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Remarks on fixed point assertions in digital topology, 8

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Abstract

This paper continues a series in which we study deficiencies in previously published works concerning fixed point assertions for digital images.

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1. Introduction

There are many beautiful results concerning fixed points for digital images. There are also many highly flawed papers concerning this topic. The current work continues that of [10, 3, 4, 6, 7, 8, 9] in discussing flaws in papers that have come to our attention since acceptance of [9] for publication.

In particular, the notion of a "digital metric space" has led many authors to attempt, in most cases either erroneously or trivially, to modify fixed point results for Euclidean spaces to digital images. This notion contains roots of all the flawed papers studied in the current paper. See [6] for discussion of why "digital metric space" does not seem a worthy topic of further research.

2. Preliminaries

Much of the material in this section is quoted or paraphrased from [6].

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers.

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A *digital image* is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation on X. Thus, a digital image is a graph. In order to model the "real world," we usually take X to be finite, although there are several papers that consider infinite digital images, e.g., for digital analogs of covering spaces. The points of X may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

2.1. **Adjacencies, continuity, fixed point.** In a digital image (X, κ), if $x, y \in X$, we use the notation $x \leftrightarrow_\kappa y$ to mean x and y are κ-adjacent; we may write $x \leftrightarrow y$ when κ can be understood. We write $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ can be understood, to mean $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies in the study of digital images are the c_u adjacencies. These are defined as follows.

Definition 2.1. Let $X \subset \mathbb{Z}^n$. Let $u \in \mathbb{Z}$, $1 \leq u \leq n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then $x \leftrightarrow_{c_u} y$ if

- $x \neq y$,
- for at most u distinct indices i, $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Definition 2.2 (see [20])**.** Let (X, κ) be a digital image. Let $x, y \in X$. Suppose there is a set $P = \{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \leq i < n$, and $x_n = y$. Then P is a *κ-path* (or just a *path* when κ is understood) in X from x to y, and n is the *length* of this path.

Definition 2.3 ([25])**.** A digital image (X, κ) is *κ-connected*, or just *connected* when κ is understood, if given $x, y \in X$ there is a κ-path in X from x to y.

Definition 2.4 ([25, 2])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is *(κ, λ)-continuous*, or *κ-continuous* if $(X, \kappa) = (Y, \lambda)$, or *digitally continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.5 ([2])**.** *A function $f : X \to Y$ between digital images (X, κ) and (Y, λ) is (κ, λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

Remarks 2.6. For $x, y \in X$, $P = \{x_i\}_{i=0}^n \subset X$ is a κ-path from x to y if and only if $f : [0, n]_\mathbb{Z} \to X$, given by $f(i) = x_i$, is (c_1, κ)-continuous. Therefore, we may also call such a function f a *(κ-)path* in X from x to y.

We use id_X to denote the identity function on X, and $C(X, \kappa)$ for the set of functions $f : X \to X$ that are κ-continuous.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. We denote by $\mathrm{Fix}(f)$ the set of fixed points of $f : X \to X$.

Let $X = \Pi_{i=1}^n X_i$. The *projection to the j^{th} coordinate* function $p_j : X \to X_j$ is the function defined for $x = (x_1, \ldots, x_n) \in X$, $x_i \in X_i$, by $p_j(x) = x_j$.

As a convenience, if x is a point in the domain of a function f, we will often abbreviate "$f(x)$" as "fx".

2.2. **Digital metric spaces.** A *digital metric space* [13] is a triple (X, d, κ), where (X, κ) is a digital image and d is a metric on X. The metric is usually taken to be the Euclidean metric or some other ℓ_p metric; alternately, d might be taken to be the shortest path metric. These are defined as follows.

- Given $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, $p > 0$, d is the ℓ_p metric if
$$d(x, y) = \left(\sum_{i=1}^{n} \mid x_i - y_i \mid^p \right)^{1/p}.$$
Note the special cases: if $p = 1$ we have the *Manhattan metric*; if $p = 2$ we have the *Euclidean metric*.
- [11] If (X, κ) is a connected digital image, d is the *shortest path metric* if for $x, y \in X$, $d(x, y)$ is the length of a shortest κ-path in X from x to y.

We say a metric space (X, d) is *uniformly discrete* if there exists $\varepsilon > 0$ such that $x, y \in X$ and $d(x, y) < \varepsilon$ implies $x = y$.

Remarks 2.7. If X is finite or

- [4] d is an ℓ_p metric, or
- (X, κ) is connected and d is the shortest path metric,

then (X, d) is uniformly discrete.

For an example of a digital metric space that is not uniformly discrete, see Example 2.10 of [6].

We say a sequence $\{x_n\}_{n=0}^{\infty}$ is *eventually constant* if for some $m > 0$, $n > m$ implies $x_n = x_m$. The notions of convergent sequence and complete digital metric space are often trivial, e.g., if the digital image is uniformly discrete, as noted in the following, a minor generalization of results of [16, 10].

