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Mizoguchi-Takahashi local contractions to Feng-Liu contractions

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Abstract

In this article, we establish that any uniformly local Mizoguchi-Takahashi contraction is actually a set-valued contraction due to Feng and Liu on a metrically convex complete metric space. Through an example, we demonstrate that this result need not hold on any arbitrary metric space. Furthermore, when the metric space is compact, we derive that any Mizoguchi-Takahashi local contraction and Nadler local contraction are equivalent. Moreover, a result related to invariant best approximation is established.

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1. Introduction

Let (Z, d) be a metric space and $CL(Z)$ denotes the collection of non-void closed subsets of Z. The set $CB(Z)$ contains all the non-void bounded closed subsets of Z. For any $P, Q \in CB(Z)$,

$$H(P,Q) = \max\left\{\sup_{u\in Q} D(u,P), \sup_{v\in P} D(v,Q)\right\} \tag{1.1}$$

is called Hausdorff metric on $CB(Z)$, where $D(u, P) = \inf\{d(u, v) : v \in P\}$. For a mapping $F : Z \to CL(Z)$, an element $z^* \in Z$ is known as a fixed point for

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the map F if $z^* \in Fz^*$. The occurrence of fixed points for set-valued mappings was first deduced by Nadler [10] in the year of 1969. Later, Mizoguchi and Takahashi [8] improved the Nadler's [10] theorem for a set-valued generalized contraction in the year 1989. The authors [8] established the below stated theorem.

Theorem 1.1 ([8]). *Suppose that (Z, d) is complete and a set-valued map $F : Z \to CB(Z)$ is such that for every $y, z \in Z$,*

$$H(Fy, Fz) \leq k(d(y, z))d(y, z), \tag{1.2}$$

where $k \in W = \{\theta : [0, \infty) \to [0, 1) | \limsup_{t \to s^+} \theta(t) < 1, \ \forall s \in [0, \infty)\}$. Then there is an element $z^* \in Fz^*$.

In fact, Suzuki [17] provided an example to deduce that every Mizoguchi-Takahashi contraction [8] is not necessarily a Nadler contraction [10] on arbitrary metric spaces in the year of 2007. However, Eldred et al. [4] observed that any Mizoguchi-Takahashi [8] contraction is actually a Nadler [10] contraction, whenever the metric space (Z, d) is metrically convex complete [4] in the year 2009. There are several extensions of the Theorem 1.1 due to Mizoguchi and Takahashi [8] in the literature, which can be found in [2, 7, 9, 11, 15].

In the year 1961, Edelstein [3] generalized the famous Banach contraction principle for uniformly local contractions [3]. Subsequently, Nadler [10] improved the result due to Edelstein [3] for set-valued uniformly local contractions. Later, Sultana and Vetrivel [15] extended the Theorem 1.1 due to Mizoguchi and Takahashi [8] for set-valued uniformly local contractions. The authors [15] derived the succeeding theorem.

Theorem 1.2 ([15]). *Assume (Z, d) is r-chainable (where $r > 0$), that is, for any $y, z \in Z$, there is sequence $(z^j)_{j=0}^N \subseteq Z$ satisfying $z^0 = y$, $z^N = z$ and $d(z^j, z^{j+1}) < r$ where $j \in \{0, \cdots, N-1\}$. Suppose that a set-valued map $F : Z \to CB(Z)$ is such that for each $y, z \in Z$ having $d(y, z) < r$ (where $r > 0$) fulfils*

$$H(Fy, Fz) \leq k(d(y, z))d(y, z), \tag{1.3}$$

where $k \in W$. Then there is an element $z^* \in Fz^*$ if (Z, d) is complete.

It is worth to note that every Mizoguchi-Takahashi local contraction [15] is not a Nadler [10] local contraction, see [16, Example 3.1]. But, Sultana and Qin [16] demonstrated that these two contractions are equivalent, whenever the metric space (Z, d) is metrically convex complete in the year 2019.

On the other hand, Feng and Liu [5] improved Nadler's [10] theorem for the mappings F from Z into $CL(Z)$ in the year of 2006, which is stated below.

