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An interpolative class of two-Lipschitz mappings of composition type

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Abstract

The paper deals with some further results concerning the class of two-Lipschitz operators. We prove first an isometric isomorphism identification of two-Lipschitz operators and Lipschitz operators. After defining and characterizing the adjoint of a two-Lipschitz operator, we prove a Schauder type theorem on the compactness of the adjoint. We study the extension of two-Lipschitz operators from the cartesian product of two complemented subspaces of a Banach space to the cartesian product of whole spaces. Also, we show that every two-Lipschitz functional defined on the cartesian product of two pointed metric spaces admits an extension with the same two-Lipschitz norm under some requirements on domaine spaces.

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1. Introduction

Inspired by Farmer and Johnson's research [17], which generalizes the notion of p-summing linear operators within the Lipschitz context, numerous scholars have introduced diverse formulations of Lipschitz operators between pointed metric spaces and Banach spaces. In a certain context, these notions extend various types of ideals of linear operators between Banach spaces (see [2], [3],

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[4], [8], [11], [15], and the references therein). On the other hand in 2009, Dubei et al., as cited in reference [16], presented the concept of two-Lipschitz mappings. These mappings are defined on the Cartesian product of two metric spaces, with the range being a Banach space that is Lipschitz separately in each variable. In specific challenging conditions, they demonstrate the bi-linearization theorem. Following that, in [25] Sánchez Pérez introduced a definition of real-valued two-Lipschitz mappings (referred to as Lipschitz bi-forms) that functioned effectively without constraints, which admits a good continuous bi-linearization between Banach spaces. It should be mentioned that the previous references did not explore the concept of ideals within this framework. From this point of view, the first author et al. in [18] introduced and studied in depth the ideals of two-Lipschitz operators between pointed metric spaces and Banach spaces. Further findings related to this subject are available in the recent publications [1] and [14].

The aim of this paper is to study the two-Lipschitz version of the classes of strongly p-summing and strongly (p, σ)-continuous linear operators, which were studied in detail in papers [6], [8], [12], [19] and [20]. After adding and demonstrating new findings concerning the concept of strongly two-Lipschitz p-summing operators, as introduced in reference [18], we proceed to construct the new ideal of two-Lipschitz operators derived from the class of strongly (p, σ)-continuous linear operators, using the composition method detailed in reference [18].

The paper is divided in four sections. After the introductory one, in Section 2 we establish terminologies and reviews the key results concerning linear, bilinear, Lipschitz, and two-Lipschitz operators. In Section 3 we present a characterization of strongly two-Lipschitz p-summing operators by integral domination and we provide a characterization using transpose operators. Finally, Section 4 is devoted to the study of the ideal of strongly (p, σ)-two-Lipschitz operators, constructed via the composition method from strongly (p, σ)-continuous linear operators. We offer an equivalent version of the Pietsch domination theorem for these mappings. In the ends of this section, following the idea of [6, Theorem 6.2] we prove the factorization theorem for these mappings. It is worth mentioning that typically, the domination and factorization theorem is proven by using full general theorems, such as those found in [22] or in [5], here, in the present paper, we have preferred to use direct proofs or employ the linearization or bi-linearization theorem.

2. Preliminary and terminologies

Throughout this paper, X, Y denoted pointed metric spaces with a distinguished point denoted by 0 and a metric that will be denoted by d. Also, E, E_1, E_2, G and F will be Banach spaces over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A Banach space E will be considered as a pointed metric space with distinguished point 0 and distance $d(x, y) = \|x - y\|$. The closed unit ball of E is denoted by B_E and the topological dual of E by E^*. The sets $\mathcal{L}(E, F)$, $\mathcal{L}^2(E_1, E_2; F)$

and $Lip_0(X,E)$ denoted the Banach spaces of all bounded linear operators, bilinear operators and Lipschitz operators respectively, endowed with the usual operator norm and the Lipschitz norm. We write $Lip_0(X,\mathbb{K}) = X^{\#}$ and we say $X^{\#}$ is the Lipschitz dual of X. It is evident that $\mathcal{L}(E,F)$ is a subspace of $Lip_0(E,F)$, and specifically, E^* is a subspace of $E^{\#}$.

We reserve the symbol $E_1 \otimes E_2$ for the 2-fold tensor product of E_1, E_2, and $E_1 \widehat{\otimes}_\pi E_2$ for their completed projective tensor product of E_1 and E_2 (see [24]). If $T \in \mathcal{L}^2(E_1, E_2; F)$, then its linearization $T_L : E_1 \widehat{\otimes}_\pi E_2 \longrightarrow F$ is the unique operator defined by $T_L(x \otimes y) = T(x,y)$ for all $x \in E_1$ and $y \in E_2$.

A molecule on X is a real-valued function m with a finite support that satisfies

$$\sum_{x \in X} m\left(x\right) = 0.$$

We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For each $m = \sum_{i=1}^{n} \alpha_i m_{x_i x_i'}$, we define the norm

$$\|m\|_{\mathcal{M}(X)} = \inf \sum_{i=1}^{n} |\alpha_i| \, d\left(x_i, x_i'\right),$$

where α_i are scalars and the infimum is taken over all representations of the molecule m. The completion of the normed space $\left(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)}\right)$ is denoted by Æ$\left(X\right)$. It is well known that the space Æ$\left(X\right)$ is the predual of $X^{\#}$, i.e., Æ$(X)^*$ and $X^{\#}$ are isometrically isomorphic (see [9]).

The maps $\delta_X : X \longrightarrow$ Æ$\left(X\right)$ is the isometric embedding from X into Æ$\left(X\right)$ defined by $\delta_X\left(x\right) = m_{x0}$. Therefor, for every $T \in Lip_0(X,E)$, there exists a unique continuous linear map $T_L :$ Æ$\left(X\right) \longrightarrow E$ such that $T = T_L \circ \delta_X$ and $\|T_L\| = Lip\left(T\right)$, (see [26, Theorem 3.6]).

