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H-partial uniform spaces and their application in the compression of digital images

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Abstract

Fixed point theorems are very important tools in different branches of mathematics. In this paper, we introduce partial uniform spaces as a generalization of metric spaces; and study some basic properties. Various examples support the theory. We prove fixed point theorems for H-partial uniform spaces by using a map called an E-distance function. Finally, we give the applications of these fixed point theorems to compress digital images.

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1. Introduction

There are lots of applications of fixed point theorems in image analysis [9, 10, 11, 18]. In this regard, many researchers have characterized the properties of digital images with tools from topology to study the properties of digital images. One such effort was done by Peters et al. in [15, 16, 17], where they introduced the concept of probe functions to characterize different feature values of a digital image. A digital image consists of a large number of small spots of colors, called pixels. When displayed on a monitor or printed on paper, pixels are so small and so closely packed together that the collective effect on

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the human eye is a continuous pattern of colors. The most common problem in digital images is decreasing their resolution (number of pixels), which we call compression of digital images. In this paper, we employ a probe function and fixed point theorems on partial uniform spaces in the compression of digital images (see also [18])

Before André Weil gave the first explicit definition of a uniform space in 1937, uniform concepts, like completeness were discussed using metric spaces. Nicolas Bourbaki provided the definition of uniform structures in terms of entourages [6]. Uniform spaces generalize metric spaces, pseudo metric spaces and topological spaces. The topology defined by the uniform structure is said to be induced by the uniformity. A topological space is said to be uniformizable, if there is a uniform structure compatible with the topology. Similar to continuous functions between topological spaces, which preserve the topological properties are the uniformly continuous functions between uniform spaces, which preserve uniform properties. The completeness property of uniform spaces opens the door to find fixed points in uniform spaces which has applications in various fields (see [18, 19]). In [1, 2, 3], Aamri et al. introduced the concept of an E-distance function on uniform spaces and utilized it to improve some well-known results of the existing literature involving both E-constructive and E-expansive maps. In [21, 22], Türkoğlu et al. proved fixed point theorems for multi-valued maps in uniform spaces (see also [4, 12]). In this paper, we study the category of partial uniform spaces as a super category of metric spaces, pseudo metric spaces and ultra metric spaces. Further, we prove fixed point theorems for a single-valued function and a multi-valued function defined for H-partial uniform spaces for the purpose of compressing digital images.

The paper is organized in the following manner: Section 2 collects some basic results and preliminaries on uniform spaces which are necessary for the development of further sections. In Section 3, we axiomatize the concept of partial uniform spaces as a generalization of various topological structures. The theory in this section builds up to prove some fundamental results required for the fixed point theorems proved in Section 4. A number of examples are given to support the new concept introduced. In Section 4, we prove two fixed point theorems: one for single-valued functions and the other for a multi-valued function in the framework of H-partial uniform spaces. In Section 5, we utilize the fixed point theorems proved in Section 4, to compress digital images using a probe function. Finally, in Section 6, we conclude the paper.

2. Preliminaries and Basic Results

In this section, we collect some useful definitions and results which are necessary to develop the theory in the subsequent sections. Throughout this paper, X is a non-empty set, $\mathcal{P}(X)$ denotes the power set of X, Γ denotes an arbitrary index set, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ denotes the set of all positive real numbers, and $\mathbb{N}$ denotes the set of all natural numbers. For any

$A \subseteq X$, the complement of A is denoted by A^c. Moreover, $A \subset B$ means that the set A is a proper subset of the set B.

Definition 2.1 ([6, 13])**.** A diagonal uniformity on the set X is a collection v of subsets of $X \times X$, which satisfies the following axioms:

(1) If $D \in v$, then $\Delta \subseteq D$, where $\Delta = \{(x, x) : x \in X\}$.
(2) If $D_1, D_2 \in v$, then $D_1 \cap D_2 \in v$.
(3) If $D \in v$, then $E \circ E \subseteq D$, for some $E \in v$.
(4) If $D \in v$, then $D^{-1} \in v$.
(5) If $D \in v$ and $D \subseteq E$, then $E \in v$.

Example 2.2. For each $\alpha \in \mathbb{R}^+$, let D_α be the subset of $\mathbb{R}^+ \times \mathbb{R}^+$ defined as $D_\alpha = \Delta \cup \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x > \alpha, y > \alpha\}$. Then the set $\{D_\alpha : \alpha \in \mathbb{R}^+\}$ forms a base for a uniformity on $\mathbb{R}^+$.

Definition 2.3 ([7])**.** A map $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a Čech closure operator on X, if it satisfies the following axioms for $A, B \subseteq X$,

$K1$. $cl(\varnothing) = \varnothing$.
$K2$. $A \subseteq cl(A)$.
$K3$. $cl(A) \cup cl(B) = cl(A \cup B)$.

Further, if cl also satisfies the following axiom,

$K4$. $cl(cl((A)) = cl(A)$.

Then it is a Kuratowski (topological) closure operator on X.

It is well-known that a Kuratowski closure operator induces a topology on X. In this paper, we will denote a topological space by (X, cl), where cl is the Kuratowski closure operator on X.

Definition 2.4 ([8, 14])**.** A topological space (X, cl) is said to be an R_0-space if, $x \in cl\{y\}$ implies $y \in cl\{x\}$.

Definition 2.5 ([1])**.** Let (X, v) be a uniform space. Then a function $\rho : X \times X \to \mathbb{R}^+$ is said to be an E-distance on X if:

(1) For any $V \in v$, if there exists $\delta > 0$ such that $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$, for some $z \in X$, then $(x, y) \in V$.
(2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$.

Definition 2.6 ([13])**.** A non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be a filter on X if :

(1) $\varnothing \notin \mathcal{F}$.
(2) If $F \in \mathcal{F}$ and $F \subseteq E$, then $E \in \mathcal{F}$.
(3) If $F, E \in \mathcal{F}$, then $F \cap E \in \mathcal{F}$.

A maximum filter on X is called an ultra-filter on X.

Definition 2.7. A metric on the set X is a map $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ which satisfies the following axioms:

$M1$. $d(x, y) \geqslant 0$, for all $x, y \in X$.
$M2$. $d(x, y) = 0$, if and only if $x = y$.

