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Butterfly points and hyperspace selections

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Abstract

If f is a continuous selection for the Vietoris hyperspace $\mathscr{F}(X)$ of the nonempty closed subsets of a space X, then the point $f(X) \in X$ is not as arbitrary as it might seem at first glance. In this paper, we will characterise these points by local properties at them. Briefly, we will show that $p = f(X)$ is a strong butterfly point precisely when it has a countable clopen base in $\overline{U}$ for some open set $U \subset X \setminus \{p\}$ with $\overline{U} = U \cup \{p\}$. Moreover, the same is valid when X is totally disconnected at $p = f(X)$ and p is only assumed to be a butterfly point. This gives the complete affirmative solution to a question raised previously by the author. Finally, when $p = f(X)$ lacks the above local base-like property, we will show that $\mathscr{F}(X)$ has a continuous selection h with the stronger property that $h(S) = p$ for every closed $S \subset X$ with $p \in S$.

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1. Introduction

All spaces in this paper are infinite Hausdorff topological spaces. Let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of a space X. We endow $\mathscr{F}(X)$ with the *Vietoris topology* τ_V, and call it the *Vietoris hyperspace* of X. Let us recall

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that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \ \text{ and } \ S \cap V \neq \varnothing, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. Also, let us recall that a map $f : \mathscr{F}(X) \to X$ is a *selection* for $\mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{F}(X)$, and f is called *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{F}(X)$. The set of all continuous selections for $\mathscr{F}(X)$ will be denoted by $\mathscr{V}_{cs}[\mathscr{F}(X)]$.

The following question was posed by the author in an unpublished note.

Question 1.1. *Let* $X = (\omega_1 + 1) \cup_{\omega_1 = 0} [0, 1]$ *be the adjunction space obtained by identifying the first uncountable ordinal* ω_1 *and* $0 \in [0, 1]$ *into a single point* $p \in X$*. Does there exist a continuous selection* $f : \mathscr{F}(X) \to X$ *with* $f(X) = p$*?*

David Buhagiar has recently proposed, in a private communication to the author, a negative solution to this question. However, his arguments were heavily dependent on stationary sets in ω_1 and the pressing down lemma.

In this paper, we will give a purely topological description of the points $p \in X$ with the property that $p = f(X)$ for some $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$. To this end, let us recall that a point $p \in X$ of a connected space X is *cut* if $X \setminus \{p\}$ is not connected or, equivalently, if $X \setminus \{p\} = U \cup V$ for some subsets $U, V \subset X$ with $\overline{U} \cap \overline{V} = \{p\}$. Extending this interpretation to an arbitrary space X, a point $p \in X$ was said to be *cut* [14], see also [6, 13], if $X \setminus \{p\} = U \cup V$ and $\overline{U} \cap \overline{V} = \{p\}$ for some subsets $U, V \subset X$. Cut points were also introduced in [3], where they were called *tie-points*. A somewhat related concept was introduced in [7], where $p \in X$ was called *countably-approachable* if it is either isolated or has a countable clopen base in $\overline{U}$ for some open set $U \subset X \setminus \{p\}$ with $\overline{U} = U \cup \{p\}$. In these terms, we consider the following two subsets of X:

$$\begin{cases} X_\Theta = \{f(X) : f \in \mathscr{V}_{cs}[\mathscr{F}(X)]\}, \\ X_\Omega = \{p \in X : p \text{ is countably-approachable}\}. \end{cases} \tag{1.1}$$

Intuitively, X_Θ can be regarded as the X-'Orbit' with respect to the 'action' of $\mathscr{V}_{cs}[\mathscr{F}(X)]$ on the hyperspace $\mathscr{F}(X)$. The non-isolated countably-approachable points were called *ω-approachable* in [7], so X_Ω is also intuitive.

Each non-isolated countably-approachable point $p \in X$ is a cut point of X, see the proof of [14, Corollary 3.2]. Going back to the adjunction space X in Question 1.1, it is evident that the point $p \in X$ is cut, but it is not countably-approachable (i.e. $p \notin X_\Omega$). The reason for the negative answer to this question (i.e. for the fact that $p \notin X_\Theta$) is now fully explained by our first main result.

Theorem 1.2. *Let* X *be a space with* $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \varnothing$*. Then*

$$X_\Omega = \{p \in X_\Theta : p \text{ is either isolated or cut}\}. \tag{1.2}$$

Regarding the proper place of Theorem 1.2, let us remark that the inclusion $X_\Omega \subset X_\Theta$ was actually established in [7, Lemma 4.2], see also [12, Lemma

2.3]. It is included in (1.2) to emphasise on the equality. The other inclusion is naturally related to the so called butterfly points. A point $p \in X$ is called *butterfly* (or, a *b-point*) [21] if $\overline{F \setminus \{p\}} \cap \overline{G \setminus \{p\}} = \{p\}$ for some closed sets $F, G \subset X$. Evidently, $p \in X$ is butterfly precisely when it is a cut point of some closed subset of X. In fact, in some sources, cut points (equivalently, tie-points) are often called *strong butterfly points*. Finally, let us agree that a space X is *totally disconnected at* $p \in X$ if the singleton $\{p\}$ is an intersection of clopen sets.

A crucial role in the proof of Theorem 1.2 will play the following result.

Theorem 1.3. *A point $p \in X_\Theta$ is butterfly if and only if it is the limit of a sequence of points of $X \setminus \{p\}$. Moreover, if X is totally disconnected at a butterfly point $p \in X_\Theta$, then p is also a cut point of X.*

Evidently, each $p \in X$ which is the limit of a nontrivial sequence of points of X is also a butterfly point. Thus, the essential contribution in the first part of Theorem 1.3 is that each butterfly point $p \in X_\Theta$ is the limit of a nontrivial sequence of points X. However, butterfly points $p \in X_\Theta$ are not necessarily cut (i.e. strong butterfly). For instance, the endpoints $0, 1 \in [0, 1]$ are noncut, they belong to $[0, 1]_\Theta$ and are both butterfly. In contrast, according to Theorem 1.2, the second part of Theorem 1.3 implies the following consequence which settles [9, Problem 4.15] in the affirmative.

Corollary 1.4. *If X is totally disconnected at a butterfly point $p \in X_\Theta$, then $p \in X_\Omega$ or, equivalently, this point is countably-approachable.*

Let us remark that [9, Problem 4.15] was stated for a space X which has a clopen π-base. A family $\mathscr{P}$ of open subsets of X is a *π-base* (called also a *pseudobase*, Oxtoby [20]) if every nonempty open subset of X contains some nonempty member of $\mathscr{P}$. In our case, we also have that $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \varnothing$ and according to [7, Corollary 2.3], such a space X must be totally disconnected.