Proposition 2.8 ([6])**.** *Let (X, d) be a metric space. If (X, d) is uniformly discrete, then any Cauchy sequence in X is eventually constant, and (X, d) is a complete metric space.*

3. Iterations of [1]

The paper [1] is concerned with comparing the rates of convergence of convergent sequences in a digital image, perhaps especially of sequences converging to a fixed point of a given function. This is a problem of greater theoretical than practical interest, as in the "real world," a digital image is finite and, usually, of small to moderate size. Also, the paper is flawed as we discuss below.

- In Definition 2.5 of [1], a digital metric space (E, μ), a function $T : E \to E$, and a sequence $\{\alpha_n\}_{n=0}^{\infty}$, where $0 \leq \alpha_n \leq 1$ are hypothesized. Statement (2.2) of this definition calls for
$$x_{n+1} = f_{T,\alpha_n}(x_n)$$

where nothing appears to define or describe f_{T,α_n}. Further, the definition proceeds with a subdefinition,

$$\epsilon_n = \mu(x_{n+1}, f_{T,\alpha_n}(x_n))$$

where μ is a metric, so, by the above, we would have $\epsilon_n = 0$. It seems unlikely that this is what the authors intended.

- The assertion of [1] labeled as Theorem 4.1 uses "$F(T)$" without definition in its hypothesis. Its usage suggests this is intended to be "F_T", defined earlier as the fixed point set of the function T.
- The argument offered as proof of Theorem 4.1 uses the symbol "δ", seemingly as a nonnegative real number. As this "proof" proceeds, it seems that δ is required to be in the interval $[0, 1)$, but this restriction is never stated. What appears to be the same "δ" appears in the arguments given as proofs of the assertions labeled Theorem 4.2 and Theorem 4.3.
- The paper's assertion labeled as Theorem 4.2 also depends on its Definition 2.5, which, as discussed above, is dubious.
- The paper's Example 4.4 assumes the function

$$T(x) = x/2 + 3$$

is defined from X to X, where X is the set of non-negative integers. Since, e.g., $T(1) = 3.5$ is non-integral, the formula given for T does not belong in a discussion of functions from X to X.

- Example 4.5 is similarly flawed, hypothesizing the function

$$T(x) = \sqrt{x^2 - 8x + 40}$$

as a function from X to X where X is the set of nonnegative integers. But, e.g., $T(1) = \sqrt{33}$ is not an integer, so the formula given for T does not belong in a discussion of functions from X to X.

Perhaps appropriate rewriting can yield valid results out of the assertions of [1]. However, as written, none of the "Main Results" of this paper can be regarded as all of well defined, proven, and valid.

4. Contractive type mappings of [15]

The paper [15] is concerned with fixed point results for self-maps T on digital images such that T is a θ-contraction (defined below).

We note [15] uses "ℓ-adjacent" for what our Definition 2.1 would call "$c_\ell - adjacent$".

4.1. Improper citations.

Improper citations exist in [15]: attributions to [16] that should be to [13], of the definition of *digital metric space* and the definition of a *digital contraction map* (not to be confused with the "contraction" that is a digital homotopy between an identity map and a constant map). Since [13] is a reference in [15], the authors of the latter should have known better.

4.2. [15]**'s Theorem 3.1.**

Definition 4.1 ([15])**.** $\Theta = \{\theta : [0, \infty) \to [0, \infty) \mid \theta$ is increasing, $\theta(t) < \sqrt{t}$ for $t > 0$, $\theta(t) = 0$ if and only if $t = 0\}$.

Definition 4.2 ([15])**.** Suppose (X, d, ℓ) is a digital metric space, $T : X \to X$, and $\theta \in \Theta$. Suppose $d(Tx, Ty) \leq \theta(d(x, y))$ for all $x, y \in X$. Then T is called a *θ-digital contraction*.

The following is Theorem 3.1 of [15].

Theorem 4.3. *Suppose (X, d, ℓ) is a digital metric space and $T : X \to X$ is a digital θ-contraction for some $\theta \in \Theta$. Then T has a unique fixed point.*

However, we note important cases for which the previous theorem reduces to triviality, as a consequence of the following.

Proposition 4.4 ([9])**.** *Let (X, d, κ) be a connected digital metric space in which*

- *d is the shortest path metric, or*
- *d is any ℓ_p metric and $\kappa = c_1$.*

Then every θ-contraction on (X, d) is a constant map.

4.3. [15]**'s Corollary 3.1.** Corollary 3.1 of [15] gives a version of the Banach Contraction Principle (defined below) for digital images. This was previously shown in [14], which is a reference of [15], so the authors of [15] should have cited [14].

Further, there are important cases for which the Banach contraction principle for digital images is a triviality.

Definition 4.5 ([14])**.** Let (X, d, κ) be a digital image. Let $0 \leq \lambda < 1$ and suppose $T : X \to X$ such that for all $x \in X$, $d(Tx, Ty) \leq \lambda d(x, y)$. Then T is a *digital contraction map*.

The Banach contraction principle for digital images is the following.