$M3.$ $d(x,y) = d(y,x)$, for all $x, y \in X$.
$M4.$ $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

The pair (X,d) is called a metric space.
The map $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is said to be a pseudo-metric on X, if it satisfies $M1$, $M3$, $M4$, and
$M5.$ $d(x,y) = 0$, if $x = y$.
Moreover, d is said to be an ultra-metric on X, if it satisfies $M1$, $M2$, $M3$, and
$M7.$ $d(x,y) \leq \max(d(x,z), d(z,y))$, for all $x, y, z \in X$.

3. Partial Uniform Spaces

In this section, we axiomatize Čech partial uniform spaces as a generalization of uniform spaces and various topological structures. Some basic topological results are proved in this section. Several examples are given to support the theory.

Definition 3.1. A collection $\mathcal{D}$ of subsets of $X \times X$ is called a Čech partial uniformity on X if, the following conditions are satisfied:

$P1.$ $\Delta \subseteq D$, $D \in \mathcal{D}$.
$P2.$ If $D_1, D_2 \in \mathcal{D}$, then $D_1 \circ D_2 \in \mathcal{D}$, where $D_1 \circ D_2 = \{(x,y) :$ there exists $z \in X$, such that $(x,z) \in D_1$ and $(z,y) \in D_2\}$.
$P3.$ If $D_1, D_2 \in \mathcal{D}$, then $D_1 \cap D_2 \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ is called a Čech partial uniform space.

Proposition 3.2. *Let $(X, \mathcal{D})$ be a Čech partial uniform space and let A be a subset of X. Then $cl_{\mathcal{D}}(A) = \{x \in X :$ for every $D \in \mathcal{D},$ there exists $y \in A,$ such that $(x,y) \in D\}$, is a Čech closure operator on X. Further, if $(X, \mathcal{D})$ satisfies the following condition:*

$P4.$ Let $x \in X$, $A \subseteq X$ and $D \in \mathcal{D}$. Then for any $y \in cl_{\mathcal{D}}(A)$, $(x,y) \in D$, implies that there exists $z \in A$, such that $(x,z) \in D$.

Then $cl_{\mathcal{D}}$ is a Kuratowski closure operator on X.

Proof. Let $(X, \mathcal{D})$ be a Čech partial uniform space. Then Axioms $K1$, $K2$ are obvious. Let $x \in cl_{\mathcal{D}}(A) \cup cl_{\mathcal{D}}(B)$, where A, B are any two subsets of X. This implies that, $x \in cl_{\mathcal{D}}(A)$ or $x \in cl_{\mathcal{D}}(B)$. So, for all $D \in \mathcal{D}$, there exists $y \in A \subseteq A \cup B$ or $y \in B \subseteq A \cup B$, such that $(x,y) \in D$. Consequently, for all $D \in \mathcal{D}$, there exists $y \in A \cup B$, such that $(x,y) \in D$. Hence, $x \in cl_{\mathcal{D}}(A \cup B)$. Conversely, let $x \in cl_{\mathcal{D}}(A \cup B)$. Then for all $D \in \mathcal{D}$ there exists $y \in A \cup B$ such that $(x,y) \in D$. Let if possible, $x \notin cl_{\mathcal{D}}(A)$ and $x \notin cl_{\mathcal{D}}(B)$. Then there exist, $D_1, D_2 \in \mathcal{D}$, such that $(x, y_1) \notin D_1$ for all $y_1 \in A$, $(x, y_2) \notin D_2$ for all $y_2 \in B$. So, $(x,y) \notin D_1 \cap D_2$ for all $y \in A \cup B$, which is a contradiction. Hence, $x \in cl_{\mathcal{D}}(A) \cup cl_{\mathcal{D}}(B)$. Moreover, let $(X, \mathcal{D})$ satisfy $P4$ and $s \in cl_{\mathcal{D}}(cl_{\mathcal{D}}(A))$. Then $s \in \{x \in X :$ for every $D \in \mathcal{D},$ there exists $y \in cl_{\mathcal{D}}(A)$ such that $(x,y) \in D\}$. Therefore by $P4$, $s \in \{x \in X :$ for every $D \in \mathcal{D},$ there exists $z \in A$ such that $(x,z) \in D\}$. Hence, $s \in cl_{\mathcal{D}}(A)$. Converse is obvious. $\square$

Definition 3.3. A collection $\mathcal{D}$ of subsets of $X \times X$ is called a partial uniformity on X, if it satisfies $P1$, $P2$, $P3$, and $P4$. The pair $(X, \mathcal{D})$ is called a partial uniform space.

Definition 3.4. Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be partial uniform spaces. Then a map $f : X \to Y$ is said to be a uniformly continuous if, for every $D' \in \mathcal{D}_Y$, there exists $D \in \mathcal{D}_X$ such that, $(f(x), f(y)) \in D'$ whenever, $(x, y) \in D$.

Definition 3.5. A partial uniformity $\mathcal{D}$ on X is called an R_0-partial uniformity on X if, it satisfies the following axiom:
$P5$. If $D \in \mathcal{D}$, then $D^{-1} \in \mathcal{D}$.

Clearly, every uniform space is a Čech partial uniform space. It may be checked that the topology generated by an R_0-partial uniform space is R_0 but not T_0, in general, as illustrated by the following example.

Example 3.6. Let $X = \{a, b, c\}$ and $D_1 = \Delta \cup \{(a, b), (b, a)\}$, $D_2 = \Delta \cup \{(a, c), (c, a), (b, c), (c, b)(a, b), (b, a)\}$. Then the collection $\mathcal{D} = \{D_1, D_2\}$ is an R_0-partial uniformity on X. The topology generated by $\mathcal{D}$ is $\{\varnothing, X, \{c\}, \{a, b\}\}$, which is an R_0-space but not a T_0-space.

Example 3.7. Let $\alpha \in \mathbb{R}^+$ and X be the set of all square matrices of order n such that their traces are α. Then the collection $\mathcal{D} = \{D \subseteq X \times X : \Delta \subseteq D\}$ forms a Čech partial uniformity on X, which is not a partial uniformity on X, in general.

Example 3.8. Let $X = \{a, b\}$ and $D_1 = \Delta \cup \{(a, b)\}$, $D_2 = \Delta \cup \{(a, b), (b, a)\}$. Then the collection $\mathcal{D} = \{D_1, D_2\}$ is a partial uniformity on X, which is not a uniformity on X, in general, and the topology generated by $\mathcal{D}$ is $\{\varnothing, X, \{b\}\}$.

