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Existence of fixed points of large MR-Kannan contractions in Banach spaces

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Abstract

The purpose of this paper is to introduce the class of large MR-Kannan contractions on Banach space that contains the classes of Kannan, enriched Kannan, large Kannan, MR-Kannan contractions and some other classes of nonlinear operators. Some examples are presented to support the concepts introduced herein. We prove the existence of a unique fixed point for such a class of operators in Banach spaces.

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1. Introduction

Let (X, d) be a metric space and $T : X \to X$ be a self operator. We denote the set $\{x \in X : T(x) = x\}$ of fixed point of T by $F(T)$. Solving a fixed point

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problem of an operator T, denote by $FPP(T)$ is to show that the set $F(T)$ is nonempty.
In 1922, Banach [7] presented the idea of the Banach fixed point theorem. We state the Banach fixed point theorem in the context of normed spaces.

Theorem 1.1 ([7]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a Banach contraction, that is an operator satisfying*

$$\|Tx - Ty\| \leq \theta\{\|x - y\|\}, \quad \forall x, y \in X, \tag{1.1}$$

with $0 < \theta < 1$. Then, T has a unique fixed point.

In 1968, Kannan [18] generalized Theorem 1.1 by presenting the idea of the Kannan fixed point operator. We state the Kannan fixed point theorem in the context of normed spaces.

Theorem 1.2 ([18]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a Kannan contraction, that is an operator satisfying*

$$\|Tx - Ty\| \leq \lambda\{\|x - Tx\| + \|y - Ty\|\}, \quad \forall x, y \in X, \tag{1.2}$$

with $0 \leq \lambda < \frac{1}{2}$. Then, T has a unique fixed point.

We present the following example that satisfies the Kannan contraction but not the Banach contraction.

Example 1.3. Let $X = \mathbb{R}$ and $T : X \to X$ be an operator defined by

$$Tx = \begin{cases} 0 & \text{if } x \leq 2, \\ -\frac{1}{4} & \text{if } x > 2. \end{cases}$$

Now, we can easily prove that for $\lambda = \frac{1}{4}$, it satisfies (1.2) such that

$$|Tx - Ty| \leq \frac{1}{4}\{|x - Tx| + |y - Ty|\}, \quad \forall x, y \in \mathbb{R}.$$

Clearly, T is discontinuous at $x = 2$. Thus, T does not satisfy the Banach contraction.

Now, we present the following example that satisfies the Banach contraction but not the Kannan contraction.

Example 1.4. Let $X = [0, 1]$ and $T : X \to X$ be an operator defined by

$$Tx = \frac{x}{2}, \quad \forall x \in X.$$

Indeed, let T be a Banach contraction, then for $\theta = \frac{1}{2}$, it satisfies (1.1) such that

$$|Tx - Ty| = \frac{1}{2}\{|x - y|\}, \quad \forall x, y \in X.$$

If T would be a Kannan contraction, then by (1.2) there exist $0 \leq \lambda < \frac{1}{2}$ such that

$$\frac{1}{2}\,|x - y| \leq \frac{1}{2}\lambda\{\,|x| + |y|\,\}, \quad \forall x, y \in X, \tag{1.3}$$

which does not satisfy for any $0 < \lambda < \frac{1}{2}$ on taking $x = 0$ and $y = 1$.

These examples show that Banach contractions and Kannan contractions are independent of each other, but the following result concludes that Banach contractions are Kannan contractions under some specific condition.

Lemma 1.5 ([15]). *Every Banach contraction with $0 \leq \theta < \frac{1}{3}$ is a Kannan contraction.*

Proof. We can refer the reader to the proof of Lemma 1 of ([15]) that leads to a conclusion. □

Many mathematicians have generalized the Kannan results, as detailed in the cited references ([4, 9, 17, 19, 21, 20, 22, 25]). One of the interesting generalization of the Kannan result was given by Dehici et al. ([15]).
In 2019, Dehici et al. [15] generalized Theorem 1.2 by presenting the idea of large Kannan contraction.
The main result of [15] in the context of normed spaces is stated as follows:

Theorem 1.6 ([15]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a large Kannan contraction, that is an operator satisfying*

$$\begin{cases} \|Tx - Ty\| < \|x - y\|, \quad \forall x, y \in X, \\ \text{with } x \neq y \text{ and for every } \epsilon > 0, \text{ there exist } \delta < \frac{1}{2} \text{ such that} \\ [x, y \in X, \ \|x - y\| \geq \epsilon] \Longrightarrow \\ \|Tx - Ty\| \leq \delta \{ \|x - Tx\| + \|y - Ty\| \}. \end{cases} \tag{1.4}$$

Then, T has a unique fixed point.

Every Kannan contraction is a large Kannan contraction, but converse is not true as it follows from Example 4 of ([15]).
In 2020, Berinde and Păcurar [11] generalized Theorem 1.6 by presenting the idea of enriched Banach contraction.
The main result of [11] is stated as follows:

Theorem 1.7 ([11]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (b, θ)-enriched Banach contraction, that is an operator satisfying*

$$\|b(x - y) + Tx - Ty\| \leq \theta\{\|x - y\|\}, \quad \forall x, y \in X, \tag{1.5}$$

with $0 \leq b < +\infty$ and $0 \leq \theta < b + 1$. Then, T has a unique fixed point.

In 2020, Berinde and Păcurar [10] generalized Theorem 1.7 by presenting the idea of enriched Kannan contraction.
The main result of [10] is stated as follows:

Theorem 1.8. [10] *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (a, b)-enriched Kannan contraction, that is an operator satisfying*

$$\|b(x - y) + Tx - Ty\| \leq a\{\|x - Tx\| + \|y - Ty\|\}, \quad \forall x, y \in X, \tag{1.6}$$

with $0 \leq b < +\infty$ and $0 \leq a < \frac{1}{2}$. Then, T has a unique fixed point.

For more results in this direction, we refer ([3, 5, 1, 2, 10, 12, 13, 16] and references therein).

Now, we impose the following question:
Question: Under which condition does an (b,θ)-enriched Banach contraction become an (a,b)-enriched Kannan contraction?