Theorem 4.6 ([14, 15])**.** *Let (X, d, κ) be a digital image. Let $T : X \to X$ be a digital contraction map. Then T has a unique fixed point.*

However, we have the following.

Theorem 4.7. *Let $T : X \to X$ be a contraction map on a connected digital image (X, d, κ). If $\kappa = c_1$ and d is an ℓ_p metric, or if d is the shortest path metric, then T is a constant map.*

Proof. For the case $\kappa = c_1$ and d is an ℓ_p metric, the assertion was shown in [10].

Now assume d is the shortest path metric, and let λ be as in Definition 4.5. Let $x \leftrightarrow_\kappa y$ in X. Since $d(Tx, Ty) \leq \lambda \cdot d(x, y) = \lambda < 1$, we must have $d(Tx, Ty) = 0$. Thus $Tx = Ty$, and it follows from the connectedness of (X, κ) that T is constant. □

Thus, for the cases discussed in Theorem 4.7, the Banach contraction principle for digital images is a triviality.

4.4. [15]**'s Theorem 3.2.** The assertion labeled Theorem 3.2 of [15] is presented with a proof that contains a major error. The assertion is as follows.

Assertion 4.8. [15] Let (X, d, ℓ) be a digital metric space and $T : X \to X$ such that $x \neq y$ implies $d(Tx, Ty) < \mu(x, y)$, where

$$\mu(x,y) = \max \left\{ \begin{array}{c} \frac{1}{2}\left[d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)} + d(Tx,Ty) + d(x,y)\right], \\ d(x,Tx)\frac{1+d(y,Ty)}{1+d(Tx,Ty)} \end{array} \right\}.$$

Then T has unique fixed point.

In order to prove the existence of a fixed point, the authors define an infinite sequence of points of X via $x_0 \in X$, $x_{n+1} = Tx_n$. They attempt to show that if the x_n are all distinct then $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence. However, a chain of equations and inequalities begins by claiming equality between $d(x_n, x_{n+1})$ and $d(x_{n-1}, x_n)$. There is no obvious reason to accept this alleged equality, and note if true, it would be counter to the goal of showing a decreasing sequence. The equation should be

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

which would make the next few lines of the "proof" correct.

However, right side of the inequality

$$\max \left\{ \begin{array}{c} \frac{1}{2}\left[d(x_n,x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)} + d(x_n,x_{n+1}) + d(x_{n-1},x_n)\frac{1+d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\right], \\ d(x_{n-1},x_n) \end{array} \right\}$$

$$\leq \max\left\{\frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})], d(x_{n-1}, x_n)\right\}$$

should be

$$\max\left\{\frac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)], d(x_{n-1}, x_n)\right\} =$$

$$\max\{[d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n-1}, x_n)], d(x_{n-1}, x_n)\}.$$

Thus, the argument fails to lead to the desired upper bound of $d(x_{n-1}, x_n)$ for $d(x_n, x_{n+1})$. Therefore, the assertion must be regarded as unproven.

5. [17]'s expansive maps

D. Jain's paper [17] is concerned with fixed point results for expansive and related digital maps.

The following definition is presented without citation in [17]. It should be attributed to [19].

Definition 5.1. Suppose that (X, d, κ) is a complete digital metric space and $S : X \to X$ is a mapping. If S satisfies $d(S(x), S(y)) \geq \alpha d(x, y)$ for all $x, y \in X$ and some $\alpha > 1$, then S is a *digital expansive mapping.*

Remarks 5.2. It is easily derived from a discussion in [25] – see also [10] – that such a function need not be digitally continuous. E.g., consider $S(x) = 2x : \mathbb{Z} \to \mathbb{Z}$ with $\alpha = 1.5$, $\kappa = c_1$. If $x \leftrightarrow_{c_1} y$ in X then $Sx \not\leftrightarrow_{c_1} Sy$.

5.1. [17]**'s Theorem 3.2.** The following is stated as Theorem 3.2 of [17].

Assertion 5.3. Let (X, d, κ) be a digital metric space. Let S be an onto continuous self map on X such that

$$d(Sx, Sy) \geq \alpha\mu(x, y)$$

where $\alpha > 1$ and

$$\mu(x, y) = \max\left\{d(x, y), \frac{d(x, Sx) + d(y, Sy)}{2}, \frac{d(x, Sy) + d(y, Sx)}{2}\right\}.$$

Then S has a fixed point.

This assertion is "almost" correct, in that Jain's argument makes use of unstated hypotheses, namely that

- (X, d, κ) is complete; and
- d is a metric (e.g., the Euclidean metric) for which $x_n \to x$ implies that for almost all n, $x_n = x$.

It may be that Jain assumed that d is the Euclidean metric ([17] does not specify d as a particular metric) that satisfies both of these properties. An example of a metric for a digital image that fails to have these properties is given in Example 2.9 of [10].

Also, the hypothesis of continuity is unclear; Jain might mean in the classic $\varepsilon - \delta$ sense - Jain's probable intent, as hinted in the "proof"; or Jain might mean in the digital sense (which would be incorrect – see Remark 5.2) and unnecessary. The argument offered in [17] as proof of Assertion 5.3 should also be corrected as discussed below.

