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On star Rothberger spaces modulo an ideal

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Abstract

In this article, we introduce the ideal star Rothberger property by coupling the notion of a star operator to that of an ideal Rothberger space, after which some of its topological characteristics are analyzed. By creating relationships between a numbers of topological features with structures similar to the ideal star Rothberger space, we reinforce the concept. In order to illustrate the differences between a number of related topological properties, we also provide several counter examples. Certain preservation-related properties under subspaces and functions are investigated. Lastly we find a way to express ideal star Rothberger space by means of families of closed sets by bringing some modifications to the SSI^I property.

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1. Introduction

Kuratowski[18] and Vaidyanathaswamy[23] began a systemic investigation of ideals in a topological space. After that, many general topologists became interested in the idea of the ideal and investigated them deeply. If a nonempty collection $I \subseteq \mathcal{P}(X)$ is closed under the subset and finite union operations then I is an ideal on the set X[13]. A subset $N \subseteq X$ is called a nowhere dense set if $int(\overline{N}) = \varnothing$ and the collection $\mathcal{N}$ of all nowhere dense subsets of the space

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X forms an ideal on X. An ideal I on (X,τ) is said to be τ-co-dense (simply co-dense) if $\tau \cap I = \{\varnothing\}$. $\mathcal{N}$ is a co-dense ideal on X.

A space X is Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} U_n$[20].

In 1991, Douwen[12] introduced the star operator. If $A \subseteq X$ and $\mathcal{U} \subseteq \mathcal{P}(X)$ then star of A with respect to $\mathcal{U}$ is given by $St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \varnothing\}$. Some recent uses of star operator can be found in [1, 2, 3, 4, 5, 6, 7, 8, 10, 22]. In 1999, Kočinac used this operator for the generalization of selection principles and Menger spaces[16, 17], in 2014 Sakai[21] introduced star Rothberger property. A space X is star Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n\}$. Sakai[21] has also studied some of its weaker forms.

Recently Bhardwaj[11] and Güldürdek[14, 15] has started some investigations on the ideal versions of Rothberger spaces. An ideal space X is I-Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} U_n \in I$. In [9], Bal has found some alternate representations for Ideal Rothberger spaces. Continuing the recent advancements in this direction we introduce and study the ideal versions of star Rothberger spaces.

2. Preliminaries

Throughout the paper a space X indicates an ideal space (X,τ,I), where τ is a topology on X and I is an ideal on X. No specific separation axiom is considered otherwise stated. $\mathcal{N}$ will denote the collection of all nowhere dense subsets of X which is also a co-dense ideal on X. For usual notations and symbols we follow [13].

Proposition 2.1 ([19])**.** *If I is an ideal on a space (X,τ) and $A \subseteq X$, then $I_A = \{I_1 \cap A : I_1 \in I\}$ is an ideal on A.*

Proposition 2.2 ([19])**.** *If I is an ideal on a space (X,τ) and $f : (X,\tau) \to (Y,\sigma)$ is a continuous function, then $f(I) = \{f(I_1) : I_1 \in I\}$ is an ideal on Y.*

Definition 2.3 ([21])**.** A space X is called weakly star Rothberger if for every sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of open covers of X, we can find a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$ and $\overline{\bigcup_{n\in\mathbb{N}} St(A_n,\mathcal{A}_n)} = X$.

Definition 2.4 ([9])**.** In an ideal space (X,τ,I), a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of subsets of X is said to have **sequential single-tonic intersection module an ideal (SSI^I)** property if for every sequence $\{F_n : n \in \mathbb{N}\}$ such that $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ we get $\bigcap_{n\in\mathbb{N}} F_n \notin I$.

3. Star Rothbergerness Modulo an Ideal

Definition 3.1. An ideal space (X,τ,I) is said to be strongly star Rothberger modulo an ideal (in short I-star Rothberger) space if for every sequence $\{\mathcal{A}_n :$

$n \in \mathbb{N}\}$ of open covers of X, we can find a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n) \in I$.

