 2 of [3]). Two of the most studied and useful hyperspaces for a continuum X are 2^X the hyperspace of all nonempty closed subsets of X and $C(X)$ the hyperspace of all connected elements of 2^X. The reader interested in hyperspaces can consult [3], [5] and [8].

Recently, in the literature have been appeared new hyperspaces, such as the *hyperspace of regular subcontinua* defined as the collection of all regular subcontinua of X and the *hyperspace of meager subcontinua* defined as the collection of all meager subcontinua of X. These hyperspaces are denoted by $D(X)$ and $M(X)$, respectively. The hyperespace $D(X)$ was defined in [9] and it is known that $D(X)$ is not always connected [9, Example 1]; and if X is a locally connected continuum, then $D(X)$ is dense, contractible and arcwise connected as a subset of $C(X)$ [9, Theorem 3.6]. Related to the compactness of $D(X)$, it is know that $D(X)$ is compact if and only if $D(X)$ is finite [9, Corollary 4.13]. The hyperspace $M(X)$ was introduced in [10] and it was proved that $M(X)$ is always connected [10, Theorem 4] but not necessarily compact [10, Theorems 7 and 8] and, if X is a locally connected continuum, then $M(X)$ is a continuum if and only if the union of all free arcs is dense in X [10, Corollary 3]. Also, it is known that if X is a smooth dendroid, then $M(X)$ is contractible [10, Theorem 17]. Readers interested in these hyperspaces can also see [11]. On the other hand, using the structure of $M(X)$, if p is a point of X, the *meager composant* of p in X is defined as $M_p^X = \bigcup\{A \in M(X) : p \in A\}$. This concept was first described by David Bellamy in [1] and after studied in [6]. We know that if X is either locally connected, hereditarily arcwise connected or irreducible of type λ, then M_p^X is closed for every $p \in X$ and the collection $\{M_p^X : p \in X\}$ is an usc decomposition [6, Corollary 8.2].

The purpose of this paper is to extend the study of the hyperspaces $D(X)$ and $M(X)$; since $D(X)$ is compact if and only if $D(X)$ is finite [9, Corollary 4.13], we are interested in the cardinality of the hyperspace $D(X)$ (see [9, Problem 4.14]) and we look for metrics spaces Y, for which there exists a continuum X such that $D(X)$ is homeomorphic to Y. We study the concept of *filament composant* of a point p (see Definition 4.4, this concept was introduced in [13] by J. R. Prajs and K. Whittington) and its relations with the concept of a meager composant of the point p. Finally, we study the contractibility of $M(X)$. In order to do this, after Preliminaries, this paper is organized as follows:

- Section 3 is related about the cardinality of $D(X)$. We prove some results (Theorems 3.2 and 3.3) that we believe can be used to prove Problem 5.9 of [9]. Also, we reduce the fact to obtain the elements of

$D(X)$ of a Graph Theory problem (see Theorem 3.16 and comments after its proof) and we prove that if X is a continuum, then $|D(X)| \notin \{2,3,4,5,8,9\}$. Also, we show in Theorem 3.25 that $D(X)$ cannot be homeomophic to the natural numbers $\mathbb{N}$.

- In Section 4, we recall the concept of filament composant and we prove that there exists a hereditarily decomposable and irreducible continuum X such that $M_x^X = Fcs_X(x)$ for each $x \in X$ (Proposition 4.9) and we show that if X is arcwise connected continuum, then there exists $p \in X$ such that $Fcs_X(p) \neq M_p^X$ (Theorem 4.12). Also interesting examples are given.
- In Section 5, we study the contractibility of $M(X)$. We prove that the hyperspace $M(X)$ of both the cylinder of a contractible continuum and the cone of every compactum space are contractible (Theorems 5.3 and 5.4). Also in Theorem 5.5 we give a solution to [10, Problem 3].

2. Preliminaries

Given a metric space X and $A \subseteq X$, we denote by $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$ and $\mathrm{diam}(A)$ the closure, interior, boundary and diameter of A, respectively. A *map* will be a continuous function. Given a continuum X, by a subcontinuum of X, we mean an element of $C(X)$. An *arc* is a continuum homeomorphic to $[0,1]$. If X is an arc and $h\colon [0,1] \to X$ is a homeomorphism, then $h(0)$ and $h(1)$ are the *end points* of X. A continuum is *arcwise connected* provided that for every pair of their points there exists an arc containing them. Given a continuum X and an arc $\alpha \subseteq X$ with end points a and b, we say that α is a *free arc* if $\alpha \setminus \{a,b\}$ is an open subset of X. A continuum X is *decomposable* if there exist two proper subcontinua A and B of X such that $X = A \cup B$. A continuum is *indecomposable* provided that it is not decomposable. Also, a continuum is called *hereditarily decomposable* (*hereditarily indecomposable*) if every nondegenerate subcontinuum is decomposable (indecomposable, respectively). A *triod* is a continuum X where there exists a proper subcontinuum Y of X such that $X \setminus Y$ has at least three components. Furthermore, X is *atriodic* provided that it does not contain any triod. A continuum X is *irreducible between a finite number of points* if there exists a finite set $F \subseteq X$ such that there is not a proper subcontinuum containing F. If F has two points, we say that X is *irreducible*. Particularly, if $F = \{p,q\}$, we will say that X is irreducible between p and q. An irreducible continuum such that every indecomposable subcontinuum has empty interior is called *continuum of type* λ. In [14, Theorem 10], it is proved the following theorem:

Theorem 2.1. *Let X be an irreducible continuum. Then, X is of type λ if and only if there exists a monotone map $f\colon X \to [0,1]$ such that $f^{-1}(t)$ has empty interior for each $t \in [0,1]$.*

Given an irreducible continuum X and a upper semicontinuous decomposition $\mathcal{D}$ of X, we say that $\mathcal{D}$ is *admissible* if D is a continuum for each $D \in \mathcal{D}$, and $\mathcal{D}$ is an arc. Furthermore, $\mathcal{D}$ is admissible minimal if $\mathrm{int}(D) = \varnothing$ for every

$D \in \mathcal{D}$. Note that by Theorem 2.1, X is of type λ if and only if there exists a minimal admissible decomposition of X. A *pseudo-arc* is a chainable and hereditarily indecomposable continuum [2, Theorem 1] (see [4] for additional information about the pseudo-arc). The *arc of pseudo-arcs* is a continuum of type λ, X, such that if $f\colon X \to [0,1]$ is the monotone map given in Theorem 2.1, $f^{-1}(t)$ is a pseudo-arc for every $t \in [0,1]$ and the admissible decomposition $\{f^{-1}(t) : t \in [0,1]\}$ is continuous.

Given continua X and Y, a map $f\colon X \to Y$, and $\varepsilon > 0$, we say that f is an ε-map provided that $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. A continuum X is said to be *arc-like* (*circle-like*) provided for any $\varepsilon > 0$ there exists an ε-map $f\colon X \to [0,1]$ ($f\colon X \to S^1$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, respectively).

3. The hyperspace of regular continua

In this section we study some properties related to the cardinality of the hyperspace of regular subcontinua $D(X)$; for instance, our main result is Theorem 3.25 where we show that it is not possible to find a continuum X such that $D(X)$ is homeomorphic to $\mathbb{N}$. We divide this section in three: in the first one, we study conditions on X to have that $D(X)$ has more than one point; in the second, we show in Theorem 3.16 an interesting condition to have that the hyperspace $D(X)$ is finite; and in the third subsection, we present necessary and sufficient conditions in order to have that $D(X)$ is discrete.

3.1. $D(X)$ is not degenerated.

It is well know that a continuum is indecomposable if and only if every proper subcontinuum has empty interior. Thus, $D(X) = \{X\}$ whenever X is an indecomposable continuum. Theorem 5.8 of [9] presents an example of a decomposable continuum X such that $D(X) = \{X\}$. The following is Problem 5.9 of [9].

Question 3.1. *Does there exist a hereditarily decomposable continuum X for which $D(X) = \{X\}$?*

Question 3.1 is still open. The following theorem characterizes when the hyperspace $D(X)$ is degenerated and could be useful to solve Question 3.1.

Theorem 3.2. *Let X be a continuum. Then, $D(X) = \{X\}$ if, and only if, for each $K \in C(X) \setminus \{X\}$, it satisfies some of the following conditions:*

1. $\operatorname{int}(K) = \varnothing$*; or*
2. *There exist two nonempty open subsets U and V of X such that* $\operatorname{int}(K) = U \cup V$ *and* $\operatorname{cl}(U) \cap \operatorname{cl}(V) = \varnothing$.