Our second main result deals with the elements of the set $X_\Theta \setminus X_\Omega$. To this end, let us recall that a point $p \in X$ is called *selection maximal* [14], see also [5, 12], if there exists a continuous selection f for $\mathscr{F}(X)$ such that $f(S) = p$ for every $S \in \mathscr{F}(X)$ with $p \in S$. In this case, the selection f is called *p-maximal*. Evidently, each selection maximal point of X belongs to X_Θ, and each point of X which has a countable clopen base belongs to X_Ω. So, consider the sets:

$$\begin{cases} X_\Theta^* = \{p \in X_\Theta : p \text{ is selection maximal}\}, \\ X_\Omega^* = \{p \in X_\Omega : p \text{ has a countable clopen base}\}. \end{cases} \tag{1.3}$$

In this paper, we will also prove the following theorem.

Theorem 1.5. *Let X be a space with $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \varnothing$. Then*

$$X_\Theta \setminus X_\Omega \subset X_\Theta^* \quad \text{and} \quad X_\Theta^* \cap X_\Omega = X_\Omega^*. \tag{1.4}$$

Theorem 1.5 is also partially known, the equality $X_\Theta^* \cap X_\Omega = X_\Omega^*$ was established in [14, Theorem 3.1 and Corollary 3.2]. It is included to emphasise on the fact that $X_\Theta \setminus X_\Omega$ is not necessarily equal to X_Θ^*. According to Theorem 1.2, the set $X_\Theta \setminus X_\Omega$ cannot contain a cut point of X. A point $p \in X$ which is not cut will be called *noncut*, see Section 6. Thus, the inclusion $X_\Theta \setminus X_\Omega \subset X_\Theta^*$ in (1.4) actually states that each noncut point $p \in X_\Theta$ is selection maximal. The crucial property to achieve this result is that the connected component of each noncut point $p \in X_\Theta$ has a clopen base (Theorem 6.1).

The paper is organised as follows. Theorem 1.3 is proved in Section 2. A condition for a point $p \in X_\Theta$ to be countably-approachable is given in Lemma 3.2 of Section 3. Based on this condition, the proof of Theorem 1.2 is accomplished in Sections 4 and 5. The final Section 6 contains the proof of Theorem 1.5.

2. Butterfly Points and Convergent Sequences

A nonempty subset S of a partially ordered set $(P, \leq)$ is *up-directed* if for every finite subset $T \subset S$ there exists $s \in S$ with $t \leq s$ for every $t \in T$. For a space X, the set $\mathscr{F}(X)$ is partially ordered with respect to the usual set-theoretic inclusion '$\subset$', and each up-directed family in $\mathscr{F}(X)$ is τ_V-convergent.

Proposition 2.1. *Each up-directed family $\mathscr{S} \subset \mathscr{F}(X)$ is τ_V-convergent to $\overline{\bigcup \mathscr{S}}$.*

Proof. Let $\mathscr{V}$ be a finite family of open subsets of X with $\overline{\bigcup \mathscr{S}} \in \langle \mathscr{V} \rangle$. Then $\bigcup \mathscr{T} \in \langle \mathscr{V} \rangle$ for some finite subfamily $\mathscr{T} \subset \mathscr{S}$. Since $\mathscr{S}$ is up-directed, it follows that $\bigcup \mathscr{T} \subset S$ for some $S \in \mathscr{S}$, and each $S \in \mathscr{S}$ with this property also belongs to $\langle \mathscr{V} \rangle$. □

Complementary to Proposition 2.1 is the following further observation about τ_V-convergence of usual sequences in the hyperspace $\mathscr{F}(X)$.

Proposition 2.2. *Let $U_n \subset X$, $n < \omega$, be a pairwise disjoint family of proper open sets. Then the sequence $S_n = X \setminus U_n$, $n < \omega$, is τ_V-convergent to X.*

Proof. Take a finite open cover $\mathscr{V}$ of X with $X \in \langle \mathscr{V} \rangle$. If $S_k \cap V_0 = \varnothing$ for some $V_0 \in \mathscr{V}$ and $k < \omega$, then $V_0 \subset U_k$. Since $\{U_n : n < \omega\}$ is pairwise disjoint, this implies that $\varnothing \neq V_0 \subset U_k \subset S_n$ for every $n \neq k$. Since $\mathscr{V}$ is finite, there exists $n_0 < \omega$ such that $S_n \cap V \neq \varnothing$ for every $V \in \mathscr{V}$ and $n \geq n_0$. In other words, $S_n \in \langle \mathscr{V} \rangle$ for every $n \geq n_0$. □

In what follows, for a set Z, let

$$\Sigma(Z) = \{S \subset Z : S \text{ is nonempty and finite}\}. \tag{2.1}$$

The following two general observations about local bases generating nontrivial convergent sequences furnish the first part of the proof of Theorem 1.3.

Proposition 2.3. *Let $p = f(X)$ be a non-isolated point for some selection $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$, and $\mathscr{B}$ be a local base at p such that $f((X \setminus B) \cup \{p\}) \in X \setminus B$ for every $B \in \mathscr{B}$. Then $X \setminus \{p\}$ contains a sequence convergent to p.*

Proof. For $B_0 \in \mathscr{B}$ and $q = f((X \setminus B_0) \cup \{p\}) \neq p$, there are disjoint open sets $O_p, O_q \subset X$ with $q \in O_q$ and $p \in O_p \subset B_0$. Hence, by continuity of f, there is a finite family $\mathscr{V}$ of open subsets of X with $(X \setminus B_0) \cup \{p\} \in \langle \mathscr{V} \rangle$ and $f(\langle \mathscr{V} \rangle) \subset O_q$. Take $B_1 \in \mathscr{B}$ such that $B_1 \subset O_p$ and $B_1 \subset \bigcap\{V \in \mathscr{V} : p \in V\}$. If $S \in \Sigma(B_1)$, then $f((X \setminus B_0) \cup S) \in X \setminus B_0$ because $(X \setminus B_0) \cup S \in \langle \mathscr{V} \rangle$ and $S \subset O_p \subset X \setminus O_q$. Since $\{(X \setminus B_0) \cup S : S \in \Sigma(B_1)\}$ is an up-directed family and $\bigcup \Sigma(B_1) = B_1$, by Proposition 2.1, $f\left((X \setminus B_0) \cup \overline{B_1}\right) \in X \setminus B_0$. Thus, by induction, there is a decreasing sequence $\{B_n\} \subset \mathscr{B}$ such that $f\left((X \setminus B_n) \cup \overline{B_{n+1}}\right) \in X \setminus B_n$ for every $n < \omega$. Then $B_n \setminus \overline{B_{n+1}}$, $n < \omega$, is a pairwise disjoint family of proper open subsets of X. Hence, by Proposition 2.2, the sequence $T_n = (X \setminus B_n) \cup \overline{B_{n+1}}$, $n < \omega$, is τ_V-convergent to X. So,

$$p = f(X) = \lim_{n \to \infty} f(T_n) \quad \text{and} \quad f(T_n) \notin B_n \ni p, \ n < \omega. \qquad \square$$

Proposition 2.4. *Let $p = f(X)$ be a butterfly point for some $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$, and $\mathscr{B}$ be a local base at p such that $f((X \setminus B) \cup \{p\}) = p$ for every $B \in \mathscr{B}$. Then $X \setminus \{p\}$ contains a sequence convergent to p.*