We will denote by $BLip_0(X,Y;E)$ the space of all two-Lipschitz operators $T : X \times Y \longrightarrow E$ in a manner that ensures the presence of a constant $C > 0$ such that

$$\|T\left(x,y\right) - T\left(x,y'\right) - T\left(x',y\right) + T\left(x',y'\right)\| \leq C d(x,x') d(y,y'),$$

and $T\left(x,0\right) = T\left(0,y\right) = 0$ for all $x, x' \in X$ and $y, y' \in Y$.

Note that $BLip_0(X,Y;E)$ is a Banach space under the norm $BLip(\cdot)$ established by the infimum of all constants $C > 0$ that fulfills the above inequality. This norm is also equal to

$$BLip\left(T\right) = \sup_{x \neq x', y \neq y'} \frac{\|T\left(x,y\right) - T\left(x,y'\right) - T\left(x',y\right) + T\left(x',y'\right)\|}{d(x,x')d(y,y')},$$

(see [18, Theorem 2.5]). For more details we refer to [1], [14], [16], [18] and [25].

Recall that with each $T \in BLip_0(X,Y;E)$, there exists a unique bilinear map $T_B :$ Æ$(X) \times$ Æ$(Y) \longrightarrow E$ such that $T = T_B \circ (\delta_X, \delta_Y)$, (see [18, Theorem 2.6]).

Since, T_B has a unique linearization referred T_L we have $T = T_B \circ (\delta_X, \delta_Y) = T_L \circ \sigma_2 \circ (\delta_X, \delta_Y)$, and $BLip\,(T) = \|T_B\| = \|T_L\|$, where $\sigma_2 : Æ(X) \times Æ(Y) \longrightarrow Æ(X) \widehat{\otimes}_\pi Æ(Y)$ is the canonical bilinear operator defined by $\sigma_2(m_{x0}, m_{y0}) = m_{x0} \otimes m_{y0}$, (see [18, Remark 2.7]).

An ideal of two-Lipschitz mappings, denoted by $\mathcal{I}_{BLip}$, is a subclass of the class $BLip_0$ such that for every pointed metric spaces X, Y, and every Banach space E, the components

$$\mathcal{I}_{BLip}(X, Y; E) := BLip_0(X, Y; E) \cap \mathcal{I}_{BLip},$$

form a vector subspace of $BLip_0(X, Y; E)$ that is invariant under the composition of a linear operator on the right and two-Lipschitz operator on the left, and it contains the two-Lipschitz operator of finite type.

Let $n \in \mathbb{N}^*$ and $1 \leq p < \infty$, we write p^* for the extended real number that satisfies $\frac{1}{p} + \frac{1}{p^*} = 1$. We denote by $\ell_p^n(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in E with the norm

$$\left\|(e_i)_{i=1}^n\right\|_p = \left(\sum_{i=1}^n \|e_i\|^p\right)^{\frac{1}{p}},$$

and by $\ell_{p,\omega}^n(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in E with the norm

$$\left\|(e_i)_{i=1}^n\right\|_{p,\omega} = \sup_{\varphi \in B_{E^*}} \left(\sum_{i=1}^n |\langle e_i, \varphi\rangle|^p\right)^{\frac{1}{p}}.$$

Now, we recall some definitions for many concepts used in this paper.
The notion of strongly (p, σ)-continuous linear and multi-linear operators was introduced by Achour et al. in [6]. Given $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$.

Definition 2.1 ([6, Definition 3.1])**.** A mapping $R \in \mathcal{L}(E, F)$ is strongly (p, σ)-continuous if there is a Banach space G, a p^*-summing operator $u \in \Pi_{p^*}(F^*, G)$ and a constant $C > 0$ such that for all $x \in E$ and $e^* \in F^*$ we have

$$|\langle R(x), e^*\rangle| \leq C \|x\| \|e^*\|^\sigma \|u(e^*)\|^{1-\sigma}. \tag{2.1}$$

We denote by $\mathcal{D}_p^\sigma(E, F)$ the class of all strongly (p, σ)-continuous operators endowed with the norm defined by

$$d_p^\sigma(R) = \inf\left\{C \pi_{p^*}(u)^{1-\sigma}\right\},$$

where the infimum is taken over all $C > 0$ and $u \in \Pi_{p^*}(F^*, G)$ such that (2.1) is investigated. This class is a Banach operator ideal (see [23, Section 4.4]). Note that if $\sigma = 0$, we obtain the ideal $\mathcal{D}_p$ of strongly p-summing operators was introduced by Cohen in [12].

The notion of strongly (p, σ)-Lipschitz operators for $1 < p < \infty$ and $0 \leq \sigma < 1$ between pointed metric space X and Banach space E was introduced by Bougoutaia et al. in [10].

Definition 2.2. Let $1 < p, q < \infty$ and $0 \leqslant \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$. A Lipschitz mapping $T : X \longrightarrow E$ is strongly (p, σ)-Lipschitz if there is a Banach space G, a p^*-summing operator $u \in \Pi_{p^*}(E^*, G)$ and a constant $C > 0$ such that for all $x, y \in X$ and $e^* \in E^*$ we have

$$|\langle T(x) - T(y), e^* \rangle| \leq C d(x, y) \|e^*\|^{\sigma} \|u(e^*)\|^{1-\sigma}. \tag{2.2}$$

We denote by $\mathcal{D}^L_{p,\sigma}(X, E)$ the class of all strongly (p, σ)-Lipschitz operators and by $d^L_{p,\sigma}(T)$ the strongly (p, σ)-Lipschitz norm which is defined by $d^L_{p,\sigma}(T) = \inf C \pi_{p^*}(u)^{1-\sigma}$, where the infimum is taken over all constants C and $u \in \Pi_{p^*}(E^*, G)$ such that (2.2) is holds.