Proof. Suppose that $D(X) = \{X\}$. Let $K \in C(X) \setminus \{X\}$ such that $\operatorname{int}(K) \neq \varnothing$. Note that if $\operatorname{cl}(\operatorname{int}(K))$ is connected, then $\operatorname{cl}(\operatorname{int}(K)) \in D(X)$ and $\operatorname{cl}(\operatorname{int}(K)) \neq X$. This contradicts that $D(X) = \{X\}$. Thus, there exist two nonempty closed subsets A and B of X such that $\operatorname{cl}(\operatorname{int}(K)) = A \cup B$. Let $U = \operatorname{int}(K) \cap A$ and $V = \operatorname{int}(K) \cap B$. It is clear that $\operatorname{cl}(U) \cap \operatorname{cl}(V) = \varnothing$. Furthermore, observe that $U = \operatorname{int}(K) \cap (X \setminus B)$ and $V = \operatorname{int}(K) \cap (X \setminus A)$. Therefore, both U and V are open subsets of X.

Conversely, note that $\mathrm{cl}(\mathrm{int}(K))$ is not connected, for every $K \in C(X)\setminus\{X\}$ such that $\mathrm{int}(K) \neq \varnothing$. Thus, $D(X) = \{X\}$. □

Proposition 4.15 of [9] shows that if $X = A_1 \cup A_2$, where A_1 and A_2 are indecomposable continua such that $|A_1 \cap A_2| = 1$, then X is a decomposable and irreducible continuum such that $|D(X)| = 3$. Next result presents families of decomposable continua where $D(X)$ is nondegenerate.

Theorem 3.3. *Let X be a decomposable continuum. If X satisfies some of the following conditions, then $|D(X)| \geq 2$.*

(1) *X is atriodic;*
(2) *X is irreducible between a finite number of points;*
(3) *X has a cut point;*

Proof. Let A and B be proper subcontinua of X such that $X = A \cup B$.

We suppose that X is atriodic. Note $X \setminus A$ has at most two components. Hence, the closure of any component of $X \setminus A$ belongs to $D(X)$. Therefore, $|D(X)| \geq 2$.

We assume 2. Let $\{p_1, \ldots, p_n\} \subseteq X$ be such that X is irreducible between $\{p_1, \ldots, p_n\}$. Suppose that $\{p_{n_1}, \ldots, p_{n_k}\} = \{p_1, \ldots, p_n\} \cap X \setminus A$. Let

$$\mathcal{J} = \{J \text{ component of } X \setminus A : J \cap \{p_{n_1}, \ldots, p_{n_k}\} \neq \varnothing\}.$$

By [7, Theorem 5.4], $\mathrm{cl}(J) \cap A \neq \varnothing$ for each $J \in \mathcal{J}$. Thus, $\{p_1, \ldots, p_n\} \subseteq A \cup (\bigcup_{J \in \mathcal{J}} J)$ and $A \cup (\bigcup_{J \in \mathcal{J}} J)$ is a subcontinuum of X. Since X is irreducible between $\{p_1, \ldots, p_n\}$, $X = A \cup (\bigcup_{J \in \mathcal{J}} J)$. Thus, $X \setminus A$ has a finite number of components and each component is open. Therefore, the closure of any component of $X \setminus A$ is regular and $|D(X)| \geq 2$.

To prove the theorem using 3, we suppose that $X \setminus \{p\}$ is not connected for some $p \in X$. Let U and V be open subsets of X such that $X \setminus \{p\} = U \cup V$. Note that $\mathrm{cl}(U) = U \cup \{p\}$ and $\mathrm{cl}(V) = V \cup \{p\}$. Furthermore, $U \cup \{p\}$ and $V \cup \{p\}$ are continua, by [7, Proposition 6.3]. Thus, $\{U \cup \{p\}, V \cup \{p\}, X\} \subseteq D(X)$ and $|D(X)| \geq 3$. □

Note that if X is either an arc-like continuum or a circle-like continuum, then X is atriodic (see [5, Corollaries 2.1.43 and 2.1.46]). Hence, next result follows from Theorem 3.3.

Corollary 3.4. *Let X be a decomposable continuum. If X is either arc-like or circle-like, then $|D(X)| \geq 2$.*

3.2. $D(X)$ **is finite.** In [9, Corollary 4.13], it is proved that $D(X)$ is compact if and only if $D(X)$ is finite. The following is Problem 4.14 of [9].

Question 3.5. *For which $n \in \mathbb{N}$, does there exist a continuum X such that $D(X)$ has exactly n elements?*

Proposition 4.5 of [9] gives examples of positive integers n for which there is a continuum X where $|D(X)| = n$. In Proposition 3.18, we summarize the results of this section showing that $|D(X)| \notin \{2, 4, 5, 8, 9\}$ for every continuum X.

Proposition 3.6. *Let X be a continuum and let $K \in D(X) \setminus \{X\}$. Then,*

(1) *if $X \setminus K$ is connected, then $|D(X)| \geq 3$;*
(2) *if $X \setminus K$ is not connected, then $|D(X)| \geq 4$.*

Proof. Suppose first that $X \setminus K$ is connected. Hence, $\mathrm{cl}(X \setminus K)$ is regular. Thus, we have that $\{K, \mathrm{cl}(X \setminus K), X\} \subseteq D(X)$ and $|D(X)| \geq 3$.

Now, suppose that there exist two open subsets U and V of X such that $X \setminus K = U \cup V$. By [7, Proposition 6.3], $U \cup K$ and $V \cup K$ are proper subcontinua of X. We show that both $U \cup K$ and $V \cup K$ are regular. Note that $U \cup \mathrm{int}(K) \subseteq \mathrm{int}(U \cup K)$. Hence, $\mathrm{cl}(U \cup \mathrm{int}(K)) \subseteq \mathrm{cl}(\mathrm{int}(U \cup K))$. Since $\mathrm{cl}(U \cup \mathrm{int}(K)) = \mathrm{cl}(U) \cup \mathrm{cl}(\mathrm{int}(K)) = \mathrm{cl}(U) \cup K = U \cup K$,

$$U \cup K \subseteq \mathrm{cl}(\mathrm{int}(U \cup K)) \subseteq U \cup K.$$

Thus, $\mathrm{cl}(\mathrm{int}(U \cup K)) = U \cup K$ and $U \cup K$ is regular. Similarly, we show that $V \cup K$ is regular. Therefore, $\{K, K \cup U, K \cup V, X\} \subseteq D(X)$ and $|D(X)| \geq 4$. □

The next result follows from Proposition 3.6.

Corollary 3.7. *There is not a continuum X such that $|D(X)| = 2$.*

Definition 3.8. Let X be a continuum. A point A of $D(X)$ is said to be *maximal* provided that if $B \in D(X)$ and $A \subsetneq B$, then $B = X$. Similarly, we say that A is *minimal* if whenever $B \in D(X)$ and $B \subseteq A$, we have that $B = A$.

Lemma 3.9. *Let X be a continuum and let $K \in D(X) \setminus \{X\}$. If K is maximal, then $\mathrm{cl}(X \setminus K)$ is minimal of $D(X)$.*

Proof. We show that $X \setminus K$ is connected. Suppose that $X \setminus K = U \cup V$ where U and V are disjoint nonempty open subsets of X. Note that $K \cup U$ is a regular continuum (see proof of Proposition 3.6) and $K \subsetneq K \cup U$. This contradicts that K is maximal. Therefore, $X \setminus K$ is connected and $\mathrm{cl}(X \setminus K) \in D(X)$.

Now, we prove that $\mathrm{cl}(X \setminus K)$ is minimal. Let $B \in D(X)$ be such that $B \subsetneq \mathrm{cl}(X \setminus K)$. We consider two cases:

1. $B \cap \mathrm{bd}(K) = \varnothing$. Hence, $B \subseteq X \setminus K$. Observe that if $X \setminus B$ is connected, then $K \subsetneq \mathrm{cl}(X \setminus B)$ and $\mathrm{cl}(X \setminus B) \in D(X)$. A contradiction. Thus, $X \setminus B = U \cup V$ where U and V are disjoint nonempty open subsets of X. Since $K \subseteq U \cup V$ and K is connected, we have that either $K \subseteq U$ or $K \subseteq V$. Suppose that $K \subseteq U$. Therefore, $K \subsetneq U \cup B$ and $U \cup B \in D(X) \setminus \{X\}$. A contradiction.

2. $B \cap \mathrm{bd}(K) \neq \varnothing$. Thus, $B \cap K \neq \varnothing$. Since $B \subsetneq \mathrm{cl}(X \setminus K)$, $B \cup K \neq X$. Furthermore, $B \cup K \in D(X)$; contradicting that K is maximal.