Example 3.9. Let $X = \{a, b\}$ and $D_1 = \Delta \cup \{(b, a)\}$, $D_2 = \Delta \cup \{(a, b), (b, a)\}$. Then the collection $\mathcal{D} = \{D_1, D_2\}$ is a partial uniformity on X, which is not an R_0-partial uniformity and hence not a uniformity on X. The topology generated by $\mathcal{D}$ is $\{\varnothing, X, \{a\}\}$.

Example 3.10. Let $D = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a \geq b\}$. Then the collection $\mathcal{D} = \{\Delta, D\}$ is a partial uniformity on $\mathbb{R}$.

Example 3.11. Let $r \in \mathbb{R}^+$. Define $D_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < r\}$. Let $\mathcal{U} = \{E \subseteq D_r : E \circ E^{-1} = E \text{ and } \Delta \subset E, r \in \mathbb{R}^+\}$; and if $E_1, E_2 \in \mathcal{U}$, then $E_1 \circ E_2 = E_2 \circ E_1$. Then $\mathcal{U}$ is an R_0-partial uniformity on $\mathbb{R}$.

Remark 3.12. (1) The intersection of two partial uniformities (R_0-partial uniformities) on a set X is a partial uniformity (R_0-partial uniformity) on X.
(2) The union of two partial uniformities on X need not be a partial uniformity on X.

We now develop some basic theory of partial uniform spaces which is required to prove our results.

Proposition 3.13. *Let $(X, \mathcal{D}_X), (Y, \mathcal{D}_Y)$ and $(Z, \mathcal{D}_Z)$ be partial uniform spaces, and $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$, and $g : (Y, \mathcal{D}_Y) \to (Z, \mathcal{D}_Z)$ be uniformly continuous maps. Then $g \circ f : (X, \mathcal{D}_X) \to (Z, \mathcal{D}_Z)$ is a uniformly continuous map.*

Proof. Let $D'' \in \mathcal{D}_Z$. Then $((g \circ f)(x), (g \circ f)(y)) \in D''$, whenever $(f(x), f(y)) \in D'$, for some $D' \in \mathcal{D}_Y$. Consequently $((g \circ f)x, (g \circ f)y) \in D''$, whenever $(x, y) \in D$, for some $D \in \mathcal{D}_X$. Thus, $g \circ f$ is a uniformly continuous map. $\square$

Proposition 3.14. *If $(X, \mathcal{D})$ and $(Y, \mathcal{D}')$ be two partial uniform spaces, and $f : X \to Y$ be a uniformly continuous map. Then f is continuous on X.*

Proof. Let $(X, \mathcal{D})$ and $(Y, \mathcal{D}')$ be two partial uniform spaces, and let $f : X \to Y$ be a uniformly continuous map. Then for each $D' \in \mathcal{D}'$, there exists $D \in \mathcal{D}$ such that $(f(x), f(y)) \in D'$, whenever $(x, y) \in D$. Suppose that $cl_{\mathcal{D}}$ and $cl_{\mathcal{D}'}$ be the closures generated by the partial uniformities $\mathcal{D}$ and $\mathcal{D}'$, respectively, and $A \subseteq X$. Then we need to show that $f(cl_{\mathcal{D}}(A)) \subseteq cl_{\mathcal{D}'}(f(A))$. For this, let $y \in f(cl_{\mathcal{D}}(A))$. Then there exists $z \in cl_{\mathcal{D}}(A)$ such that $f(z) = y$. Thus for all $D \in \mathcal{D}$, there exists $x \in A$ such that $(z, x) \in D$. Since f is uniformly continuous, for any $D' \in \mathcal{D}'$, there exists $f(x) \in f(A)$ such that $(f(z), f(x)) \in D'$. Consequently, $f(z) = y \in cl_{\mathcal{D}'}(f(A))$. Hence, $f(cl_{\mathcal{D}}(A)) \subseteq cl_{\mathcal{D}'}(f(A))$. $\square$

Definition 3.15. The category **PUS** consists of objects as partial uniform spaces and morphisms as uniformly continuous maps. Similarly, the category $\mathbf{R_0PUS}$ consists of objects as R_0-partial uniform spaces and morphisms as uniformly continuous maps.

Clearly, $\mathbf{R_0PUS}$ is a full subcategory of **PUS**.

Proposition 3.16. *The category* **UMET** *of ultra-metric spaces and continuous maps is a full subcategory of the category* $\mathbf{R_0PUS}$.

Proof. Let (X, d) be an ultra-metric space, and let $\mathcal{D}$ be the collection of all subsets of $X \times X$, defined as $\mathcal{D} = \{D_\varepsilon : \varepsilon > 0\}$, where $D_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$, for every $\varepsilon > 0$. Then $\mathcal{D}$ satisfies $P1, P3$ and $P5$ clearly. For $P2$, let $D_{\varepsilon_1} \in \mathcal{D}$ and $D_{\varepsilon_2} \in \mathcal{D}$. Then $D_{\varepsilon_1} \circ D_{\varepsilon_2} = D_{\varepsilon_1}$, if $\varepsilon_1 \geqslant \varepsilon_2$, and $D_{\varepsilon_1} \circ D_{\varepsilon_2} = D_{\varepsilon_2}$, if $\varepsilon_2 \geqslant \varepsilon_1$. Finally let $(x, y) \in D_\varepsilon, y \in cl_{\mathcal{D}}(A), A \subseteq X$. Then $(x, y) \in D_\varepsilon, (y, z) \in D_\varepsilon$, for some $z \in A$. This implies that $d(x, z) < \varepsilon$, for some $z \in A$. That is $(x, z), \in D_\varepsilon$, for some $z \in A$. $\square$

Definition 3.17. Let $(X, \mathcal{D})$ be a partial uniform space. Then a function $f : X \times X \to \mathbb{R}^+$ is said to be an E-distance function on X if, the following conditions are satisfied:

(1) For any $D \in \mathcal{D}$, if there exists $\delta > 0$, such that $\rho(x, z) \leq \delta$ and $\rho(z, y) \leq \delta$, for some $z \in X$, then $(x, y) \in D$.
(2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$.
(3) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

Definition 3.18. A partial uniformity $\mathcal{D}$ on X is called an H-partial uniformity on X if, it satisfies the following axiom:

$P6$. $\bigcap\{D : D \in \mathcal{D}\} = \Delta$.