Errors in Jain's "proof" are

- The inequality "$0 \geq k\mu(x, y)$" at line 6 of page 106 is stated without justification. It does not clearly follow from what precedes.
- Two lines later, we see the claim that "S is continuous", which had not been hypothesized and we will not assume.

Theorem 5.4 below is a correct version of Assertion 5.3, in which we omit the hypotheses of d being complete and S being continuous. Also, since $\mu(x, y) \geq d(x, y)$, we can substitute the latter for $\mu(x, y)$.

Theorem 5.4. *Let (X, d, κ) be a digital metric space, where d is any ℓ_p metric or the shortest path metric. Let S be an onto self map on X such that*

$$d(Sx, Sy) \geq \alpha d(x, y)$$

where $\alpha > 1$. Then S has a fixed point.

Proof. Note our choice of d makes (X, d, κ) complete, and $d(x_n, x_0) \to 0$ implies $x_n = x_0$ for almost all n.

We will modify Jain's argument as is useful. Jain's argument starts with $x_0 \in X$. Since S is onto, there exists $x_1 \in S^{-1}(x_0)$, and, inductively, $x_n \in S^{-1}(x_{n-1})$.

Jain correctly shows $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, although this can be established much more briefly, as follows.

$$d(x_{n-1}, x_n) = d(Sx_n, Sx_{n+1}) \geq \alpha d(x_n, x_{n+1}).$$

So

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \leq$$
$$(1/\alpha)^n d(x_0, x_1) + \cdots + (1/\alpha)^{n+m-1} d(x_0, x_1) =$$
$$(1/\alpha)^n d(x_0, x_1) \left[1 + (1/\alpha) + \cdots + (1/\alpha)^{m-1}\right] \to_{n \to \infty} 0.$$

Hence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

By Remark 2.7 and Proposition 2.8, there exists $x \in X$ such that $x_n = x$ for almost all n. Therefore, for some n, $S(x_{n+1}) = x_n = x_{n+1}$, so $x_n \in \mathrm{Fix}(S)$. □

We also note the following.

Theorem 5.5. *Let (X, κ) be a finite digital image of more than one point. For any metric d, there is no function S that is as described in Theorem 5.4.*

Proof. Since X is finite but has more than one point, there exist distinct $x_0, x_1 \in X$ such that $d(x_0, x_1) = diam_d(X)$. Suppose there exists S as described in Theorem 5.4. Then

$$d(Sx_0, Sx_1) \geq \alpha \cdot \mu(x_0, x_1) \geq \alpha \cdot d(x_0, x_1) > d(x_0, x_1),$$

contrary to our choice of x_0, x_1. The assertion follows. □

5.2. [17]'s Theorem 3.3.

The following is stated as Theorem 3.3 of [17].

Assertion 5.6. Let (X, d, κ) be a complete digital metric space and let $S : X \to X$ be an onto self map that is continuous. Suppose S satisfies

$$d(Sx, Sy) \geq \alpha\mu$$

where $\alpha > 1$ and

$$\mu = \max\left\{d(x, y), \frac{d(x, Sx) + d(y, Sy)}{2}, d(x, Sy), d(y, Sx)\right\}$$

then S has a fixed point.

But notice that μ is greater than or equal to the expression used for μ in Assertion 5.3. Therefore, we can make the same modifications that we made to Assertion 5.3, which gives us Theorem 5.4.

6. [18]'s β–ψ contractive maps

The notion of a "β–ψ contractive map" appears to have originated in [27] using the name "α–ψ contractive map". [18] uses both "α–ψ contractive map" and "β–ψ contractive map" for the same notion.

6.1. **Fundamentals of β–ψ contractive maps.**

Definition 6.1 ([18, 27])**.** Ψ is the set of functions $\psi : [0, \infty) \to [0, \infty)$ such that

i) ψ is nondecreasing, and

ii) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} \nu_k$, such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + \nu_k$$

for $k \geq k_0$ and all $t \in \mathbb{R}^+$.

Note [18] does not define the symbol "$\mathbb{R}^+$"; according to [27], it represents $[0, \infty)$.

Lemma 6.2 ([27])**.** *$\psi \in \Psi$ implies*

(1) *For all $t \in \mathbb{R}^+$, $\psi^n(t) \to_{n \to \infty} 0$.*
(2) *$\psi(t) < t$ for all $t > 0$.*
(3) *ψ is continuous at 0.*
(4) *$\sum_{n=1}^{\infty} \psi^n(t)$ converges, for all $t \in \mathbb{R}^+$.*

Definition 6.3 ([27])**.** Let $T : X \to X$, $\alpha : X \times X \to [0, \infty)$. We say T is *α-admissible* if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 6.4. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$. If $T : X \to X$ such that $x, y \in X$ implies

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

then T is an *$\alpha - \psi$ contractive map.*

6.2. [18]**'s Theorem 3.3.** Theorem 3.3 of [18] states the following.