Therefore, $\mathrm{cl}(X \setminus K)$ is minimal of $D(X)$. □

Proposition 3.10. *Let X be a continuum. If M_1 and M_2 are different maximal points of $D(X)$, then $X = M_1 \cup M_2$.*

Proof. Observe that if M_1 is maximal, then $\mathrm{cl}(X \setminus M_1)$ belongs to $D(X)$, by Lemma 3.9. Since $M_2 \setminus M_1 \neq \varnothing$, we have that $M_2 \cup \mathrm{cl}(X \setminus M_1) \in D(X) \setminus \{X\}$. Since M_2 is maximal, $\mathrm{cl}(X \setminus M_1) \subseteq M_2$. Therefore, $X = M_1 \cup M_2$. □

Theorem 3.11. *Let X be a continuum and let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $D(X)$ such that $\lim_{n\to\infty} K_n = K$, for some $K \in C(X)$. If $K_n \subseteq K$ for each $n \in \mathbb{N}$, then $K \in D(X)$.*

Proof. We will see that $\mathrm{cl}(\mathrm{int}(K)) = K$. It is clear that $\mathrm{cl}(\mathrm{int}(K)) \subseteq K$. We will show that $K \subseteq \mathrm{cl}(\mathrm{int}(K))$. Let $x \in K$. Let U be an open subset of X such that $x \in U$. Since $\lim_{n\to\infty} K_n = K$, there exists $j_0 \in \mathbb{N}$ such that $K_{j_0} \cap U \neq \varnothing$. Since K_{j_0} is regular, $\mathrm{int}(K_{j_0}) \cap U \neq \varnothing$. Furthermore, $\mathrm{int}(K_{j_0}) \cap U \subseteq \mathrm{int}(K) \cap U$. Thus, $U \cap \mathrm{int}(K) \neq \varnothing$ and $x \in \mathrm{cl}(\mathrm{int}(K))$. Therefore, $K \subseteq \mathrm{cl}(\mathrm{int}(K))$ and $K \in D(X)$. □

Corollary 3.12. *Let X be a continuum and let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $D(X)$. If $K_n \subseteq K_{n+1}$ for each $n \in \mathbb{N}$, then $\lim_{n\to\infty} K_n$ belongs to $D(X)$.*

Proof. Note that $\lim_{n\to\infty} K_n = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} K_n)$ (see [3, 4.16, p.27]). Thus, our result follows from Theorem 3.11. □

Corollary 3.13. *Let X be a continuum. If $D(X)$ is discrete and $A \in D(X)$, then there exists a maximal set $K \in D(X)$ such that $A \subseteq K$.*

Proof. Let $\mathcal{L} = \{M \in D(X) : A \subseteq M\}$. Since $D(X)$ is discrete, there is not an increasing chain in $\mathcal{L}$, by Corollary 3.12. Thus, there exists a maximal point $K \in D(X)$ such that $A \subseteq K$. □

Proposition 3.14. *Let X be a continuum such that $D(X)$ is discrete. If N_1 and N_2 are different minimal points of $D(X)$, then $N_i \cap \mathrm{int}(N_j) = \varnothing$ where $\{i, j\} = \{1, 2\}$.*

Proof. Suppose that $N_1 \cap \mathrm{int}(N_2) \neq \varnothing$. Since N_1 is regular, $\mathrm{int}(N_1) \cap \mathrm{int}(N_2) \neq \varnothing$. Let $Y = N_1 \cup N_2$. Observe that $Y \in D(X)$. Since $D(X)$ is discrete, $D(Y)$ is discrete and there exists a maximal M of $D(Y)$ such that $N_1 \subseteq M$, by Corollary 3.13. Thus, $N = \mathrm{cl}(Y \setminus M)$ is minimal, by Lemma 3.9. Since $\mathrm{int}(N_1) \cap \mathrm{int}(N_2) \neq \varnothing$ and $N_1 \subseteq M$, we have that $N \subsetneq N_2$. This contradicts the fact that N_2 is minimal. Therefore, $N_1 \cap \mathrm{int}(N_2) = \varnothing$. Similarly we show that $N_2 \cap \mathrm{int}(N_1) = \varnothing$. □

Proposition 3.15. *Let X be a continuum such that $D(X)$ is discrete. If N is minimal of $D(X)$ and $A \in D(X)$ is such that $A \cap \mathrm{int}(N) \neq \varnothing$, then $N \subseteq A$.*

Proof. Suppose that there exists $A \in D(X)$ such that $A \cap \mathrm{int}(N) \neq \varnothing$ and $N \setminus A \neq \varnothing$. Note that $A \cup N \in D(X)$. Since $D(X)$ is discrete, $D(A \cup N)$ is discrete. Thus, there exists a maximal M of $D(A \cup N)$ such that $A \subseteq M$, by Corollary 3.13. Furthermore, by Lemma 3.9, $\mathrm{cl}((A \cup N) \setminus M)$ is minimal of $D(A \cup N)$. Since $A \cap \mathrm{int}(N) \neq \varnothing$ and $A \subseteq M$, we have that $\mathrm{cl}((A \cup N) \setminus M) \subseteq N$ and $\mathrm{cl}((A \cup N) \setminus M) \neq N$. A contradiction. Therefore, $N \subseteq A$ for every $A \in D(X)$ such that $A \cap \mathrm{int}(N) \neq \varnothing$. □

Theorem 3.16. *Let X be a continuum such that $D(X)$ is discrete. Then, $D(X)$ is finite if and only if there exist minimal sets $N_1, \ldots, N_n$ in $D(X)$ such that $\mathrm{int}(N_i) \cap \mathrm{int}(N_j) = \varnothing$ whenever $i \neq j$, and $X = \bigcup_{i=1}^{n} N_i$.*

Proof. Suppose that $D(X)$ is finite. If $D(X) = \{X\}$, then X is minimal. Hence, suppose that there exists K_1 in $D(X) \setminus \{X\}$. By Corollary 3.13, we may suppose that K_1 is maximal. Note that $N_1 = \text{cl}(X \setminus K_1)$ is minimal in $D(X)$, by Lemma 3.9. If K_1 is minimal, we have that $X = K_1 \cup N_1$ where $\text{int}(K_1) \cap \text{int}(N_1) = \varnothing$. Thus, suppose that K_1 is not minimal. Let K_2 be maximal in $D(K_1)$ and let $N_2 = \text{cl}(K_1 \setminus K_2)$. By Lemma 3.9, N_2 is minimal in $D(K_1)$ and hence, minimal in $D(X)$. It is clear that $X = N_1 \cup N_2 \cup K_2$, where $\text{int}(N_1), \text{int}(N_2)$ and $\text{int}(K_2)$ are pairwise disjoint. If K_2 is minimal, then we finish the proof. Thus, since $D(X)$ is finite, there exists K_{n-1} such that K_{n-1} is both maximal and minimal in $D(K_{n-2})$ where $X = N_1 \cup \cdots \cup N_{n-1} \cup K_{n-1}$ and the interiors of $N_1, \ldots, N_{n-1}$ and K_{n-1} are pairwise disjoint subsets of X. Therefore, if $N_n = K_{n-1}$, then there exist minimal sets $N_1, \ldots, N_n$ in $D(X)$ such that $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$ whenever $i \neq j$, and $X = \bigcup_{i=1}^{n} N_i$.

Conversely, suppose that $X = \bigcup_{i=1}^{n} N_i$ where $N_1, \ldots, N_n$ are minimal of $D(X)$ such that $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$ whenever that $i \neq j$. Let $K \in D(X)$. Observe that by Proposition 3.15,

$$K = \bigcup \{N_i : \text{int}(N_i) \cap K \neq \varnothing\}. \tag{3.1}$$

Therefore, $D(X)$ is finite. □

Let X be a continuum such that $D(X)$ is finite, and let $N_1, \ldots, N_k$ be the minimal subsets of X such that $X = \bigcup_{i=1}^{k} N_i$ and $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$ whenever $i \neq j$. By (3.1), $|D(X)| = |\mathcal{L}(X)|$ where

$$\mathcal{L}(X) = \left\{ \bigcup_{i \in F} N_i : \bigcup_{i \in F} N_i \in \mathcal{C}(X) \text{ and } F \subseteq \{1, \ldots, k\} \right\}.$$

We illustrate X by a finite graph where each vertex v_i represents the continuum N_i, and two vertices v_i and v_j have an edge between them whenever $N_i \cap N_j \neq \varnothing$. For instance, if $n \in \{2, 3\}$, then

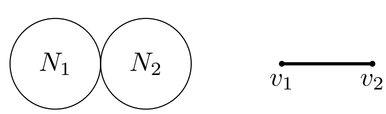


Figure 1. $X = N_1 \cup N_2$

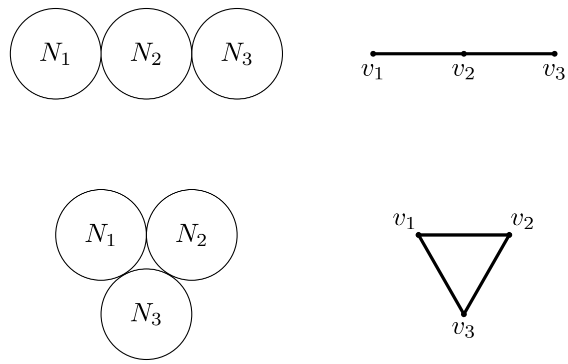


Figure 2. $X = N_1 \cup N_2 \cup N_3$

Thus, if $X = N_1 \cup N_2$, then $D(X) = \{N_1, N_2, X\}$; and if $X = N_1 \cup N_2 \cup N_3$, then either $D(X) = \{N_1, N_2, N_3, N_1 \cup N_2 N_2 \cup N_3, X\}$ or $D(X) = \{N_1, N_2, N_3, N_1 \cup N_2, N_2 \cup N_3, N_1 \cup N_3, X\}$. Therefore, if $n \in \{1, 2, 3\}$, then $|D(X)| \in \{1, 3, 6, 7\}$.