Definition 3.19. A Čech partial uniformity $\mathcal{D}$ on X is called a completely regular partial uniformity on X if, the following condition is satisfied:

$P7$. $D \circ D^{-1} \subseteq D$, $D \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ is called a completely regular partial uniform space.

It may be observed that a completely regular partial uniformity consists of symmetric entourages. In [20], we have shown that the topology generated by a completely regular partial uniform space is completely regular.

Remark 3.20. A completely regular partial uniform space $(X, \mathcal{D})$ is a partial uniform space because the axiom $P4$ follows from the axiom $P7$: Let $(x, y) \in D, y \in cl_{\mathcal{D}}(A), A \subseteq X$. Then $(x, y) \in D$ and $(y, z) \in D$, for some $z \in A$. Consequently, $(x, z) \in D \circ D \subseteq D, z \in A$.

Proposition 3.21. *The category* **MET** *of metric spaces and continuous maps is a full subcategory of the category* $\mathbf{R_0CPUS}$ *consisting of* R_0*-completely regular partial uniform spaces and uniformly continuous maps.*

Proof. Let (X, d) be a metric space and $r \in \mathbb{R}^+$. Define $D_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : d(x, y) < r\}$. Let $\mathcal{U} = \{E \subseteq D_r : E \circ E^{-1} = E$ and $\Delta \subset E, r \in \mathbb{R}^+\}$; and $E_1, E_2 \in \mathcal{U}$ implies $E_1 \circ E_2 = E_2 \circ E_1$. Then we can easily show that $\mathcal{U}$ is an R_0-completely regular partial uniformity on X. □

Corollary 3.22. *The category* **PMET** *of pseudo-metric spaces and continuous maps is a full subcategory of the category* $\mathbf{R_0PUS}$.

Corollary 3.23. *The category* **MET** *is a full subcategory of the category* $\mathbf{R_0PUS}$.

Example 3.24. Let $X = \{a, b, c\}$ and $D_1 = \Delta \cup \{(a, b), (b, a)\}$, $D_2 = \Delta \cup \{(a, c), (c, a), (b, c), (c, b)(a, b), (b, a)\}$. Then the collection $\mathcal{D} = \{D_1, D_2\}$ is a completely regular partial uniformity on X which is not a uniformity on X.

Completely regular partial uniform spaces are different from uniform spaces: In Example 3.24, $(X, \mathcal{D})$ is a completely regular partial uniform space which is not a uniform space. Moreover, a uniform space need not be completely regular partial uniform space as shown in the following example.

Example 3.25. For each $\alpha \in \mathbb{R}^+$, let D_α be the subset of $\mathbb{R}^+ \times \mathbb{R}^+$ defined as $D_\alpha = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : |x - y| < \alpha\}$. Then the set $\{D_\alpha : \alpha \in \mathbb{R}^+\}$ forms a uniformity on $\mathbb{R}^+$ which is not a completely regular partial uniformity on $\mathbb{R}^+ : (1, 2) \in D_1 \circ D_1^{-1}$ but $(1, 2) \notin D_1$.

In the next example, we construct a completely regular H-partial uniform space.

Example 3.26. Let A be a non-empty finite subset of $\mathbb{R}$, and $D_A = (\mathbb{R} - \{A\}) \times (\mathbb{R} - \{A\}) \cup \{(x, x) : x \in A\}$. Then the collection $\mathcal{D} = \{\mathbb{R} \times \mathbb{R}\} \cup \{D_A : A$ is a non-empty finite subset of $\mathbb{R}\}$ is a completely regular H-partial uniformity on $\mathbb{R}$: Clearly $\Delta \subseteq D_A$ for all non-empty finite subset A of $\mathbb{R}$. For $P2$, let $(x,y) \in D_A \circ D_B, x \neq y$, where A and B are two non-empty finite subsets of $\mathbb{R}$, such that $A \cap B \neq \varnothing$. Then $(x,z) \in D_A$ and $(z,y) \in D_B$, for some $z \in \mathbb{R}$. This implies that $x,z \notin A$ and $z,y \notin B$. Consequently, $x,y,z \notin A \cap B$. Hence, $(x,y) \in D_{A\cap B}$. Hence, $D_A \circ D_B \subseteq D_{A\cap B}$. Conversely, let $(x,y) \in D_{A\cap B}$. Then $x,y \notin A \cap B$. That is, we have the following cases: If $x \notin A, y \notin A$, then $(x,y) \in D_A \subseteq D_A \circ D_B$. If $x \notin B, y \notin B$, then $(x,y) \in D_B \subseteq D_A \circ D_B$. If $x \notin A, y \notin B$. Then we can find some $z \notin A \cup B$ such that $(x,z) \in D_A$ and $(z,y) \in D_B$. Consequently, $(x,y) \in D_A \circ D_B$. Similarly, if $x \notin B, y \notin A$, then also $D_{A\cap B} \subseteq D_A \circ D_B$. Hence, $D_A \circ D_B = D_{A\cap B}$. Thus, if $A \cap B \neq \varnothing$, then $D_A \circ D_B = D_{A\cap B} \in \mathcal{D}$. Further, if $A \cap B = \varnothing$, then $D_A \circ D_B = \mathbb{R} \times \mathbb{R} \in \mathcal{D}$. For $P3$, let $(x,y) \in D_A \cap D_B$, where A and B are two finite subsets of $\mathbb{R}$. Then for $x \neq y$, $x,y \notin A$ and $x,y \notin B$. Consequently, $x,y \notin A \cup B$. Hence, $(x,y) \in D_{A\cup B}$. Conversely, let $(x,y) \in D_{A\cup B}$. Then $x,y \notin A \cup B$. This implies that, $x,y \notin A$ and $x,y \notin B$. Consequently, $(x,y) \in D_A$ and $(x,y) \in D_B$. Hence, $(x,y) \in D_A \cap D_B$. Thus, $D_A \cap D_B = D_{A\cup B} \in \mathcal{D}$. Axiom $P6$ is obvious. Finally, for $P7$, we have $D_A \circ D_A^{-1} = D_A \circ D_A = D_A$.