In 2023, Anjum et al. [6] generalized Theorem 1.8 by presenting the idea of (ψ, a)-MR-Kannan type contraction.
Before presenting the main result of MR Kannan type contraction, we need the following result from ([6]):
Let

$$\zeta = \{\psi : X \to \mathbb{R} : \psi(x) \neq -1, \forall x \in X\}.$$

The main Theorem of [6] is stated as follows.

Theorem 1.9 ([6]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (ψ, a)-MR Kannan type contraction, that is an operator satisfying*

$$\left\| \frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y))} \right\| \leq a\left(\left| \frac{1}{1 + \psi(x)} \right| \|x - Tx\| + \left| \frac{1}{1 + \psi(y)} \right| \|y - Ty\| \right), \tag{1.7}$$

for all $x, y \in X$, with $0 \leq a < \frac{1}{2}$ and $\psi \in \zeta$. Then, T has a unique fixed point.

Remark 1.10.
(i) If $\psi(x) = 0$, for all $x \in X$ in contractive condition (1.7), we obtain Theorem 1.2.
(ii) If $\psi(x) = b > 0$, for all $x \in X$ in contractive condition (1.7), we obtain Theorem 1.8.

The aim of this paper is manifold. Firstly, we define the new class of operator called large MR-Kannan contraction, which includes the Kannan, Enriched Kannan, Large Kannan, and MR Kannan contractions. Secondly, we prove the existence of a unique fixed point for such a class of operators. Thirdly, we generalized the main result by introducing the class of some real-valued control functions and the existence of a unique fixed point for this class of operators. Finally, the existence of a unique fixed point for such a class of operators in the non-necessarily continuous sense.

2. Main Result

To answer the above question, we start this section with the following lemma:

Lemma 2.1. *Every (b,θ)-enriched Banach contraction with $0 \leq \theta < \frac{b+1}{3}$ is an (a,b)-enriched Kannan contraction.*

Proof. It follows from ([14]) that for $\lambda = \frac{1}{b+1}$, the (b,θ)-enriched Banach contraction (1.5) is equivalent to following Banach contraction:

$$\|T_\lambda x - T_\lambda y\| \leq c \|x - y\|, \quad \forall x, y \in X, \tag{2.1}$$

where $\lambda\theta = c$. Since $0 \leq \theta < \frac{b+1}{3}$, then $c \in [0, 1/3)$.
Also, it follows from ([10]) that for $\lambda = \frac{1}{b+1}$, (a,b)-enriched Kannan contraction (1.6) is equivalent to following Kannan Contraction:

$$\|T_\lambda x - T_\lambda y\| \leq a\big\{ \|x - T_\lambda x\| + \|y - T_\lambda y\| \big\}. \tag{2.2}$$

where $a \in [0, \frac{1}{2})$.
Hence the conclusion follows from Lemma (1.5). □

2.1. **Large MR-Kannan Contractions.**
Let us begin with the definition that follows.

Definition 2.2. Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$ be a large (ψ, δ)-MR Kannan contraction satifying

$$\begin{cases} \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| < \|x-y\|, \quad \forall x, y \in X, \\ \text{where } \psi \in \zeta, \text{ with } x \neq y \text{ and for every } \epsilon > 0, \text{ there exists } \delta < \frac{1}{2} \text{ such that} \\ [x, y \in X, \quad \|x-y\| \geq \epsilon] \Longrightarrow \\ \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| \leq \delta \left\{ \left\| \frac{x-Tx}{1+\psi(x)} \right\| + \left\| \frac{y-Ty}{1+\psi(y)} \right\| \right\}. \end{cases} \tag{2.3}$$

Now, we present the following example that satisfies the large (ψ, a)-MR Kannan contraction but not Kannan, Large Kannan, (a, b)-enriched Kannan and (ψ, a)-MR Kannan contractions.

Example 2.3. Let $X = \mathbb{R}$ and an operator $T : X \to X$ be defined by

$$T(x) = \frac{-x^3 + x^5}{1 + x^4}, \quad \forall x \in X.$$

(i) If T would be a large Kannan contraction, then from contractive condition (1.4), we have

$$\begin{cases} \left| \frac{-x^3+x^5}{1+x^4} - \frac{-y^3+y^5}{1+y^4} \right| < |x-y| \\ \text{for every } \epsilon > 0, \text{ there exist } \delta < \frac{1}{2} \text{ such that} \\ [x, y \in X, |x-y| > \epsilon] \Longrightarrow \\ \left| \frac{-x^3+x^5}{1+x^4} - \frac{-y^3+y^5}{1+y^4} \right| \leq \delta \left\{ \left| x - \frac{-x^3+x^5}{1+x^4} \right| + \left| y - \frac{-y^3+y^5}{1+y^4} \right| \right\}, \end{cases}$$

which is equivalent to the form

$$\begin{cases} \left| \frac{-x^3+x^5}{1+x^4} - \frac{-y^3+y^5}{1+y^4} \right| < |x-y| \\ \text{for every } \epsilon > 0, \text{ there exist } \delta < \frac{1}{2} \text{ such that} \\ [x, y \in X, |x-y| > \epsilon] \Longrightarrow \\ \left| \frac{-x^3+x^5}{1+x^4} - \frac{-y^3+y^5}{1+y^4} \right| \leq \delta \left\{ \left| \frac{x+x^3}{1+x^4} \right| + \left| \frac{y+y^3}{1+y^4} \right| \right\}. \end{cases} \tag{2.4}$$

On taking $x = 2$ and $y = 0$, this leads to the conclusion that it does not satisfy (2.4) for any $\delta < \frac{1}{2}$.

(ii) If T would be an (a, b)-enriched Kannan contraction, then contractive condition (1.6) becomes

$$\left| b(x-y) + \frac{-x^3+x^5}{1+x^4} - \frac{-y^3+y^5}{1+y^4} \right| \leq a \left\{ \left| \frac{x+x^3}{1+x^4} \right| + \left| \frac{y+y^3}{1+y^4} \right| \right\}. \tag{2.5}$$

On taking $x = 2$ and $y = 0$, contractive condition (2.5) reduces to

$$|-34b + 24| \leq 10a, \tag{2.6}$$

this leads to the conclusion that it does not satisfy (2.6) for any $0 \leq a < \frac{1}{2}$ and $0 \leq b < +\infty$.