The following result is not difficult to prove.

Proposition 3.17. *Let X be a continuum such that $D(X)$ is discrete. Then, the following are equivalent:*

(1) *There exists $A \in D(X)$ such that A is both maximal and minimal;*
(2) *There exists exactly two minimal sets in $D(X)$;*
(3) $|D(X)| = 3$.

Now, we analyze the case $n = 4$. Let $X = \bigcup_{i=1}^{4} N_i$. The continuum X can be as we show in the Figure 3, up to homeomorphisms.

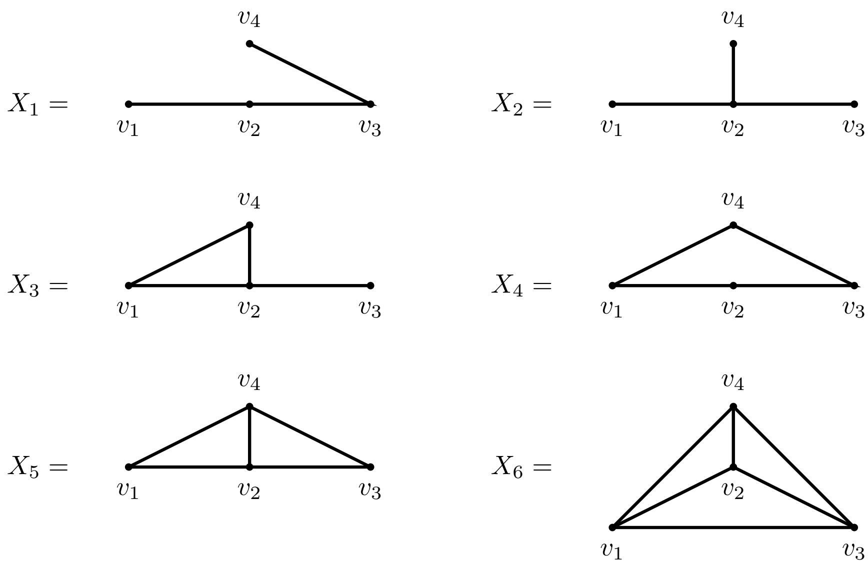
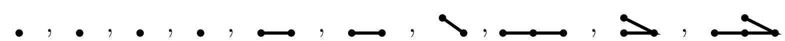


Figure 3. $X = \bigcup_{i=1}^{4} N_i$

Then, observe that:

$\mathcal{L}(X_1) = \{$  $\}$.

Thus, respectively with each graph of $\mathcal{L}(X_1)$, we have that

$D(X_1) = \{N_1, N_2, N_3, N_4, N_1 \cup N_2, N_2 \cup N_3, N_3 \cup N_4, N_1 \cup N_2 \cup N_3, N_2 \cup N_3 \cup N_4, X\}$.

Therefore, $D(X_1)$ has exactly 10 points. In a similar way, it is not difficult to see that $|D(X_2)| = 11, |D(X_3)| = 12, |D(X_4)| = 13, |D(X_5)| = 14$ and $|D(X_6)| = 15$. Note that if $n \geq 5$, then $|D(X)| \geq 15$. Hence, we have the following proposition:

Proposition 3.18. *Let X be a continuum. Then, $|D(X)| \notin \{2, 4, 5, 8, 9\}$.*

Furthermore, similarly to [9, Proposition 4.15], we have the following result.

Proposition 3.19. *Let X be a continuum such that $D(X)$ is finite and let $N_1, \ldots, N_n$ be the minimal sets where $X = \bigcup_{i=1}^{n} N_i$. Then,*

(1) $|D(X)| = \frac{n(n+1)}{2}$ *whenever, $N_i \cap N_j \neq \varnothing$ if and only of $|i - j| \leq 1$;*
(2) $|D(X)| = n(n-1)+1$ *whenever, $N_i \cap N_j \neq \varnothing$ if and only of $|i - j| \leq 1$ or $|i - j| = n - 1$;*
(3) $|D(X)| = 2^n - 1$ *whenever $N_i \cap N_j \neq \varnothing$ for every $i, j \in \{1, \ldots, n\}$.*

Remark 3.20. Given $X = N_1 \cup \cdots \cup N_n$, find the cardinality of $\mathcal{L}(X)$ is a problem of Theory of Graphs that can be solved partially with a simple program in Python as we show:

```
n=input()
m=set()
for g in graphs.nauty_geng(str(n)):
    if g.is_connected():
        d=[]
        for k in g.connected_subgraph_iterator():
            d=d+[k]
        for i in range(len(d)):
            for j in range(i+1,len(d)):
                if d[i].vertex_iterator==d[j].vertex_iterator:
                    del d[j]
                    j=j-1
        m=m.union({len(d)})
print(m)
```

Where m is the set of all possible values of the cardinality of $\mathcal{L}(X)$. Thus, running the program for $n \in \{2,3,4,5,6,7,8\}$, we conclude that it is not possible to have a continuum X such that $|D(X)| = 16$. Furthermore, if $k \in \{17,\ldots,255\}$, then there exists a continuum X such that $|D(X)| = k$.

We finish this section with a natural question.

Question 3.21. *If $k \geq 256$, then does there exist a continuum X such that $|D(X)| = k$?*

3.3. $D(X)$ **is discrete.** If X is a simple closed curve (X is homeomorphic to S^1), then it is not difficult to see that $D(X) = C(X) \setminus F_1(X)$. Thus, $D(X)$ is homeomorphic to $\{z \in \mathbb{C} : |z| < 1\}$ (see [3, Example 5.2]). As we showed in Section 3.2, some finite sets can be represented as $D(X)$ for some continuum X. We are interested in giving an answer of the following problem.

Problem 3.22. *Characterize the family of metric spaces S for which there exists a continuum X such that $D(X) \cong S$.*

In order to give partial answers to Problem 3.22, in this section we study when $D(X)$ is discrete.

Theorem 3.23. *Let X be a continuum. If $D(X)$ has infinitely many maximal points, then X is not an isolated point of $D(X)$.*

Proof. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of different maximal points of $D(X)$. Since $C(X)$ is compact and $D(X) \subseteq C(X)$, we have that there exists a subsequence $(M_{n_i})_{i\in\mathbb{N}}$ of $(M_n)_{n\in\mathbb{N}}$ such that $\lim_{i\to\infty} M_{n_i} = M$, for some $M \in C(X)$.

We see that $M = X$. Suppose that $X \setminus M \neq \varnothing$. Let U be an open subset of X such that $\mathrm{cl}(U) \cap M = \varnothing$. It is clear that $M \in \langle X \setminus \mathrm{cl}(U)\rangle$. Thus, there exists $k \in \mathbb{N}$ such that $M_{n_i} \in \langle X \setminus \mathrm{cl}(U)\rangle$ for each $i \geq k$. Hence, $U \subseteq X \setminus M_{n_i}$ for each $i \geq k$. This contradicts Proposition 3.10. Therefore, $M = X$. □

Theorem 3.24. *Let X be a continuum. If $D(X)$ is a discrete infinite set, then $D(X)$ has infinitely many maximal points.*

Proof. Let $K_0 = X$. By Corollary 3.13, we can choose K_1 a maximal point of $D(K_0)$. Since $D(K_1) \subseteq D(K_0)$, $D(K_1)$ is discrete. Let $N_1 = \text{cl}(K_0 - K_1)$. By Lemma 3.9, N_1 is minimal in $D(K_0)$ and hence, N_1 is minimal in $D(K_0)$.

Claim I. There exists a subcontinuum K_2 of X, such that:

(1) $K_2 \subsetneq K_1$ and $D(K_2)$ is discrete;
(2) K_2 maximal in $D(K_1)$;
(3) $N_2 = \text{cl}(K_1 \setminus K_2)$ is minimal in $D(K_1)$;
(4) $\text{int}(N_1) \cap \text{int}(N_2) = \varnothing$.

In order to proof (1) and (2), suppose that $D(K_1) = \{K_1\}$. Then K_1 is minimal in $D(K_0)$. By Lemma 3.9, $D(K_0) = \{K_1, \text{cl}(X \setminus K_1), K_0\}$, which is a contradiction. Hence, by Corollary 3.13, there exists $K_2 \subsetneq K_1$ maximal in $D(K_1)$. Since $D(K_2) \subseteq D(K_1)$, $D(K_2)$ is discrete. On the other hand, by Lemma 3.9, $N_1 = \text{cl}(K_0 - K_1)$ is minimal in $D(K_1)$, which proves (3). Finally, since $K_{j-1} \subsetneq K_{i-1}$, we have that $N_j \subseteq K_{j-1}$. Hence, $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$. This proves (4).