Proposition 3.2. *Every I-Rothberger space is a I-star Rothberger space.*

Proof. Let (X, τ, I) be an I-Rothberger space and $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be an arbitrary sequence of open covers of X. Then there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} A_n \in I$. Since $A_n \subseteq St(A_n, \mathcal{A}_n)$ for all $n \in \mathbb{N}$, $X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n) \in I$. Thus X is I-star Rothberger. □

Corollary 3.3. *Every Rothberger space is I-star Rothberger space.*

Example 3.4. There exists an I-star Rothberger space which is not I-Rothberger. Consider the ideal space (X, τ, I) where $X = \mathbb{R}$, τ is generated by the base $\mathcal{B} = \{\varnothing, \mathbb{Q}\} \cup \{U_x = \mathbb{Q} \cup \{x\} : x \in (\mathbb{R} \setminus \mathbb{Q})\}$ and $I = \mathcal{P}(A)$ with $A = (\mathbb{R} \setminus \mathbb{Q}) \cap \{(-\infty, -1) \cup [1, \infty)\}$. Let $\{\mathcal{U}_n = \mathcal{U} : n \in \mathbb{N}\}$ be a sequence of open covers of X where $\mathcal{U} = \{U_x = \mathbb{Q} \cup \{x\} : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Now for any sequence $\{U_{x_n} : n \in \mathbb{N}\}$ with each $U_{x_n} \in \mathcal{U}_n$ we get $((\mathbb{R} \setminus \mathbb{Q}) \cap [-1, 1]) \setminus \bigcup_{n\in\mathbb{N}} U_{x_n} \neq \varnothing$. It follows that $X \setminus \bigcup_{n\in\mathbb{N}} U_{x_n} \notin I$. Thus X is not I-Rothberger.

We now show that X is I-star Rothberger. Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. For each n select $A_n \in \mathcal{A}_n$ such that $\mathbb{Q} \subseteq A_n$. The sequence $\{A_n : n \in N\}$ witnesses that X is I-star Rothberger.

Proposition 3.5. *Every star Rothberger space is I-star Rothberger space.*

Proof. The proof follows directly as $\varnothing \in I$. Hence omitted. □

Example 3.6. There exist a T_0 I-star Rothberger space which is not star Rothberger.

Let $X = \mathbb{R}$, $\mathcal{B} = \{\{1, 2, ..., n\} : n \in \mathbb{N}\} \cup \{\{x\} : x \in \mathbb{R} \setminus \mathbb{N}\} \cup \{\varnothing\}$ and τ be the the topology generated by $\mathcal{B}$. Thus (X, τ) is a T_0 topological space. With the ideal $I = \mathcal{P}(\mathbb{R} \setminus \mathbb{N})$ defined on X, (X, τ, I) forms an ideal space.
Consider the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers where, $\mathcal{U}_n = \mathcal{B}$ for all $n \in \mathbb{N}$. For every selection of the sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$, we will have $\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n) \subseteq \mathbb{N} \cup \{x_n : n \in \mathbb{N}\}$ where $x_n \in U_n \setminus \mathbb{N}$ and $x_n \in \mathbb{R} \setminus \mathbb{N}$. Clearly, $|\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n)| \leq \omega_o$, the first infinite cardinal. So, $\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n) \neq X$ and hence, (X, τ, I) is not a star Rothberger space.

On the other hand, for every sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of open covers of X, we can select $A_1 \in \mathcal{A}$, such that $1 \in A_1$ and choose $A_n \in \mathcal{A}_n$ (for $n = 2, 3, 4, ...$) randomly. So, $\mathbb{N} \subseteq St(A_1, \mathcal{A}_1) \subseteq \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n)$. Thus, $X \setminus \bigcap_{n\in\mathbb{N}} (A_n, \mathcal{A}_n) \subseteq \mathbb{R} \setminus \mathbb{N} \in I$. Therefore, (X, τ, I) is an I-star Rothberger pace.

Theorem 3.7. *The following statements are equivalent:*
1. X is $\mathcal{N}$-star Rothberger.
2. X is weakly star Rothberger.

Proof. Let (X, τ) be an $\mathcal{N}$-star Rothberger space and $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then there exists a sequence $\{A_n : n \in \mathbb{N}\}$ with $A_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n) \in \mathcal{N}$. Therefore,

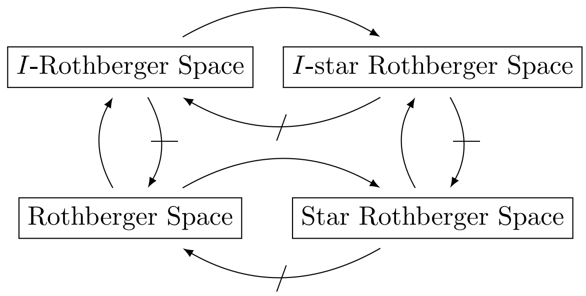

FIGURE 1. Relation among several related spaces.