(iii) If T would be a Kannan contraction, then contractive condition (1.2) reduces to

$$\left|\frac{-x^3 + x^5}{1 + x^4} - \frac{-y^3 + y^5}{1 + y^4}\right| \leq \lambda \left\{ \left|\frac{x + x^3}{1 + x^4}\right| + \left|\frac{y + y^3}{1 + y^4}\right| \right\}. \tag{2.7}$$

On taking $x = 2$ and $y = 0$, this leads to the conclusion that it does not satisfy (2.7) for any $0 \leq \lambda < \frac{1}{2}$.

(iv) On the other hand, T is a large MR-Kannan contraction. Indeed, suppose that for $\psi(x) = \frac{-x^4}{1+x^4}$, for all $x \in X$ and clearly, $\psi \in \zeta$.
We have

$$\begin{aligned}
\left|\frac{x\psi(x) + Tx}{1 + \psi(x)} - \frac{y\psi(y) + Ty}{1 + \psi(y)}\right| &= \left|\frac{x\left(\frac{-x^4}{1+x^4}\right) + \left(\frac{-x^3+x^5}{1+x^4}\right)}{1 + \left(\frac{-x^4}{1+x^4}\right)} - \frac{y\left(\frac{-y^4}{1+y^4}\right) + \left(\frac{-y^3+y^5}{1+y^4}\right)}{1 + \left(\frac{-y^4}{1+y^4}\right)}\right| \\
&= \left|\frac{\left(\frac{-x^5-x^3+x^5}{1+x^4}\right)}{\left(\frac{1+x^4-x^4}{1+x^4}\right)} - \frac{\left(\frac{-y^5-y^3+y^5}{1+y^4}\right)}{\left(\frac{1+y^4-y^4}{1+y^4}\right)}\right| \\
&= \left|x^3 - y^3\right| \\
&= |x - y| \left|x^2 + y^2 + xy\right| \\
&\leq (|x| + |y|) \left|x^2 + y^2 + xy\right|,
\end{aligned} \tag{2.8}$$

if we take $x \geq 0$, then $|x| = x \leq x + x^3 = x - (-x^3) = |x - T_\phi x|$ and the set defined by

$$\omega = \left\{(x, y) \in [-1, 1]^2 : \left|x^2 + y^2 + xy\right| + \frac{1}{2}|x - y|^2 \leq \frac{1}{2}\right\},$$

then, (2.8) becomes

$$|T_\phi x - T_\phi y| \leq (|x - T_\phi x| + |y - T_\phi y|) \left|\frac{1 - |x - y|^2}{2}\right|,$$

for some $\epsilon > 0$, if we have $|x - y| \geq \epsilon$, for all $x, y \in \omega$, then

$$|T_\phi x - T_\phi y| \leq (|x - T_\phi x| + |y - T_\phi y|) \left|\frac{1 - \epsilon^2}{2}\right|,$$

which is equivalent to the form as

$$\left|\frac{x\psi(x)+Tx}{1+\psi(x)}-\frac{x\psi(x)+Tx}{1+\psi(x)}\right| \leq \left(\left|\frac{x-Tx}{1+\psi(x)}\right|+\left|\frac{y-Ty}{1+\psi(y)}\right|\right)\left|\frac{1-\epsilon^2}{2}\right|, \quad (2.9)$$

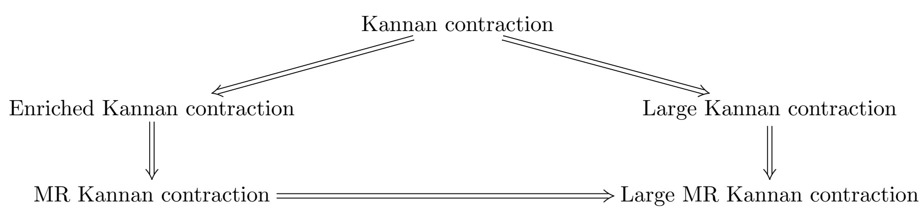
this leads to the conclusion that for $\delta(\epsilon)=\frac{1-\epsilon^2}{2}$, (2.9) satifies that T is a large MR-Kannan contraction.

For this ψ, T does not satisfy a (ψ, a)-MR Kannan contraction and contractive condition (1.7) reduces to

$$\left|x^3-y^3\right| \leq a\{\left|x+x^3\right|+\left|y+y^3\right|\}. \quad (2.10)$$

On taking $x=2$ and $y=0$, this leads to the conclusion that it does not satisfy (2.10) for any $0 \leq a < \frac{1}{2}$.

From the conclusion of the above example, we can draw the following diagram:



Before proceeding with the proof of the large (ψ, δ)-MR Kannan contraction, we discuss the following results from ([6]):
Let

$$\beta = \phi : X \to \mathbb{R} : \phi(x) \neq 0, \quad \forall x \in X,$$

An operator $T_\phi : X \to X$ is defined by where $\phi \in \beta$.

Lemma 2.4 ([6])**.** *Let $T_\phi : X \to X$ be a generalized averaged operator defined by*

$$T_\phi = (1-\phi(x))x + \phi(x)Tx, \quad \forall x \in X,$$

has a property that $F(T) = F(T_\phi)$, where $\phi \in \beta$. We would like to direct the reader's attention to the fact that the term generalized averaged operator refers to a specific type of admissible perturbations [23, 26]. It is worth noting that the class of generalized averaged operators includes the class of averaged operators (a term coined in [8]). This is demonstrated by considering $\lambda \in (0,1)$ and defining $\phi(x = \lambda$ for all $x \in X$.

We start with the following result.