Obviously, in Example 3.26, $\mathcal{D}$ is not a uniformity on $\mathbb{R}$. Let us study the effect of the axiom $P6$ on the topology generated by a completely regular H-partial uniform space. Infact, the topology comes out to be Hausdorff. To prove this, we need the following definition.

Definition 3.27. Let $\{(X_i, \mathcal{D}_i) : i = 1,2,3,\ldots,n\}$ be a family of n partial uniform spaces and $X = \Pi X_i$. Then the family $\mathcal{D}$ of subsets D of $X \times X$, corresponding to a unique pair of n entourages $D_j \in \mathcal{D}_i, i,j = 1,2,\cdots,n$ (i.e., choosing one D_j from each $\mathcal{D}_i$), where

$$D = \{((x_1^1, x_2^1, \ldots, x_n^1), (x_1^2, x_2^2, \ldots, x_n^2)) : (x_i^1, x_i^2) \in D_j, D_j \in \mathcal{D}_i, i = 1,2,3,\ldots,n\}$$

is a partial uniformity on X, which is called the product partial uniformity on X. The pair $(X, \mathcal{D})$ is called the product partial uniform space.

Theorem 3.28. *The topology generated by a completely regular H-partial uniformity is Hausdorff.*

Proof. Let $(X, \mathcal{D})$ be a completely regular H- partial uniform space. Then $D_i \times D_j = \{((a,b),(c,d)) : (a,c) \in D_i, (b,d) \in D_j\}$. Define $\mathcal{D}^* = \{D_i \times D_j : D_i, D_j \in \mathcal{D}\}$. Then $\mathcal{D}^*$ is the product partial uniformity on $X \times X$. Let $(x,y) \in cl_{\mathcal{D}^*}(\Delta)$. Then for all $D_i \times D_j \in \mathcal{D}^*$, there exists $z \in X$ such that $((x,y),(z,z)) \in D_i \times D_j$. So for $D_i \times D_i$, there exists $z \in X$ such that $((x,y),(z,z)) \in D_i \times D_i$. This implies that, $(x,z) \in D_i, (y,z) \in D_i$. Consequently, $(x,y) \in D_i \circ D_i^{-1} = D_i$. This is true for all $D_i \in \mathcal{D}$. As a result, $(x,y) \in \bigcap\{D_i : D_i \in \mathcal{D}\} = \Delta$. So, $x = y$, therefore $(x,y) \in \Delta$. Hence $cl_{\mathcal{D}}(\Delta) = \Delta$. ☐

Definition 3.29 (cf. [1, 5])**.** Let $(X, \mathcal{D})$ be a partial uniform space and ρ be an E-distance function on X. Then

(1) A sequence $< x_n >$ in the partial uniform space $(X, \mathcal{D})$ is said to be a ρ-Cauchy sequence if, for each positive real number ε, there exists a positive integer, m, such that $\rho(x_n, x_m) < \varepsilon$, $n \geq m$.
(2) X is said to be S-complete if, for each ρ-Cauchy sequence $< x_n >$, there exists $x \in X$, satisfying $\lim_{n\to\infty} \rho(x_n, x) = 0$.
(3) X is said to be ρ-Cauchy complete if, for each ρ-Cauchy sequence $< x_n >$, there exists $x \in X$ satisfying $\lim_{n\to\infty} x_n = x$.
(4) A map $f : X \to X$ is said to be ρ-continuous if, $\lim_{n\to\infty} \rho(x_n, x) = 0$, implies $\lim_{n\to\infty} \rho(f(x_n), f(x)) = 0$.
(5) A map $f : X \to X$ is said to be continuous if, $\lim_{n\to\infty} x_n = x$, implies $\lim_{n\to\infty} f(x_n) = f(x)$.

Definition 3.30 (cf. [5]). A partial uniform space $(X, \mathcal{D})$ is said to be precompact or totally bounded if, there exists a finite set $S = \{x_1, x_2, \cdots, x_n\}$ such that $X = \cup_{x\in S} cl_{\mathcal{D}}\{x\}$.

Definition 3.31 (cf. [13]). Let $(X, \mathcal{D})$ be a partial uniform space. Then

(1) a net $\{x_a : a \in \Gamma\}$ is said to be a Cauchy net if, for each $D \in \mathcal{D}$, there exists $a_0 \in \Gamma$ such that $(x_a, x_b) \in D$ for $a, b > a_0$.
(2) A filter $F = \{V_a : a \in \Gamma\}$ is said to be a Cauchy filter if, for each $D \in \mathcal{D}$, there exists $V_a \in F$ such that $(x_a, y_a) \in D$, $x_a, y_a \in V_a$.
(3) A net $\{x_a : a \in \Gamma\}$ is said to be convergent to a point $x_0 \in X$ if, for each $D \in \mathcal{D}$, there exists $a_0 \in \Gamma$ such that $(x_a, x_0) \in D$, for all $a > a_0$.
(4) A filter $F = \{V_a : a \in \Gamma\}$ is said to be convergent to a point $x_0 \in X$ if, for each $D \in \mathcal{D}$, there exists $V_a \in F$ such that $(x_a, x_0) \in D$, for all $x_a \in V_a$.
(5) $(X, \mathcal{D})$ is said to be complete if, each Cauchy net (Cauchy filter) in X converges to a point in X.
(6) A point x_0 is said to be a cluster point of the net $\{x_a : a \in \Gamma\}$ in X if, for all $D \in \mathcal{D}, (x_a, x_0) \in D$ for infinitely many values of a.
(7) A point $x \in X$ is said to be a limit point of $A \subseteq X$, if for all $D \in \mathcal{D}$ there exists $y \in A - \{x\}$ such that $(x, y) \in D$. The set of all limit points of A is called the derived set of A and is denoted by $Der(A)$.