Continuing with these arguments, inductively, we can construct a sequence $(K_n)_{n\in\mathbb{N}}$ in $D(X)$ and a sequence $(N_n)_{n\in\mathbb{N}}$ where $N_{n+1} = \text{cl}(K_n \setminus K_{n+1})$ such that:

(1) $K_{n+1} \subsetneq K_n$ and $D(K_{n+1})$ is discrete for each $n \in \mathbb{N}$;
(2) K_{n+1} is maximal of $D(K_n)$ for each $n \in \mathbb{N}$.
(3) N_{n+1} is minimal in $D(K_n)$ and hence, N_{n+1} is minimal in $D(X)$ for each $n \in \mathbb{N}$;
(4) $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$ for each $i \neq j$.

Let $\mathcal{N} = \{N_n : n \in \mathbb{N}\}$ and let

$$\mathcal{M} = \left\{\bigcup \mathcal{S} : \mathcal{S} \subseteq \mathcal{N} \text{ is finite and } \bigcup \mathcal{S} \text{ is connected}\right\}.$$

Note that $\mathcal{M} \subseteq D(X)$. Since $D(X)$ is discrete, by Corollary 3.12, for each $S \in \mathcal{M}$ there exists $M \in \mathcal{M}$ maximal in $\mathcal{M}$ such that $S \subseteq M$. Let

$$\mathcal{M}' = \{S \in \mathcal{M} : S \text{ is maximal in } \mathcal{M}\}.$$

It is clear that $\bigcup \mathcal{M}' = \bigcup \mathcal{N}$, which implies that $\mathcal{M}'$ is a partition of $\bigcup \mathcal{N}$. Since $\mathcal{N}$ is countable infinite and every element of $\mathcal{M}'$ is a finite union of elements of $\mathcal{N}$, we have that $\mathcal{M}'$ is also a countable infinite set. Let $\mathcal{M}' = \{S_n : n \in \mathbb{N}\}$.

Claim II. For each $n \in \mathbb{N}$, $L_n = \text{cl}(X \setminus S_n) \in D(X)$.

Let $n \in \mathbb{N}$ and let $N_{i_1}, \ldots, N_{i_m}$ be in $\mathcal{N}$ such that $S_n = \bigcup_{j=1}^{m} N_{i_j}$. We may assume that $i_1 < \cdots < i_m$. It is clear that $X = K_{i_m} \cup (\bigcup_{j=1}^{i_m} N_j)$. Hence, $L_n = K_{i_m} \cup (\bigcup\{N_j : j \in \{1, \ldots, i_m\} \setminus \{i_1, \ldots, i_m\}\})$. Since S_n belongs to $\mathcal{M}'$ and $\text{int}(N_i) \cap \text{int}(N_j) = \varnothing$ for each $i \in \{i_1, \ldots, i_m\}$ and $j \in \{1, \ldots, i_m\} \setminus \{i_1, \ldots, i_m\}$, we conclude that L_n is a subcontinuum of X which belongs to $L_n \in D(X)$. This proves Claim II.

By Corollary 3.13, for each $n \in \mathbb{N}$, there exists a maximal element M_n in $D(X)$ such that $L_n \subseteq M_n$. Since $X \setminus M_i \subseteq S_i$ and $S_1, S_2, \dots$ are pairwise disjoint, we have that $M_i \neq M_j$ whenever $i \neq j$. Therefore, $D(X)$ has infinitely many maximal sets. □

The following theorem follows from Theorems 3.23 and 3.24.

Theorem 3.25. *There is not a continuum X such that $D(X)$ is homeomorphic to $\mathbb{N}$.*

Question 3.26. *Does there exist a continuum X such that $D(X)$ is homeomorphic to either $\mathbb{Q}$ or $\mathbb{I}$?*

4. Meager composants and filament composants

In this section, we study some problems related to the hyperspace of meager subcontinua. We use the following notation: Given a point p of a continuum X, the meager composant of p is defined by: $M_p^X = \bigcup M_p(X)$, where $M_p(X) = \{A \in M(X) : p \in A\}$. The following is [6, Proposition 2.5].

Proposition 4.1. *If X is a continuum, then $\mathcal{M}_X = \{M_p^X : p \in X\}$ is a partition of X.*

In this section we propose several open questions. Some of these were raised by Professor David Bellamy in a workshop held in the city of Puebla, Mexico, on July 2002. The authors have not found any published manuscript with them.

Question 4.2. *Does there exist a continuum X and two points $p, q \in X$ such that M_p^X is dense and M_q^X is nowhere dense in X?*

Question 4.3. *For every continuum X, is M_p^X a F_σ-set for each $p \in X$? Is it possible that $\mathcal{M}_X = \{M_p^X : p \in X\}$ is either finite non-degenerate or a countable set?*

Question 4.4. *If X is a continuum such that M_p^X is closed for every $p \in X$, then is $\mathcal{M}_X = \{M_p^X : p \in X\}$ an upper semicontinuous decomposition of X?*

The following concepts were introduced in [13] by J. R. Prajs and K. Whittington.

Definition 4.5. Let X be a continuum and let K be a subcontinuum of X. We say that K is a *filament* provided that there exists a neighborhood N of K in X such that the component of K in N has empty interior. Given $p \in X$, the *filament composant of* p in X is defined as:

$$Fcs_X(p) = \bigcup \{A \in C(X) : A \text{ is a filament and } p \in A\}.$$

Next result follows from definition.

Proposition 4.6. *Let X be a continuum. Then $A \in M(X)$ for every filament A of X. Hence, $Fcs_X(p) \subseteq M_p^X$ for each $p \in X$.*

We have the following remark from definitions.

Remark 4.7. Let X be a continuum and let $p \in X$. Then:

(1) If X is locally connected at p, then $Fcs_X(p) = \varnothing$.
(2) If X is an indecomposable continuum, the $M_p^X = Fcs_X(p)$.
(3) If $Fcs_X(p)$ is nonempty, then $Fcs_X(p)$ has uncountable many points.

It is natural to rise the following problem:

Problem 4.8. *Characterize continua X for which $M_p^X = Fcs_X(p)$ for every $p \in X$ (for some $p \in X$, respectively).*

In the next result, we show a continuum X such that $M_p^X = Fcs_X(p)$ for every $p \in X$ and X is not indecomposable (see (2) of Remark 4.7).

We denote by $\mathcal{C}$ to the Cantor set in $[0,1]$ constructed under the classical way; that is $\mathcal{C} = \bigcap_{n\in\mathbb{N}} A_n$ where $A_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$, $A_2 = A_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}))$ and in general, having A_{n-1}, A_n is obtained by removing the open middle thirds form each of the 2^{n-1} closed intervals that make up A_{n-1}.

Proposition 4.9. *There exists a hereditarily decomposable and irreducible continuum X such that $M_x^X = Fcs_X(x)$ for each $x \in X$.*

Proof. Let T be the simple triod $T = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$ and let $Y = T \times \mathcal{C}$. Let $a = (1,0), b = (-1,0)$ and $c = (0,1)$ be the end points of T.

We define the following equivalence relation on Y. Given $(x,t),(x',s) \in Y$, we say that $(x,t) \sim (x',s)$ if and only if:

- $x = x' = a$ and $|t-s| = 1/3^{3i+1}, i \in \mathbb{N}$;
- $x = x' = b$ and $|t-s| = 1/3^{3i+2}, i \in \mathbb{N}$; or
- $x = x' = c$ and $|t-s| = 1/3^{3i+3}, i \in \mathbb{N}$.

Observe that $\sim$ is an upper semicontinuous decomposition and hence, $X = Y/\sim$ is a continuum. Note that:

(1) X is hereditarily decomposable and irreducible. Thus by definition, X is a continuum of type λ.
(2) If $f\colon X \to [0,1]$ is the monotone map such that $f^{-1}(t)$ has empty interior for each $t \in [0,1]$ (see Theorem 2.1), then $f^{-1}(t)$ is either a simple triod or the union of two simple triods, which are attached by one of their end points. Therefore, $M_x^X = f^{-1}(f(x))$ for every $x \in X$.

By Proposition 4.6, $Fcs_X(x) \subseteq f^{-1}(f(x))$ for each $x \in X$. In order to prove that $f^{-1}(f(x)) \subseteq Fcs_X(x)$, observe that every arc is a filament. Since $f^{-1}(f(x))$ is a finite union of arcs, $f^{-1}(f(x)) \subseteq Fcs_X(x)$. Therefore, $M_x^X = Fcs_X(x)$ for each $x \in X$. □

In Theorem 4.12, we show that if X is a continuum such that $M_x^X = Fcs_X(x)$ for each $x \in X$, then X cannot be arcwise connected.