Theorem 2.5. *Let $(\theta, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a large (ψ, δ)-MR Kannan contraction. Then, T has a unique fixed point.*

Proof. Let $\phi(x) = \frac{1}{\psi(x)+1}$, for all $x \in X$ and $\phi \in \beta$. Then, from the contraction condition (2.3), we have

$$\left\|\phi(x)\left(\left(\frac{1}{\phi(x)} - 1\right)x + Tx\right) - \phi(y)\left(\left(\frac{1}{\phi(y)} - 1\right)y + Ty\right)\right\| \leq \delta\big(\|\phi(x)(x - Tx)\|\big) + \big(\|\phi(y)(y - Ty)\|\big)$$

$$\left\|\phi(x)\frac{(1 - \phi(x)x + \phi(x)Tx}{\phi(x)} - \phi(y)\frac{(1 - \phi(y)y + \phi(y)Ty}{\phi(y)}\right\| \leq \delta\big(\|\phi(x)(x - Tx)\|\big) + \big(\|\phi(y)(y - Ty)\|\big).$$

This can be expressed in the following equivalent form:

$$\|T_\phi x - T_\phi y\| \leq a\,\|x - T_\phi x\| + \|y - T_\phi y\|\,, \quad \forall\ x, y \in X. \tag{2.11}$$

if an integer $m \geq 1$ such that $T_\phi^m x_0 = T_\phi^{m+1} x_0$, for all $x_0 \in X$, then $T_\phi(T_\phi^m x_0) = T_\phi^m x_0$.
Now, suppose that $T_\phi^n x_0 \neq T_\phi^{n+1} x_0$ for $n \geq 1$. As T_ϕ is a large (ψ, δ)-MR Kannan contraction, then

$$\left\|T_\phi^{n+1}x_0 - T_\phi^n x_0\right\| < \left\|T_\phi^n x_0 - T_\phi^{n-1}x_0\right\| < \ldots < \|T_\phi x_0 - x_0\|\,.$$

This shows that $\eta_n = \left\|T_\phi^{n+1}x_0 - T_\phi^n x_0\right\|$ is a strictly decreasing sequence. Thus, $\lim_{n\to+\infty} \eta_n = \gamma \geq 0$. if we take $\gamma > 0$, then for every $n \geq 1$, we have

$$\left\|T_\phi^{n+1}x_0 - T_\phi^n x_0\right\| \geq \gamma.$$

As a result, for $\delta < \frac{1}{2}$ we have

$$\begin{aligned}\left\|T_\phi^{n+1}x_0 - T_\phi^{n+2}x_0\right\| &= \left\|T_\phi^n x_0 - T_\phi^{n+1}x_0\right\| \\ &\leq \delta\left[\left\|T_\phi^n x_0 - T_\phi^{n+1}x_0\right\| + \left\|T_\phi^{n+1}x_0 - T_\phi^{n+2}x_0\right\|\right].\end{aligned}$$

It concludes that

$$(1 - \delta)\left\|T_\phi^{n+1}x_0 - T_\phi^{n+2}x_0\right\| \leq \delta\left\|T_\phi^n x_0 - T_\phi^{n+1}x_0\right\|$$

so, we have

$$\begin{aligned}\left\|T_\phi^{n+1}x_0 - T_\phi^{n+2}x_0\right\| &\le \frac{\delta}{1-\delta}\left\|T_\phi^{n}x_0 - T_\phi^{n+1}x_0\right\|\\ &\le \left(\frac{\delta}{1-\delta}\right)^2\left\|T_\phi^{n}x_0 - T_\phi^{n+1}x_0\right\|\\ &.\\ &.\\ &.\\ &\le \left(\frac{\delta}{1-\delta}\right)^n\left\|T_\phi x_0 - T_\phi^{2}x_0\right\|\\ &\le \left(\frac{\delta}{1-\delta}\right)^{n+1}\left\|x_0 - T_\phi x_0\right\|. \end{aligned} \tag{2.12}$$

As $\delta < \frac{1}{2}$, we deduce that $k = \frac{\delta}{1-\delta}$. Consequently, it follows from (2.12) that

$$\lim_{n\to+\infty}\left\|T_\phi^{n}x_0 - T_\phi^{n+1}x_0\right\| = 0, \tag{2.13}$$

that is a contradiction. Therefore, $\gamma = 0$. Now, we have to prove that $\{x_n\}$ defined by $x_n = T_\phi^n x_0$ is a Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, for $\epsilon > 0$ and subsequences of integers $(N_k), (n_k), (m_k)$ such that

$$N_k \to +\infty, m_k > n_k > N_k, \tag{2.14}$$

and

$$\epsilon \le \|x_{mk} - x_{nk}\|. \tag{2.15}$$

As T_ϕ is a large (ψ, δ)-MR Kannan operator, by utilizing (2.15), for $\delta < \frac{1}{2}$ we have

$$\epsilon < \|x_{mk} - x_{nk}\| = \|T_\phi x_{mk-1} - T_\phi x_{nk-1}\| \le \delta\left[\ \|x_{mk-1} - x_{mk}\| + \|x_{nk-1} - x_{nk}\|\ \right].$$

Taking $k \to +\infty$, from (2.13), we have

$$\lim_{k\to+\infty}\|x_{mk-1} - x_{mk}\| = \lim_{k\to+\infty}\|x_{nk-1} - x_{nk}\| = 0.$$

Thus, $\lim_{k\to+\infty}\|x_{mk} - x_{nk}\| = 0$, which contradicts our assumption. Hence, $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, then $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} T_\phi^n x_0 = l$, for all $l \in X$. Since T_ϕ is continuous. Thus, l is a fixed point of T_ϕ.

On the contrary, suppose that l' is another fixed point of T_ϕ and $l \ne l'$. Then, for $\epsilon_0 > 0$ we have $l - l' \ge \epsilon_0$. As T_ϕ is a large (ψ, δ)-MR Kannan operator, there exist $\delta_0 < \frac{1}{2}$ such that

$$\left\|l - l'\right\| = \left\|T_\phi(l) - T_\phi(l')\right\| \le \delta_0\left[\|l - T_\phi(l)\| + \left\|l' - T_\phi(l')\right\|\right].$$

Hence, we obtain $\left\| l - l^{'} \right\| = 0$, it contradicts our assumption. So, we have $l = l^{'}$. □

We deduce a Theorem 1.6 as a corollary from Theorem 2.5.

