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A generalization of strongly irreducible ideals with a view towards rings of continuous functions

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Abstract

In this article, we introduce and study the concept of a semi-strongly irreducible ideal, a natural generalization of a strongly irreducible ideal. We say an ideal I of a commutative ring R is semi-strongly irreducible if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J^2 \subseteq I$ or $K^2 \subseteq I$. After some general results, the article focuses on semi-strongly irreducible ideals in rings of continuous functions.

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1. Introduction

An ideal I of a commutative ring R is called *strongly irreducible* if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. Obviously, an ideal I is strongly irreducible if and only if for all $x, y \in R$, $Rx \cap Ry \subseteq I$ implies that $x \in I$ or $y \in I$. Prime ideals are strongly irreducible. Every ideal in a valuation domain is strongly irreducible. Strongly irreducible ideals were first studied by Fuchs, [13], under the name *primitive ideals*. Apparently, the name "*strongly irreducible*" was first used by Blair in [7]. In [8, p. 177, Exercise 34], the strongly irreducible ideals are called *quasi prime*. We refer

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the reader to [4], [18], and [23] for more information about strongly irreducible ideals. Throughout this paper, all rings are commutative with $1 \neq 0$. A ring R is called *reduced* if it has no non-zero nilpotent elements. Everything needed about rings can be found in [21].

In the following, we introduce the concept of a semi-strongly irreducible ideal, as a generalization of the notion of a strongly irreducible ideal.

Definition 1.1. We say an ideal I of a ring R is *semi-strongly irreducible* if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J^2 \subseteq I$ or $K^2 \subseteq I$.

Obviously, every strongly irreducible ideal is semi-strongly irreducible. However, the converse is not true (even in a Noetherian domain). Take $I = (x^2, xy, y^2)$ in $K[x, y]$ where K is a field. From $I = (x, y^2) \cap (x^2, y)$, we deduce that I is not strongly irreducible. It is not hard to see that I is a semi-strongly irreducible ideal.

An ideal I of a ring R is said to be *irreducible* if I is not the intersection of two ideals of R that properly contain it. It is known that every strongly irreducible ideal is irreducible, see [18, Lemma 2.2(1)] for example. The example in the above paragraph shows that a semi-strongly irreducible ideal need not be irreducible.

This note aims to investigate semi-strongly irreducible ideals with a view towards rings of continuous functions.

2. Semi-strongly irreducible ideals

We begin with a few results about semi-strongly irreducible ideals.

Lemma 2.1. *Let I be an ideal in a ring R. The following statements hold.*

1. *If I is a strongly irreducible ideal, then I^2 is semi-strongly irreducible.*
2. *If I is a semi-strongly irreducible ideal of R, then I is a prime ideal if and only if I is semiprime.*
3. *For each proper ideal I of R, there is a minimal semi-strongly irreducible ideal over I.*
4. *If I is a semi-strongly irreducible ideal in R containing an ideal H, then I/H is a semi-strongly irreducible ideal of R/H. Moreover, if R is an arithmetical ring (that is, a ring in which for every three ideals I, J and K, we have $I + (J \cap K) = (I + J) \cap (I + K)$), then the converse also holds.*
5. *Every semi-strongly irreducible ideal of a von Neumann regular ring is strongly irreducible.*

Proof. (1) Assume J and K are two ideals of R such that $J \cap K \subseteq I^2$ (and so $J \cap K \subseteq I$). Since I is a strongly irreducible ideal, we infer that either $J \subseteq I$ or $K \subseteq I$. From this, we deduce that $J^2 \subseteq I^2$ or $K^2 \subseteq I^2$, as desired.
(2) If I is a prime ideal, then there is nothing to prove. Assume that I is a semiprime ideal and $JK \subseteq I$. A result due to Fuchs [11] states that an ideal I of a commutative ring R is semiprime if and only if it contains the intersection