Theorem 1.3 ([5]). *Suppose that (Z, d) is a complete metric space and a set-valued map $F : Z \to CL(Z)$ is such that for each $y \in Z$, there is $z \in I^y_{c \in (0,1)} = \{x \in Fy : cd(x, y) \leq D(y, Fy)\}$ such that*

$$D(z, Fz) \leq \alpha d(y, z) \text{ where } 0 \leq \alpha \leq c < 1. \tag{1.4}$$

Then there is an element $z^* \in Fz^*$ *if* $h : Z \to \mathbb{R}$ *by* $h(z) = D(z, Fz)$ *is lower semicontinuous.*

In this manuscript, inspired by Eldred et. al [4] and Sultana-Qin [16], we establish that any set-valued contraction due to Feng-Liu [5] is actually a Mizoguchi-Takahashi local contraction [16] on a metrically convex metric space. We have provided an example to illustrate that this result is not always true on any arbitrary metric space. The fixed points for the set-valued mappings fulfil the equation (1.3) are also studied on metrically convex spaces. Furthermore, the equivalence of Mizoguchi-Takahashi local contraction [16] and Nadler local contraction [10] is established in a compact metric space. An invariant approximation result is also discussed through our main theorem.

2. Notations and Definitions

This segment contains some mathematical definitions and notations, that are necessary up to the end of this paper. For the metric space (Z, d), let u, v be any two points in Z. Then a point w in Z, where $u \neq w \neq v$, is called metrically [4] between u and v if $d(u, v) = d(u, w) + d(w, v)$. The space (Z, d) is called metrically convex [4] if for every pair of elements u and v in Z, there is an element $w \in Z$ such that w is metrically between u and v. For example, if we consider $Z = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}$ along with usual metric d, then (Z, d) becomes a metrically convex metric space. The below-stated lemma is regarding the property of metrically convex space, which is necessary to establish our main theorem.

Lemma 2.1 ([4])**.** *Assume that* (Z, d) *is metrically convex complete and* $u, v \in Z$ *are any arbitrary points. Then there is* $[a_1, a_2] \subset \mathbb{R}$ *and an isomorphism* $\varphi : [a_1, a_2] \to Z$ *such that* $\varphi(a_1) = u$ *and* $\varphi(a_2) = v$.

Again for the metric space (Z, d), consider a self map Ψ on Z. Then Ψ is defined as uniformly local contraction [3] if for each $y, z \in Z$ having $d(y, z) < r$ fulfils $d(\Psi(y), \Psi(z)) \leq \alpha d(y, z)$, where $0 \leq \alpha < 1$ and $r > 0$. In 2012, Samet et al. [13] introduced the notion of $\alpha - \psi$-contractive and α-admissible mappings. We observe that any uniformly local contraction is actually a $\alpha - \psi$-contractive map. Furthermore, for $y, z \in Z$, a r-chain [3] between y and z means, there is a finite sequence $y = z_0, z_1, \cdots, z_N = z$ in Z, such that for every $j \in \{1, 2, \cdots, N\}$, $d(z_{j-1}, z_j) < r$. The space (Z, d) is stated as r-chainable [3] whenever there is a r-chain between every points y and z in Z.

3. Main Result

Now we commence our main theorem about that any Mizoguchi-Takahashi local contraction [15] is actually a Feng-Liu contraction [5] on a metrically convex metric space (Z, d), that is, the mappings that meet the equation (1.3) also satisfy the equation (1.4) whenever the metric space (Z, d) is metrically convex.

Theorem 3.1. *Suppose that (Z,d) is a complete metric space having metrically convexity property and a set-valued map $F : Z \to CB(Z)$ is such that for $y,z \in Z$ having $d(y,z) < r$ (where $r > 0$),*

$$H(Fy,Fz) \leq k(d(y,z))d(y,z) \tag{3.1}$$

where $k \in W$. Then F becomes a Feng-Liu contraction, that is, for each $y^{'} \in Z$, there is $z^{'} \in I_c^{y^{'}}$ fulfils,

$$D(z^{'},Fz^{'}) \leq \alpha d(y^{'},z^{'}) \text{ where } 0 \leq \alpha < c < 1.$$