Corollary 2.6. *Let T be a large Kannan contaction. Then, T has a unique fixed point.*

Proof. If we take $\psi(x) = 0$, for all $x \in X$ in the contraction condition (2.3), then it reduces to Theorem 1.6. Consequently, the result follows from Theorem 2.5. □

It follows from Rakotch ([24]) that the set

$$\{f : (0, +\infty) \to [0, 0.5), f(t_n) \mapsto 0.5 \Rightarrow t_n \to 0 (n \to +\infty)\}$$

is represented by $\prod$.
We can now prove the generalised version of Theorem 2.5, stated as follows:

Theorem 2.7. *Let $T : X \to X$ be a large (ψ, δ)-MR Kannan contraction, that is an operator satisfying*

$$\left\{ \begin{array}{l} \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| < \|x - y\|, \quad \forall x, y \in X \\ \textit{with } x \neq y, \textit{ and if for all } \epsilon > 0, \textit{ there exists } f_\epsilon \in \prod \textit{ such that} \\ [x, y \in X, \quad \|x - y\| \geq \epsilon] \Longrightarrow \\ \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| \leq f_\epsilon(\|x - y\|) \left[\left\| \frac{x-Tx}{1+\psi(x)} \right\| + \left\| \frac{y-Ty}{1+\psi(y)} \right\| \right]. \end{array} \right. \tag{2.16}$$

Then T has a unique fixed point.

Proof. It follows from Theorem 2.5 that we have

$$\|T_\phi x - T_\phi y\| \leq a \|x - T_\phi x\| + \|y - T_\phi y\|, \quad \forall\, x, y \in X. \tag{2.17}$$

if an integer $m_0 \geq 1$ such that $T_\phi^{m_0} x_0 = T_\phi^{m_0+1} x_0$, for all $x_0 \in X$, then $T_\phi(T_\phi^{m_0} x_0) = T_\phi^{m_0} x_0$.
Now, suppose that $T_\phi^n x_0 \neq T_\phi^{n+1} x_0$ for $n \geq 1.1$. The sequence $\{x_n\}$ defined by $x_n = T_\phi^n x_0$

$$\|x_n - x_{n+1}\| = \left\| T_\phi^n x_0 - T_\phi^{n+1} x_0 \right\| < \left\| T_\phi^n x_0 - T_\phi^{n-1} x_0 \right\| = \|x_{n-1} - x_n\|.$$

This shows that $\eta_n = \|x_n - x_{n+1}\|$ is a strictly decreasing sequence. Thus, $\lim_{n \to +\infty} \eta_n = \gamma \geq 0$. if we take $\gamma > 0$, there exist $f_\gamma \in \prod$, we have

$$\|x_n - x_{n+1}\| = \left\| T_\phi^n x_0 - T_\phi^{n+1} x_0 \right\| \leq f_\gamma(\|x_{n-1} - x_n\|)[\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|].$$

Consequently, we obtain

$$\frac{\|x_n - x_{n+1}\|}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} < f_\gamma(\|x_{n-1} - x_n\|) < \frac{1}{2}.$$

Taking $n \to +\infty$, we have

$$\frac{\|x_n - x_{n+1}\|}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} = \frac{\gamma}{2\gamma} = \frac{1}{2} \leq \|x_{n-1} - x_n\| < \frac{1}{2}, \tag{2.18}$$

that is a contradiction. Therefore, $\gamma = 0$. Now, we have to prove that $\{x_n\}$ defined by $x_n = T_\phi^n x_0$ is a Cauchy sequence.
On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, for $\epsilon > 0$ and subsequences of integers $(N_k), (n_k), (m_k)$ such that

$$N_k \to +\infty, m_k > n_k > N_k, \tag{2.19}$$

and

$$\epsilon \leq \|x_{mk} - x_{nk}\| = \|T_\phi x_{mk-1} - T_\phi x_{nk-1}\| \tag{2.20}$$

Hence, $x_{mk-1} \neq x_{nk-1}$. From our assumption and the relation (2.20), for $f_\epsilon \in \prod$ we have

$$\begin{aligned}\epsilon_0 &\leq \|x_{nk} - x_{mk}\| = \|T_\phi x_{nk-1} - T_\phi x_{mk-1}\| \\ &\leq f_\epsilon(\|x_{nk-1} - x_{mk-1}\|)[\|x_{nk-1} - x_{nk}\| + \|x_{mk-1} - x_{mk}\|] \\ &\leq \frac{1}{2}[\|x_{nk-1} - x_{nk}\| + \|x_{mk-1} - x_{mk}\|].\end{aligned}$$

Taking $k \to +\infty$, we have

$$\epsilon_0 \leq \lim_{k \to +\infty} \|x_{nk} - x_{mk}\| = 0,$$

which contradicts our assumption. Hence, $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, then $\lim_{n \to +\infty} x_n = z_0$, for all $z_0 \in X$.
Now, we have to prove that z_0 is a fixed point for T. By triangular inequality for an integer $n \geq 1$, we have

$$\begin{aligned}\|z_0 - T_\phi z_0\| &\leq \|z_0 - x_{n+1}\| + \|T_\phi z_0 - x_{n+1}\| \\ &= \|z_0 - x_{n+1}\| + \|T_\phi z_0 - T_\phi x_n\|.\end{aligned}$$

let $x_n \neq z_0$. so, we have

$$\begin{aligned}\|z_0 - T_\phi z_0\| &\leq \|z_0 - x_{n+1}\| + \|T_\phi z_0 - x_{n+1}\| \\ &\leq \|z_0 - x_{n+1}\| + \frac{1}{2}(\|x_n - x_{n+1}\| + \|z_0 - T_\phi z_0\|).\end{aligned}$$

consequently, we obtain

$$0 \leq \frac{1}{2}\|z_0 - T_\phi z_0\| \leq \|z_0 - x_{n+1}\| + \frac{1}{2}\|x_n - x_{n+1}\|.$$

Taking $n \to +\infty$, we get

$$0 \leq \frac{1}{2}\|z_0 - T_\phi z_0\| \leq \lim_{n \to +\infty} \|z_0 - x_{n+1}\| + \frac{1}{2}\lim_{n \to +\infty} \|x_n - x_{n+1}\| = 0.$$

Thus, z_0 is a fixed point for T_ϕ.
On the contrary, suppose that z_1 is another fixed point of T_ϕ and $z_0 \neq z_1$.