Proof. By definition, $\overline{F \setminus \{p\}} \cap \overline{G \setminus \{p\}} = \{p\}$ for some closed sets $F, G \subset X$. Set $U = F \setminus \{p\}$ and $V = G \setminus \{p\}$, and take $B_0 \in \mathscr{B}$. Since $f((X \setminus B_0) \cup \{p\}) = p$ and $p \in B_0 \cap \overline{U}$, there is $x_0 \in B_0 \cap U$ such that $f((X \setminus B_0) \cup \{x_0\}) = x_0$. For the same reason, taking $B_1 \subset B_0 \setminus \{x_0\}$, there is a point $x_1 \in B_1 \cap V$ with $f((X \setminus B_1) \cup \{x_1\}) = x_1$. Hence, by induction, there exists a sequence $\{B_n\} \subset \mathscr{B}$ and a sequence $\{x_n\} \subset X$ such that $B_{n+1} \subset B_n \setminus \{x_n\}$ and

$$f((X \setminus B_{2n}) \cup \{x_{2n}\}) = x_{2n} \in U \text{ and } f((X \setminus B_{2n+1}) \cup \{x_{2n+1}\}) = x_{2n+1} \in V.$$

Since $T_n = (X \setminus B_n) \cup \{x_n\}$, $n < \omega$, is an increasing sequence of closed sets, it is τ_V-convergent. Evidently, $\lim_{n \to \infty} x_n = \lim_{n \to \infty} f(T_n) \in \overline{U} \cap \overline{V} = \{p\}$. $\square$

Let $\mathscr{F}_2(X) = \{S \subset X : 1 \leq |S| \leq 2\}$. A selection σ for $\mathscr{F}_2(X)$ is called a *weak selection* for X. It generates a relation $\leq_\sigma$ on X defined for $x, y \in X$ by $x \leq_\sigma y$ if $\sigma(\{x, y\}) = x$ [17, Definition 7.1]. This relation is both *total* and *antisymmetric*, but not necessarily *transitive*. We write $x <_\sigma y$ whenever $x \leq_\sigma y$ and $x \neq y$, and use the standard notation for the intervals generated by $\leq_\sigma$. For instance, $(\leftarrow, p)_{\leq_\sigma}$ will stand for all $x \in X$ with $x <_\sigma p$; $(\leftarrow, p]_{\leq_\sigma}$ for that of all $x \in X$ with $x \leq_\sigma p$; the intervals $(p, \rightarrow)_{\leq_\sigma}$, $[p, \rightarrow)_{\leq_\sigma}$, etc., are defined in a similar way. The intervals $(\leftarrow, p)_{\leq_\sigma}$ and $(p, \rightarrow)_{\leq_\sigma}$, $p \in X$, form a subbase for a natural topology $\mathscr{T}_\sigma$ on X, called a *selection topology* [11].

A weak selection σ for X is *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{F}_2(X)$, equivalently if for every $p, q \in X$ with $p <_\sigma q$, there are open sets $U, V \subset X$ such that $p \in U$, $q \in V$ and $x <_\sigma y$ for every $x \in U$ and $y \in V$, see [11, Theorem 3.1]. Thus, if σ is continuous and $p \in X$, then the intervals $(\leftarrow, p)_{\leq_\sigma}$ and $(p, \rightarrow)_{\leq_\sigma}$ are open in X and $(\leftarrow, p]_{\leq_\sigma}$ and $[p, \rightarrow)_{\leq_\sigma}$ are closed in X, see [17]. However, the converse is not necessarily true [11, Example 3.6], see also [15, Corollary 4.2 and Example 4.3]. The following property is actually known, it will be found useful also in the rest of this paper.

Proposition 2.5. *Let X be a space which has a continuous weak selection σ and is totally disconnected at a point $p \in X$. If Δ_p is one of the intervals $(\leftarrow, p)_{\leq_\sigma}$ or $(p, \rightarrow)_{\leq_\sigma}$, and p is the limit of a sequence of points of Δ_p, then p is a countable intersection of clopen subsets of $\overline{\Delta_p}$.*

Proof. According to [5, Theorem 4.1], see also [10, Remark 3.5], p is a G_δ-point in $\overline{\Delta_p}$ with respect to the selection topology $\mathscr{T}_\sigma$. Hence, since X is totally disconnected at this point, it follows from [15, Proposition 5.6] that p is also a countable intersection of clopen subsets of $\overline{\Delta_p}$. □

The remaining part of the proof of Theorem 1.3 now follows from the following observation.

Proposition 2.6. *Let X be a space which is totally disconnected at a point $p \in X$. If X has a continuous weak selection and p is the limit of a nontrivial convergent sequence, then p is a cut point of X.*

Proof. Let σ be a continuous weak selection for X. Then, by condition, p is the limit of a sequence of points of $\Delta_p \subset X$, where Δ_p is one of the intervals $(\leftarrow, p)_{\leq_\sigma}$ or $(p, \rightarrow)_{\leq_\sigma}$. Hence, by Proposition 2.5, there is a decreasing sequence $\{H_n\}$ of clopen subsets of $\overline{\Delta_p}$ and a sequence $\{x_n\} \subset \Delta_p$ convergent to p such that $\bigcap_{n<\omega} H_n = \{p\}$ and $x_n \in H_n$, $n < \omega$. Taking subsequences if necessary, we can further assume $x_n \in S_n = H_n \setminus H_{n+1}$ for all $n < \omega$. Then $U = \bigcup_{n<\omega} S_{2n} \subset \Delta_p \subset X \setminus \{p\}$ is an open set with $\overline{U} = U \cup \{p\}$ because $\{x_{2n}\} \subset U$. Accordingly, for the set $V = X \setminus \overline{U} \subset X \setminus \{p\}$ we also have that $\overline{V} = V \cup \{p\}$ because $\{x_{2n+1}\} \subset V$. Thus, p is a cut point of X. □

3. Countably-Approachable Points

For a space X, the *components* (called also *connected components*) are the maximal connected subsets of X. They form a closed partition $\mathscr{C}$ of X, and each element $\mathscr{C}[x] \in \mathscr{C}$ containing a point $x \in X$ is called the *component* of this point.

Proposition 3.1. *Let X be a space and $T, Z \in \mathscr{F}(X)$ be such that Z is connected. If $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$ and $q = f(T \cup D)$ for some $D \in \mathscr{F}(Z)$, then $f(T \cup S) \in \mathscr{C}[q]$ for every $S \in \mathscr{F}(Z)$.*

Proof. Define a continuous map $f_T : \mathscr{F}(Z) \to X$ by $f_T(S) = f(T \cup S)$ for every $S \in \mathscr{F}(Z)$. Then $Q = f_T(\mathscr{F}(Z))$ is a connected subset of X because $\mathscr{F}(Z)$ is τ_V-connected, see [17, Theorem 4.10]. Accordingly, $Q \subset \mathscr{C}[q]$ because $q \in Q$. □

We now have the following relaxed condition for countably-approachable points.