Following the concept introduced by Achour et al. in [6] for multilinear operators, if $1 < p, q < \infty$ and $0 \leqslant \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$. A bilinear mapping $T : E_1 \times E_2 \longrightarrow F$ is strongly (p, σ)-continuous if there is a constant $C > 0$ such that for all $(x_i)_{i=1}^n \subset E_1$, $(y_i)_{i=1}^n \subset E_2$ and $(e_i^*)_{i=1}^n \subset F^*$,

$$\sum_{i=1}^{n} |\langle T(x_i, y_i), e_i^* \rangle| \leq C \left(\sum_{i=1}^{n} \|x_i\|^q \|y_i\|^q \right)^{\frac{1}{q}} \sup_{\varphi \in B_{E^{**}}} \left(\sum_{i=1}^{n} \left(|\langle e_i^*, \varphi \rangle|^{1-\sigma} \|e_i^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}}.$$

In this case, we define the bilinear strongly (p, σ)-continuous norm of T by $d^2_{p,\sigma}(T) = \inf C$. The class $\mathcal{D}^2_{p,\sigma}$ of strongly (p, σ)-continuous bilinear operators is a Banach bilinear ideal with its norm $d^2_{p,\sigma}(\cdot)$. For $\sigma = 0$, we have $\mathcal{D}^2_{p,0} = \mathcal{D}^2_p$ the class of Cohen strongly p-summing bilinear operators (see [7]).

We point out that there is a Pietsch's Domination Theorem which states that T is strongly (p, σ)-continuous bilinear operator.

Theorem 2.3 ([6, Theorem 4.3])**.** *A bilinear operator $T : E_1 \times E_2 \longrightarrow F$ is strongly (p, σ)-continuous if and only if there is a constant $C > 0$ and a regular Borel probability measure ν on $B_{F^{**}}$ (with the weak star topology) such that for every $x \in E_1, y \in E_2$ and $e^* \in F^*$, the inequality*

$$|\langle T(x, y), e^* \rangle| \leq C \|x\| \|y\| \left(\int_{B_{E^{**}}} \left(|\langle e^*, \varphi \rangle|^{1-\sigma} \|e^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi) \right)^{\frac{1-\sigma}{p^*}},$$

holds.

3. Further results on strongly two-Lipschitz p-summing operators

The first author et al. in [18] characterized the class of strongly two-Lipschitz p-summing operators which is constructed by the composition method starting from $(\mathcal{D}_p, \| \cdot \|)$.

In this section we give father characterizations and results for strongly two-Lipschitz p-summing operators.

Definition 3.1. For $1 < p \leq \infty$. A two-Lipschitz operator $T : X \times Y \longrightarrow E$ is strongly two-Lipschitz p-summing if there exist a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that for all $x, x' \in X$, $y, y' \in Y$ and $e^* \in E^*$ we have

$$|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \leq d(x,x')d(y,y') \|S(e^*)\| . \tag{3.1}$$

The set of all strongly two-Lipschitz p-summing operators from $X \times Y$ to E is denoted by $\mathcal{D}_p^{BL}(X,Y;E)$. If $T \in \mathcal{D}_p^{BL}(X,Y;E)$, we set $d_p^{BL}(T) = \inf\{\pi_{p^*}(S)\}$ where the infimum is taken over all Banach spaces G and operators S such that the inequality (3.1) holds.

The following inclusion result is a direct consequence of Definition 3.1. Note that for $1 < p \leqslant q \leqslant \infty$, every q^*-summing operator is p^*-summing operator.

Corollary 3.2. *Let $1 < p \leqslant q \leqslant \infty$ we have $\mathcal{D}_q^{BL}(X,Y;E) \subseteq \mathcal{D}_p^{BL}(X,Y;E)$ and $d_q^{BL}(T) \leqslant d_p^{BL}(T)$ for all $T \in \mathcal{D}_q^{BL}(X,Y;E)$.*

Using the bi-linearization theorem, we can prove the Pietsch domination theorem for the class $\mathcal{D}_p^{BL}$.

Theorem 3.3. *Let $T \in BLip_0(X,Y;E)$. Then the following statements are equivalent.*

1. $T \in \mathcal{D}_p^{BL}(X,Y;E)$.
2. *There exist a constant $C > 0$ and a regular Borel probability measure ν on $B_{E^{**}}$ such that for every $x, x' \in X, y, y' \in Y$ and $e^* \in E^*$ we have*

$$|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \tag{3.2}$$
$$\leq Cd(x,x')\, d(y,y') \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}} .$$

Furthermore, $d_p^{BL}(T) = \inf C$, where the infimum is taken over all constants C either in (3.2).