Then, for $\epsilon = \|z_0 - z_1\|$, there exist $f_{\frac{\epsilon}{2}} \in [0, \frac{1}{2})$ such that

$$0 < \frac{\epsilon}{2} < \|z_0 - z_1\| = \|T_\phi z_0 - T_\phi z_1\| \leq f_{\frac{\epsilon}{2}}(\|z_0 - z_1\|)(\|z_0 - T_\phi z_0\| + \|z_1 - T_\phi z_1\|)$$
$$\leq \frac{1}{2}(\|z_0 - T_\phi z_0\| + \|z_1 - T_\phi z_1\|) = 0.$$

It contradicts our assumption. So, we have $z_0 = z_1$. □

Definition 2.8. Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$ be asymptotically regular, that is an operator satisfying

$$\lim_{n\to+\infty} \left\|T^n x - T^{n+1}x\right\| = 0. \tag{2.21}$$

for all $x \in X$.

The set

$$\{f : (0, +\infty) \to [0, 1), f(t_n) \mapsto 1 \Rightarrow t_n \to 0(n \to +\infty)\}$$

is represented by $\coprod$.

Now, we can drop the contractive condition $\left\|\frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)}\right\| < \|x - y\|$, for all $x, y \in X$, with $x \neq y$, and replace it by T is continuos and asymptotically regular.

We can now prove the generalised version of Theorem 2.5, stated as follows:

Theorem 2.9. *Let $T : X \to X$ be a large (ψ, δ)-MR Kannan contraction and T be a continuous and asymptotically regular, that is an operator satsisfying*

$$\begin{cases} \text{for all } \epsilon > 0 \text{ there exists } f_\epsilon \in \coprod \text{ such that} \\ [x, y \in X, \quad \|x - y\| \geq \epsilon] \Longrightarrow \\ \left\|\frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)}\right\| \leq f_\epsilon(\|x - y\|)\left[\left\|\frac{x-Tx}{1+\psi(x)}\right\| + \left\|\frac{y-Ty}{1+\psi(y)}\right\|\right]. \end{cases} \tag{2.22}$$

Then T has a unique fixed point.

Proof. It follows from Theorem 2.5 that we have

$$\|T_\phi x - T_\phi y\| \leq a \|x - T_\phi x\| + \|y - T_\phi y\|, \quad \forall\ x, y \in X. \tag{2.23}$$

if an integer $m_0 \geq 1$ such that $T_\phi^{m_0} x_0 = T_\phi^{m_0+1} x_0$, for all $x_0 \in X$, then $T_\phi(T_\phi^{m_0} x_0) = T_\phi^{m_0} x_0$.

Now, suppose that $T_\phi^n x_0 \neq T_\phi^{n+1} x_0$ for $n \geq 1$. We have to prove that $\{x_n\}$ defined by $x_n = T_\phi^n x_0$ is a Cauchy sequence.

On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, for $\epsilon > 0$ and subsequences of integers $(N_k), (n_k), (m_k)$ such that

$$N_k \to +\infty, m_k > n_k > N_k, \quad \textit{and} \quad \|x_{mk} - x_{nk}\| \geq \epsilon_0. \tag{2.24}$$

Hence, by triangular inequality and our assumption, we have

$$\begin{aligned}\|x_{mk}-x_{nk}\| &\leq \|x_{mk}-x_{mk+1}\|+\|x_{mk+1}-x_{nk+1}\|+\|x_{nk+1}-x_{nk}\| \\ &\leq \|x_{mk}-x_{mk+1}\|+f_{\epsilon_0}(\|x_{nk}-x_{mk}\|)[\|x_{nk}-x_{nk+1}\|+ \\ &\qquad \|x_{mk}-x_{mk+1}\|]+\|x_{nk+1}-x_{nk}\|,\end{aligned}$$

where $f_{\epsilon_0}\in\coprod$. Eventually,

$$\begin{aligned}[1-f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)]\,\|x_{mk}-x_{nk}\| &\leq \|x_{mk}-x_{nk}\| \\ &\leq [1+f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)] \\ &\quad (\|x_{nk}-x_{nk+1}\|+\|x_{mk}-x_{mk+1}\|).\end{aligned}$$

Dividing right side by

$$[1-f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)](\|x_{nk}-x_{nk+1}\|+\|x_{mk}-x_{mk+1}\|),$$

As, $\|x_{mk}-x_{nk}\|\geq\epsilon_0$, we have

$$\begin{aligned}\frac{\epsilon_0}{\|x_{nk}-x_{nk+1}\|+\|x_{mk}-x_{mk+1}\|} &\leq \frac{\|x_{mk}-x_{nk}\|}{\|x_{nk}-x_{nk+1}\|+\|x_{mk}-x_{mk+1}\|} \\ &\leq \frac{1+f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)}{1-f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)}.\end{aligned}$$

Taking $k\to+\infty$, we have

$$\lim_{k\to+\infty}\frac{1+f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)}{1-f_{\epsilon_0}(\|x_{mk}-x_{nk}\|)}=+\infty.$$

Hence,

$$\lim_{n,m\to+\infty}\sup f_{\epsilon_0}(\|x_m-x_n\|)=1. \tag{2.25}$$

As $f_{\epsilon_0}\in\coprod$, we concluded that

$$\lim_{n,m\to+\infty}(\|x_m-x_n\|)=0,$$

it contradicts our assumption. Hence, $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, then $\lim_{n\to+\infty}x_n=z_0^{'}$, for all $z_0^{'}\in X$. Since T_ϕ is continuous. Thus, $z_0^{'}$ is a fixed point of T_ϕ.