Lemma 3.2. *Let X be a space, $p = f(H)$ for some $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$ and $H \in \mathscr{F}(X)$, and $U \subset X \setminus \{p\}$ be an open set with $\overline{U} = U \cup \{p\}$. Also, let $\{H_n\} \subset \mathscr{F}(X)$ be a sequence which is τ_V-convergent to H such that for every $n < \omega$,*

$$f(H_n) \in H_n \cap U \subset H_{n+1} \cap U \quad \text{and} \quad H_n \cap U \text{ is clopen.} \tag{3.1}$$

Then p is countably-approachable.

Proof. The sets $L_n = H_n \cap U$ and $F_n = L_n \cup (H_{n+1} \setminus U)$, $n < \omega$, will play a crucial role in this proof. According to (3.1), $f(H_n) \in L_n \subset L_{n+1}$ for every $n < \omega$. Since $\lim_{n\to\infty} f(H_n) = f(H) = p \notin U$, taking a subsequence if necessary, we can assume that

$$f(H_{n+1}) \in L_{n+1} \setminus L_n = H_{n+1} \setminus F_n \quad \text{for every } n < \omega. \tag{3.2}$$

Moreover, let us observe that

$$\{F_n\} \subset \mathscr{F}(X) \text{ is } \tau_V\text{-convergent to } H. \tag{3.3}$$

Indeed, by (3.1), $\{L_n\} \subset \mathscr{F}(X)$ is τ_V-convergent to $L = \overline{\bigcup_{n<\omega} L_n} \subset H$. Take a finite open cover $\mathscr{V}$ of H with $H \in \langle\mathscr{V}\rangle$, and set $\mathscr{V}_L = \{V \in \mathscr{V} : V \cap L \neq \varnothing\}$. Then there is $k < \omega$ such that $L_n \in \langle\mathscr{V}_L\rangle$ and $H_n \in \langle\mathscr{V}\rangle$ for every $n \geq k$. If $n \geq k$ and $L_n \cap W = \varnothing$ for some $W \in \mathscr{V}$, then $W \notin \mathscr{V}_L$ and, therefore, $(H_{n+1} \setminus U) \cap W \neq \varnothing$. Accordingly, $F_n = L_n \cup (H_{n+1} \setminus U) \in \langle\mathscr{V}\rangle$.

Now, as in the proof of [7, Lemma 4.4], for every $n < \omega$ we will construct a closed set $T_n \subset X$ and a nonempty clopen set $S_n \subset L_{n+1} \setminus L_n$ such that

$$F_n \subset T_n \subset H_{n+1} \setminus S_n \quad \text{and} \quad f(T_n \cup \{x\}) = x, \text{ for every } x \in S_n. \tag{3.4}$$

Briefly, $F_n \subset H_{n+1}$ and by (3.1), $H_{n+1} \setminus F_n = L_{n+1} \setminus L_n$ is clopen. Moreover, by (3.2), $f(H_{n+1}) \in H_{n+1} \setminus F_n$ and, therefore, $q = f(F_n \cup E) \in E$ for some finite set $E \subset H_{n+1} \setminus F_n$. Accordingly, we also have that $\mathscr{C}[q] \subset H_{n+1} \setminus F_n$. Thus, setting $D = E \cap \mathscr{C}[q]$, $K = E \setminus \mathscr{C}[q]$ and $T_n = F_n \cup K$, it follows that $D \subset \mathscr{C}[q] \subset H_{n+1} \setminus T_n$. Hence, by Proposition 3.1, $f(T_n \cup \{y\}) = y$ for every $y \in \mathscr{C}[q]$ because $f(T_n \cup D) = f(F_n \cup E) = q$. Finally, since K is a finite set and $H_{n+1} \setminus F_n$ is clopen, $\mathscr{C}[q] \subset S$ for some clopen set $S \subset H_{n+1} \setminus T_n$. Thus, the sets T_n and $S_n = \{x \in S : f(T_n \cup \{x\}) = x\}$ are as required in (3.4).

To finish the proof, it only remains to show that $\{S_n\} \subset \mathscr{F}(X)$ is τ_V-convergent to $\{p\}$, see [7, Section 4]. So, take an open set W containing p and a finite family $\mathscr{V}$ of open sets such that $H \in \langle\mathscr{V}\rangle$ and $f(\langle\mathscr{V}\rangle) \subset W$. Then by condition and the property in (3.3), there is $k < \omega$ with $F_n, H_n \in \langle\mathscr{V}\rangle$ for every $n \geq k$. Accordingly, for $n \geq k$ and $x \in S_n$, it follows from (3.4) that $x = f(T_n \cup \{x\}) \in W$ because $F_n \subset T_n \cup \{x\} \subset H_{n+1}$ implies that $T_n \cup \{x\} \in \langle\mathscr{V}\rangle$. The proof is complete. □

4. Approaching Trivial Components

The *quasi-component* $\mathscr{Q}[p]$ of a point $p \in X$ is the intersection of all clopen subsets of X containing this point. Evidently, $\mathscr{C}[p] \subset \mathscr{Q}[p]$ for every $p \in X$, but the converse is not necessarily true. However, these components coincide for spaces with continuous weak selections, see [12, Theorem 4.1]. Hence, in this case, X is totally disconnected at $p \in X$ precisely when $\mathscr{C}[p] = \{p\}$ is trivial.

Here, we will prove the special case of Theorem 1.2 when the component of X at $p \in X_\Theta$ is trivial. So, throughout this section, $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$ is a fixed selection such that $p = f(X)$ is a *cut point* of X, and X is *totally disconnected at* p. In this setting, the trivial case is when p is a G_δ-point of X.

Proposition 4.1. *If p is a countable intersection of clopen subsets of X, then it is countably-approachable.*

Proof. By condition, $U = X \setminus \{p\} = \bigcup_{n<\omega} H_n$ for some increasing sequence $\{H_n\} \subset \mathscr{F}(X)$ of clopen sets. Hence, the property follows from Lemma 3.2 by taking $H = X$. □

The rest of this section deals with the nontrivial case when p is not a countable intersection of clopen sets. To this end, we shall say that a pair (U, V) of subsets of X is a *p-cut* of X if $X \setminus \{p\} = U \cup V$ and $\overline{U} \cap \overline{V} = \{p\}$.

Proposition 4.2. *If p is not a countable intersection of clopen subsets of X, then X has a p-cut (U, V) such that*

(i) *p is a countable intersection of clopen subsets of $\overline{U}$,*
(ii) *V doesn't contain a sequence convergent to p.*

Proof. Since $p \in X_\Theta$ is a cut point, by Theorem 1.3, it is the limit of a sequence of points of $X \setminus \{p\}$. Therefore, p is the limit of a sequence of points of $U \subset X$, where U is one of the intervals $(\leftarrow, p)_{\leq_f}$ or $(p, \rightarrow)_{\leq_f}$. Hence, by Proposition 2.5, p is a countable intersection of clopen subsets of $\overline{U}$. This implies that $V = X \setminus \overline{U}$ is not clopen in X because p is not a countable intersection of clopen subsets of X. For the same reason, V doesn't contain a sequence convergent to p. Accordingly, this p-cut (U, V) of X is as required. □

The following two mutually exclusive cases finalise the proof of Theorem 1.2 when $\mathscr{C}[p] = \{p\}$. They are based on two alternatives for the selection f with respect to the p-cut (U, V), constructed in Proposition 4.2, the set $Y = \overline{V}$ and a fixed increasing sequence $\{T_n\} \subset \mathscr{F}(X)$ of clopen sets with $\bigcup_{n<\omega} T_n = U$.