Proof. Our main goal is to show that there is certain α, where $0 \leq \alpha < c < 1$ is such that for all $y \in Z$, $z \in I_c^y$ meets $D(z,Fz) \leq \alpha d(y,z)$. Suppose that

$$M = \{m > 0 : \sup\{k(d(y,z)) : d(y,z) \in [0,m]\} = 1\}. \tag{3.2}$$

Then there are two possibilities that could happen. Firstly, we assume that the set M is void. Then for each positive m, the supremum value of $k(d(y,z))$ is strictly less than 1, where $d(y,z) \in [0,m]$. Since $r > 0$ is fixed, we assume that

$$\sup\{k(d(y,z)) : d(y,z) \in [0,r]\} = \alpha < 1. \tag{3.3}$$

Evidently, $\alpha \in [0,1)$. Let $y^{'} \in Z$ be a fixed element. Then there is an element $z^{'} \in I_c^{y^{'}}$ where $c > \alpha$, due to the fact that $F(y^{'}) \subset CL(Z)$. Consequently, for chosen $y^{'}$ and $z^{'}$, an isometry $\varphi_1 : [t_1,t_2] \to Z$ exists such that $\varphi_1(t_1) = y^{'}$ and $\varphi_1(t_2) = z^{'}$ (where $[t_1,t_2] \subset \mathbb{R}$) according to the Lemma 2.1. Now for a fixed positive real number $q < r$, we can always find $L \in \mathbb{N}$ satisfying $t_1 + Lq < t_2$ and $t_2 \leq t_1 + (L+1)q$. Consequently,

$$\begin{aligned} d(y^{'},z^{'}) &= d(\varphi_1(t_1),\varphi_1(t_2)) = |t_1 - t_2| \\ &= |t_1 - (t_1+q)| + |(t_1+q) - (t_1+2q)| + \cdots + |(t_1+Lq) - t_2| \\ &= d(y^{'},\varphi_1(t_1+q)) + d(\varphi_1(t_1+q),\varphi_1(t_1+2q)) + \cdots + \\ &\quad d(\varphi_1(t_1+Lq),z^{'}). \end{aligned} \tag{3.4}$$

Again, it is simple to observe that $d(z^{'},\varphi_1(t_1+Lq)) \leq q < r$ and $d(\varphi_1(t_1 + nq),\varphi_1(t_1+(n+1)q)) = q < r$ for every $0 \leq n < L$, that is every terms of right hand side of the equation (3.4) is less than r. Therefore for chosen $y^{'} \in Z$ and $z^{'} \in I_c^{y^{'}}$, it yields

$$\begin{aligned} D(z^{'},Fz^{'}) &\leq H(Fy^{'},Fz^{'}) \\ &\leq H(Fy^{'},F(\varphi_1(t_1+q))) + \cdots + H(F(\varphi_1(t_1+Lq)),Fz^{'}) \\ &\leq \alpha[d(y^{'},\varphi_1(t_1+q)) + \cdots + d(\varphi_1(t_1+Lq),z^{'})] \text{ [by (3.1) and (3.3)]} \\ &\leq \alpha d(y^{'},z^{'}). \qquad \text{[using (3.4)]} \end{aligned}$$

Next, we assume that the set M is non-void and m_0 is the infimum of M. If possible, let $m_0 = 0$. Then we are able to find two sequences $\{d(y_n,z_n)\}_n$,

$\{k(d(y_n, z_n))\}_n$ such that $d(y_n, z_n) \to 0$ and $k(d(y_n, z_n)) \to 1$, which contradicts to the criteria of the map $k \in W$. Hence $m_0 \neq 0$. Then for a chosen positive real $q_0 < m_0$, we have

$$\sup\{k(d(x_1, x_2)) : d(x_1, x_2) \in [0, q_0]\} = \alpha_0 < 1. \tag{3.5}$$

Let $y^{'} \in Z$ be a fixed element. Subsequently, there is $z^{'} \in I_c^{y^{'}}$, where $c > \alpha_0$. Then for that $y^{'}$ and $z^{'}$, there is an isometry $\varphi_2 : [s_1, s_2] \to Z$ such that $\varphi_2(s_1) = y^{'}$ and $\varphi_2(s_2) = z^{'}$, where $[s_1, s_2] \subset \mathbb{R}$. Consequently, for certain fixed positive number $a < \min\{q_0, r\}$, we can find $L^{'} \in \mathbb{N}$ satisfying $s_1 + L^{'}a < s_2$ and $s_2 \leq s_1 + (L^{'} + 1)a$. Evidently,

$$\begin{aligned} d(y^{'}, z^{'}) &= d(\varphi_2(s_1), \varphi_2(s_2)) = |s_1 - s_2| \\ &= |s_1 - (s_1 + a)| + |(s_1 + a) - (s_1 + 2a)| + \cdots + |(s_1 + L^{'}a) - s_2| \\ &= d(y^{'}, \varphi_2(s_1 + a)) + d(\varphi_2(s_1 + a), \varphi_2(s_1 + 2a)) + \cdots + \\ &\quad d(\varphi_2(s_1 + L^{'}a), z^{'}). \end{aligned} \tag{3.6}$$