$int(X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n)) = \varnothing$. Thus $\overline{\bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n)} = X$. Hence, X is weakly star Rothberger space.

Conversely, let X be weakly star Rothberger space and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Then there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ and $\overline{\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n)} = X$. So, $X \setminus \overline{\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n)} = \varnothing$. Thus, $int(X \setminus \bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n)) = \varnothing$. Therefore, $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{U}_n) \in \mathcal{N}$ and hence X is an $\mathcal{N}$-star Rothberger space. □

Theorem 3.8. *A space X is I-star Rothberger space for some co-dense ideal I if and only if X is weakly star Rothberger space.*

Proof. Let, X be I-star Rothberger where I is a co-dense ideal of X. Suppose $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is an arbitrary sequence of open covers of X. Therefore, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n) = I_1 \in I$. If there exists a non-empty open set $O \subseteq I$, then $O \in I$, which contradict the fact that I is co-dense. Therefore, $int(X \setminus \bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n)) = \varnothing$, i.e., $\overline{\bigcup_{n\in\mathbb{N}} St(A_n, \mathcal{A}_n)} = X$. Hence X is weakly star Rothberger.

Conversely, suppose X is weakly star Rothberger then by Theorem 3.7, X is $\mathcal{N}$-star Rothberger and $\mathcal{N}$ is a co-dense ideal of X. Hence the theorem. □

Proposition 3.9. *If I and J are two ideal on the same space X and $I \subseteq J$ then I-star Rothberger criteria implies the J-star Rothberger criteria for X.*

Proof. The proof follows directly, hence omitted. □

Example 3.10. Closed subspace (A, τ_A) of a I-star Rothberger space (X, τ) may not be a I_A- star Rothberger space.

Let $X = \mathbb{R}$, $\mathcal{B} = \{\varnothing, \{0\}, \mathbb{R}\} \cup \{\{0, x\} : x \in \mathbb{R}\}$ and τ be the topology generated by $\mathcal{B}$. Consider the ideal $I_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q})$. From the construction of the topology, for every open cover $\mathcal{U}$ of X we can select an $U \in \mathcal{U}$ such that $0 \in U$ and so it will give $St(U, \mathcal{U}) = X$. Thus for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers, we have a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ and

$\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) = X$, i.e., $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) = \varnothing \in I_{\mathbb{Q}}$. Hence (X,τ) is an $I_{\mathbb{Q}}$-star Rothberger space.

Now consider the closed set $A = \mathbb{R}\setminus\{0\}$. Then (A,τ_A) is a subspace topology where $\mathcal{B}_A = \{\varnothing, A\} \cup \{\{x\} : x \in A\}$ generates τ_A. Evidently $I_A = A \cap I_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q} \setminus \{0\})$ and so $I_A \subseteq I_{\mathbb{Q}}$. Consider the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of τ_A- open covers of A where $\mathcal{U}_n = \{\{x\} : x \in A\}$, for all $x \in \mathbb{N}$. For every selection $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)$ is countable i.e., $A \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)$ is uncountable. So, $A\setminus\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) \nsubseteq \mathbb{Q} \subseteq \mathbb{Q}\setminus\{0\}$. Thus, $A \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) \notin I_A$ and hence (A,τ_A) cannot be an I_A-star Rothberger space.

Theorem 3.11. *If (X,τ) is an I-star Rothberger space and A is a clopen subset of X, then (A,τ_A) is an I_A-star Rothberger space.*

Proof. Let (X,τ) be an I-star Rothberger space and (A,τ_A) is a clopen subspace of X. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of τ_A- open covers of A. Then $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of open covers of X, where $\mathcal{V}_n = \{X \setminus A\} \cup \mathcal{U}_n$ for all $n \in \mathbb{N}$. But X is an I-star Rothberger. So, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ such that $V_n \in \mathcal{V}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) = I_1(\text{say}) \in I$. Thus $\bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) \cup I_1 = X$.

Case-I

If $V_n = X\setminus A$ for all $n \in \mathbb{N}$ then $St(V_n,\mathcal{V}_n) = X\setminus A$. So, $X\setminus\bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) = A = I_1 \in I$. Therefore, every subset of A belong to I_A.

Now, for every sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$, $A \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) \subseteq A$. So, $A \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) \in I_A$ and hence, (A,τ_A) is an I_A- star Rothberger space.