Proposition 4.3. *Suppose that for every $S \in \mathscr{F}(Y)$ with $p \notin S$,*

$$f(T_n \cup \{p\} \cup S) \neq p \quad \text{for all but finitely many } n < \omega. \tag{4.1}$$

Then p is countably-approachable.

Proof. We proceed as in the proof of Proposition 2.3. Namely, take a local base $\mathscr{B}$ at p in Y and $B_0 \in \mathscr{B}$ with $S_0 = Y \setminus B_0 \neq \varnothing$. Then by (4.1), there exists $n_0 \geq 0$ such that $f(T_{n_0} \cup \{p\} \cup S_0) \neq p$. Next, using continuity of f, take $B_1 \in \mathscr{B}$ such that $B_1 \subset B_0$ and $f(T_{n_0} \cup K \cup S_0) \in T_{n_0} \cup S_0$ for every $K \in \Sigma(B_1)$, see (2.1). Hence, by Proposition 2.1, we also have that $f\left(T_{n_0} \cup \overline{B_1} \cup S_0\right) \in T_{n_0} \cup S_0$. We can repeat the construction with $S_1 = Y \setminus B_1$

and some $n_1 > n_0$. Thus, by induction, there exists a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and a decreasing sequence $\{B_k\} \subset \mathscr{B}$ such that for $S_k = Y \setminus B_k$, $k < \omega$,

$$f\left(T_{n_k} \cup \overline{B_{k+1}} \cup S_k\right) \in T_{n_k} \cup S_k \quad \text{for every } k < \omega. \tag{4.2}$$

By Proposition 2.2, the sequence $\overline{B_{k+1}} \cup S_k$, $k < \omega$, is τ_V-convergent to Y because $\left\{B_k \setminus \overline{B_{k+1}} : k < \omega\right\}$ is a pairwise disjoint family of proper open subsets of Y. Moreover, $\{T_{n_k}\}$ is τ_V-convergent to $\overline{U}$ being a subsequence of $\{T_n\}$. Hence, $H_k = T_{n_k} \cup \overline{B_{k+1}} \cup S_k$, $k < \omega$, is τ_V-convergent to X. Accordingly, $p = f(X) = \lim_{k\to\infty} f(H_k)$. However, by (4.2), $f(H_k) \neq p$ for every $k < \omega$. Therefore, by (ii) of Proposition 4.2, $f(H_k) \in U$ for all but finitely many $k < \omega$. Thus, by Lemma 3.2, the point p is countably-approachable. □

Proposition 4.4. *Suppose that there exists $S \in \mathscr{F}(Y)$ with $p \notin S$, and a subsequence $\left\{T_{n_j}\right\}$ of $\{T_n\}$ such that*

$$f\left(T_{n_j} \cup \{p\} \cup S\right) = p \quad \textit{for all } j < \omega. \tag{4.3}$$

Then p is countably-approachable.

Proof. Evidently, we can assume that (4.3) holds for all $n < \omega$. Next, using Theorem 1.3 and (ii) of Proposition 4.2, take a sequence $\{x_n\} \subset U$ which is convergent to p and $x_n \in T_n$ for every $n < \omega$. Since f is continuous and the sequence $T_0 \cup \{x_n\} \cup S$, $n < \omega$, is τ_V-convergent to $T_0 \cup \{p\} \cup S$, it follows from (4.3) that $f(T_0 \cup \{x_{n_0}\} \cup S) = x_{n_0}$ for some $n_0 < \omega$. We can repeat this with T_{n_0}. Namely, the sequence $T_{n_0} \cup \{x_n\} \cup S$, $n > n_0$, is τ_V-convergent to $T_{n_0} \cup \{p\} \cup S$. Hence, for the same reason, $f(T_{n_0} \cup \{x_{n_1}\} \cup S) = x_{n_1}$ for some $n_1 > n_0$. Thus, by induction, there are subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{T_{n_k}\}$ of $\{T_n\}$ such that $f(T_{n_k} \cup \{x_{n_{k+1}}\} \cup S) = x_{n_{k+1}}$ for every $k < \omega$. Then $H_k = T_{n_k} \cup S$, $k < \omega$, is a τ_V-convergent sequence with $\lim_{k\to\infty} f(H_k) = p$, because $H_k \subset H_k \cup \{x_{n_{k+1}}\} \subset H_{k+1}$ for every $k < \omega$. Furthermore, by (ii) of Proposition 4.2, $f(H_k) \in U$ for all but finitely many $k < \omega$. Therefore, just like before, Lemma 3.2 implies that p is countably-approachable. □

5. Approaching Nontrivial Components

Here, we will finalise the proof of Theorem 1.2 with the remaining case when X is not totally disconnected at p. To this end, let us recall that a space X is *weakly orderable* if there exists a coarser orderable topology on X with respect to some linear order on it (called *compatible* for X). The weakly orderable spaces were introduced by Eilenberg [4], and are often called "Eilenberg orderable".

Each connected space Z with a continuous weak selection σ is weakly orderable with respect to $\leq_\sigma$, see [17, Lemmas 7.2]. The following simple observation was implicitly present in the proof of [12, Theorem 1.5]. In this observation, and what follows, $\mathrm{nct}(Z)$ are the noncut points of a connected space Z, and $\mathrm{ct}(Z)$ — the cut points of Z.

Proposition 5.1. *Let X be a space and $p = f(X)$ for some $f \in \mathscr{V}_{cs}[\mathscr{F}(X)]$. Then $p \in \mathrm{nct}(\mathscr{C}[p])$.*

Proof. Set $Z = \mathscr{C}[p]$ and assume that $p \in H = \mathrm{ct}(Z)$. Since H is open in X (see [8, Corollary 2.7]) and f is continuous, $f(\langle \mathscr{U} \rangle) \subset H \subset Z$ for some finite open cover $\mathscr{U}$ of X. Take a finite set $T \subset X \setminus Z$ with $Y = T \cup Z \in \langle \mathscr{U} \rangle$. Then it follows from Proposition 3.1 that $g(S) = f(T \cup S) \in S$ for every $S \in \mathscr{F}(Z)$. Accordingly, $g : \mathscr{F}(Z) \to Z$ is a continuous selection with $g(Z) \in H = \mathrm{ct}(Z)$. However, this is impossible because Z is weakly orderable with respect to $\leq_g$ and $g(Z)$ is the first $\leq_g$-element of Z, see [17, Lemmas 7.2 and 7.3] and [8, Corollary 2.7]. ☐

In the rest of this section, $p \in X_\Theta$ is a cut point such that the component $\mathscr{C}[p]$ is not a singleton. In this case, by Proposition 5.1, p is a noncut point of $\mathscr{C}[p]$. Thus, we can also fix a p-cut (U, V) of X such that $\mathscr{C}[p] \subset \overline{V}$. Accordingly, $Y = \overline{U}$ is totally disconnected at p. In this setting, the remaining part of the proof of Theorem 1.2 consists of showing that p is countably-approachable in Y. To this end, we will first show that $\overline{V}$ can itself be assumed to be connected.