Proof. Let we assume that $T \in \mathcal{D}_p^{BL}(X,Y;E)$. Proposition 4.11 in [18] implies that $T_B \in \mathcal{D}_p^2(\text{Æ}(X), \text{Æ}(Y); E)$. By Theorem 3.1 in [21] we have $(T_B)^* \in \Pi_{p^*}\left(E^*, \mathcal{L}^2(\text{Æ}(X), \text{Æ}(Y))\right)$ with $\pi_{p^*}\left((T_B)^*\right) = d_p^2(T_B)$, then there exists a probability measure ν on $\mathrm{B}_{E^{**}}$ such that for all $x, x' \in X$ $y, y' \in Y$ and $e^* \in E^*$, we have

$$\begin{aligned}
&|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \\
&\quad = |\langle T_B(m_{x,x'}, m_{y,y'}), e^*\rangle| \\
&\quad \leq \pi_{p^*}\left((T_B)^*\right) \|m_{x,x'}\| \|m_{y,y'}\| \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}} \\
&\quad = \pi_{p^*}\left((T_B)^*\right) d(x,x')\, d(y,y') \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}} .
\end{aligned}$$

Hence, we deduce that (3.2) holds and

$$\inf C \leq \pi_{p^*}\left((T_B)^*\right) = d_p^2(T_B) = d_p^{BL}(T) .$$

Conversely, by referring to [18, Corrolary 4.12], it is sufficient to show that $T_L \in \mathcal{D}_p\big(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\big)$. For $\varepsilon > 0$, $m^1 \in \mathcal{M}(X)$ and $m^2 \in \mathcal{M}(Y)$, we can choose a representation of m^1 and m^2 as $m^1 = \sum_{i=1}^n \alpha_i m_{x_i,x_i'}$ and $m^2 = \sum_{j=1}^r \beta_j m_{y_j,y_j'}$ such that

$$\sum_{i=1}^n |\alpha_i| \, d(x_i, x_i') \leq \left\|m^1\right\|_{\mathcal{M}(X)} + \varepsilon \quad \text{and} \quad \sum_{j=1}^r |\beta_j| \, d(y_i, y_i') \leq \left\|m^2\right\|_{\mathcal{M}(Y)} + \varepsilon.$$

Using the fact that (3.2) holds, we infer that

$$\begin{aligned}
&\left|\left\langle T_L\left(m^1 \otimes m^2\right), e^*\right\rangle\right| \\
&\quad \leq \sum_{i=1}^n |\alpha_i| \sum_{j=1}^r |\beta_j| \left|\left\langle T(x_i, y_j) - T\left(x_i, y_j'\right) - T(x_i', y_j) + T\left(x_i', y_j'\right), e^*\right\rangle\right| \\
&\quad \leq C \sum_{i=1}^n |\alpha_i| \, d(x_i, x_i') \sum_{j=1}^r |\beta_j| \, d\left(y_j, y_j'\right) \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}} \\
&\quad \leq C\big(\left\|m^1\right\|_{\mathcal{M}(X)} + \varepsilon\big)\big(\left\|m^2\right\|_{\mathcal{M}(Y)} + \varepsilon\big) \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\left|\left\langle T_L\left(m^1 \otimes m^2\right), e^*\right\rangle\right| \leq C \left\|m^1 \otimes m^2\right\| \left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1}{p^*}}.$$

Hence, we deduce from [12, Theorem 2.3.1] that $T_L \in \mathcal{D}_p\big(Æ(X)\widehat{\otimes}_\pi Æ(Y), E\big)$ and $d_{p,\sigma}^{BL}(T) \leq \inf C$. ☐

Recently in [14], Dahia defined and characterized the transpose of a two-Lipschitz operator. Let X, Y be pointed metric spaces, E be Banach space and a two-Lipschitz map $T : X \times Y \longrightarrow E$. The transpose of T is the linear operator

$$T^t : E^* \longrightarrow BLip_0(X, Y), \quad e^* \longmapsto T^t(e^*) : X \times Y \longrightarrow \mathbb{K},$$

with $T^t(e^*)(x, y) = e^*\big(T(x, y)\big)$. In addition $\|T^t\| = BLip(T)$.

The following theorem relates strongly two-Lipschitz p-summing operator with its transpose.

Theorem 3.4. *Let $1 < p \leq \infty$ and let $T \in BLip_0(X, Y; E)$. Then T is strongly two-Lipschitz p-summing if and only if T^t is p^*-summing. In this case, $d_p^{BL}(T) = \pi_{p^*}(T^t)$.*

Proof. Suppose that T is strongly two-Lipschitz p-summing. Then by Corollary 4.13 in [18], for all $\varepsilon > 0$ there is a Banach space G, an operator $u \in \mathcal{D}_p(G, E)$ and $R \in BLip(X, Y; G)$ such that $T = u \circ R$ and $\|u\|_{\mathcal{D}_p} BLip(R) \leq \varepsilon + d_p^{BL}(T)$. According to [12, Therem 2.2.2] we have $u^* \in \Pi_{p^*}(E^*, G^*)$. Then, by the ideal property $T^t = R^t \circ u^* \in \Pi_{p^*}\big(E^*, BLip_0(X, Y)\big)$. Moreover,

$$\pi_{p^*}\left(T^t\right) \leq \pi_{p^*}(u^*) \left\|R^t\right\| = \|u\|_{\mathcal{D}_p} BLip(R) \leq \varepsilon + d_p^{BL}(T).$$

Conversely, Suppose that T^t is p^*-summing, then, there exists a regular Borel probability measure ν on $B_{E^{**}}$ such that

$$BLip\left(T^t\left(e^*\right)\right) \leq \pi_{p^*}\left(T^t\right)\left(\int_{B_{E^{**}}} \left|\left\langle e^*, \varphi\right\rangle\right|^{p^*} d\nu\left(\varphi\right)\right)^{\frac{1}{p^*}},$$

for each $e^* \in E^*$. Therefore, for all $x, x' \in X$ and $y, y' \in Y$ we get

$$\begin{aligned}&\left|\left\langle T^t\left(e^*\right)(x,y) - T^t\left(e^*\right)(x,y') - T^t\left(e^*\right)(x',y) + T^t\left(e^*\right)(x',y'), e^*\right\rangle\right| \\ &\leq \pi_{p^*}\left(T^t\right) d\left(x,x'\right) d\left(y,y'\right)\left(\int_{B_{E^{**}}} \left|\left\langle e^*, \varphi\right\rangle\right|^{p^*} d\nu\left(\varphi\right)\right)^{\frac{1}{p^*}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\left\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\right\rangle\right| \\ &\leq \pi_{p^*}\left(T^t\right) d\left(x,x'\right) d\left(y,y'\right)\left(\int_{B_{E^{**}}} \left|\left\langle e^*, \varphi\right\rangle\right|^{p^*} d\nu\left(\varphi\right)\right)^{\frac{1}{p^*}}.\end{aligned}$$

By Theorem 3.3, we deduce that $T \in \mathcal{D}_p^{BL}(X, Y; E)$ and $d_p^{BL}(T) \leq \pi_{p^*}\left(T^t\right)$. □

It is well known that every strongly p-summing operator is weakly compact [12, Corollary 2.2.5]. Therefore, by [18, Corollary 4.4, Remark 4.5], we can provide the following result.