Lemma 4.10. *Let X be a continuum. If $p \in X$, then $Fcs_X(p)$ has empty interior.*

Proof. By [13, Proposition 1.8], $Fcs_X(p)$ is a countable union of filament subcontinua of X. By Theorem 4.6, every filament has empty interior. Therefore, by Baire's Theorem, $Fcs_X(p)$ has empty interior. □

Proposition 4.11. *Let X be a continuum. If there exists $p \in X$ such that M_p^X has nonempty interior, then $Fcs_X(p) \neq M_p^X$.*

Proof. Consider $p \in X$ such that M_p^X has nonempty interior. Thus, $Fcs_X(p) \neq M_p^X$, by Lemma 4.10. □

Theorem 4.12. *If X is an arcwise connected continuum, then there exists $p \in X$ such that $Fcs_X(p) \neq M_p^X$.*

Proof. Let X be an arcwise connected continuum. Observe that if there is a free arc L contained in X with end points a and b, then for any $p \in L \setminus \{a, b\}$, we have $\{p\} = M_p^X$ and $Fcs_X(p) = \varnothing$. Hence, we may assume that X does not contain free arcs. In this case, we obtain that $M_p^X = X$ for every $p \in X$ and by Lemma 4.10, we conclude that $M_p^X \neq Fcs_X(p)$ for every $p \in X$. □

Question 4.13. *Does there exist an arcwise connected continuum X and $p \in X$ such that $M_p^X = Fcs_X(p)$?*

Given a continuum X, let $Fcs(X) = \{Fcs_X(x) : x \in X\}$. Note that if X is locally connected at some $p \in X$, then there is not a filament K of X such that $p \in K$; i.e., $Fcs_X(p) = \varnothing$. Thus, $Fcs(X)$ is not in general a partition of X. The following definition was taken from [12].

Definition 4.14. A continuum X is *filament additive* provided that for each two filament subcontinua K and L with nonempty intersection, the union $K \cup L$ is filament.

Observe that if $X = \mathrm{cl}_{\mathbb{R}^2}\{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$, then $\{0\} \times [0, 1]$ and $\{0\} \times [-1, 0]$ are filament subcontinua of X, but $\{0\} \times [-1, 1]$ is not a filament. Thus, X is not filament additive.

Theorem 4.15. *Let X be a continuum. If X is filament additive and $Fcs_X(x) \neq \varnothing$ for each $x \in X$, then $Fcs(X)$ is a partition.*

Proof. Since $Fcs_X(x) \neq \varnothing$, $x \in Fcs_X(x)$ for each $x \in X$ and hence, $\bigcup Fcs(X) = X$. Let $p, q \in X$ such that $Fcs_X(p) \cap Fcs_X(q) \neq \varnothing$. We will see that $Fcs_X(p) = Fcs_X(q)$. Let $z \in Fcs_X(p)$. Then there exists a filament L such that $p, z \in L$. Since $Fcs_X(p) \cap Fcs_X(q) \neq \varnothing$, there is $w \in Fcs_X(p) \cap Fcs_X(q)$. Let M, N be filaments such that $w, p \in M$ and $w, q \in N$. Since X is filament additive, $K = L \cup M \cup N$ is a filament such that $z, q \in K$. Thus, $z \in Fcs_X(q)$ and $Fcs_X(p) \subseteq Fcs_X(q)$. A similar argument shows that $Fcs_X(q) \subseteq Fcs_X(p)$. Therefore, $Fcs_X(p) = Fcs_X(q)$ and $Fcs(X)$ is a partition. □

Note that the continuum X defined in Proposition 4.9 is such that $Fcs(X)$ is a partition, but X is not filament additive. Also, observe that if $Fcs(X)$ is a partition, then $Fcs(X)$ is not trivial, by Lemma 4.10.

Problem 4.16. *Characterize continua X such that $Fcs(X)$ is a partition of X.*

The following result shows a continuum X such that $Fcs(X)$ is a partition, but $Fcs_X(x) \neq M_x^X$ for any $x \in X$.

Proposition 4.17. *There exists a continuum X such that it satisfies the following conditions:*

(1) *X is homogeneouos;*
(2) *$Fcs(X)$ is a partition of X;*
(3) *X is filament additive;*
(4) *$\{M_x^X : x \in X\}$ is a continuous decomposition of X;*
(5) *$Fcs_X(x) \subsetneq M_x^X$ for each $x \in X$.*

Proof. Let X be the circle of pseudo-arcs and let $f\colon X \to S^1$ be the monotone open map such that $f^{-1}(t)$ is a pseudo-arc with empty interior. It is well known that X is homogemeouos and $\{f^{-1}(z) : z \in S^1\}$ is a continuous decomposition of X. By [6, Lemma 3.2], we have that $A \in M(X)$ if and only if $A \subseteq f^{-1}(z)$ for some $z \in S^1$. Hence, $\{M_x^X : x \in X\} = \{f^{-1}(z) : z \in S^1\}$. Finally, observe that $Fcs_X(x)$ is the composant of the pseudo-arc $f^{-1}(z)$ where $x \in f^{-1}(z)$; i.e., $Fcs_X(x) \subsetneq M_x^X$ for each $x \in X$. Thus, X is filament additive and $Fcs(X)$ is a partition, by Theorem 4.15. Therefore, the circle of pseudo-arcs satisfies all the conditions of the proposition. ☐

Recall that a subcontinuum A of a continuum X is called *terminal* provided that for any subcontinuum B of X such that $A \cap B \neq \varnothing$, then $A \subset B$ or $B \subset A$.

Theorem 4.18. *Let X be a continuum and let Y be a terminal subcontinuum of X. Then, Y is decomposable if and only if there exists $p \in Y$ such that $Fcs_X(p) = Y$.*

Proof. Let M and N be proper subcontinua of Y such that $Y = M \cup N$. Let $p \in M \cap N$ and let $x \in Y \setminus \{p\}$. Suppose that $x \in M$. Let U be a neighborhood of M in X such that $Y \setminus \mathrm{cl}(U) \neq \varnothing$. Since Y is terminal, if C is the component of U such that $M \subseteq C$, then C has empty interior. Thus, $x \in Fcs_X(p)$. This shows that $Y = Fcs_X(p)$.

Conversely, suppose that Y is indecomposable. It is not difficult to show that $Fcs_X(x) = \kappa_Y(x)$ for each $x \in Y$, where $\kappa_Y(x)$ is the composant of Y containing x. Therefore, $Fcs_X(x) \neq Y$ for any $x \in Y$. ☐

Theorem 4.19. *Let X be a continuum and let Y be a terminal subcontinuum of X. If Y is non irreducible, then $Y = Fcs_X(p)$ for each $p \in Y$.*

Proof. Suppose that Y is non irreducible. Let $p \in Y$ and let $x \in Y \setminus \{p\}$. Since Y is non irreducible, there exists a proper subcontinuum A of Y such that $p, x \in A$. Let U be a neighborhood of A in X such that $Y \setminus \mathrm{cl}(U)$. It is clear that the component of A in U has empty interior and hence, $x \in Fcs_X(p)$. This shows that $Y = Fcs(p)$. ☐

Next corollaries follows from Theorems 4.18 and 4.19.

Corollary 4.20. *Let X be a compactification of the ray with remainder Y. Then, Y is decomposable if and only if there exists $p \in Y$ such that $Fcs_X(p) = Y$.*

Corollary 4.21. *Let X be a compactification of the ray with remainder Y. If Y is non irreducible, then $Y = Fcs_X(p)$ for each $p \in Y$.*

Now, we present some examples giving partial answers to the following question.

Question 4.22. *Given a metric space Y, does there exist a continuum X such that $Fcs(X)$ is homeomorphic to Y?*

Example 4.23. Note that the continuum X presented in Proposition 4.9 satisfies that $Fcs(X)$ is homeomorphic to $[0,1]$. Furthermore, if Z is the quotient space of the continuum X, where the only nondegenerate element is by identifying the point $(a,0)$ with the point $(a,1)$, then $Fcs(Z) \cong S^1$.

Example 4.24. In the Euclidean space $\mathbb{R}^3$ consider $S' = \{(x,y,0) : x^2+y^2 = 1\}$ and $S'' = \{(x,y,1) : x^2+y^2 = 1\}$. Let

$$\mathcal{X}_0 = S' \cup \Big\{(\cos(\tfrac{1}{t}), \sin(\tfrac{1}{t}), t) : t \in (0, \tfrac{1}{2}]\} \cup \{(\cos(\tfrac{1}{1-t}), \sin(\tfrac{1}{1-t}), t) : t \in [\tfrac{1}{2}, 1)\Big\} \cup S''.$$

Observe that $\mathcal{X}_0$ is a compactification of the real line $\mathbb{R}$, contained in the cylinder $\mathcal{L} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$ with $S' \cup S''$ as a remainder. It is easy to verify that $Fcs(p) = S'$ for every $p \in S'$, $Fcs(p) = S''$ for every $p \in S''$ and $Fcs(p) = \varnothing$ in other case. This implies that:

$$Fcs(X) = \{Fcs(p) : p \in \mathcal{X}_0\} = \{S', S''\}.$$

To the next example consider the following.