Proposition 5.2. *Let $f : \mathscr{F}(X) \to X$ be a continuous selection with $f(X) = p$. Then there exists a nondegenerate connected subset $Z \subset \mathscr{C}[p]$ such that $p \in Z$ and $X_* = Y \cup Z$ has a continuous selection $f_* : \mathscr{F}(X_*) \to X_*$ with $f_*(X_*) = p$.*

Proof. Since $H = \mathscr{C}[p]$ has a continuous weak selection and $p \in \mathrm{nct}(H)$, the space H is weakly orderable with respect to a linear order $\leq$ such that $p \leq x$ for every $x \in H$, see [17, Lemma 7.2] and [8, Corollary 2.7]. Accordingly, each closed interval $Z_x = [p, x]_\leq \in \mathscr{F}(H)$, $x \in \mathrm{ct}(H)$, is a connected subset of H, see [16, Theorem 1.3]. Moreover, if $T = \overline{V \setminus H}$, then $f(Y \cup H \cup T) = f(X) = p \in H$. Thus, by Propositions 3.1 and 5.1,

$$f(Y \cup Z_x \cup T) \in \mathrm{nct}(Z_x) \cup T = \{p, x\} \cup T \quad \text{for every } x \in \mathrm{ct}(H). \tag{5.1}$$

Evidently, the resulting family $\mathscr{S} = \{Y \cup Z_x \cup T : x \in \mathrm{ct}(H)\}$ is up-directed. Therefore, by Proposition 2.1, it is τ_V-convergent to $\overline{\bigcup \mathscr{S}} = X$. Hence, by (5.1), $f(Y \cup Z_q \cup T) = p$ for some $q \in \mathrm{ct}(H)$ because $\lim_{S \in \mathscr{S}} f(S) = f(X) = p$. Finally, let $Z = Z_q$, $X_* = Y \cup Z$ and $\mathscr{T} = \{S \in \mathscr{F}(X_*) : f(S \cup T) \in T\}$. Then $\mathscr{T}$ is a τ_V-clopen set in $\mathscr{F}(X_*)$ because T is clopen in $X_* \cup T$. So, we may define a continuous selection $f_* : \mathscr{F}(X_*) \to X_*$ by letting for $S \in \mathscr{F}(X_*)$ that

$$f_*(S) = \begin{cases} f(S) & \text{if } S \in \mathscr{T}, \quad \text{and} \\ f(S \cup T) & \text{if } S \notin \mathscr{T}. \end{cases}$$

Since $f(X_* \cup T) = f(Y \cup Z \cup T) = f(Y \cup Z_q \cup T) = p \notin T$, we get that $X_* \notin \mathscr{T}$. Accordingly, we also have that $f_*(X_*) = f(X_* \cup T) = p$. ☐

Since the space $X_* = Y \cup Z$ in Proposition 5.2 has all properties of X relevant to our case, we can identify X with this space. In this refined setting, the fixed p-cut (U, V) of X has the extra property that $Z = \overline{V}$ is connected, while $Y = \overline{U}$ is the same as before.

Proposition 5.3. *If $f : \mathscr{F}(X) \to X$ is a continuous selection with $f(X) = p$, then $f(Y) = p$. Moreover,*

$$f(Y \cup \{q\}) = p \quad \textit{for every } q \in Z. \tag{5.2}$$

Proof. Since Z is connected, it follows from Proposition 3.1 that $f(Y \cup S) \in Z$ for every $S \in \mathscr{F}(Z)$. Accordingly, $f(Y) = f(Y \cup \{p\}) = p$. Regarding (5.2), we argue by contradiction. Namely, assume that

$$f(Y \cup \{q\}) = q \quad \text{for some } q \in Z \text{ with } q \neq p. \tag{5.3}$$

Next, as in the proof of Proposition 5.2, using that Z is weakly orderable and $p \in \operatorname{nct}(Z)$, take a compatible linear order $\leq$ on Z such that $p \leq z$ for every $z \in Z$. Then by (5.3), $p < q$ and we now have that

$$f(Y \cup S) \in S, \quad \text{whenever } S \in \mathscr{F}\left([z, \rightarrow)_{\leq}\right) \text{ for some } z > p. \tag{5.4}$$

Briefly, for $z > p$ and $S \in \mathscr{F}\left([z, \rightarrow)_{\leq}\right)$, it follows that either $S \subset [q, \rightarrow)_{\leq}$ or $q \in [z, \rightarrow)_{\leq}$. Since all $\leq$-intervals of Z are connected, (5.4) follows from (5.3) and Proposition 3.1.

This now implies that the continuous map $g(T) = f(Y \cup T)$, $T \in \mathscr{F}(Z)$, is a selection for $\mathscr{F}(Z)$. Indeed, for $T \in \mathscr{F}(Z)$ with $T \neq \{p\}$, set

$$\mathscr{S} = \left\{T \cap [z, \rightarrow)_{\leq} : z \in T \setminus \{p\}\right\}.$$

Since the family $\mathscr{S}$ is up-directed in $\mathscr{F}(Z)$, by Proposition 2.1, it is τ_V-convergent to $\overline{\bigcup \mathscr{S}}$. Moreover, by (5.4), $g(S) = f(Y \cup S) \in S$ for every $S \in \mathscr{S}$. Therefore, $g(T) \in \overline{\bigcup \mathscr{S}} \subset T$ because $Y \cup \left(\overline{\bigcup \mathscr{S}}\right) = Y \cup T$. However, according to [17, Lemma 7.3], $\mathscr{F}(Z)$ has at most two continuous selections — taking the minimal element, or taking the maximal element of each $T \in \mathscr{F}(Z)$. Therefore, $g(T) = \min_{\leq} T$ for every $T \in \mathscr{F}(Z)$ because $g(Z) = f(Y \cup Z) = f(X) = p = \min_{\leq} Z$. But this is impossible because by (5.3), $q = g(\{q\}) = f(Y \cup \{q\}) = \min_{\leq}\{p, q\} = p$. ☐

The following final observation completes the proof of Theorem 1.2.

Proposition 5.4. *If $f : \mathscr{F}(X) \to X$ is a continuous selection with $f(X) = p$, then $Y \setminus \{p\}$ contains a sequence convergent to p. In particular, p is a cut point of Y, and is therefore also countably-approachable.*

Proof. Let $\mathscr{B}$ be a local base at p in Y with $Y \notin \mathscr{B}$. Then there exists $B_0 \in \mathscr{B}$ such that

$$f\left((Y \setminus B) \cup \{p\}\right) \in Y \setminus B \quad \text{for every } B \in \mathscr{B} \text{ with } B \subset B_0. \tag{5.5}$$

Indeed, assume that (5.5) fails, and let $\mathscr{B}_*$ be the collection of all $B \in \mathscr{B}$ such that $f\left((Y \setminus B) \cup \{p\}\right) = p$. Then $\mathscr{B}_*$ is also a local base at $p \in Y$. Hence, $\mathscr{S} = \{Y \setminus B : B \in \mathscr{B}_*\}$ is an up-directed cover of $Y \setminus \{p\}$ and by Proposition 2.1, it is τ_V-convergent to Y. Moreover, by assumption, $f(S \cup \{p\}) = p$ for every $S \in \mathscr{S}$. Therefore, by Proposition 3.1, we also have that $f(S \cup \{q\}) = q$ for every $q \in Z$ and $S \in \mathscr{S}$. Accordingly, $f(Y \cup \{q\}) = \lim_{S \in \mathscr{S}} f(S \cup \{q\}) = q$ for every $q \in Z$. However, by (5.2) of Proposition 5.3, this is impossible.