On the contrary, suppose that $z_1^{'}$ is another fixed point of T_ϕ and $z_1^{'}\neq z_0^{'}$. Then, for $\left\|z_1^{'}-z_0^{'}\right\|=\epsilon_1>\frac{\epsilon_1}{2}$, there exist $f_{\frac{\epsilon}{2}}\in\coprod$ such that

$$[x,y\in X,\quad \|x-y\|\geq\frac{\epsilon_1}{2}]\Longrightarrow$$

$$\|x-y\|\leq f_{\frac{\epsilon_1}{2}}\,\|x-y\|\,[\|x-Tx\|+\|y-Ty\|].$$

Thus, it concludes that

$$\left\|z_1^{'}-z_0^{'}\right\|\leq f_{\epsilon_1}\left(\left\|z_1^{'}-z_0^{'}\right\|\right)\left[\left\|z_1^{'}-Tz_1^{'}\right\|+\left\|z_0^{'}-Tz_0^{'}\right\|\right],$$

It contradicts our assumption. So, we have $z_0^{'}=z_1^{'}$. □

2.2. **Large MR-Kannan Contractions in the Non (Necessarily) continuos Sense.**
Let us begin with the following definition.

Definition 2.10. Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$ be a large (ψ, δ)-MR Kannan contraction (in the non-necessarily continuous sense), that is an operator satisfying

$$\begin{cases} \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| < \frac{1}{2}\left(\left\| \frac{x-Tx}{1+\psi(x)} \right\| + \left\| \frac{y-Ty}{1+\psi(y)} \right\| \right), \\ \text{for all } x, y \in X, \text{ with } x \neq y, \text{ and if for all } \epsilon > 0, \text{ there exists } \delta < \frac{1}{2} \text{ such that} \\ [x, y \in X, \quad \|x - y\| \geq \epsilon] \Longrightarrow \\ \left\| \frac{x\psi(x)+Tx}{1+\psi(x)} - \frac{y\psi(y)+Ty}{1+\psi(y)} \right\| \leq \delta \left[\left\| \frac{x-Tx}{1+\psi(x)} \right\| + \left\| \frac{y-Ty}{1+\psi(y)} \right\| \right]. \end{cases} \tag{2.26}$$

Theorem 2.11. *Let T be a large (ψ, δ)-MR Kannan contraction (in the non-necessarily continuous sense). Then, $F(T)$ is a singleton set.*

Proof. It follows from Theorem 2.5 that we have

$$\|T_\phi x - T_\phi y\| \leq a \|x - T_\phi x\| + \|y - T_\phi y\|, \quad \forall\ x, y \in X. \tag{2.27}$$

Uniqueness of fixed point: On the contrary, suppose that $x_0, x_1 \in X$ with $x_0 \neq x_1$. So,

$$0 \leq \|x_0 - x_1\| = \|T_\phi x_0 - T_\phi x_1\| < \frac{1}{2}(\|x_0 - T_{phi} x_0\| + \|x_1 - T_\phi x_1\|),$$

it contradicts our assumption. So, $x_0 = x_1$.
Existance of fixed point: if an integer $m \geq 1$ such that $T_\phi^m x_0 = T_\phi^{m+1} x_0$, for all $x_0 \in X$, then $T_\phi(T_\phi^m x_0) = T_\phi^m x_0$.
Now, suppose that $x_n = T_\phi^n x_0 \neq T_\phi^{n+1} x_0 = x_{n+1}$ for $n \geq 1.1$. As T_ϕ is a large (ψ, δ)-MR Kannan contraction, then

$$\begin{aligned} \|x_n - x_{n+1}\| = \left\| T_\phi T_\phi^{n-1} x_0 - T_\phi T_\phi^n x_0 \right\| &< \frac{1}{2}\left(\left\| T_\phi^{n-1} x_0 - T_\phi^n x_0 \right\| + \left\| T_\phi^n x_0 - T_\phi^{n+1} x_0 \right\| \right) \\ &= \frac{1}{2}\left(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\| \right) \end{aligned}$$

This shows that $\epsilon_n = \|x_{n-1} - x_n\|$ is a strictly decreasing sequence. Thus, $\lim_{n \to +\infty} \|x_n - x_{n+1}\| = \epsilon_0 \geq 0$.
As, $\epsilon_n = \|x_n - x_{n+1}\|$ is a decreasing sequence. So, for all $n \geq 1$ we have $\epsilon_0 < \|x_n - x_{n+1}\|$. Assuming that $\epsilon_0 > 0$ for $0 < \delta_0 < \frac{1}{2}$, we have

$$\begin{aligned} \|x_n - x_{n+1}\| = \left\| T_\phi T_\phi^{n-1} x_0 - T_\phi T_\phi^n x_0 \right\| &< \frac{1}{2}\left(\left\| T_\phi^{n-1} x_0 - T_\phi^n x_0 \right\| + \left\| T_\phi^n x_0 - T_\phi^{n+1} x_0 \right\| \right) \\ &\delta_0(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|), \end{aligned}$$

it concludes

$$\|x_n - x_{n+1}\| \leq \frac{\delta_0}{1 - \delta_0} \|x_{n-1} - x_n\|.$$

Inductively, we deduce that

$$\|x_n - x_{n+1}\| \leq \left(\frac{\delta_0}{1-\delta_0}\right)^n \|x_{n-1} - x_n\|,$$

where $\frac{\delta_0}{1-\delta_0} = c < 1$. Hence, it concludes that $\lim_{n\to+\infty}(c)_n = 0 \Rightarrow \|x_n - x_{n+1}\| = 0$, it contradicts our assumption. Thus, $\epsilon_0 = 0$.

Now, we have to prove that $\{x_n\}$ defined by $x_n = T_\phi^n x_0$ is a Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, for $\alpha_0 > 0$ and subsequences of integers $(N_k), (n_k), (m_k)$ such that

$$N_k \to +\infty, m_k > n_k > N_k, \quad and \quad \|x_{mk} - x_{nk}\| \geq \alpha_0, \tag{2.28}$$

leading to the conclusion that $x_{mk-1} \neq x_{nk-1}$.