> Let (X, d, ρ) be a complete digital metric space and let $T : X \to X$ be a $\beta - \psi$ contractive map such that
> (1) T is β-admissible.
> (2) There exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$.
> (3) T is digitally continuous.
> Then T has a fixed point.

The assertion is correct, but, together with its "proof," is substantially flawed, as follows.

- We will show below that the assumptions of completeness and continuity are unnecessary under "usual" conditions.
- There are multiple incorrect references: The reference to "(6)" should be to "(1)" in the definition of $\beta - \psi$ contractive map; "Using (5)" should be "Using (4)"; and "lemma 1.2" should be "Lemma 2.2".

Thus, we can state the following version of Theorem 3.3 of [18].

Theorem 6.5. *Let (X, d, ρ) be a connected digital metric space, where X is finite or d is an ℓ_p metric or the shortest path metric. Let $T : X \to X$ be a $\beta - \psi$ contractive map such that*

(1) *T is β-admissible.*
(2) *There exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$.*

Then T has a fixed point.

Proof. Note our assumptions about (X, d) imply completeness, by Remark 2.7 and Proposition 2.8.

We use much of the argument of [18]. Let $x_{n+1} = Tx_n$ for all $n \geq 0$. By assumption,

$$\beta(x_0, x_1) = \beta(x_0, Tx_0) \geq 1.$$

A simple induction, using the fact that T is β-admissible, lets us know

$$\beta(x_n, x_{n+1}) = \beta(Tx_{n-1}, Tx_n) \geq 1 \text{ for } n > 0.$$

Then $n > 0$ implies

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \beta(x_{n-1}, x_n) \cdot d(Tx_{n-1}, Tx_n)$$

$$\leq \psi(d(x_{n-1}, x_n)).$$

and a simple induction yields

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

By Lemma 6.2(1), we have $d(x_n, x_{n+1}) \to_{n\to\infty} 0$. By Proposition 2.8, for almost all n, $x_n = x_{n+1} = T(x_n)$, so $x_n \in \mathrm{Fix}(T)$. □

6.3. [18]**'s Theorem 3.5.** Despite a reference in the argument offered as proof of [18]'s Theorem 3.5 to a Theorem 3.4 in the same paper, there is no Theorem 3.4 in [18].

The following is stated as Theorem 3.5 of [18].

> Let (X, d, ρ) be a complete digital metric space. Let $T : X \to X$ be a digital $\beta - \psi$ contractive map satisfying
> (i) T is β-admissible;
> (ii) There exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$;
> (iii) If $\{x_n\}_{n=0}^{\infty} \subset X$ such that $\beta(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to_{n\to\infty} x \in X$ then $\beta(x_n, x) \geq 1$ for almost all n.
> Then there exists $u \in \mathrm{Fix}(T)$.

The authors suggest that the purpose of this assertion is to be a version of their Theorem 3.3 without the requirement of continuity. We note, however, that our version of their Theorem 3.3, namely our Theorem 6.5, requires neither a continuity assumption nor item (iii) of the current assertion.

6.4. [18]**'s Examples 3.6 and 3.7.** The authors wish to demonstrate uses of their previous assertions in these examples. However, the metric spaces considered are not digital images, as they are not subsets of any $\mathbb{Z}^n$.

7. [21]'s fixed point assertion for pairs of functions

[21] uses Ψ as the symbol for a function, not the set of functions so labeled in [18]. In [21], $\Psi : [0, \infty) \to [0, \infty)$ is continuous, and $\Psi(t) = 0$ if and only if $t = 0$.

The author also assumes a function $\phi : [0, \infty) \to [0, \infty)$ that is lower semi-continuous such that $\Psi(t) = 0$ if and only if $t = 0$. It seems likely that the latter is meant to be $\phi(t) = 0$ if and only if $t = 0$.

7.1. **"Theorem" 3.1.** The following is stated as Theorem 3.1 of [21].

Assertion 7.1. Let (X, d, ρ) be a complete digital metric space. Let N be a nonempty closed subset of X. Let $P, Q : N \to N$, $G, H : N \to X$ such that $Q(N) \subset H(N)$ and for all $x, y \in X$,

$$\Psi(d(Px, Qy)) \geq \phi(d_{G,H}(x, y)) + \frac{1}{2}\Psi(d_{G,H}(x, y) + \phi(d_{G,H}(x, y)) \tag{7.1}$$

where

$$d_{G,H}(x, y)) = \max \left\{ \begin{array}{c} d(x, y), d(Gx, Hy), d(Gx, Px), d(Hy, Qy), \\ \frac{1}{3}d((Gx, Qy) + (Hy, Px)) \end{array} \right\} \tag{7.2}$$

and

$$d_{P,Q}(x, y) = \max \left\{ \begin{array}{c} d(x, y), d(Gx, Hy), d(Gx, Px), d(Hy, Qy), \\ \frac{1}{4}d((Gx, Qy) + (Hy, Px)) \end{array} \right\} \tag{7.3}$$

Then $\{P, G\}$ and $\{Q, H\}$ have a unique point of coincidence in X. Moreover, if $\{P, G\}$ and $\{Q, H\}$ are self-mappings, then P, Q, G, and H have a unique fixed point in X.