Having already established (5.5), it follows from Propositions 5.3 and 2.3 that p is the limit of a sequence of points of $Y\setminus\{p\}$. Hence, by Proposition 2.6, p is a cut point of Y. Therefore, by the already proven case of Theorem 1.2, the point p is also countably-approachable in Y. Since $Y=\overline{U}=U\cup\{p\}$ and $U\subset X\setminus\{p\}$ is open, p is countably-approachable in X as well. $\square$

6. Point-Maximal Selections

Recall that a point $p\in X$ in an arbitrary space X is *noncut* if it is not a cut point. The prototype of such points can be traced back to Michael's nowhere cuts defined in [18]. In his terminology, a subset $A\subset X$ *nowhere cuts* X [18] if A has an empty interior (i.e. A is *thin*) and whenever $p\in A$ and U is a neighbourhood of p in X, then $U\setminus A$ does not split into two disjoint open sets both having p in their closure. Evidently, the singleton $\{p\}$ nowhere cuts X for each noncut point $p\in X$. A slight variation of this concept was considered in [2] (under the name 'does not cut') and in [19] (under the name 'nowhere disconnects').

As commented in the Introduction, the equality $X^*_\Theta\cap X_\Omega=X^*_\Omega$ in (1.4) of Theorem 1.5 is known, see (1.1) and (1.3). Here, we will prove the following refined version of this theorem showing that the members of $X_\Theta\setminus X_\Omega$ possess a similar property with respect to the connected components. To this end, let us recall that a (closed) subset $C\subset X$ has a *clopen base* if for each neighbourhood U of C there exists a clopen set $H\subset X$ with $C\subset H\subset U$. In case $C=\{p\}$ is a singleton, we simply say that X is *zero-dimensional* at $p\in X$.

Theorem 6.1. *Let X be a space with $\mathscr{V}_{cs}[\mathscr{F}(X)]\neq\varnothing$ and $p\in X_\Theta\setminus X_\Omega$. Then $\mathscr{C}[p]$ has a clopen base and $p\in X^*_\Theta$.*

Evidently, the essential case in Theorem 6.1 is when $\mathscr{C}[p]$ is not a clopen set, otherwise the property follow easily from known results and Proposition 3.1. Thus, in the rest of this section, $\mathscr{C}[p]$ will be assumed to be not clopen.

The next lemma covers the case of $\mathscr{C}[p]=\{p\}$ in Theorem 6.1.

Lemma 6.2. *Let X be a space with $\mathscr{V}_{cs}[\mathscr{F}(X)]\neq\varnothing$. If X is totally disconnected at some point $p\in X_\Theta\setminus X_\Omega$, then X is zero-dimensional at p and $p\in X^*_\Theta$.*

The proof of this lemma is base on the following two simple observations.

Proposition 6.3. *Let $f:\mathscr{F}(X)\to X$ be a continuous selection, $p\in X$ with $\mathscr{C}[p]=\{p\}$, and $K\in\mathscr{F}(X)$ be such that $p\notin K$ and $f(K\cup S)=p$ for every closed set $S\subset X$ with $p\in S$. Then X has a clopen base at p.*

Proof. We follow the idea in the proof of [12, Theorem 1.4], see also [1]. Take an open set $U\subset X$ with $p\in U\subset X\setminus K$, and set $F=X\setminus U$. Since $f(F)\neq p$, there exists a clopen set $T\subset X$ with $f(F)\in T$ and $p\notin T$. Then $f^{-1}(T)$ is a τ_V-clopen subset $\mathscr{F}(X)$. Take a maximal chain $\mathscr{M}\subset f^{-1}(T)$ with $F\in\mathscr{M}$. Then $M=\overline{\bigcup\mathscr{M}}$ is the maximal element of $\mathscr{M}$, and therefore M is clopen in X because $f^{-1}(T)$ is τ_V-clopen. Moreover, $K\subset F\subset M$ because $F\in\mathscr{M}$.

Finally, M doesn't contain p because $f(M) \neq p$. Indeed, $p \in M$ will imply that $f(M) = f(K \cup M) = p$, but this is impossible. Thus, $H = X \setminus M$ is a clopen set with $p \in H \subset U$. ☐

Proposition 6.4. *Let $f : \mathscr{F}(X) \to X$ be a continuous selection, $p \in X$ with $\mathscr{C}[p] = \{p\}$, and $K \in \mathscr{F}(X)$ be such that $p \notin K$ and $f(K \cup S) = p$ for every closed set $S \subset X$ with $p \in S$. Then $p \in X^*_\Theta$.*

Proof. Since $f(K \cup \{p\}) = p \notin K$, there is a finite family $\mathscr{V}$ of open subsets of X such that $K \cup \{p\} \in \langle \mathscr{V} \rangle$ and $f(\langle \mathscr{V} \rangle) \subset X \setminus K$. Then by Proposition 6.3, there exists a clopen set H such that $p \in H \subset X \setminus K$ and $H \subset \bigcap\{V \in \mathscr{V} : p \in V\}$. Accordingly, $f(K \cup S) \in S$ for every $S \in \mathscr{F}(H)$. We can now define a continuous selection $h : \mathscr{F}(X) \to X$ by letting for $S \in \mathscr{F}(X)$ that

$$h(S) = \begin{cases} f(S) & \text{if } S \cap H = \varnothing, \text{ and} \\ f(K \cup S_H) & \text{if } S_H = S \cap H \neq \varnothing. \end{cases}$$

Then h is p-maximal. Indeed, $p \in S \in \mathscr{F}(X)$ implies that $p \in S_H = S \cap H$ and by the property of K, we have that $h(S) = f(K \cup S_H) = p$. ☐

Proof of Lemma 6.2. According to Propositions 6.3 and 6.4, it suffices to show that there exists $K \in \mathscr{F}(X)$ such that

$$p \notin K \quad \text{and} \quad f(K \cup S) = p, \ \text{ for every } S \in \mathscr{F}(X) \text{ with } p \in S. \tag{6.1}$$