of two ideals whenever it contains their product. From this, $J \cap K \subseteq I$. Since I is a semi-strongly irreducible ideal, we deduce that either $J^2 \subseteq I$ or $K^2 \subseteq I$. Since I is semiprime ideal, we conclude that either $J \subseteq I$ or $K \subseteq I$, as desired.
(3) Let $\Lambda = \{J : J$ is a semi-strongly irreducible ideal of R containing $I\}$. Since every maximal ideal is semi-strongly irreducible, $\Lambda \neq \varnothing$. By Zorn's lemma Λ has a minimal element with respect to $\supseteq$.
(4) For the first assertion, let J and K be ideals of R such that $J/H \cap K/H \subseteq I/H$. Then $J \cap K \subseteq I$, and since I is semi-strongly irreducible it follows that either $J^2 \subseteq I$ or $K^2 \subseteq I$. Therefore, either $(J/H)^2 \subseteq I/H$ or $(K/H)^2 \subseteq I/H$, i.e., I/H is semi-strongly irreducible. For the last assertion in (4), let $J \cap K \subseteq I$. Then $H + (J \cap K) = (H + J) \cap (H + K) \subseteq I$ and consequently $(H + J)/H \cap (H + K)/H \subseteq I/H$. Since I/H is semi-strongly irreducible, we infer that either $(H+J)^2 \subseteq I$ or $(H+K)^2 \subseteq I$, and so either $J^2 \subseteq I$ or $K^2 \subseteq I$. Thus I is semi-strongly irreducible.
(5) Let us first recall that a ring R is said to be *von Neumann regular* if for every $a \in R$ there is an $x \in R$ for which $a = a^2x$. It is known that a ring R is von Neumann regular if and only if every ideal of R is an idempotent, see [21, Ex. 10.19]. By this fact, we infer that every semi-strongly irreducible ideal of a von Neumann regular ring is strongly irreducible. □

We mention here that if J and K are semi-strongly irreducible ideals of a ring R, then $J \cap K$ and JK need not be semi-strongly irreducible of R. For example, in the ring of integers $\mathbb{Z}$, $2\mathbb{Z}$ and $3\mathbb{Z}$ are prime (so are semi-strongly irreducible) but $2\mathbb{Z} \times 3\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not semi-strongly irreducible.

Theorem 2.2. *Let $R = R_1 \times R_2$, where R_1 and R_2 are two rings. Let J be a proper ideal of R. The following statement are equivalent:*

(1) *J is a semi-strongly irreducible ideal.*
(2) *Either $J = I_1 \times R_2$ for some semi-strongly irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some semi-strongly irreducible ideal I_2 of R_2.*

Proof. (1) ⇒ (2) Assume (1). Let $J = I_1 \times I_2$ be an ideal of $R_1 \times R_2$. First, we show that either $I_1 = R_1$ or $I_2 = R_2$. Assume, for a contradiction, $I_1 \neq R_1$ and $I_2 \neq R_2$. Take the ideal $(R_1 \times 0) \cap (0 \times R_2) \subseteq J$. This implies that either $(R_1 \times 0) \subseteq J$ or $(0 \times R_2) \subseteq J$, a contradiction. Now suppose that $J = I_1 \times R_2$ where I_1 is an ideal of R_1. We show that I_1 is semi-strongly irreducible. Assume that $K \cap H \subseteq I$ where K and H are two ideals of R_1. From this, we deduce that $(K \times R_2) \cap (H \times R_2) \subseteq I_1 \times R_2 = J$. Since J is semi-strongly irreducible, either $K^2 \times R_2 \subseteq J$ or $H^2 \times R_2 \subseteq J$. Thus, either $K^2 \subseteq I$ or $H^2 \subseteq I$, as desired. A similar proof works for when $J = R_1 \times I_2$ where I_2 is an ideal of R_2.
(2) ⇒ (1) Straightforward. □