This assertion has the following deficiencies.

- If d is a "usual" metric, (X, d) is a discrete topological space, so all subsets of X are closed. Either the requirement of N being closed is unnecessary, or the author meant something else.
- The statement of the assertion has $Q(N) \subset H(N)$, but the first line of the "proof" says $Q(N) \subset G(N)$ and $P(N) \subset H(N)$.
- Statements (7.1), (7.2), and (7.3) are expected to be true for all $x, y \in X$, but if $N \neq X$ then Px, Qy are undefined for $x, y \in X \setminus N$.
- In each of (7.2) and (7.3), it seems likely that "$d((Gx, Qy)+(Hy, Px))$" should be "$d(Gx, Qy) + d(Hy, Px)$".
- $d_{P,Q}$, defined in (7.3), is not mentioned in (7.1).
  - It seems that either $d_{P,Q}$ is intended to be part of (7.1) or else it should be deleted.
  - It appears that correction of the second lines of both (7.2) and (7.3) would leave $d_{P,Q}(x, y) \leq d_{G,H}(x, y)$ for all $x, y \in X$. Therefore, perhaps $d_{P,Q}$ is unnecessary.
- What does it mean to say "$\{P, G\}$ and $\{Q, H\}$ have a unique point of coincidence"? This notion is undefined in [21]. Does it mean each pair has a point of coincidence?

- It is claimed in the "proof" that
$$\Psi(y_{2n+1}, y_{2n+2}) = \Psi(Px_{2n}, Qx_{2n+1}) \leq$$
$$\Psi(d_{G,H}(x_{2k}, x_{2k+1})) - \phi(d_{G,H}(x_{2k}, x_{2k+1})).$$
It appears the symbol "d" is missing from the left side of this inequailty. Further, there is no explanation of this statement, and there is no obvious derivation of it from (7.1) even after the seemingly appropriate corrections.

We conclude that whatever this assertion is meant to say, is unproven.

7.2. **Example 3.2.** Example 3.2 of [21] is meant to illustrate the paper's "Theorem" 3.1, discussed above. The example uses $X = [4, 40]$, which is not a subset of any $\mathbb{Z}^n$ and thus cannot be the set underlying a digital image. Also, H is undefined for some members of X and multiply defined for others; e.g., $H(9)$ is undefined, while $H(x)$ is defined both by $17 + x$ and by 16 for $13 \leq x \leq 14$.

7.3. **Corollary 3.3.** The following is stated as Corollary 3.3 of [21]. Despite being labeled a corollary, the assertion has no clear relation to previous assertions in [21], and, in fact, is false.

Assertion 7.2. Let P and Q be self mappings of a complete digital metric space (X, d, ρ) into itself. Suppose $P(X) \subset Q(X)$. If there exists $\alpha \in (0, 1)$ and a positive integer k such that $d(P^k(x), P^k(y)) \leq \alpha d(Qx, Qy)$ for all $x, y \in X$, then P and Q have a unique common fixed point.

To show Assertion 7.2 is false, consider the following.

Example 7.3. Let $X = [0, 1]_{\mathbb{Z}}$, $d(x, y) = |x - y|$. Let $P(x) = 0$, $Q(x) = 1 - x$. Clearly, $P(X) \subset Q(X)$, and for all $x, y \in X$,
$$d(P^1(x), P^1(y)) = 0 \leq 0.5d(Qx, Qy)$$
but Q has no fixed point.

8. [22]'s common fixed points

Theorem 5 of [22] says the following.

Theorem 8.1. *Consider two commuting and* [sic] *self-mappings f and g on a complete digital metric space (K, d, p) with coefficient $\alpha \in (0, 1)$ such that f is continuous, $g(K) \subset f(K)$, and*
$$d(gx, gy) \leq \alpha d(fx, fy) \qquad \textit{for all } x, y \in K.$$
Then f and g have a unique common fixed point in K.

However, this is unoriginal; indeed, stronger results appear in older papers. Theorem 3.1.4 of [24] shows the assertion above can be part of an "if and only if" theorem. Theorem 5.3 of [5] improves on Theorem 3.1.4 of [24] by showing that the hypotheses of completeness and continuity are unnecessary, and the "commuting" condition can be replaced by a weaker restriction of "weakly commuting".

9. [23]'s fixed point theorems

9.1. [23]'s Theorem 3.1.

Definition 9.1 ([15]). Let Θ be the set of functions $\theta : [0, \infty) \to [0, \infty)$ such that θ is increasing, $\theta(t) < \sqrt{(t)}$ for $t > 0$, and $\theta(t) = 0$ if and only if $t = 0$.

Theorem 3.1 of [23] states the following.

Theorem 9.2. *Let (X, d, β) be a digital metric space, $T : X \to X$, and $\theta \in \Theta$. Suppose*

$$d(Tx, Ty) \leq \theta(d(x, y)) \text{ for all } x, y \in X. \tag{9.1}$$

Then T has a unique fixed point.