Case-II

If $V_n = U_n$ for some $U_n \in \mathcal{U}_n$ and for all $n \in \mathbb{N}$, then $St(V_n,\mathcal{V}_n) = St(U_n,\mathcal{U}_n)$ for all $n \in \mathbb{N}$. So, $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) = X \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) = I_1 \in I$. Thus, $A\setminus\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) = A\cap I_1 \in I_A$ and hence, (A,τ_A) is an I_A- star Rothberger space.

Case-III

If $V_i = X \setminus A$ for some $i \in \mathbb{N}$ and $V_j = U_j \in \mathcal{U}_j$ for some $j \in \mathbb{N}$ then we construct a new sequence $\{W_n : n \in \mathbb{N}\}$ such that

$$W_n = \begin{cases} U_n = V_n & \text{if } V_n \neq X \setminus A \\ U_n \in \mathcal{U}_n \subseteq \mathcal{V}_n & (\text{ any } U_n \in \mathcal{U}_n) \text{ if } V_n = X \setminus A. \end{cases}$$

$A \setminus \bigcup_{n\in\mathbb{N}} St(W_n,\mathcal{U}_n) = A \setminus \bigcup_{n\in\mathbb{N}} St(W_n,\mathcal{V}_n) \subseteq A \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) \subseteq X \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) = I_1 \in I$. Therefore, $A \setminus \bigcup_{n\in\mathbb{N}} St(W_n,\mathcal{U}_n) \subseteq I_1 \in I_A$. Thus, $A \setminus \bigcup_{n\in\mathbb{N}} St(W_n,\mathcal{U}_n) \in I_A$ and hence (A,τ_A) is an I-star Rothberger space. □

Theorem 3.12. *Let $f : (X,\tau) \to (Y,\delta)$ be a continuous bijection and I be an ideal on X. If (X,τ) is I-star Rothberger then (Y,δ) is $f(I)$-star Rothberger.*

Proof. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y. We take $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ for all $n \in \mathbb{N}$. Since f is a continuous surjection, $\{\mathcal{U}_n : n \in$

$\mathbb{N}\}$ is a sequence of open covers of X. But X is an I-star Rothberger space. Therefore, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n) \in I$. So, $f(X \setminus \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)) \in f(I)$. Thus $Y \setminus f(\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)) \in f(I)$. [Since, f is surjective]. ...(a)

Let $y \in Y \setminus (\bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n))$. So, $y \in Y$ and $y \notin \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n)$. Implies that $y \notin St(V_n,\mathcal{V}_n)$ for any $n \in \mathbb{N}$. Therefore, $y \notin V$ whenever $V_n \cap V \neq \varnothing$ for all $V \in \mathcal{V}_n$ and for any $n \in \mathbb{N}$. Corresponding to $V, V_n \in \delta$, there exists a $U, U_n \in \tau$ such that $U = f^{-1}(V)$ and $U_n = f^{-1}(V_n)$ for any $n \in \mathbb{N}$. Therefore, $y \notin f(U)$ whenever $f(U_n) \cap f(U) \neq \varnothing$ for all $U \in \mathcal{U}_n$ and for any $n \in \mathbb{N}$. So, $f^{-1}(y) \notin U$ whenever $f(U_n \cap U) \neq \varnothing$ for all $U \in \mathcal{U}_n$ and for any $n \in \mathbb{N}$. [Since, f is injective, $f(U_n \cap U) = f(U_n) \cap f(U)$]. Thus, $f^{-1}(y) \notin U$ whenever $U_n \cap U \neq \varnothing$ for all $U \in \mathcal{U}_n$ and for any $n \in \mathbb{N}$. So, $f^{-1}(y) \notin \bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)$. Thus, $y \in Y \setminus f(\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n))$. Therefore, $Y \setminus (\bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n)) \subseteq Y \setminus f(\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n))$. By equation (a), $Y \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) \subseteq Y \setminus f(\bigcup_{n\in\mathbb{N}} St(U_n,\mathcal{U}_n)) \in f(I)$. So, $Y \setminus \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{V}_n) \in f(I)$. Thus the sequence $\{V_n : n \in \mathbb{N}\}$ witnesses the $f(I)$- star Rothbergerness of the space (Y,δ). □

Definition 3.13. A sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of family of subsets of X is said to have **modified sequential single-tonic intersection modulo an ideal property ($MSSI^IP$)** if for all sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$ either $E_n \cup F = X$ for some $F \in \mathcal{H}_n$ or $\bigcap_{n\in\mathbb{N}}(\cap\mathcal{H}_n) \notin I$.