Let R be a ring and let S be a multiplicatively closed subset of R. For each ideal I of the ring $S^{-1}R$, we consider

$$I^c = \left\{x \in R : \frac{x}{1} \in I\right\} = I \cap R \quad \text{and} \quad C = \left\{I^c : I \text{ is an ideal of } S^{-1}R\right\}.$$

Theorem 2.3. *Let R be a ring and S be a multiplicatively closed subset of R. Then there is a one-to-one correspondence between the semi-strongly irreducible ideals of $S^{-1}R$ and the semi-strongly irreducible ideals of R contained in C which do not meet S.*

Proof. The proof is an analogue of [4, Theorem 3.1]. We write out all the detail for the convenience of the reader. Let I be a semi-strongly irreducible ideal of $S^{-1}R$. Obviously, $I^c \neq R$, $I^c \in C$, and $I^c \cap S = \varnothing$. Let $A \cap B \subseteq I^c$, where A and B are ideals of R. Then we have $(S^{-1}A) \cap (S^{-1}B) = S^{-1}(A \cap B) \subseteq S^{-1}(I^c) = I$. Hence, $S^{-1}A^2 \subseteq I$ or $S^{-1}B^2 \subseteq I$, and so $A^2 \subseteq (S^{-1}A^2)^c \subseteq I^c$ or $B^2 \subseteq (S^{-1}B^2)^c \subseteq I^c$. Thus, I^c is a semi-strongly irreducible ideal of R. Conversely, let I be a semi-strongly irreducible ideal of R, $I \cap S = \varnothing$, and $I \in C$. Since $I \cap S = \varnothing$, $S^{-1}I \neq S^{-1}R$. Let $A \cap B \subseteq S^{-1}I$, where A and B are ideals of $S^{-1}R$. Then $A^c \cap B^c = (A \cap B)^c \subseteq (S^{-1}I)^c$. Now since $I \in C$, $(S^{-1}I)^c = I$. So $A^c \cap B^c \subseteq I$. Consequently, $(A^c)^2 \subseteq I$ or $(B^c)^2 \subseteq I$. Thus, $A^2 = S^{-1}((A^c)^2) \subseteq S^{-1}I$ or $B^2 = S^{-1}((B^c)^2) \subseteq S^{-1}I$. Therefore, $S^{-1}I$ is a semi-strongly irreducible ideal of $S^{-1}R$. □

Let R and T be two rings, let J be an ideal of T and let $f : R \to T$ be a ring homomorphism. According to [10], the following ring construction called the amalgamation of R with T along J with respect to f is a subring of $R \times T$ defined by

$$R \bowtie^f J := \{(r, f(r) + j) | r \in R \,, j \in J\} .$$

This construction generalizes amalgamated duplication of a ring along an ideal that introduced and studied by D'Anna and Fontana in [9], which is the subring of $R \times R$ given by

$$R \bowtie I := \{(r, r + i) | r \in R \,, i \in I\} .$$

Our next results establish the transfer of semi-strongly irreducible ideals in amalgamation of rings.

Theorem 2.4. *Let R and T be two rings and $f : R \to T$ be a ring homomorphism. For an ideal I of R and an ideal J of T, the ideal $I \bowtie^f J$ is a semi-strongly irreducible ideal of $R \bowtie^f J$ if and only if I is a semi-strongly irreducible ideal of R.*

Proof. Assume that $I \bowtie^f J$ is a semi-strongly irreducible ideal of $R \bowtie^f J$. Let K and L be two ideals of R satisfy $K \cap L \subseteq I$. Thus, $(K \bowtie^f J) \cap (L \bowtie^f J) \subseteq I \bowtie^f J$. By our assumption, we deduce that either $(K \bowtie^f J)^2 \subseteq I \bowtie^f J$ or $(L \bowtie^f J)^2 \subseteq I \bowtie^f J$ and so either $K^2 \subseteq I$ or $L^2 \subseteq I$. This means that I is a semi-strongly irreducible ideal of R. Conversely, assume that I is a semi-strongly irreducible ideal of R. Let H be an ideal of $R \bowtie^f J$ and set $I_H = \{a \in R | (a, f(a) + j) \in H \text{ for some } j \in J\}$. Let $H_1 \cap H_2 \subseteq I \bowtie^f J$. Obviously, $I_{H_1} \cap I_{H_2} \subseteq I$. By our assumption, we infer that either $I_{H_1}^2 \subseteq I$ or $I_{H_2}^2 \subseteq I$. and hence we conclude that either $H_1^2 \subseteq I \bowtie^f J$ or $H_2^2 \subseteq I \bowtie^f J$, as desired. □