Given $a, b \in [0,1]$ such that $a < b$, let

$$\mathcal{X}_{[a,b]} = \{(x,y,(b-a)z+a) : (x,y,z) \in \mathcal{X}_0\}$$

and for each $d \in [0,1]$ let

$$S_d = \{(x,y,d) : x^2+y^2 = 1\}.$$

Observe that $S_0 = S'$ and $S_1 = S''$, also note that $\mathcal{X}_{[a,b]}$ is a copy of $\mathcal{X}_0$ contained in the cylinder $\mathcal{L}$ between the planes $z = a$ and $z = b$. Also, note that $S_a = \{(x,y,(b-a)z+a) : (x,y,z) \in S'\}$ and $S_b = \{(x,y,(b-a)z+a) : (x,y,z) \in S''\}$ are just a translation of S' and S'', respectively; and $S_a \cup S_b$ is the remainder of $\mathcal{X}_{[a,b]}$. Finally, it follows from construction that, if $0 \le a < b < c \le 1$, then $\mathcal{X}_{[a,b]} \cap \mathcal{X}_{[b,c]} = S_b$.

Example 4.25. Given $n \ge 1$, for each $k \in \{0, \dots, n\}$. Consider the continuum in $\mathbb{R}^3$ defined by:

$$X_n = \bigcup_{k=1}^{n} \mathcal{X}_{[\frac{k-1}{n}, \frac{k}{n}]}.$$

It is clear that $Fcs(X_n) = \{S_0, S_{\frac{1}{n}}, \ldots, S_1\}$ which has exactly $n+1$ elements.

Example 4.26. In the euclidean space, let $Y = \bigcup_{n\in\mathbb{N}} \mathcal{X}_{[\frac{1}{n+1},\frac{1}{n}]}$ and consider:

$$X_\infty = \{(zx, zy, z) : (x, y, z) \in Y\} \cup \{(0, 0, 0)\}.$$

Since X_∞ is locally connected at $(0, 0, 0)$, it is easy to verify that $Fcs(X_\infty) = \{B_n : n \in \mathbb{N}\}$ where $B_n = \{(zx, zy, z) \in \mathbb{R}^3 : (x, y, z) \in S_{\frac{1}{n}}\}$ for each $n \in \mathbb{N}$; which is homeomorphic to the harmonic sequence.

Example 4.27. In the euclidean space, let

$$\mathcal{X}_\infty = [\bigcup_{n\in\mathbb{N}} \mathcal{X}_{[\frac{1}{n+1},\frac{1}{n}]}] \cup S_0.$$

It is easy to verify that $Fcs(\mathcal{X}_\infty) = \{S_{\frac{1}{n}} : n \in \mathbb{N}\} \cup \{S_0\}$ which is homeomorphic to closure of the harmonic sequence.

Example 4.28. Let $\mathcal{C}$ be the middle third Cantor set contained in $[0, 1]$. Let

$$\mathcal{X}_\mathcal{C} = \{S_c : c \in \mathcal{C}\} \cup \{\mathcal{X}_{[a,b]} : a, b \in \mathcal{C} \text{ and } [a, b] \cap \mathcal{C} = \{a, b\}\}$$

Observe that $Fcs(\mathcal{X}_\mathcal{C}) = \{S_c : c \in \mathcal{C}\}$ which is homeomorphic to the Cantor set.

From the ideas used to construct Examples 4.23 - 4.28, we can conclude that for every compact countable metric space Y, it is possible to construct a continuum X such that $Y \cong Fcs(X)$.

To end this part, observe that there exist continua X for which $M_x^X = \{x\}$ for each $x \in X$, such as finite graphs. Hence, given a continuum X, we define

$$\mathcal{F}_X = \{p \in X : M_p = \{p\}\}.$$

The following example shows that for every compact K in $[0, 1]$, there exists a continuum X such that $\mathcal{F}_X \cong K$.

Example 4.29. Let T be the arc of pseudo-arcs and let $f : T \to [0, 1]$ be the monotone map such that $\mathcal{D} = \{f^{-1}(t) : t \in [0, 1]\}$ is the continuous decomposition of T such that $f^{-1}(t)$ is a pseudo-arc, for each $t \in [0, 1]$. Let $K \in 2^{[0,1]}$ and let

$$\mathcal{D}' = \{\{x\} \subseteq T : f(x) \in [0, 1] \setminus K\} \cup \{f^{-1}(t) \subseteq T : t \in K\}.$$

Notice that $(\mathcal{D}', \tau_{\mathcal{D}'})$ is an upper semicontinuous decomposition of T and $\mathcal{F}_{\mathcal{D}'} = \{f^{-1}(x) : x \in K\}$; i.e., $\mathcal{F}_{\mathcal{D}'} \cong K$.

Problem 4.30. *Given a metric space Y, does there exist a continuum X such that $\mathcal{F}_X$ is homeomorphic to Y?*

Question 4.31. *Is the set $\mathcal{F}_X$ always a F_σ-set of X?*

5. Contractibility of M(X)

In this section, we study the contractibility of the hyperspace $M(X)$. In [10], it is raised the following questions:

Question 5.1. *Let X be a continuum. If $M(X)$ is contractible, then does it follow that X is contractible?*

Question 5.2. *Let X be a continuum. If X is contractible, then does it follow that $M(X)$ is contractible?*

In this section, Theorem 5.5 shows that if X is the arc of pseudoarcs, then $M(X)$ is contractible, giving a negative answer to Question 5.1. Furthermore, we provide partial answers to Question 5.2.

Given a map between continua $f\colon X \to Y$, in the proof of the following result, we denote by $C(f)\colon C(X) \to C(Y)$ the induced map defined by $C(f)(A) = f(A)$ for each $A \in C(X)$ [3, 77.1].

Theorem 5.3. *Let X be a continuum. If X is contractible, then $M(X \times [0,1])$ is contractible.*

Proof. Since X is contractible, there exist $p \in X$ and $g\colon X \times [0,1] \to X$ a map such that $g(x,0) = x$ and $g(x,1) = p$ for each $x \in X$. Let $h\colon X \times [0,1] \times [0,1/2] \to X \times [0,1]$ be defined for each $(x,s,t) \in X \times [0,1] \times [0,1/2]$ by:

$$h(x,s,t) = (x, s - 2ts).$$

Let $f\colon X \times \{0\} \times [1/2,1] \to X \times \{0\}$ be defined for each $(x,0,t) \in X \times \{0\} \times [1/2,1]$ by:

$$f(x,0,t) = (g(x,2t-1),0).$$

Notice that $f(x,0,1/2) = (x,0)$ and $f(x,0,1) = (p,0)$ for each $(x,0) \in X \times \{0\}$.

Let $\pi_0\colon X \times [0,1] \to X \times \{0\}$ be the projection defined by $\pi_0(x,t) = (x,0)$ for each $(x,t) \in X \times [0,1]$.

Let $H_1\colon M(X \times [0,1]) \times [0,1/2] \to M(X \times [0,1])$ be defined by:

$$H_1(A,t) = C(h)(A \times \{t\}) \text{ for each } (A,t) \in M(X \times [0,1]) \times [0,1/2].$$

We will show that H_1 is well defined. Let $(A,t) \in M(X \times [0,1]) \times [0,1/2]$. We show that $H_1(A,t) \in M(X \times [0,1])$. Notice that $H_1(A,t) \in C(X \times [0,1])$. We see that $\operatorname{int}(H_1(A,t)) = \varnothing$. Observe that $H_1(A,\frac{1}{2}) \subseteq X \times \{0\}$ and hence, $\operatorname{int}(H_1(A,\frac{1}{2})) = \varnothing$. Suppose that $t \neq 1/2$. Let $(x,r) \in H_1(A,t)$. Then there exists $(y,s) \in A$ such that $(x,r) = h(y,s,t) = (y, s-2ts)$. Since $A \in M(X \times [0,1])$, there exists a sequence $((y_n,s_n))_{n\in\mathbb{N}}$ in $(X \times [0,1]) \setminus A$ such that $\lim_{n\to\infty}(y_n,s_n) = (y,s)$. Let $((y_n, s_n - 2ts_n))_{n\in\mathbb{N}}$ be a sequence in $X \times [0,1]$. We need to show that $(y_n, s_n - 2ts_n) \notin H_1(A,t)$. If $(y_n, s_n - 2ts_n) \in H_1(A,t)$ for some n, then there exists $(x',s') \in A$, such that $h(x',s',t) = (y_n, s_n - 2ts_n)$. Thus $(x', s' - 2ts') = (y_n, s_n - 2ts_n)$ and we have that both $x' = y_n$ and $s' = s_n$. This contradicts the fact that $(y_n,s_n) \notin A$. Therefore, $(y_n, s_n - 2ts_n) \notin H_1(A,t)$ for each $n \in \mathbb{N}$. Furthermore, $\lim_{n\to\infty}(y_n, s_n - 2ts_n) = (x,r)$. Thus, $(x,r) \notin$

$\text{int}(H_1(A,t))$ and $H_1(A,t) \in M(X \times [0,1])$. Therefore, H_1 is well defined. It is clear that H_1 is a map.