Again, it is simple to check that $d(z^{'}, \varphi_2(s_1+L^{'}a)) \leq a$ and $d(\varphi_2(s_1+na), \varphi_2(s_1+(n+1)a)) = a$ for every $0 \leq n < L^{'}$, that is each terms of right hand side of the equation (3.6) is less than or equal to a. Therefore for that chosen $y^{'} \in Z$ and $z^{'} \in I_c^{y^{'}}$, it yields

$$\begin{aligned} D(z^{'}, Fz^{'}) \leq & H(Fy^{'}, Fz^{'}) \\ \leq & H(Fy^{'}, F(\varphi_2(s_1 + a))) + \cdots + H(F(\varphi_2(s_1 + L^{'}a)), Fz^{'}) \\ \leq & \alpha_0[d(y^{'}, \varphi_2(s_1 + a)) + \cdots + d(\varphi_2(s_1 + L^{'}a), z^{'})] [\text{by (3.1) and (3.5)}] \\ \leq & \alpha_0 d(y^{'}, z^{'}) \qquad [\text{using (3.6)}]. \end{aligned}$$

Thus F becomes a Feng-Liu contraction. □

In the succeeding theorem, we establish the fixed points for a Mizoguchi-Takahashi local contraction on a metrically convex space. Indeed, this result follows from our above mentioned Theorem 3.1 and the Theorem 1.3 due to Feng and Liu [5]. Although, this theorem is established by Sultana and Qin [16, Corollary 3.1] in the year 2019.

Theorem 3.2 ([16]). *Suppose that (Z, d) is metrically convex complete and a set-valued map $F : Z \to CB(Z)$ meets the equation (1.3). Then presence of $z^* \in Z$ such that $z^* \in Fz^*$ can be assured.*

Proof. Since F meets the equation (1.3) and Z is metrically convex, then by Theorem 3.1, the map F fulfils the Feng-Liu contraction. As a consequence, for each $y \in Z$, there is $z \in I_c^y$ fulfils

$$D(z, Fz) \leq \alpha d(y, z) \text{ where } 0 \leq \alpha < c < 1.$$

Now our aim is to establish that $h : Z \to \mathbb{R}$ by $h(z) = D(z, Fz)$ is lower semicontinuous. Let us assume $\{q_n\}_n \in Z$ is such that $q_n \to q$. Then we achieve an $l \in \mathbb{N}$ such that $d(q_n, q) < r$, for each $n \geq l$. Now for every $n \geq 1$ and $z \in Fq_n$,

$$\begin{aligned} D(q, Fq) &\leq d(q, q_n) + d(q_n, z) + D(z, Fq) \\ &\leq d(q, q_n) + d(q_n, z) + H(Fq_n, Fq) \\ &\leq d(q, q_n) + D(q_n, Fq_n) + H(Fq_n, Fq). \end{aligned} \tag{3.7}$$

As for all $n \geq l$, $d(q_n, q) < r$, then the last inequation leads to

$$\begin{aligned} D(q, Fq) &\leq d(q, q_n) + D(q_n, Fq_n) + k(d(q_n, q))d(q_n, q) \\ &< d(q, q_n) + D(q_n, Fq_n) + d(q_n, q). \end{aligned}$$

Taking $n \to \infty$, we conclude that $D(q, Fq) \leq \liminf_{n\to\infty} D(q_n, Fq_n)$. Now applying the Theorem 1.3, we are able to find $z^* \in Z$ such that $z^* \in Fz^*$. $\square$

In 2009, Eldred et. al [4, Theorem 2.4] also established that Mizoguchi-Takahashi contractions [8] are actually Nadler contractions [10], whenever the metric space is compact. In the below stated theorem, we establish the equivalence between the Mizoguchi-Takahashi local contraction [15] and the Nadler local contraction [10] in a metric space which is compact.