Corollary 3.5. *Let $1 < p \leq \infty$ and $T \in BLip_0(X, Y; E)$. Then, every strongly two-Lipschitz p-summing is two-Lipschitz weakly compact.*

4. Strongly (p, σ)-two-Lipschitz operators

We are about to build a new two-Lipschitz operator ideal using the composition technique, originating from the exiting operator ideal.

Definition 4.1. Let X, Y be pointed metric spaces and E be a Banach space, and let $1 < p, q < \infty$ and $0 \leqslant \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$. A two-Lipschitz mapping $T : X \times Y \longrightarrow E$ is called strongly (p, σ)-two-Lipschitz if there is a constant $C > 0$, a Banach space G and a p^*-summing linear operator $S : E^* \longrightarrow G$ such that

$$\begin{aligned}&\left|\left\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\right\rangle\right| \\ &\leq C d\left(x,x'\right) d\left(y,y'\right)\left\|e^*\right\|^{\sigma}\left\|S\left(e^*\right)\right\|^{1-\sigma},\end{aligned} \tag{4.1}$$

for all $x, x' \in X, y, y' \in Y$ and $e^* \in E^*$. We denote by $\mathcal{D}_{p,\sigma}^{BL}(X, Y; E)$ the linear space of all strongly (p, σ)-two-Lipschitz mappings from $X \times Y$ to E and $d_{p,\sigma}^{BL}(\cdot)$ represents the norm defined as the infimum of all $C\pi_{p^*}(S)^{1-\sigma}$, where C and $S \in \Pi_{p^*}(E^*, G)$ satisfied the inequality (4.1).

Remark 4.2.

(1) For $\sigma = 0$ we have $\mathcal{D}_{p,0}^{BL}(X, Y; E) = \mathcal{D}_p^{BL}(X, Y; E)$, the class of strongly two-Lipschitz p-summing operators.

(2) It is obvious that in the case $T \in Lip_0(X,E)$ we get the well-known concept of strongly (p,σ)-Lipschitz operators.

The subsequent proposition provide validation that the class being study genuinely upon the concept of bilinearity.

Proposition 4.3. *Let X, Y and E be Banach spaces, and let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$. If $T \in \mathcal{L}^2(X,Y;E)$, then $T \in \mathcal{D}^{BL}_{p,\sigma}(X,Y;E)$ if and only if $T \in \mathcal{D}^2_{p,\sigma}(X,Y;E)$. Furthermore, $d^{BL}_{p,\sigma}(T) = d^2_{p,\sigma}(T)$.*

Proof. Suppose that $T \in \mathcal{D}^{BL}_{p,\sigma}(X,Y;E)$, for $\varepsilon > 0$ choose a constant $C > 0$, a Banach space G and a p^*-summing operator $S : E^* \longrightarrow G$ such that (4.1) holds and $C\pi_{p^*}(S)^{1-\sigma} \leq \varepsilon + d^{BL}_{p,\sigma}(T)$. According to [6, Proposition 4.5] it is enough to show that $T_L : X\widehat{\otimes}_\pi Y \longrightarrow E$ is strongly (p,σ)-continuous. Indeed for all $u \in X\widehat{\otimes}_\pi Y$, with representation, $u = \sum_{i=1}^n x_i \otimes y_i$ and $e^* \in E^*$ we have

$$\begin{aligned} |\langle T_L(u), e^*\rangle| &= \sum_{i=1}^n |\langle T(x_i,y_i), e^*\rangle| \\ &\leq C\|e^*\|^\sigma \|S(e^*)\|^{1-\sigma} \sum_{i=1}^n \|x_i\|\|y_i\|. \end{aligned}$$

Taking the infimum over all representation of u, we find that $T_L \in \mathcal{D}^\sigma_p\left(X\widehat{\otimes}_\pi Y, E\right)$ and

$$d^2_{p,\sigma}(T) = d^\sigma_p(T_L) \leq C\pi_{p^*}(S)^{1-\sigma} \leq \varepsilon + d^{BL}_{p,\sigma}(T).$$

In order to establish the reverse, take $T \in \mathcal{D}^2_{p,\sigma}(X,Y;E)$. By Theorem 2.3 there is a constant $C > 0$ and a regular Borel probability measure ν on $B_{E^{**}}$ such that

$$\begin{aligned} &|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \\ &= |\langle T(x-x', y-y'), e^*\rangle| \\ &\leq C\|x-x'\|\|y-y'\| \left(\int_{B_{E^{**}}} \left(|\langle e^*,\varphi\rangle|^{1-\sigma}\|e^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}}. \end{aligned}$$

We follow the steps of the second implication of [18, Theorem 4.10] to achieve what is required. □