Let $H_2\colon M(X \times [0,1]) \times [1/2,1] \to M(X \times [0,1])$ be defined by:

$$H_2(A,t) = C(f)(C(\pi_0)(A) \times \{t\}), \text{ for each } (A,t) \in M(X \times [0,1]) \times [1/2,1].$$

We see that H_2 is well defined. Let $(A,t) \in M(X\times[0,1])\times[1/2,1]$. Notice that $H_2(A,t) \subseteq X \times \{0\}$. Thus, $\text{int}(H_2(A,t)) = \varnothing$ and $H_2(A,t) \in M(X \times [0,1])$. Observe that H_2 is a map.

Finally, let $H\colon M(X \times [0,1]) \times [0,1] \to M(X \times [0,1])$ be defined for each $(A,t) \in M(X \times [0,1])$ by:

$$H(A,t) = \begin{cases} H_1(A,t), & \text{if } t \in [0,1/2]; \\ H_2(A,t), & \text{if } t \in [1/2,1]. \end{cases}$$

Note that if $A \in M(X \times [0,1])$, then $H_1(A,1/2) = h(A \times \{1/2\}) = \pi_0(A) = f(\pi_0(A) \times \{1/2\}) = H_2(A,1/2)$. Thus, H is a map. Furthermore, $H(A,0) = A$ y $H(A,1) = X \times \{0\}$ for each $A \in M(X \times [0,1])$. Therefore, $M(X \times [0,1])$ is contractible. □

Given a topological space Y, recall that the *cone over* Y, which is denoted by $\text{Cone}(Y)$, is the quotient space obtained from $Y \times [0,1]$ by shrinking $Y \times \{1\}$ to a point. Note that $\text{Cone}(Y)$ is contractible, for every compactum Y. Hence, the following result gives a partial answer to Question 5.2.

Theorem 5.4. *Let Y be a compactum. Then, $M(\text{Cone}(Y))$ is contractible.*

Proof. Let $q\colon Y \times [0,1] \to \text{Cone}(Y)$ be the quotient map, where $\text{Cone}(Y) = (Y \times [0,1])/(Y \times \{0\})$. We denote $v_Y = Y \times \{0\}$. Let $g\colon \text{Cone}(Y) \times [0,1] \to \text{Cone}(Y)$ be defined by:

$$g(\chi,t) = \begin{cases} v_Y, & \text{if either } \chi = v_Y \text{ or } t = 1; \\ q(x, s - ts), & \text{if } \chi = q(x,s),\ s \neq 0 \text{ and } t \neq 1. \end{cases}$$

We see that g is a map and that $g(\chi,t) = v_Y$ if, and only if, $\chi = v_Y$ or $t = 1$.

Let $H\colon M(\text{Cone}(Y))\times[0,1] \to M(\text{Cone}(Y))$ be defined by $H(A,t) = C(g)(A\times\{t\})$. We will show that H is well defined. Let $(A,t) \in M(\text{Cone}(Y)) \times [0,1]$. Notice that $H(A,t) \in C(\text{Cone}(Y))$. Hence, we have to prove that $H(A,t) \in M(\text{Cone}(Y))$. If $t = 1$, then $H(A,1) = \{v_Y\}$. Hence, $H(A,1) \in M(\text{Cone}(Y))$. Assume that $t \in [0,1)$. Let $\chi \in H(A,t)$. There exists $\gamma \in A$, such that $g(\gamma,t) = \chi$. Since $A \in M(\text{Cone}(Y))$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\text{Cone}(Y)\setminus A$ such that $\lim_{n\to\infty}(\gamma_n) = \gamma$. We can suppose that $\gamma_n \neq v_Y$, for all n. Thus, there exists $(y_n,s_n) \in A\times(0,1]$ such that $\gamma_n = q(y_n,s_n) = \{(y_n,s_n)\}$ for every n. We have that $\lim_{n\to\infty} g(\gamma_n,t) = \chi$. If we show that $g(\gamma_n,t) \notin H(A,t)$ for all n, then $\chi \notin \text{int}(H(A,t))$. Thus, $H(A,t) \in M(\text{Cone}(Y))$. Indeed, if $g(\gamma_n,t) \in H(A,t)$ for some n, then $q(y_n, s_n - ts_n) \in H(A,t)$. Hence, there exists $\gamma' \in A$, such that $q(y_n, s_n - ts_n) = g(\gamma',t)$. Notice that $\gamma' \neq v_Y$. Then $\gamma' = q(y',s')$, for some $(y',s') \in X\times(0,1]$. Hence, $q(y_n, s_n - ts_n) = q(y', s' - ts')$. Then, $\{(y_n, s_n - ts_n)\} = \{(y', s' - ts')\}$. Thus, $y_n = y'$ and $s_n - ts_n = s' - ts'$.

Hence, $y_n = y'$ and $s_n = s'$. For that, $\gamma_n = \gamma$. Then, $\gamma_n \in A$, a contradiction. Thus, $g(\gamma_n, t) \notin H(A, t)$ for all n. Thus, we have the result. □

Example 5.5. There exists a continuum X such that $M(X)$ is contractible, but X is not contractible.

Proof. Let X be the arc of pseudo-arcs. Note that X is not arcwise connected and hence, X is not contractible.

We show that $M(X)$ is contractible. Let $f\colon X \to [0,1]$ be the monotone map such that $\mathcal{D} = \{f^{-1}(t) : t \in [0,1]\}$ is the minimal admissible decomposition of X. We know that $\mathcal{D}$ is a continuous decomposition where $f^{-1}(t)$ is a pseudo-arc for every $t \in [0,1]$. Furthermore, it is not difficult to see that:

Claim. If $A \in M(X)$, then $f(A) = \{s_A\}$ for some $s_A \in [0,1]$; i.e., $A \subseteq f^{-1}(s)$ for some $s \in [0,1]$.

Let $w\colon C(X) \to [0,1]$ be a Whitney map such that $w(f^{-1}(t)) = \frac{1}{2}$ for each $t \in [0,1]$ (see [3, Theorem 23.3]). Let $h\colon X \times [0,1] \to C(X)$ be defined for each $(x,t) \in X \times [0,1]$ by $h(x,t) = u(F_w(x,t))$, where $u\colon C(C(X)) \to C(X)$ is the union map, and $F_w(x,t) = \{A \in C(X) : x \in A, w(A) = t\}$. Since X has the property of Kelley, F_w is a map by [3, Proposition 20.11]. Thus, h is a map. Let $H_1 : M(X) \times [0,1/2] \to M(X)$ be defined for each $(A,t) \in M(X) \times [0,\frac{1}{2}]$ as follow:

$$H_1(A,t) = u(C(h)(A \times \{t\})).$$

Observe that $H_1(A,t) \in C(X)$. Since $t \leq \frac{1}{2}$ and $f^{-1}(t)$ is terminal, $K \subseteq f^{-1}(t)$ for each $K \in F_w(A,t)$. Thus, $H_1(A,t) \subseteq f^{-1}(t)$ and $\operatorname{int}(H_1(A,t)) = \varnothing$. This shows that H_1 is well defined. It is clear that H_1 is a map.

Let $H_2\colon M(X) \times [\frac{1}{2},1] \to M(X)$ be defined for all $(A,t) \in M(X) \times [\frac{1}{2},1]$ by:

$$H_2(A,t) = f^{-1}((2-2t)s_A), \text{ where } f(A) = \{s_A\}.$$

Since $\mathcal{D}$ is a continuous decomposition, the function $\phi\colon [0,1] \to \mathcal{D}$ defined by $\phi(t) = f^{-1}(t)$ is a homeomorphism. Hence, H_2 is a map. Finally, let $H\colon M(X) \times [0,1] \to M(X)$ be defined for each $(A,t) \in M(X) \times [0,1]$ by:

$$H(A,t) = \begin{cases} H_1(A,t), & \text{if } t \in [0,\frac{1}{2}]; \\ H_2(A,t), & \text{if } t \in [\frac{1}{2},1]. \end{cases}$$

Observe that $H_1(A,\frac{1}{2}) = H_2(A,\frac{1}{2})$ for each $A \in M(X)$. Thus, H is a map. Furthermore, $H(A,0) = H_1(A,0) = A$ and $H(A,1) = H_2(A,1) = f^{-1}(0)$ for every $A \in M(X)$. Therefore, $M(X)$ is contractible. □

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