Theorem 3.3. *Let (Z, d) be compact and a set-valued map $F : Z \to CB(Z)$ meets the equation (1.3). Then for each $y, z \in Z$ having $d(y, z) < r$ (where $r > 0$),*

$$H(Fy, Fz) \leq \alpha\, d(y, z) \quad \textit{where} \;\; 0 \leq \alpha < 1.$$

Proof. Let $0 < q < r$ be a fixed real number. Now consider a real number m, which is the supremum of $\frac{H(Fy,Fz)}{d(y,z)}$, where the elements y and z lies in Z having $0 < d(y, z) \leq q$. We want to show that $m < 1$. If possible, let $m = 1$. There is $\{(y_n, z_n)\}_n \in Z \times Z$ meeting $d(y_n, z_n) \in (0, q]$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{H(Fy_n, Fz_n)}{d(y_n, z_n)} = 1. \tag{3.8}$$

On the account of $d(y_n, z_n) \leq q < r$, we have $\frac{H(Fy_n,Fz_n)}{d(y_n,z_n)} \leq k(d(y_n, z_n)) < 1$. Taking $n \to \infty$, it yields that $k(d(y_n, z_n)) \to 1$. If $d(y_n, z_n)$ converges to 0, then it contradicts the criteria of $k \in W$. Hence we assert that $\{d(y_n, z_n)\}_n$ converges to a positive real number.

As $\{(y_n, z_n)\}_n \in Z \times Z$ and Z is compact, then there is $\{(y_{n_l}, z_{n_l})\}_{l\in\mathbb{N}}$ such that $(y_{n_l}, z_{n_l}) \to (y^*, z^*)$. Consequently, $d(y_{n_l}, z_{n_l}) \to d(y^*, z^*)$. Since $d(y_n, z_n) \in (0, q]$, then $d(y^*, z^*) \in (0, q]$, that is $d(y^*, z^*) \leq q < r$. Now

$$\begin{aligned} \lim_{l\to\infty} \frac{H(Fy_{n_l}, Fz_{n_l})}{d(y_{n_l}, z_{n_l})} &= \frac{H(Fy^*, Fz^*)}{d(y^*, z^*)} \\ &\leq \frac{k(d(y^*, z^*))d(y^*, z^*)}{d(y^*, z^*)} \qquad [\because\; d(y^*, z^*) < r] \\ &= k(d(y^*, z^*)) < 1. \end{aligned}$$

This contradicts the equation (3.8). Thus we can conclude that $m < 1$, that is there is an non-negative α such that $m = \alpha < 1$. Therefore for each $y, z \in Z$ having $d(y, z) < r$, we have

$$H(Fy, Fz) \leq k(d(y, z))d(y, z) \leq \alpha d(y, z) \quad \text{where } \alpha \in [0, 1).$$

□

Many mathematicians are interested in dealing with the invariant best approximation problems through fixed points; see [1, 12]. In [6, 14], the authors established invariant best approximation theorems for set-valued mappings. In the following, using Theorem 1.3 due to Feng-Liu [5] and our main Theorem 3.1, we derive an invariant best approximation theorem for a set-valued mapping which meets the criteria (1.3) on a metrically convex space. Let P be a non-void subset of a normed linear space $(Z, \|.\|)$ and $p \in Z$. Then a collection of elements $y \in P$ such that $D(p, P) = \|y - p\|$ is described as $B_P(p)$ and it is stated as collection of best P-approximate of p over P.

Theorem 3.4. *Let P be a subset of a metrically convex normed linear space $(Z, \|.\|)$ and a set-valued map $F : P \to CB(P)$ be such that for every $y, z \in P$ having $d(y, z) < r$ fulfils (1.3). Suppose that the below-stated criteria*

(i) *the set $B_P(p)$ is a complete subset of P,*
(ii) *for every $x \in B_P(p)$, $\sup_{y \in Fx} \|y - p\| \leq \|x - p\|$.*

Then there is $z^* \in B_P(p) \cap \{z \in Z : z \in Fz\}$ if the function $h : B_P(p) \to \mathbb{R}$ by $h(z) = D(z, Fz)$ is lower semicontinuous.