From the decreasing sequence $\epsilon_n = \|T_\phi x_{nk-1} - T_\phi x_{mk-1}\|$ and our assumption, we have

$$\begin{aligned}\alpha_0 \leq \|x_{mk} - x_{nk}\| &= \|T_\phi x_{mk-1} - T_\phi x_{nk-1}\| \\ &\leq \frac{1}{2}(\|x_{nk-1} - x_{nk}\| + \|x_{mk-1} - x_{mk}\|) \\ &\leq \|x_{nk-1} - x_{nk}\|.\end{aligned}$$

By taking $\lim_{k\to+\infty}$, it concludes that

$$\alpha_0 \leq \lim_{k\to+\infty} \|x_{mk} - x_{nk}\| \leq \lim_{k\to+\infty} \|x_{nk} - x_{nk-1}\| = 0,$$

it contradicts our assumption. Thus, $\{x_n\}$ is a Cauchy sequence.

Because X is a Banach space, then $\lim_{n\to+\infty} x_n = z_0$. By triangular inequality for an integer $n \geq 1$, we have

$$\begin{aligned}\|z_0 - T_\phi z_0\| &\leq \|z_0 - x_{n+1}\| + \|x_{n+1} - T_\phi z_0\|. \\ &\|x_{n+1} - z_0\| + \|T_\phi x_n - T_\phi z_0\|.\end{aligned}$$

Taking $x_n \neq z_0$, we have

$$\begin{aligned}\|z_0 - T_\phi z_0\| &\leq \|z_0 - x_{n+1}\| + \|T_\phi z_0 - x_{n+1}\| \\ &\leq \|z_0 - x_{n+1}\| + \frac{1}{2}(\|x_n - x_{n+1}\| + \|z_0 - T_\phi z_0\|).\end{aligned}$$

Equivalently

$$0 \leq \frac{1}{2}\|z_0 - T_\phi z_0\|) \leq \|z_0 - x_{n+1}\|) + \frac{1}{2}\|x_n - x_{n+1}\|)$$

Taking $x_n \neq z_0$, we have

$$0 \leq \frac{1}{2}\|z_0 - T_\phi z_0\|) \leq \|z_0 - x_{n+1}\|) + \frac{1}{2}\|x_n - x_{n+1}\|) = 0.$$

Thus, $T_\phi z_0 = z_0$. It completes our proof. □

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References

[1] M. Abbas, R. Anjum, and V. Berinde, Equivalence of certain iteration processes obtained by two new classes of operators, Mathematics 9, no. 18 (2021), 2292.

[2] M. Abbas, R. Anjum, and V. Berinde, Enriched multivalued contractions with applications to differential inclusions and dynamic programming, Symmetry 13, no. 8 (2021), 1350.

[3] M. Abbas, R. Anjum, and H. Iqbal, Generalized enriched cyclic contractions with application to generalized iterated function system, Chaos, Solitons and Fractals 154 (2022), 111591.

[4] M. Abbas, R. Anjum, and S. Riasat, Fixed point results of enriched interpolative Kannan type operators with applications, Appl. Gen. Topol. 23, no. 2 (2022), 391–404.

[5] R. Anjum and M. Abbas, Common Fixed point theorem for modified Kannan enriched contraction pair in Banach spaces and its applications, Filomat 35, no. 8 (2021), 2485–2495.

[6] R. Anjum, M. Abbas, and H. Işık, Completeness problem via fixed point theory, Complex Anal. Oper. Theory 17 (2023), 85.

[7] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta mathematicae 3, no. 1 (1922), 133–181.

[8] J. B. Baillon, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, Houston J. Math. 4 (1978), 1–9.

[9] R. Batra, R. Gupta, and P. Sahni, A new extension of Kannan contractions and related fixed point results, The Journal of Analysis 28 (2020), 1143–1154.

[10] V. Berinde, and M. Păcurar, Kannan's fixed point approximation for solving split feasibility and variational inequality problems, Journal of Computational and Applied Mathematics 386 (2020), 377–427.

[11] V. Berinde, and M. Păcurar, Approximating fixed points of enriched contractions in Banach spaces, Journal of Fixed Point Theory and Applications 22 (2020), 1–10.

[12] V. Berinde, and M. Păcurar, Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces, Carpathian Journal of Mathematics 37, no. 2 (2021), 173–184.

[13] V. Berinde, and M. Păcurar, Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces, Journal of Fixed Point Theory and Applications 22 (2021), 1–16.

[14] V. Berinde, and M. Păcurar, Approximating fixed points of enriched contractions in Banach spaces, Journal of Fixed Point Theory and Applications 22 (2020), 1–10.

[15] A. Dehici, M. B. Mesmouli, and E. Karapinar, On the fixed points of large-Kannan contraction mappings and applications, Applied Mathematics E-Notes 19 (2019), 535–551.

[16] J. Górnicki, Various extensions of Kannan's fixed point theorem, Journal of Fixed Point Theory and Applications 20, no. 1 (2018), 1–12.

[17] N. Haokip, and N. Goswami, Some fixed point theorems for generalized Kannan type mappings in b-metric spaces, Proyecciones (Antofagasta) 38, no. 4 (2019), 763–782.

[18] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.

[19] E. Karapinar, Revisiting the Kannan type contractions via interpolation, Advances in the Theory of Nonlinear Analysis and its Application 2, no. 2 (2018), 85–87.

[20] E. Karapinar, R. Agarwal, and H. Aydi, Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces, Mathematics 6, no. 11 (2018), 256.

[21] E. Karapinar, O. Alqahtani, and H. Aydi, On interpolative Hardy-Rogers type contractions, Symmetry 11, no. 1 (2018), 8.
[22] S. K. Malhotra, J. B. Sharma, and S. Shukla, Fixed points of generalized Kannan type α-admissible mappings in cone metric spaces with Banach algebra, Theory and Applications of Mathematics and Computer Science 7, no. 1 (2017), 1.
[23] A. Petrusel, and I. A. Rus, An abstract point of view on iterative approximation schemes of fixed points for multivalued operators, J. Nonlinear Sci. Appl. 6, no. 2 (2013), 97–107.
[24] E. Rakotch, A note on contractive mappings, Proceedings of the American Mathematical Society 13, no. 3 (1962), 459–465.
[25] M. Rossafi, and H. Massit, Some fixed point theorems for generalized Kannan type mappings in rectangular b-metric spaces, Nonlinear Funct. Anal. Appl. 27 (2022), 663–677.
[26] I. A. Rus, An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations, Fixed Point Theory 13, no. 1 (2012), 179–192.