Example 4.4. Let $1 < p < \infty, 0 \leq \sigma < 1, f \in X^\#$ and $R : Y \longrightarrow E$ be a strongly (p,σ)-Lipschitz operator. The mapping

$$T : X \times Y \longrightarrow E, \quad T(x,y) = f(x)R(y),$$

is a strongly (p,σ)-two-Lipschitz operator with $d^{BL}_{p,\sigma}(T) \leq Lip(f) d^L_{p,\sigma}(R)$. Indeed, since R is strongly (p,σ)-Lipschitz, there is a Banach space G, an operator $u \in \Pi_{p^*}(E^*, G)$ and $C > 0$ such that (2.1) holds, and for all $\varepsilon > 0$

$$C\pi_{p^*}(u)^{1-\sigma} \leq \varepsilon + d^L_{p,\sigma}(R).$$

For all $x, x' \in X$, $y, y' \in Y$, $e^* \in E^*$ we have

$$\begin{aligned}&|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \\ &= |f(x) - f(x')|\,|\langle R(y) - R(y'), e^*\rangle| \\ &\leq Lip(f)\big(\varepsilon + d^L_{p,\sigma}(R)\big)d(x,x')d(y,y')\,\|e^*\|^{\sigma}\,\|u(e^*)\|^{1-\sigma}.\end{aligned}$$

This implies, $T \in \mathcal{D}^{BL}_{p,\sigma}(X,Y;E)$ and

$$d^{BL}_{p,\sigma}(T) \leq Lip(f)d^L_{p,\sigma}(R).$$

We show in what follows that the two-Lipschitz ideal generated by the composition method from the operator ideal $\mathcal{D}^{\sigma}_p$ coincide with the space of strongly (p,σ)-two-Lipschitz operators.

Proposition 4.5. *Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$ and let $T \in BLip_0(X,Y;E)$. Then T is strongly (p,σ)-two-Lipschitz if and only if its bi-linearization T_B is strongly (p,σ)-continuous.*
Moreover, $d^{BL}_{p,\sigma}(T) = d^2_{p,\sigma}(T_B)$.

Proof. Assume that $T \in \mathcal{D}^{BL}_{p,\sigma}(X,Y;E)$. Then for all $\varepsilon > 0$, the two-Lipschitz operator T satisfying (4.1) such that $C\pi_{p^*}(S)^{1-\sigma} \leq \varepsilon + d^{BL}_{p,\sigma}(T)$. Therefore, for each $m^1 \in \mathcal{M}(X)$, $m^2 \in \mathcal{M}(Y)$ and $\varepsilon > 0$, we can choose a representation of m^1 and m^2 as $m^1 = \sum_{i=1}^n \alpha_i m_{x_i,x_i'}$ and $m^2 = \sum_{j=1}^r \beta_j m_{y_j,y_j'}$ such that

$$\sum_{i=1}^n |\alpha_i|\, d(x_i,x_i') \leq \|m^1\|_{\mathcal{M}(X)} + \varepsilon \text{ and } \sum_{j=1}^r |\beta_j|\, d(y_i,y_i') \leq \|m^2\|_{\mathcal{M}(Y)} + \varepsilon,$$

we have

$$\begin{aligned}&\left|\left\langle T_B\left(m^1,m^2\right), e^*\right\rangle\right| \\ &\quad= \sum_{i=1}^n |\alpha_i| \sum_{j=1}^r |\beta_j| \left|\left\langle T(x_i,y_j) - T\left(x_i,y_j'\right) - T(x_i',y_j) + T\left(x_i',y_j'\right), e^*\right\rangle\right| \\ &\quad\leq C\sum_{i=1}^n |\alpha_i|\, d(x_i,x_i') \sum_{j=1}^r |\beta_j|\, d\left(y_j,y_j'\right) \|e^*\|^{\sigma} \|S(e^*)\|^{1-\sigma} \\ &\quad\leq C\big(\|m^1\|_{\mathcal{M}(X)} + \varepsilon\big)\big(\|m^2\|_{\mathcal{M}(Y)} + \varepsilon\big) \|e^*\|^{\sigma} \|S(e^*)\|^{1-\sigma}.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0$ we have

$$\left|\left\langle T_B\left(m^1,m^2\right), e^*\right\rangle\right| \leq C\|m^1\|_{\mathcal{M}(X)} \|m^2\|_{\mathcal{M}(Y)} \|e^*\|^{\sigma} \|S(e^*)\|^{1-\sigma}.$$

The domination theorem for the p^*-summing operator S, provides a regular Borel probability measure ν on $B_{E^{**}}$ such that

$$\|S(e^*)\|^{1-\sigma} \leq \pi_{p^*}(S)^{1-\sigma}\left(\int_{B_{E^{**}}} |\langle e^*, \varphi\rangle|^{p^*} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}}. \tag{4.2}$$

Therefore, from Theorem 2.3, T_B being strongly (p,σ)-continuous and

$$d^2_{p,\sigma}(T_B) \leq C\pi_{p^*}(S)^{1-\sigma} \leq \varepsilon + d^{BL}_{p,\sigma}(T).$$

Conversely, let $T_B \in \mathcal{D}^2_{p,\sigma}(Æ(X), Æ(Y); E)$ and let $x, x' \in X, y, y' \in Y$ and $e^* \in E^*$. By Theorem 2.3, there is a regular Borel probability measure ν on $B_{E^{**}}$ such that

$$\begin{aligned}
&|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \\
&= |\langle T_B(m_{x,x'}, m_{y,y'}), e^*\rangle| \\
&\leq d^2_{p,\sigma}(T_B) \|m_{x,x'}\|_{\mathcal{M}(X)} \|m_{y,y'}\|_{\mathcal{M}(Y)} \left(\int_{B_{E^{**}}} \left(|\langle e^*, \varphi\rangle|^{1-\sigma} \|e^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}} \\
&= d^2_{p,\sigma}(T_B) d(x,x') d(y,y') \left(\int_{B_{E^{**}}} \left(|\langle e^*, \varphi\rangle|^{1-\sigma} \|e^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

A similar argument, analogous to the one conducted in the proof of the second implication of Theorem 4.10 in [18], reveals that $T \in \mathcal{D}^{BL}_{p,\sigma}(X, Y; E)$ and $d^{BL}_{p,\sigma}(T) \leq d^2_{p,\sigma}(T_B)$. □

Now, we establish the Pietsch's domination theorem for this class.