Theorem 2.5. *Let R be a ring in which 2 is invertible. The following statements are equivalent for an ideal I:*

(1) *I is a semi-strongly irreducible ideal.*
(2) *For all $x, y \in R$, $Rx \cap Ry \subseteq I$ implies that either $x^2 \in I$ or $y^2 \in I$.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (1) Assume $J \cap K \subseteq I$ for some ideals J, K of R and $J^2 \nsubseteq I$. Thus, there exists $z = \sum_{i=1}^{n} x_i y_i \in J^2 \setminus I$ where $x_i, y_i \in J$ for $1 \leq i \leq n$. From this, there exist $x, y \in J$ such that $xy \notin I$. Since $4xy = (x+y)^2 - (x-y)^2$, we infer that either $(x+y)^2 \notin I$ or $(x-y)^2 \notin I$. Without loss of generality, we may assume that $(x+y)^2 \notin I$. From $J \cap K \subseteq I$, we have $R(x+y) \cap Rk \subseteq I$ and so $k^2 \in I$ for each $k \in K$. This implies that $k_1 k_2 = 2^{-1}((k_1+k_2)^2 - k_1^2 - k_2^2) \in I$ for $k_1, k_2 \in K$. This means that $K^2 \subseteq I$, as desired. □

Remark 2.6. Following [6], an ideal I of a ring R is called *2-prime* if whenever $a, b \in R$ and $ab \in I$, then either $a^2 \in I$ or $b^2 \in I$. Let S be a ring in which 2 is invertible. Theorem 2.5 shows that every 2-prime ideal of S is semi-strongly irreducible.

Lemma 2.7. *Let R be a ring. The following statement are equivalent:*

(1) *Every ideal of R is a semi-strongly irreducible ideal.*
(2) *For every pair of ideals I and J of R, we have either $J^2 \subseteq I$ or $I^2 \subseteq J$.*

Proof. (1) $\Rightarrow$ (2) Let I and J be two ideals of R. Assume (1). The ideal $I \cap J$ is a 2-strongly irreducible ideal. From $I \cap J \subseteq I \cap J$, we deduce that either $I^2 \subseteq I \cap J$ or $J^2 \subseteq I \cap J$. Hence, we infer that either $I^2 \subseteq J$ or $J^2 \subseteq I$, as desired.
(2) $\Rightarrow$ (1) Let I be an ideal of R. Assume that $J \cap K \subseteq I$ where J and K are two ideals of R. Assume (2). We have either $J^2 \subseteq K$ or $K^2 \subseteq J$. Hence, we have $J^2 \subseteq I$ or $K^2 \subseteq I$, as desired. □

To state the next corollary, we will need the following lemma.

Lemma 2.8. *Let R be a ring where 2 is invertible. The following statement are equivalent:*

(1) *Every ideal of R is a semi-strongly irreducible ideal.*
(2) *For every pair of elements x and y of R, we have either $x|y^2$ or $y|x^2$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in R$. Assume (1). The ideal $Rx \cap Ry$ is a semi-strongly irreducible ideal. From $Rx \cap Ry \subseteq Rx \cap Ry$, we deduce that either $x^2 \subseteq Rx \cap Ry$ or $y^2 \subseteq Rx \cap Ry$. Hence, we infer that either $x^2 \in Ry$ or $y^2 \in Rx$. Thus, we have either $y|x^2$ or $x|y^2$, as desired.
(2) $\Rightarrow$ (1) Let I be an ideal of R. Assume that $Rx \cap Ry \subseteq I$ for $x, y \in R$. Assume (2). We have either $Ry^2 \subseteq Rx$ or $Rx^2 \subseteq Ry$. Hence, we have $Ry^2 \subseteq I$ or $Rx^2 \subseteq I$. Theorem 2.5, completes the proof. □