Proposition 3.32. *If $(X, \mathcal{D})$ is a partial uniform space which is totally bounded, then it is separable.*

Proof. Let $(X, \mathcal{D})$ be a partial uniform space which is totally bounded. Then there exists a finite set $S = \{x_1, x_2, \cdots, x_n\}$ such that $X = \bigcup_{x_i\in S} cl_{\mathcal{D}}\{x_i\}$. Let $a \in X$. Then $a \in cl_{\mathcal{D}}\{x_i\}$ for some $x_i \in S$. So, $(a, x_i) \in D$, for all $D \in \mathcal{D}$. It is obvious that $a \in cl_{\mathcal{D}}\{S\}$. Therefore $X = cl_{\mathcal{D}}\{S\}$. Thus, S is a countable dense subset of X. □

Proposition 3.33. *Every convergent net (filter) in a completely regular partial uniform space $(X, \mathcal{D})$ is a Cauchy net (Cauchy filter).*

Proof. Let $(X, \mathcal{D})$ be a completely regular partial uniform space and let $n = \{x_a : a \in \Gamma\}$ be a convergent net which converges to $x_0 \in X$. Then for each $D \in \mathcal{D}$, there exists $a_0 \in \Gamma$ such that $(x_a, x_0) \in D$, for all $a > a_0$. Further, if $b > a_0$, then $(x_b, x_0) \in D$. Consequently, $(x_a, x_b) \in D \circ D^{-1}$, $a, b > a_0$. Since $D \circ D^{-1} \subseteq D$, therefore $(x_a, x_b) \in D$, $a, b > a_0$. Thus n is a Cauchy net. Similarly, let $\mathcal{F} = \{V_a : a \in \Gamma\}$ be a filter which is convergent to a point $x_0 \in X$. So, for each $D \in \mathcal{D}$, there exists $V_a \in \mathcal{F}$ such that $(x_a, x_0) \in D$, for all $x_a \in V_a$. It is obvious that, for $x_b \in V_a$, $(x_b, x_0) \in D$. Thus $(x_0, x_b) \in D^{-1}$, $x_b \in V_a$. Consequently, $(x_a, x_b) \in D \circ D^{-1}$, for $x_b, x_a \in V_a$. Thus $(x_a, x_b) \in D$, $x_b, x_a \in V_a$. Hence, $\mathcal{F}$ is a Cauchy filter in X. □

Proposition 3.34. *A Cauchy net* (*Cauchy filter*) *in a completely regular partial uniform space* $(X, \mathcal{D})$ *converges to its cluster point.*

Proof. Let $(X, \mathcal{D})$ be a completely regular partial uniform space and let $n = \{x_a : a \in \Gamma\}$ be a Cauchy net in X. Let x_0 be a cluster point of n. So, for any $D \in \mathcal{D}$, $(x_a, x_0) \in D$ for infinitely many $a \in \Gamma$. Since n is a Cauchy net, therefore for each $D \in \mathcal{D}$, there exists $a_0 \in \Gamma$ such that $(x_a, x_b) \in D$, for $a, b > a_0$. Thus $(x_b, x_a) \in D^{-1}$, for $a, b > a_0$. Consequently, $(x_b, x_0) \in D^{-1} \circ D = D \circ D^{-1} \subseteq D$, for $b > a_0$, $D \in \mathcal{D}$. Therefore $(x_b, x_0) \in D$, for $b > a_0$, $D \in \mathcal{D}$. Thus n converges to its cluster point. The proof for the Cauchy filter follows similarly. □

Corollary 3.35. *In a completely regular H-partial uniform space, every Cauchy net* (*Cauchy filter*) *converges to its unique cluster point.*

Proposition 3.36. *Every closed subset of a complete partial uniform space is complete.*

Proof. Let $(X, \mathcal{D})$ be a complete partial uniform space and F be a closed subset of X. Now let $n = \{x_a : a \in \Gamma\}$ be a Cauchy net in F. Then n converges to a point $x_0 \in X$. So it must be a cluster point of F. Since F be a closed subset of X, therefore $x_0 \in F$. Thus, the net n converges to a point $x_0 \in F$. Hence, F is complete. □

4. Fixed Point Theorems on H-Partial Uniform Spaces

In this section, we prove two fixed point theorems: one for a single-valued function and the other for a multi-valued function, defined on H-partial uniform spaces.

Definition 4.1 ([5])**.** A multi-valued operator $T : X \to \mathcal{P}(X)$ is said to be order closed if, for monotonic sequences $< \mu_n >$ and $< \nu_n >$ of X, $\mu_n \to \mu_0$, $\nu_n \to \nu_0$ and $\nu_n \in T(\mu_n)$, implies $v_0 \in T(u_0)$.

Definition 4.2 ([5])**.** Let $(X, \mathcal{D})$ be an H-partial uniform space and ρ be the E-distance function on X. A multi-valued operator $T : X \to \mathcal{P}(X)$ is called ρ-order closed if, for monotonic sequences $< \mu_n >$ and $< \nu_n >$ in X, $\lim_{n\to\infty} \rho(\mu_n, \mu_0) = \lim_{n\to\infty} \rho(\nu_n, \nu_0) = 0$ and $\nu_n \in T(\mu_n)$, implies $\nu_0 \in T(\mu_0)$.

Lemma 4.3 (cf. [1, 5])**.** *Let $(X, \mathcal{D})$ be an H-partial uniform space and ρ be an E-distance function on X, let $< x_n >$ and $< y_n >$ be two arbitrary sequences in X, and $< a_n >$, $< b_n >$ be two sequences in $\mathbb{R}^+$ converging to zero. Then for $x, y, z \in X$, the following axioms hold:*

(1) *If $\rho(x_n, y) \leq a_n$ and $\rho(x_n, z) \leq b_n$, $n \in \mathbb{N}$, then $y = z$ in other words, if $\rho(x, y) = 0 = \rho(x, z)$, then $y = z$.*
(2) *If $\rho(x_n, y_n) \leq a_n$, $n \geq m$, then $< x_n >$ is a ρ-Cauchy sequence in X.*

Proof. (1) Let $< a_n >$, $< b_n >$ be two sequences in $\mathbb{R}^+$ converging to zero. Then for any positive real number, say δ, there exists a positive integer, say m, such that $|a_n| < \delta, n > m$. By using the condition (1) we have, $\rho(x_n, y) < \delta$ and $\rho(x_n, z) < \delta$. So for any $D \in \mathcal{D}$, there exists δ such that $\rho(x_n, y) < \delta$ and $\rho(x_n, z) < \delta$. Consequently, $(y, z) \in D$, for all $D \in \mathcal{D}$. By $P6$, $y = z$.
(2) Obvious.