Theorem 4.6. *Let $1 < p, q < \infty$ and $0 \leqslant \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$ and let $T \in BLip_0(X, Y; E)$. Then the following statements are equivalent.*

(1) $T \in \mathcal{D}^{BL}_{p,\sigma}(X, Y; E)$.

(2) *There exist a constant $C > 0$ and a regular Borel probability measure ν on $B_{E^{**}}$ such that for all $x, x' \in X, y, y' \in Y$ and $e^* \in E^*$ we have*

$$\begin{aligned}
&|\langle T(x,y) - T(x,y') - T(x',y) + T(x',y'), e^*\rangle| \qquad (4.3)\\
&\leq C d(x,x') d(y,y') \left(\int_{B_{E^{**}}} \left(|\langle e^*, \varphi\rangle|^{1-\sigma} \|e^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

Furthermore, $d^{BL}_{p,\sigma}(T) = \inf C$, where the infimum is taken over all constants C either in (4.3).

Proof. (1) $\Longrightarrow$ (2) Suppose that $T \in \mathcal{D}^{BL}_{p,\sigma}(X, Y; E)$. Combining (4.1) and (4.2) we conclude that (4.3) is holds.
(2) $\Longrightarrow$ (1) Starting from (4.3), we prove that T_B belongs to $\mathcal{D}^2_{p,\sigma}(Æ(X), Æ(Y); E)$. Let $m^1 = \sum_{i=1}^n \alpha_i m_{x_i,x'_i} \in \mathcal{M}(X)$ and $m^2 = \sum_{j=1}^r \beta_j m_{y_j,y'_j} \in \mathcal{M}(Y)$, then

$$\begin{aligned}
&\left|\left\langle T_B\left(m^1, m^2\right), e^*\right\rangle\right| \\
&\leq \sum_{i=1}^n |\alpha_i| \sum_{j=1}^r |\beta_j| \left|\left\langle T_B\left(m_{x_i,x'_i}, m_{y_j,y'_j}\right), e^*\right\rangle\right| \\
&= \sum_{i=1}^n |\alpha_i| \sum_{j=1}^r |\beta_j| \left|\left\langle T(x_i, y_j) - T\left(x_i, y'_j\right) - T\left(x'_i, y_j\right) + T\left(x'_i, y'_j\right), e^*\right\rangle\right| \\
&\leq C \sum_{i=1}^n |\alpha_i| d(x_i, x'_i) \sum_{j=1}^r |\beta_j| d\left(y_j, y'_j\right) \\
&\qquad \left(\int_{B_{E^{**}}} \left(|\langle e^*, \varphi\rangle|^{1-\sigma} \|e^*\|^\sigma\right)^{\frac{p^*}{1-\sigma}} d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}}.
\end{aligned}$$

Taking the infimum over all representation of m^1 and m^2 we get

$$\begin{aligned}&\left|\left\langle T_B\left(m^1,m^2\right),e^*\right\rangle\right|\\&\leq C\left\|m^1\right\|_{\mathcal{M}(X)}\left\|m^2\right\|_{\mathcal{M}(Y)}\left(\int_{B_{E^{**}}}\left(\left|\langle e^*,\varphi\rangle\right|^{1-\sigma}\left\|e^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}d\nu(\varphi)\right)^{\frac{1-\sigma}{p^*}},\end{aligned}$$

and the result following from Theorem 2.3. □

Let $1<p<r<\infty$, since every absolutely r^*-summing operator is p^*-summing, we have the following corollary.

Corollary 4.7. *Let $1<p<r<\infty$, and $0\leqslant\sigma<1$. Then*

$$\mathcal{D}^{BL}_{r,\sigma}(X,Y;E)\subset\mathcal{D}^{BL}_{p,\sigma}(X,Y;E).$$

Moreover, $d^{BL}_{p,\sigma}(T)\leq d^{BL}_{r,\sigma}(T)$ for all $T\in\mathcal{D}^{BL}_{r,\sigma}(X,Y;E)$.

The proof of the following result is derived from Proposition 4.5 and [6, Proposition 4.5].

Corollary 4.8. *Let $1<p,q<\infty$ and $0\leq\sigma<1$ such that $\frac{1}{q}+\frac{1-\sigma}{p^*}=1$. Then T is strongly (p,σ)-two-Lipschitz if and only its linearization T_L is strongly (p,σ)-continuous linear operator. In this case $d^{BL}_{p,\sigma}(T)=d^{\sigma}_{p}(T_L)$.*

We obtain the following corollary as a straightforward consequence of the preceding corollary and [18, Proposition 3.6].

Corollary 4.9. *$\left(\mathcal{D}^{BL}_{p,\sigma},d^{BL}_{p,\sigma}(\cdot)\right)$ is a two-Lipschitz operator ideal generated by the composition method from the Banach operator ideal $\mathcal{D}^{\sigma}_{p}$, i.e.,*

$$\mathcal{D}^{BL}_{p,\sigma}(X,Y;E)=\mathcal{D}^{\sigma}_{p}\circ BLip_0(X,Y;E),$$

for all pointed metric spaces X,Y and Banach space E.

From [6, Remark 3.3] it follows that if u is strongly (p,σ)-continuous linear operator then u^* is (p^*,σ)-absolutely continuous. The following result is proved in a similar way Theorem 3.4.