Theorem 9.2 is, some would say, correctly proven in [23]; others would say that where the argument offered as proof reaches the correct inequality

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1})$$

and claims that this shows $x_n \in \text{Fix}(T)$ for almost all n, that this conclusion should be established by an easy but absent argument using (9.1).

We observe important cases for which Theorem 9.2 reduces to triviality.

Proposition 9.3. *Let X and T be as in Theorem 9.2. Suppose (X, d, β) is connected. If d is the shortest path metric, or if $\beta = c_1$ and d is any ℓ_p metric, then T is a constant function.*

Proof. Given $x \leftrightarrow y$ in (X, β), we have

$$d(Tx, Ty) \leq \theta(d(x, y)) = \theta(1) < \sqrt{1} = 1$$

so $d(Tx, Ty) = 0$, i.e., $Tx = Ty$. The assertion follows from connectedness. $\square$

9.2. [23]'s Theorem 3.2.

[23] states its "Theorem" 3.2 as another attempt to obtain what we have called Assertion 4.8. The argument, and its errors, are similar to those of [15].

Like [15], [23]'s attempt to show $\{d(x_n, x_{n+1})\}$ is a decreasing sequence has a chain of comparisons beginning with an incorrect claim that $d(x_n, x_{n+1})$ is equal to $d(x_{n-1}, x_n)$, where it should say $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$.

As in [15], correcting this error lets us proceed through subsequent lines of the argument, until we come to the claimed inequality

$$\max \left\{ \begin{array}{c} \frac{1}{2} \left[d(x_n, x_{n+1}) \frac{1+d(x_{n-1}, x_n)}{1+d(x_{n-1}, x_n)} + d(x_n, x_{n+1}) + d(x_{n-1}, x_n) \frac{1+d(x_n, x_{n+1})}{1+d(x_n, x_{n+1})} \right], \\ d(x_{n-1}, x_n) \end{array} \right\}$$

$$\leq \max \left\{ \frac{1}{2} [d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})], d(x_{n-1}, x_n) \right\},$$

an error that appeared in the "proof" of [15], discussed above in section 4.4, correction of which does not lead to the desired conclusion.

10. [26]'s Dass-Gupta contraction

[26] studies a digital version of the Dass-Gupta contraction [12].

10.1. [26]**'s Theorem 3.** Theorem 3 of [26], as written, is not correctly proven, although with a small number of minor changes, a correct result can be obtained. The following is stated as Theorem 3 of [26].

Assertion 10.1. Let (F, Φ, γ) be a complete digital metric space. Let $K : F \to F$ be a mapping that satisfies the rational contraction condition

$$\Phi(Ku, Kv) \leq \frac{\xi_1 \Phi(v, Kv)[1 + \Phi(u, Ku)]}{1 + \Phi(u, v)} + \xi_2 \Phi(u, v) \text{ for all } u, v \in F,$$

where $\xi_1, \xi_2 > 0$ and $\xi_1 + \xi_2 < 1$. Then K has a unique fixed point.

The argument offered as proof of Assertion 10.1 in [26] is marred by the following. For $u_0 \in F$, the sequence $\{u_n = Ku_{n-1}\}$ and the constant $\eta = \frac{\xi_2}{1-\xi_1}$ are defined. Then, in the argument for existence of a fixed point:

- The inequality

$$\Phi(u_n, u_{n+k}) \leq \frac{\eta^n}{1 - \eta} \Phi(u_0, u_1)$$

  is derived. It is argued that this inequality shows $\{u_n\}$ is a Cauchy sequence. But this line of reasoning requires $\eta < 1$, hence $\xi_2 < 1 - \xi_1$, which is not hypothesized.
- It is claimed that K is continuous. This was neither hypothesized nor proven, and, we show below, is unnecessary.

And in the argument for the uniqueness of a fixed point:

- If μ and λ are fixed points, "$\Phi(\mu, \lambda) = (K\mu, K\lambda)$" should be "$\Phi(\mu, \lambda) = \Phi(K\mu, K\lambda)$".
- "$0 < \lambda < 1$" should be "$0 < \xi_2 < 1$".

Therefore, as written, Assertion 10.1 is unproven.

We modify Assertion 10.1 and its "proof" to obtain the following.

Theorem 10.2 ([26])**.** *Let (F, Φ, Γ) be a digital metric space, and $K : F \to F$ a mapping satisfying*

$$\Phi(Ku, Kv) \leq \frac{\xi_1 \Phi(v, Kv)[1 + \Phi(u, Ku)]}{1 + \Phi(u, v)} + \xi_2 \Phi(u, v) \tag{10.1}$$

for all $u, v \in F$, where $\xi_1, \xi_2 > 0$, and

$$\frac{\xi_2}{1 - \xi_1} < 1. \tag{10.2}$$

Then K has a unique fixed point.

Note the inequality (10.2) does not appear in [26], but seems to be necessary in the proof.