□

Lemma 4.4. *Let $(X, \mathcal{D})$ be an H-partial uniform space, ρ be an E-distance function on X and $\Phi : X \to \mathbb{R}^n$ be a vector-valued function such that $\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \cdots, \phi_n(x)) \in \mathbb{R}^n$, where $\phi_i : X \to \mathbb{R}$ is a real-valued function for each $i = 1, 2, \ldots, n$. Define a relation $\preceq$ on X as given below:*

$$x \preceq y \text{ if and only if } x = y \text{ or } \rho(x, y) \leq \phi_i(x) - \phi_i(y), \text{ for all } i = 1, 2, \ldots, n.$$

Then $\preceq$ is a partial order relation on X induced by Φ.

Proof. Clearly, the relation $\preceq$ is reflexive and transitive. To prove that $\preceq$ is antisymmetric, let $x \preceq y$ and $y \preceq x$. Then either $x = y$ or $\rho(x, y) \leq \phi_i(x) - \phi_i(y)$ and $\rho(y, x) \leq \phi_i(y) - \phi_i(x)$. If $x = y$, then the proof follows. If $x \neq y$, then $\rho(y, x) + \rho(x, y) = 0$ because $\rho(y, x) = \rho(x, y)$. Consequently, $\rho(x, y) = 0$. Since $\rho(y, y) \leq \rho(y, x) + \rho(x, y) = 0$, by Lemma 4.3, $x = y$. □

Below is the fixed point theorem proved for single-valued functions in the framework of H-partial uniform spaces. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, is the usual norm defined on $\mathbb{R}^n$. Also, $x \leq y$ if and only if $x_i \leq y_i$, for all $i = 1, 2, \ldots, n$.

Theorem 4.5. *Let $(X, \mathcal{D})$ be an H-partial uniform space and ρ be the E-distance function on X such that X is S-complete. Let $\Phi : X \to \mathbb{R}^n$ be a function which is bounded below. If $\preceq$ is a partial order relation induced by Φ (defined in Lemma 4.4) and $f : X \to X$ is a ρ-continuous non-decreasing function with $x_0 \preceq f(x_0)$, for some $x_0 \in X$. Then f has a fixed point in X.*

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq f(x_0)$. Define a sequence $< x_n >$ in X, such that $x_n = f(x_{n-1})$, for $n = 1, 2, \cdots$. Since f is non-decreasing, therefore $f(x_0) \preceq f(x_1) \preceq f(x_2) \preceq \cdots$, which implies $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$. Thus, the sequence $< x_n >$ is non-decreasing. Since $\preceq$ is a partial order relation, so $\cdots \leqslant \Phi(x_2) \leqslant \Phi(x_1) \leqslant \Phi(x_0)$. It implies that the sequence $< \Phi(x_n) >$ is non-increasing in $\mathbb{R}^n$. Since Φ is bounded below in $\mathbb{R}^n$, therefore

$< \Phi(x_n) >$ is a convergent sequence and hence a Cauchy sequence in $\mathbb{R}^n$. Then for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\Phi(x_n) - \Phi(x_m)\| < \varepsilon$, $n, m \geq n_0$, where $\|.\|$ is the usual norm on $\mathbb{R}^n$. Since $\rho(x_n, x_m) \leqslant \|\Phi(x_n) - \Phi(y_m)\|$, we have $\rho(x_n, x_m) < \varepsilon$ for, $m, n > n_0$. As a result, $< x_n >$ is a ρ-Cauchy sequence. So by S-completeness of X and ρ-continuity of f, there exists $z \in X$ such that $\lim_{n\to\infty} \rho(x_n, z) = \lim_{n\to\infty} \rho(f(x_n), f(z)) = 0$. Hence, $f(z) = z$, by Lemma 4.3. $\square$

Now, we prove a fixed point theorem for a multi-valued function in the skeleton of H-partial uniform spaces.

Theorem 4.6. *Let $(X, \mathcal{D})$ be an H-partial uniform space and ρ be the E-distance function on X such that X is S-complete. Let $\Phi : X \to \mathbb{R}^n$ be a function which is bounded below. Suppose that $\preceq$ is a partial order induced by Φ (defined in Lemma 4.4) and $f : X \to \mathcal{P}(X) - \{\varnothing\}$ is a ρ-order closed multi-valued operator with $x_0 \preceq f(x_0)$, for some $x_0 \in X$. Further, let $x, y \in X$ with $x \preceq y$ implies for every $a \in f(x)$, there exists $b \in f(y)$ such that $a \preceq b$. Then f has a fixed point in X.*

Proof. Since $f(x)$ is non-empty for each $x \in X$, therefore there exists $x_1 \in f(x_0)$ such that $x_0 \preceq x_1$. Similarly there exists $x_2 \in f(x_1)$ such that $x_1 \preceq x_2$. By continuing this process, we get a non-decreasing sequence $< x_n >$ which satisfies $x_{n+1} \in f(x_n)$. Since $\preceq$ as a partial order relation, therefore $\cdots \leqslant \Phi(x_n) \leqslant \cdots \leqslant \Phi(x_2) \leqslant \Phi(x_1) \leqslant \Phi(x_0)$, i.e., $< \Phi(x_n) >$ is a non-increasing sequence in $\mathbb{R}^n$. Since Φ is bounded below, therefore the sequence $< \Phi(x_n) >$ is a Cauchy sequence in $\mathbb{R}^n$. Then for each $\varepsilon > 0$, there exists a positive integer m such that $\|\Phi(x_n) - \Phi(x_m)\| < \varepsilon$, $n \geq m$. Since $x_m \preceq x_n$, therefore either $x_n = x_m$ or consequently, $\rho(x_n, x_m) < \varepsilon$. Therefore $< x_n >$ is a ρ-Cauchy sequence. Thus, by the S-completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} \rho(x_n, z) = 0 = \lim_{n\to\infty} \rho(x_{n+1}, z)$. As a result, $z \in f(z)$ because f is ρ-ordered closed. $\square$

5. Application: Digital Image Compression

A good digital image requires large number of bytes to represent itself. So, a popular problem in the field of image analysis is digital image compression. Image compression is a process that makes image files smaller. Image compression most often works by removing bytes of information from the image, in a way that takes up less storage space and without degrading image quality below an acceptable threshold. This process is often done by fixed point theorems in the framework of digital images. In this section, we will apply separately Theorem 4.5 and Theorem 4.6 to compress a digital image X, shown in Figure 2(A). In [15, 16, 17], Peters et al. introduced probe functions to represent various feature values, viz., color, edge, etc., of a digital image. Probe function further describes the perceptual neighbourhood of a pixel in a digital image which in turn solve several problems in digital image analysis. Let us define a probe function first. In this section, we assume X is a set of all pixels of a digital

image.
A function $\Phi : X \to \mathbb{R}^n$ is called a probe function (object description), defined as

$$\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \cdots, \phi_n(x)),\ x \in X.$$

This description function gives the different n features $\phi_1(x), \phi_2(x), \phi_3(x), \cdots, \phi_n(x)$ of an object (pixel) $x \in X$, say colour, edge, etc.
Next, we define the 8-neighbourhood of a pixel $x \in X$.