Badawi [5, Theorem 1] proved that the prime ideals of a ring R are linearly ordered if and only if for every pair of elements x and y of R, there is is an

$n \geq 1$ such that $x|y^n$ or $y|x^n$. In view of Lemma 2.8, we make the following observation.

Corollary 2.9. *If every ideal of a ring R is semi-strongly irreducible, then the prime ideals of R are linearly ordered.*

The concept of weakly irreducible ideal, which is a generalization of strongly irreducible ideal, was introduced and investigated by Samiei and Fazaeli Moghimi [22]. They defined a nonzero proper ideal I of R to be a *weakly irreducible* ideal of R, if for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$. It is easy to check that every semi-strongly irreducible ideal is weakly irreducible.

In view of [5, Theorem 1] and [22, Theorem 3.5], we have the following.

Corollary 2.10. *The following statement are equivalent:*

(1) *Every ideal of R is weakly irreducible.*
(2) *For $x, y \in R$, there is an $n \geq 1$ such that either $x|y^n$ or $y|x^n$.*

We close this section with a result about pm-rings. Let us recall that a ring R is called a *pm-ring* (also known as *Gelfand ring*) if every prime ideal is contained in a unique maximal ideal. Examples of pm-rings include von Neumann regular rings and rings of continuous functions.

Corollary 2.11. *The following statements are equivalent for a reduced ring R:*

(1) *R is a pm-ring.*
(2) *Every weakly irreducible ideal is contained in a unique maximal ideal.*
(3) *Every semi-strongly irreducible ideal is contained in a unique maximal ideal.*
(4) *Every strongly irreducible ideal is contained in a unique maximal ideal.*

Proof. (1) ⇒ (2) First, let us recall a fact. A ring R is a pm-ring if and only if for each pair of distinct maximal ideals M_1 and M_2 there exist $a \notin M_1$, $b \notin M_2$ such that $ab = 0$. Let I be a weakly irreducible ideal. Assume that $I \subseteq M_1$ and $I \subseteq M_2$ where M_1, M_2 are two maximal ideals of R. From this, there exist $x \notin M_1$, $y \notin M_2$ such that $xy = 0$. Since R is reduced, we have $Rx \cap Ry = 0$. Hence, we conclude that either $Rx \subseteq \sqrt{I} \subseteq M_1$ or $Ry \subseteq \sqrt{I} \subseteq M_2$. That is a contradiction.
(2) ⇒ (3) ⇒ (4) Clear.
(4) ⇒ (1) It follows from the fact that every prime ideal is strongly irreducible. □

3. Applications to $C(X)$

In this section, we concern ourselves with rings of real-valued continuous functions on a topological space. Throughout, topological spaces are assumed to be Tychonoff, that is, completely regular Hausdorff, while $C(X)$ will denote the ring of real-valued continuous functions on a space X. The notation, terminology and results of the Gillman-Jerison text [16] will be used always. The reader is referred to [20] and [14] for more details regarding $C(X)$ and its subrings.

Following [17], an ideal I of a ring R is called *pseudoprime* if for $a, b \in R$, $ab = 0$ implies, $a \in I$ or $b \in I$. Trivially, all prime ideals are pseudoprime. Any ideal containing a pseudoprime ideal is pseudoprime. In particular, any ideal containing a prime ideal is pseudoprime. The converse of this fact is also true for $C(X)$, see [17, Theorem 4.1]. Note that the ideal $6\mathbb{Z}$ of the ring of integers $\mathbb{Z}$ is a pseudoprime but not prime.