Proof. We use ideas of [26], with modifications to correct and abbreviate the argument as are desirable.

Let $u_0 \in F$, $u_n = Ku_{n-1}$ for all $n \in \mathbb{N}$.

Then

$$\Phi(u_n, u_{n+1}) = \Phi(Ku_{n-1}, Ku_n) \leq$$

$$\frac{\xi_1\Phi(u_n,Ku_n)[1+\Phi(u_{n-1},Ku_{n-1})]}{1+\Phi(u_{n-1},u_n)}+\xi_2\Phi(u_{n-1},u_n)=$$
$$\frac{\xi_1\Phi(u_n,u_{n+1})[1+\Phi(u_{n-1},u_n)]}{1+\Phi(u_{n-1},u_n)}+\xi_2\Phi(u_{n-1},u_n)=$$
$$\xi_1\Phi(u_n,u_{n+1})+\xi_2\Phi(u_{n-1},u_n).$$

Thus

$$\Phi(u_n,u_{n+1})\leq\frac{\xi_2}{1-\xi_1}\Phi(u_{n-1},u_n).$$

An easy induction allows us to conclude that

$$\Phi(u_n,u_{n+1})\leq\left(\frac{\xi_2}{1-\xi_1}\right)^n\Phi(u_0,u_1).$$

So either $u_{n+1}=u_n$, in which case $u_n\in\mathrm{Fix}(K)$; or, (by (10.2)), $\{\Phi(u_n,u_{n+1})\}$ is a sequence decreasing to 0, so $u_n=u_{n+1}$ for $n\geq n_0$, for some $n_0\in\mathbb{N}$. Thus $u_{n_0}\in\mathrm{Fix}(K)$.

To show the uniqueness of our fixed point, suppose $u,v\in\mathrm{Fix}(K)$. By (10.1),

$$\Phi(u,v)=\Phi(Ku,Kv)\leq 0+\xi_2\Phi(u,v).$$

So $\Phi(u,v)=0$, i.e., $u=v$. □

10.2. [26]**'s Example 3.2.** This example uses $F=[0,1]$, which is not a subset of $\mathbb{Z}^n$ for any n. Thus F is not appropriate for use as a digital image.

11. [28]'s assertions with rational inequalities

The paper [28] claims fixed point results for self maps satisfying certain rational inequalities on digital images. However, these assertions are not well defined.

The assertion labeled "Theorem" 3.1 of [28] is as follows.

Assertion 11.1. Let (X,d,κ) be a digital metric space. Let S,T be self maps on X satisfying

$$d(S(x,y),T(u,v))\leq\alpha[d(x,u)+d(y,v)]+\beta[d(x,S(x,y))+d(u,T(u,v)]+$$
$$\gamma[d(x,T(u,v))+d(u,S(x,y))+\delta\left(\tfrac{d(x,S(x,y))d(u,T(u,v))}{d(x,u)+d(y,u)}\right)$$
$$+\eta\left[\tfrac{[d(x,u)+d(y,u)][d(x,S(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right]+$$
$$\zeta\left[\frac{d(u,S(x,y)+d(x,T(u,v))}{1+d(u,T(u,v))d(u,S(x,y)}\right]\tag{11.1}$$

for all $u,v,x,y\in X$ and

$$2(\alpha+\beta+\eta)+4(\gamma+\zeta)+\delta<1.$$

Then S and T have a common fixed point.

Several items are undefined in inequality (11.1):

- S and T are supposed to be defined on X, but throughout the inequality they appear as if defined on $X \times X$.
- The line beginning "γ[" is missing a matching "]" – should it come after the $d(u, S(x,y))$ term, or perhaps at the end of the line?
- In the same line, the last term has a denominator of 0 when $x = u = y$.
- Shouldn't the coefficients $\alpha, \beta, \gamma, \delta, \eta, \zeta$ have lower bounds (perhaps 0)?

Also, the "proof" of this "theorem" begins with sequences described as follows: x_0, y_0 are arbitrary members of X. Then

> "$x_{n+1} = S(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$, and $x_{n+2} = T(x_{n+1}, y_{n+1})$, $y_{n+2} = S(y_{n+1}, x_{n+1})$ for $n \in \mathbb{N}$."

Accordingly, we might have $x_{n+2} = T(x_{n+1}, y_{n+1})$, or we might have $x_{n+2} = x_{(n+1)+1} = S(x_{n+1}, y_{n+1})$. Similarly, y_{n+2} appears defined in two ways that seem incompatible (unless $S = T$).

We also observe that the "Corollaries" 3.2 through 3.6 of [28] all share one or more of the flaws discussed above. We conclude that whatever is intended by each of Assertion 11.1 and its "Corollaries" in [28] is unproven.

12. Further remarks

We have discussed several papers that seek to advance fixed point assertions for digital metric spaces. Many of these assertions are incorrect, incorrectly proven, or reduce to triviality; consequently, all of these papers should have been rejected or required to undergo extensive revisions. This reflects badly not only on the authors, but also on the referees and editors that approved their publication.

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