To this end, let $\mathscr{O}$ be the collection of all open subsets containing p, and $\mathscr{B} \subset \mathscr{O}$ be that one of those $B \in \mathscr{O}$ for which $f((X \setminus B) \cup \{p\}) \neq p$. If $\mathscr{B}$ is a local base at p, then by Proposition 2.3, $X \setminus \{p\}$ contains a sequence convergent to p. Hence, by Proposition 2.6, p must be a cut point of X. However, by Theorem 1.2, this is impossible because $p \notin X_\Omega$. Accordingly, there exists $U \in \mathscr{O}$ such that $K = X \setminus U \neq \varnothing$ and the family $\mathscr{V} = \{V \in \mathscr{O} : V \subset U\}$ doesn't contain any member of $\mathscr{B}$, namely $f((X \setminus V) \cup \{p\}) = p$ for every $V \in \mathscr{V}$. To see that this K is as in (6.1), take a closed set $S \subset X$ with $p \in S$, and set $\mathscr{L} = \{S \setminus V : V \in \mathscr{V}\}$. Then $X \setminus (K \cup L) \in \mathscr{V}$, $L \in \mathscr{L}$, and therefore $f(K \cup L \cup \{p\}) = p$ for every $L \in \mathscr{L}$. Moreover, by Proposition 2.1, $\mathscr{H} = \{K \cup L \cup \{p\} : L \in \mathscr{L}\}$ is τ_V-convergent to $\overline{\bigcup \mathscr{H}} = \overline{K \cup S} = K \cup S$ being an up-directed cover of $K \cup S$. Since f is continuous, this implies that $f(K \cup S) = p$ and the proof is complete. ☐

The other case of Theorem 6.1 is covered by the following lemma.

Lemma 6.5. *Let X be a space with $\mathscr{V}_{cs}[\mathscr{F}(X)] \neq \varnothing$, and $p \in X_\Theta \setminus X_\Omega$ be such that $\mathscr{C}[p] \neq \{p\}$. Then $\mathscr{C}[p]$ has a clopen base and $p \in X^*_\Theta$.*

In this lemma, according to Theorem 1.2 (see also Proposition 5.1), p is both a noncut point of X and a noncut point of $\mathscr{C}[p]$. Since $\mathscr{C}[p]$ is not clopen in X, it has another noncut point $q \in \mathscr{C}[p]$ defined by the property that $q \in \overline{X \setminus \mathscr{C}[p]}$. In particular, q is a cut point of X. Thus, in this case, $U = X \setminus \mathscr{C}[p]$ and $V = \mathscr{C}[p] \setminus \{q\}$ form a q-cut of X such that $Y = \overline{U}$ is totally disconnected at q and $Z = \overline{V} = \mathscr{C}[p]$. In this setting, Y and Z have the following properties.

Proposition 6.6. *There exists a nonempty finite set $K \subset U$ such that for every closed set $S \subset Y$, the map $f_{(K,S)}(T) = f(K \cup S \cup T)$, $T \in \mathscr{F}(Z)$, is a continuous p-maximal selection for $\mathscr{F}(Z)$.*

Proof. Since $f(X) = p \in V$, there exists a finite open cover $\mathscr{W}$ of X with $X \in \langle \mathscr{W} \rangle$ and $f(\langle \mathscr{W} \rangle) \subset V$. Take a finite set $K \subset U$ such that $K \cap W \neq \varnothing$ for every $W \in \mathscr{W}$ with $W \cap Y \neq \varnothing$. Then K has the property that

$$f(K \cup S \cup Z) = p \quad \text{for every closed set } S \subset Y. \tag{6.2}$$

Indeed, in this case, $f(K \cup S \cup Z) \in V \subset \mathscr{C}[p]$ because $K \cup S \cup Z \in \langle \mathscr{W} \rangle$. Hence, by Proposition 5.1, $f(K \cup S \cup Z) = p$ because $q \notin V$.

For a closed subset $S \subset Y$, the map $f_{(K,S)}$ is continuous and by Proposition 3.1, $f_{(K,S)}(T) = f(K \cup S \cup T) \in Z$ for every $T \in \mathscr{F}(Z)$. Hence, by (6.2) and [17, Lemmas 7.2 and 7.3], it only suffices to show that $f_{(K,S)}$ is a selection for $\mathscr{F}(Z)$. If $T \in \mathscr{F}(Z)$ and $q \notin S$, then $f_{(K,S)}(T) \in T$ because $K \cup S \subset U \subset X \setminus Z$. Otherwise, if $q \in S$, we set $F = S \setminus \{q\}$ and distinguish the following two cases:

(i) If F is a closed set, as remarked above, $f_{(K,F)}$ is a selection for $\mathscr{F}(Z)$. Therefore, by (6.2), $f_{(K,F)}$ is 'q-minimal' in the sense that $f_{(K,F)}(T) = q$ precisely when $T = \{q\}$ because $q \in \operatorname{nct}(\mathscr{C}[p])$, see [17, Lemma 7.3] and [8, Corollary 2.7]. In other words, $f_{(K,S)}(T) = f_{(K,F)}(T \cup \{q\}) \in T$ for every $T \in \mathscr{F}(Z)$.

(ii) If F is not closed, by **(i)**, $f_{(K,E)}(T) = f(K \cup E \cup T) \in T$ for every $E \in \Sigma(F)$, see (2.1). Moreover, by Proposition 2.1, $\mathscr{H} = \{K \cup E : E \in \Sigma(F)\}$ is an up-directed family which is τ_V-convergent to $K \cup S$. Accordingly,

$$f_{(K,S)}(T) = f(K \cup S \cup T) = \lim_{H \in \mathscr{H}} f(H \cup T) \in \overline{T} = T. \qquad \square$$

Proof of Lemma 6.5. According to Proposition 6.6, there exists a nonempty finite set $K \subset U = X \setminus \mathscr{C}[p]$ such that $f(K \cup S \cup \{q\}) = q$ for every closed set $S \subset Y = \overline{U}$. Since Y is totally disconnected at q, it follows from Proposition 6.3 that q has a clopen base in Y. This implies that $\mathscr{C}[p]$ has a clopen base in X. To show the remaining part of this lemma, as in the proof of Proposition 6.4, take a clopen set $H \subset Y$ such that $q \in H \subset Y \setminus K$ and $f(K \cup S) \in S$ for every $S \in \mathscr{F}(H)$. Then $L = H \cup Z$ is a clopen subset of X with the same property. Indeed, take any $S \in \mathscr{F}(L)$. If $S \subset Y$, then $S \subset H$ and therefore $f(K \cup S) \in S$. If $S \setminus Y \neq \varnothing$, set $D = S \cap Y$ and $T = S \cap Z$. Then by Proposition 6.6, $f(K \cup S) = f(K \cup D \cup T) \in T \subset S$. Hence, just like before, we can define a continuous p-maximal selection $h : \mathscr{F}(X) \to X$ by

$$h(S) = \begin{cases} f(S) & \text{if } S \cap L = \varnothing, \text{ and} \\ f(K \cup S_L) & \text{if } S_L = S \cap L \neq \varnothing. \end{cases}$$

Indeed, if $p \in S \in \mathscr{F}(X)$, then $p \in S_L = S \cap L$. Moreover, $D_L = S_L \cap Y$ is closed in Y and $p \in T_L = S_L \cap Z \in \mathscr{F}(Z)$. According to Proposition 6.6, $h(S) = f(K \cup S_L) = f(K \cup D_L \cup T_L) = p$. $\square$

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