Theorem 4.10. *Let $1<p,q<\infty$ and $0\leq\sigma<1$ such that $\frac{1}{q}+\frac{1-\sigma}{p^*}=1$. Then T is strongly (p,σ)-two-Lipschitz if and only if its transpose T^t is (p^*,σ)-absolutely continuous. In this case, $d^{BL}_{p,\sigma}(T)=\pi_{p^*,\sigma}(T^t)$.*

Remark 4.11. If E is a reflexive Banach space, then every strongly (p,σ)-two-Lipschitz operator is two-Lipschitz compact. Indeed, since $T^t:E^*\longrightarrow BLip_0(X,Y)$ is (p^*,σ)-absolutely continuous, we can conclude from [13, Corollary 2.1.22] that T^t is compact and the result follows from [14, Theorem 2.10].

The following results gives the relationship between the classes of strongly two-Lipschitz p-summing and strongly (p,σ)-two-Lipschitz operators.

Proposition 4.12. *Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$. Then*

$$\mathcal{D}_p^{BL}(X, Y; E) \subset \mathcal{D}_{p,\sigma}^{BL}(X, Y; E).$$

Moreover, for all $T \in \mathcal{D}_p^{BL}(X, Y; E)$ we obtain

$$d_{p,\sigma}^{BL}(T) \leq d_p^{BL}(T).$$

Proof. Let $T \in \mathcal{D}_p^{BL}(X, Y; E)$. Then from Theorem 3.4 it follows that its transpose T^t is p^*-summing, and by [20, Proposition 4.2], T^t is (p^*, σ)-absolutely continuous and $\pi_{p^*,\sigma}(T^t) \leq d_p^{BL}(T)$. Therefore, by using Theorem 4.10, we can obtain the result. □

This paragraph aims to characterize strongly (p, σ)-two-Lipschitz operators by a factorization theorem. In fact, this theorem is an exact match to its counterpart with regard to strongly (p, σ)-continuous multilinear operators given in [6]. Let X, Y be two pointed metric spaces, E be a Banach space and $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$ and let ν be a regular Borel probability measure on $B_{E^{**}}$. Let $\iota_{E^*} : E^* \longrightarrow C(B_{E^{**}})$ be the isometric embedding defined by $\iota_{E^*}(e^*) = < e^*, \cdot >$. Let $J_{p,\sigma} : \iota_{E^*}(E^*) \longrightarrow L_{p,\sigma}(\nu)$ be the projection onto the quotient. Note that this last space is studied and detailed in [6]. Let $K : X \times Y \longrightarrow BLip_0(X, Y)^*$ the mapping given by $K(x, y)(f) = f(x, y)$. In fact, K is a two-Lipschitz operator with $BLip(K) = 1$. (see [14, Example 2.2 (b)]). It is clear that if we consider the natural embedding $k_E : E \hookrightarrow E^{**}$, then $k_E \circ T = (T^t)^* \circ K$ for all $T \in BLip_0(X, Y; E)$.

Now, we present the factorization theorem for the class of strongly (p, σ)-two-Lipschitz operators.

Theorem 4.13. *Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{q} + \frac{1-\sigma}{p^*} = 1$ and let $T \in BLip_0(X, Y; E)$. The following statements are equivalent:*

(1) $T \in \mathcal{D}_{p,\sigma}^{BL}(X, Y; E)$.

(2) *There exist a regular Borel probability measure ν on $B_{E^{**}}$ and a two-Lipschitz mapping $R : X \times Y \longrightarrow \left(L_{p^*,\sigma}(\nu)\right)^*$ such that the following diagram commutes,*

$$\begin{array}{ccccc} X \times Y & \xrightarrow{T} & E & \xrightarrow{k_E} & E^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle \iota_{E^*}^*} \\ \left(L_{p^*,\sigma}(\nu)\right)^* & & \xrightarrow{J_{p^*,\sigma}^*} & & \left(\iota_{E^*}(E^*)\right)^*. \end{array}$$

i.e., $k_E \circ T = \iota_{E^*}^* \circ J_{p^*,\sigma}^* \circ R$.

Proof. As mentioned previously, the proof is inspired by the proof of [6, Theorem 6.2]. However, taking into a count that the space $L_{p^*,\sigma}(\nu)$ is the closure of $(J_{p^*,\sigma} \circ \iota_{E^*})(E^*)$, it is worth mentioning that R is defined by

$$R(x, y)\big(J_{p^*,\sigma} \circ \iota_{E^*}(e^*)\big) := u\big(J_{p^*,\sigma} \circ \iota_{E^*}(e^*)\big)(x, y),$$

where $u : L_{p^*,\sigma}(\nu) \longrightarrow BLip_0(X,Y)$ is the bounded linear operator provided by the factorization theorem concerning the (p^*,σ)-absolutely continuous linear operator T^t with $T \in \mathcal{D}^{BL}_{p,\sigma}(X,Y;E)$. A direct computation shows that R is two-Lipschitz with $BLip(R) \leq \|u\|$. □

In particular, if $\sigma = 0$, then we present the factorization theorem for strongly two-Lipschitz p-summing operators.

Theorem 4.14. *Let $1 < p \leq \infty$ and let $T \in BLip_0(X,Y;E)$. The following statements are equivalent:*

(1) $T \in \mathcal{D}^{BL}_{p}(X,Y;E)$,

(2) *There exist a regular Borel probability measure ν on $B_{E^{**}}$ and a two-Lipschitz mapping $R : X \times Y \longrightarrow (L_{p^*}(\nu))^*$ such that*

$$k_E \circ T = \iota^*_{E^*} \circ J^*_{p^*} \circ R.$$

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