Theorem 3.14. *The following statements are equivalent:*

(1) (X,τ) is I-star Rothberger space

(2) If a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of closed sets have $MSSI^IP$ then there exists a $n_0 \in \mathbb{N}$ such that $\cap\mathcal{F}_{n_0} \neq \varnothing$.

Proof. Let (X,τ) be an I- star Rothberger space and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of family of closed sets, where $\cap\mathcal{F}_n = \varnothing$ for all $n \in \mathbb{N}$. We take $\mathcal{G}_n = \{X \setminus F : F \in \mathcal{F}_n\}$ for all $n \in \mathbb{N}$. Clearly, $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers for X. But X is I-star Rothberger. Therefore, there exists a sequence $\{G_n : n \in \mathbb{N}\}$ such that $G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} St(G_n,\mathcal{G}_n) \in I$. So, $X \setminus \big(\bigcup_{n\in\mathbb{N}} \bigcup\{G \in \mathcal{G}_n : G \cap G_n \neq \varnothing\}\big) \in I$. Therefore, $X \setminus (\bigcup_{n\in\mathbb{N}} \bigcup\{X \setminus F : F \in \mathcal{F}_n$ and $(X \setminus F) \cap (X \setminus E_n) \neq \varnothing\}) \in I$, where $G_n = X \setminus E_n$ for all $n \in \mathbb{N}$. So, $\bigcap_{n\in\mathbb{N}} \bigcap\{F \in \mathcal{F}_n : X \setminus (F \cup E_n) \neq \varnothing\} = \bigcap_{n\in\mathbb{N}} \bigcap\{F \in \mathcal{F}_n : E_n \cup F \neq X\} \in I$.

Now, we take $\mathcal{H}_n = \{F \in \mathcal{F}_n : E_n \cup F \neq X\}$. So $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ are two sequences such that $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$ with $E_n \cup F \neq X$ for any $F \in \mathcal{H}_n$ and $\bigcap_{n\in\mathbb{N}}(\cap\mathcal{H}_n) \in I$, which contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has $MSSI^IP$ property. Therefore, there must exists at least one $n_o \in \mathbb{N}$ such that $\cap\mathcal{F}_{n_0} \neq \varnothing$.

Conversely, let condition (2) hold and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a arbitrary sequence of open covers of X. We take $\mathcal{F}_n = \{X \setminus U : U \in \mathcal{U}_n\}$ for all $n \in \mathbb{N}$. Clearly, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of families of closed sets such that $\cap\mathcal{F}_n = \varnothing$ for all $n \in \mathbb{N}$. Thus by contra positivity of condition (2), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ cannot have the $MSSI^I$ property. Therefore there must exist sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ where $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subseteq \mathcal{F}_n$ such that $E_n \cup F \neq X$ for any $F \in \mathcal{H}_n$ and $\bigcap_{n\in\mathbb{N}}(\cap\mathcal{H}_n) \in I$. We take $U_n = X \setminus E_n$ for all $n \in \mathbb{N}$ and $\mathcal{G}_n = \{U = X \setminus F : F \in \mathcal{H}_n\}$. Clearly, $\{U_n : n \in \mathbb{N}\}$ and $\{\mathcal{G}_n : n \in \mathbb{N}\}$ are two sequences such that $U_n \in \mathcal{V}_n$ and $\mathcal{G}_n \subseteq \mathcal{V}_n$ for all $n \in \mathbb{N}$ with $(X \setminus U_n) \cup (X \setminus U) \neq X$ for any $U \in \mathcal{G}_n$ for all $n \in \mathbb{N}$. So, $X \setminus (U_n \cap U) \neq X$ for any $U \in \mathcal{G}_n$ for all $n \in \mathbb{N}$. Therefore, $U_n \cap U \neq \varnothing$ for any $U \in \mathcal{G}_n$ for all $n \in \mathbb{N}$. So, $St(U_n, \mathcal{V}_n) \supseteq St(U_n, \mathcal{G}_n) = \cup\mathcal{G}_n$ for all $n \in \mathbb{N}$. So, $\bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{V}_n) \supseteq \bigcup_{n\in\mathbb{N}}(\cup\mathcal{G}_n)$ and so $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{V}_n) \subseteq X \setminus \left(\bigcup_{n\in\mathbb{N}}(\cup\mathcal{G}_n)\right) = \bigcap_{n\in\mathbb{N}}(\cap\mathcal{H}_n) \in I$. Thus, $X \setminus \bigcup_{n\in\mathbb{N}} St(U_n, \mathcal{V}_n) \in I$ and hence (X, τ) is I-star Rothberger space. □

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