Lemma 3.1. *Every semi-strongly irreducible ideal of $C(X)$ is pseudoprime.*

Proof. Let I be a semi-strongly irreducible ideal of $C(X)$. Suppose that $fg = 0$ for $f, g \in C(X)$. Hence, $f^{\frac{1}{3}} g^{\frac{1}{3}} = 0$. It is easy to see $(f^{\frac{1}{3}}) \cap (g^{\frac{1}{3}}) = 0$. Thus, we infer that $0 = (f^{\frac{1}{3}}) \cap (g^{\frac{1}{3}}) \subseteq I$. Since I is semi-strongly irreducible, we conclude that either $f^{\frac{2}{3}} \in I$ or $g^{\frac{2}{3}} \in I$. Thus, either $f \in I$ or $g \in I$, as desired. □

Remark 3.2. Lemma 2.8 implies that $C(X)$ always contains an ideal which is not a semi-strongly irreducible ideal, unless $C(X) = \mathbb{R}$. For this, let $|X| > 1$ and take $x, y \in X$. Define $f \in C(X)$ such that $f(x) = 1$ and $f(y) = -1$. Now consider two elements $f + |f|$ and $f - |f|$. Clearly, neither $(f - |f|)\big|(f + |f|)^2$ nor $(f + |f|)\big|(f - |f|)^2$. In fact, if $(f - |f|)^2 = (f + |f|)h$ for some $h \in C(X)$, then $(f - |f|)^3 = 0$ implies $f = |f|$, a contradiction. Now Lemma 2.8 states that $C(X)$ has an ideal which is not semi-strongly irreducible. In particular the zero ideal of $C(X)$, where $|X| > 1$, is not semi-strongly irreducible, since $(f - |f|)(f + |f|) = 0$.

Proposition 3.3. *The following statements are equivalent:*

(1) *Every ideal of $C(X)$ is semi-strongly irreducible.*
(2) *Every ideal of $C(X)$ is weakly irreducible.*
(3) $|X| = 1$.
(4) $C(X) = \mathbb{R}$.

Proof. Clearly (3) and (4) are equivalent and (4) implies (1) and (2), because $C(X) = \mathbb{R}$ is a field. It is enough to show that (1) implies (3) and also (2) implies (3). Suppose on the countrary, that $|X| > 1$. Then using the Remark 3.2, the zero ideal is not semi-strongly irreducible, a contradiction. This shows (1) implies (3). Again if assume that $|X| > 1$, then applying the function f as in the Remark 3.2, we have $(f - |f|) \cap (f + |f|) \subseteq (0)$, but neither $(f - |f|)^n = 0$ nor $(f + |f|)^n = 0$ for all $n \geq 1$, since $f(x) + |f(x)| = 2$ and $f(y) - |f(y)| = -2$. Thus (2) implies (3) and we are done. □

Remark 3.4. Following [12], a proper ideal I of R is called *quasi-primary* if $\sqrt{I}$ is prime. As it mentioned in [22], every quasi-primary ideal is weakly irreducible. By [17, Theorem 4.1], an ideal I of $C(X)$ is pseudoprime if and only if it is quasi-primary. From this, every pseudoprime ideal of $C(X)$ is weakly irreducible.

Our next goal is to characterize spaces X for which every pseudoprime ideal of $C(X)$ is semi-strongly irreducible. To begin our investigations in this direction, we recall a definition from [19] and make a definition. A space X is an *SV-space* if for every prime ideal P of the ring $C(X)$, the ordered integral domain $C(X)/P$ is a valuation ring (i.e., of any two nonzero elements of $C(X)/P$, one divides the other).

Definition 3.5. We say an integral domain R is *semi-valuation* if every pair of ideals I and J of R, we have either $J^2 \subseteq I$ or $I^2 \subseteq J$. A space X is a *semi-SV-space* if for every prime ideal P of the ring $C(X)$, $C(X)/P$ is a semi-valuation ring.