Definition 5.1. Let X be a digital image. Then a pixel q is in 8-neighbours of a given pixel p if p and q either share an edge or a vertex. The set of all pixels in the 8-neighbourhood of p is denoted by $N_8(p)$.

Figure 1 represents 8-neighbourhood of pixel p. Now we consider the digital image X in Figure 2(A). In Example 5.1, we employ Theorem 4.6 to compress the digital image X which can be taken as the set of all pixels representing X.

Example 5.2. Consider the digital image in X in Figure 2(A). Take a small portion 'A' in X. Define a descriptive distance $d : A \times A \to \mathbb{R}^+$ by,

$$\begin{aligned} d(x,y) &= \sqrt{(\phi_1(x) - \phi_1(y))^2 + (\phi_2(x) - \phi_2(y))^2 + \cdots + (\phi_n(x) - \phi_n(y))^2} \\ &= \|\Phi(x) - \Phi(y)\|. \end{aligned}$$

Here $\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \cdots, \phi_n(x))$ is the probe function, where $x \in A$.
Let $U_\varepsilon = \{(x,y) \in A \times A : d(x,y) < \varepsilon, \varepsilon > 0\}$. Then $\mathcal{U} = \{E \subseteq U_\varepsilon : E \circ E^{-1} = E, r \in \mathbb{R}^+\}$; and $E_1, E_2 \in \mathcal{U}$ implies $E_1 \circ E_2 = E_2 \circ E_1$, is an H-partial uniformity on A.
Define an E-distance function $\rho : A \times A \to \mathbb{R}^+$ by,
$\rho(x,y) = \min\{|\phi_1(x) - \phi_1(y))|, |\phi_2(x) - \phi_2(y)|, |\phi_3(x) - \phi_3(y)|, \cdots, |\phi_n(x) - \phi_n(y)|\}$.

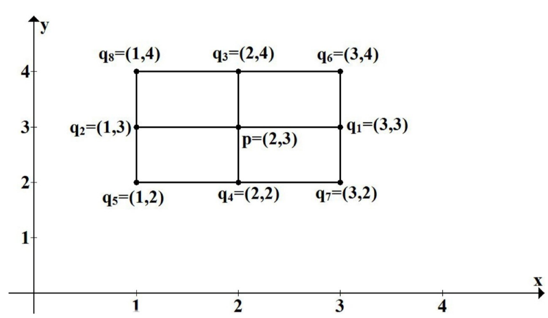
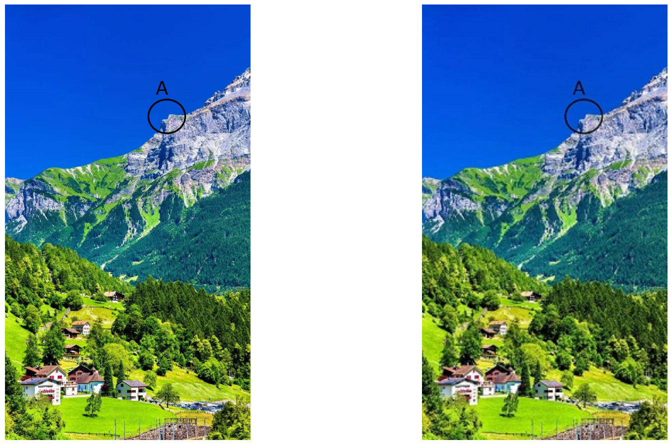


FIGURE 1. 8-neighbourhood of p.

Further, define a relation $\preceq$ on A as,
$x \preceq y$ if and only if $x = y$ or $\rho(x,y) \leqslant \phi_i(x) - \phi_i(y)$, $x, y \in A$, and for all



(A) Original image X (B) Compressed image

FIGURE 2. Original image X (set of pixels) and its compressed version.

$i = 1, 2, \dots, n$.
Then $\preceq$ is a partial order on A. Let $f : A \to \mathcal{P}(A) - \{\varnothing\}$ be defined as,
$f(x) = y$, such that $\|\Phi(x) - \Phi(y)\| \leqslant 2$, $x \in A$. By Theorem 4.6, the function f has a fixed point. The compressed part of the portion A of X is the set of all fixed points of A. Consequently, the compressed image of X is the set of all fixed points of various portions of the image obtained via this method, which is represented in Figure 2(B).

In the next example, we give the application of the Theorem 4.5 in the compression of the image X.

Example 5.3. Again, consider the portion A of the digital image X, shown in Figure 2(A). Define the H-partial uniformity $\mathcal{U}$, the E-distance function ρ and partial ordering $\preceq$ similarly as in Example 5.1. Let $f : A \to A$ be defined by,

$$f(x) = y, \text{ such that, } y \in N_8(x), x \in A, \text{ and } \|\Phi(x) - \Phi(y)\| = \min_{z \in N_8(x)} \|\Phi(x) - \Phi(z)\|.$$

Note that if, for $z_1, z_2 \in N_8(x)$, $\|\Phi(x) - \Phi(z_1)\| = \|\Phi(x) - \Phi(z_2)\|$, then we take either $f(x) = z_1$ or $f(x) = z_2$, i.e., we choose only one value from z_1 and z_2, so on.

Thus, by Theorem 4.5, the function f has a fixed point. The compressed part of the portion A of X is the set of all fixed points of A. Finally, collecting all the fixed points of various portions of the image, in this way, gives the compressed image Figure 2(B) of X.

6. Conclusion

In this paper, we study partial uniform spaces as topological structures different from uniform spaces. Further the category $\mathbf{R_0PUS}$ of R_0-partial uniform spaces and uniformly continuous maps is a super category of the categories **MET**, **UMET** and **PMET**. Finally, two fixed point theorems are proved in the framework of the new structure introduced for the purpose of compressing digital images.

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