Proof. As Z is metrically convex and the set-valued map $F : P \to CB(P)$ fulfils (1.3) for each $y, z \in P$ having $d(y, z) < r$, then by Theorem 3.1, for each $y \in B_P(p) \subseteq P$, there is $z \in I_c^y$ meets $D(z, Fz) \leq \alpha d(y, z)$, where $0 \leq \alpha < c < 1$. Now let $x \in B_P(p)$ and $z \in Fx$. As $x \in B_P(p)$, then $D(p, P) = \|p - x\|$. Again from (ii) we have

$$\|z - p\| \leq \sup_{y \in Fx} \|y - p\| \leq \|x - p\| = D(p, P).$$

Thus we can conclude that $\|z - p\| = D(p, P)$. Therefore $z \in B_P(p)$. Hence $Fx \subseteq B_P(p)$ for each $x \in B_P(p)$, that is $F(B_P(p)) \subseteq B_P(p)$. Immediately it observe that Fx is closed for any $x \in B_P(p)$, on the account of $Fx \subset CL(P)$, for each $x \in P$. Consequently $F|_{B_P(p)} : B_P(p) \to CL(B_P(p))$. Thus F fulfils all the criteria of Theorem 1.3 on the set $B_P(p)$ and hence

$$w^* \in \{z \in B_P(p) : z \in F|_{B_P(p)} z\} = \{z \in Z : z \in Fz\} \cap B_P(p).$$

□

The succeeding example illustrates that any uniformly local contraction (1.3) due to Sultana and Vetrivel [15] does not follow Feng-Liu contraction [5] on any arbitrary metric space.

Example 3.5. Consider $Z = [0, \frac{1}{2}] \cup \{n \in \mathbb{N} : n \geq 2\}$ and $d : Z \times Z \to \mathbb{R}$ which is defined by

$$d(y,z)=\begin{cases}0 & \text{if } y=z,\\ |y-z| & \text{if } 0\leq y,z\leq \frac{1}{2},\\ y+z & \text{if one of } y,z\notin[0,\frac{1}{2}].\end{cases}$$

Indeed (Z,d) is not a metrically convex metric space. Let us consider two elements 0 and 2 in the set Z, then from the definition of the metric d, we have $d(0,2)=2$. Now, if (Z,d) is a metrically convex metric space, then there is a point $r\notin\{0,2\}$ such that, $d(0,2)=d(0,r)+d(r,2)$, which leads to the the conclusion that $r=0$, this is a contradiction. Therefore (Z,d) is not a metrically convex metric space. Now define a map $F:Z\to CB(Z)\subseteq CL(Z)$ such that

$$Fz=\begin{cases}\left\{\frac{1}{2}z^2\right\} & \text{if } z\in[0,\frac{1}{2}],\\ \{7,2z-1\} & \text{if } z\in\{2,3,4,\cdots\}.\end{cases}$$

Our goal is to show that F is not a Feng-Liu contraction but a Mizoguchi-Takahashi local contraction. Now for $3\in Z$, $F3=\{7,5\}$. Then there are two cases that arise.

(i) Choose $7\in\{7,5\}$, then $F7=\{7,13\}$. Now we have $D(7,F7)=\inf\{d(7,7),d(7,13)\}=\inf\{14,20\}=14$. Again $d(3,7)=10$. Therefore

$$D(7,F7)>d(3,7).$$

(ii) Choose $5\in\{7,5\}$, then $F5=\{7,9\}$. Now we see that $D(5,F5)=\inf\{d(5,7),d(5,9)\}=\inf\{12,14\}=12$. Again $d(3,5)=8$. Therefore

$$D(5,F5)>d(3,5).$$

Hence for $y=3\in Z$ and every $z\in F3$, we obtain that

$$D(z,Fz)>d(y,z).$$

This indicates that F is not a Feng-Liu contraction. On the other side, when $d(y,z)<\frac{1}{2}$, then it is easy to visualize that $y,z\in[0,\frac{1}{2}]$. Now

$$\begin{aligned}H(Fy,Fz) &= H\left(\left\{\frac{1}{2}y^2\right\},\left\{\frac{1}{2}z^2\right\}\right)\\ &\leq \frac{1}{2}(y+z)d(y,z).\end{aligned}$$

Consider a map $k:[0,\infty)\to[0,1)$ by

$$k(s)=\begin{cases}\frac{3}{4} & \text{if } s\in[0,1/2)\\ 0 & \text{if } s\in[1/2,\infty).\end{cases}$$

Therefore $k\in W$ and hence for $d(y,z)<\frac{1}{2}$,

$$H(Fy,Fz)\leq k(d(y,z))d(y,z).$$

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