Hereafter we assume that $C(X)$ satisfies the property: $(J \cap K) + P = (J + P) \cap (K + P)$ for two ideals J, K and a prime ideal P. The next theorem is the counterpart of [15, Proposition 4.6] and [2, Proposition 4.14].

Theorem 3.6. *Let X be a topological space. The following statements are equivalent:*

(1) *X is a semi-SV-space.*
(2) *Every pseudoprime ideal of $C(X)$ is semi-strongly irreducible.*

Proof. (1) ⇒ (2) Let X be a semi-SV-space and let I be a pseudoprime ideal of $C(X)$. Assume that $J \cap K \subseteq I$ for ideals J and K of $C(X)$. By [17, Theorem 4.1], there is a prime ideal P where $P \subseteq I$. Clearly, $(J + P)/P \cap (K + P)/P \subseteq I/P$. By hypothesis, $C(X)/P$ is a semi-valuation ring and so either $(K + P)^2/P \subseteq (J + P)/P$ or $(J + P)^2/P \subseteq (K + P)/P$. Without loss of generality, we may assume that $(K + P)^2/P \subseteq (J + P)/P$. This yields $(K + P)^2/P \subseteq I/P$ and so $(K + P)^2 \subseteq I$. Since $P^2 = P \subseteq I$, we infer that $K^2 \subseteq I$, as desired.
(2) ⇒ (1) Let P be a prime ideal of $C(X)$. Suppose that $P \subseteq I$ and $P \subseteq J$ are two ideals of $C(X)$. By [17, Theorem 4.1], we infer that $I \cap J$ is a pseudoprime ideal. By hypothesis, $I \cap J$ is semi-strongly irreducible. This yields either $I^2 \subseteq I \cap J$ or $J^2 \subseteq I \cap J$. Hence, we have $P = P^2 \subseteq I^2 \subseteq J$ or $P = P^2 \subseteq J^2 \subseteq I$. This means that $C(X)/P$ is a semi-valuation ring. □

Obviously, every SV-space is a semi-SV-space. We do not know whether there is a semi-SV-space that is not an SV-space. In this direction, we make the following.

Corollary 3.7. *Let X be a topological space. The following statements are equivalent:*

(1) *X is an SV-space.*
(2) *X is a semi-SV-space such that every semi-strongly irreducible ideal of $C(X)$ is strongly irreducible.*
(3) *X is a semi-SV-space such that every semi-strongly irreducible ideal of $C(X)$ is 2-prime.*

Proof. (1) ⇒ (2) Assume (1). It is clear that X is a semi-SV-space. In view of [15, Proposition 4.6], we deduce that every pseudoprime ideal of $C(X)$ is

strongly irreducible. By Lemma 3.1, we infer that every semi-strongly irreducible of $C(X)$ is strongly irreducible, as desired.
(2) $\Rightarrow$ (1) Assume (2). Theorem 3.6 yields every pseudoprime ideal of $C(X)$ is semi-strongly irreducible. Thus, every pseudoprime ideal of $C(X)$ is strongly irreducible. Using [15, Proposition 4.6], we conclude that X is an SV-space.
(1) $\Rightarrow$ (3) It suffices to show that every semi-strongly irreducible ideal of $C(X)$ is 2-prime. In [1, Theorem 5.7(2)], it is shown that a space X is an SV-space if and only if every pseudoprime ideal of $C(X)$ is 2-prime. Assume (1). With the help of Lemma 3.1 and [1, Theorem 5.7(2)], we deduce that every semi-strongly irreducible ideal of $C(X)$ is 2-prime, as desired.
(3) $\Rightarrow$ (1) Assume X is a semi-SV-space. By Theorem 3.6, we conclude that every pseudoprime ideal is 2-prime. Using [1, Theorem 5.7(2)], we deduce that X is an SV-space. □

A ring R is a *Bézout ring* if every finitely generated ideal is principal. A subspace S of X is called C^**-embedded* in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$, where $C^*(X)$ is the subring of $C(X)$ consisting of all members of $C(X)$. A space X is called an F*-space* if every cozero-set in X is C^*-embedded. It is known that $C(X)$ is a Bézout ring if and only if X is an F-space, see [16] and [3] for more details. It is known that every F-space is an SV-space but not conversely.

In the next result for $p \in \beta X$, O^p is the set $\{f \in C(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, where βX is the Stone-Čech compactification of X and $Z(f) = \{x \in X : f(x) = 0\}$, which is called the *zero-set* of f. In fact, O^p is a z-ideal (an ideal I in $C(X)$ is called a z*-ideal* if $f \in C(X)$ and $Z(f) = Z(g)$ for some $g \in I$, then $f \in I$) in $C(X)$, see [16, 2.9 and 7.12].

Corollary 3.8. *Let X be a topological space. The following statements are equivalent:*

1. *X is an F-space.*
2. *Every ideal in $C(X)$ is an intersection of semi-strongly irreducible ideals.*
3. *Every irreducible ideal in $C(X)$ is a semi-strongly irreducible ideal.*
4. *O^p is semi-strongly irreducible for each $p \in \beta X$.*

Proof. (1) $\Rightarrow$ (2) Assume (1). By [17, Theorem 6.2], every ideal in $C(X)$ is an intersection of pseudoprime ideals. Since every F-space is a semi-SV-space, by Theorem 3.6, we infer that, every ideal in $C(X)$ is an intersection of semi-strongly irreducible ideals.
(2) $\Rightarrow$ (1) Assume (2). Lemma 3.1 yields every ideal in $C(X)$ is an intersection of pseudoprime ideals. Theorem 6.2 in [17] completes the proof.
(2) $\Rightarrow$ (3) It follows from the fact that an irreducible ideal is not the intersection of two ideals that properly contain it.
(3) $\Rightarrow$ (1) Assume (3). By Lemma 3.1, we deduce that every irreducible ideal of $C(X)$ is pseudoprime. Using [15, Proposition 4.8], we infer that X is an F-space.

(1) $\Rightarrow$ (4) If X is an F-space, then each O^p is prime by Theorem 14.25 in [16], and hence O^p is semi-strongly irreducible.
(4) $\Rightarrow$ (1) By [16, 14.25], X is an F-space if and only if O^p is a prime ideal for each $p \in \beta X$. If each O^p, where $p \in \beta X$ is semi-strongly irreducible, then O^p is pseudoprime by Lemma 3.1, and hence each O^p is prime by Theorem 2.9 in [16], and hence X is an F-space. □

Recall that a space X is said to be a *P-space*, if every zero-set of X is open. It is known that $C(X)$ is a von Neumann regular ring if and only if X is a P-space, see [16, 4J and 14.29] for more details.

Corollary 3.9. *Let X be a topological space. The following statements are equivalent:*

(1) *X is a P-space.*
(2) *X is an F-space and every semi-strongly irreducible ideal of $C(X)$ is semiprime.*
(3) *X is an SV-space and every semi-strongly irreducible ideal of $C(X)$ is semiprime.*
(4) *X is a semi-SV-space and every semi-strongly irreducible ideal of $C(X)$ is semiprime.*

Proof. (1) $\Rightarrow$ (2) First, we note that every P-space is an F-space. The result follows from the fact that a commutative ring R is von Neumann regular if and only if every ideal of R is semiprime, see [21, Ex. 10.19].
The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear.
(4) $\Rightarrow$ (1) From Theorem 3.6, we have every pseudoprime ideal of $C(X)$ is semi-strongly irreducible. From (4), we also deduce that every pseudoprime ideal of $C(X)$ is semiprime. We note that an ideal I of $C(X)$ is pseudoprime if and only if $\sqrt{I}$ is prime, see [17, Theorem 4.1]. From this, we conclude that every pseudoprime ideal of $C(X)$ is prime. By [15, Lemma 3.29], we deduce that X is a P-space. □

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