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Vector-Valued Fuzzy Metric Spaces and Fixed Point Theorems

Satish Shukla [1], Nikita Dubey [1] and Juan-José Miñana [2,*]

[1] Department of Mathematics, Shri Vaishnav Institute of Science, Shri Vaishnav Vidyapeeth Vishwavidyalaya, Gram Baroli Sanwer Road, Indore 453331, India; satishmathematics@svvv.edu.in (S.S.); nikitadubey@svvv.edu.in (N.D.)
[2] Departamento de Matemática Aplicada, Universitat Politècnica de València, Calle Paranimf, 1, 46730 Gandia, Spain
* Correspondence: juamiapr@mat.upv.es

Abstract: The purpose of this paper is to generalize the concept of classical fuzzy set to vector-valued fuzzy set which can attend values not only in the real interval $[0, 1]$, but in an ordered interval of a Banach algebra as well. This notion allows us to introduce the concept of vector-valued fuzzy metric space which generalizes, extends and unifies the notion of classical fuzzy metric space and complex-valued fuzzy metric space and permits us to consider the fuzzy sets and metrics in a larger domain. Some topological properties of such spaces are discussed and some fixed point results in this new setting are proved. Multifarious examples are presented which clarify and justify our claims and results.

Keywords: vector-valued fuzzy set; vector-valued fuzzy metric space; strongly minihedral cone; fuzzy contractive mapping; fixed point

MSC: 54H25; 47H10; 03E72

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1. Introduction

A fuzzy set is considered as a class of objects whose grade of membership lies not only in the set $\{0, 1\}$ but in the interval $[0, 1]$ [1]. The approach of fuzzy sets is very useful in the study of systems of uncertain nature. Kramosil and Michalek used this approach to define a fuzzy version of classical metrics and introduced a notion of fuzzy metric space in [2]. Later on, George and Veeramani [3] suggested some modifications in the definition of fuzzy metric introduced in [2] and showed that with the modified definition the fuzzy metrics generate a topology which is first countable and Hausdorff. Then, some works contributed to the study of both aforesaid notions of fuzzy metrics. Among them, it is worth noting [4] in which the authors proved that fuzzy metrics given by George and Veeramani are metrizable. Moreover, on account of the one exposed in [4], one can derive such a conclusion also for fuzzy metrics introduced by Kramosil and Michalek. So, from the topological point of view fuzzy metrics and classical metrics are the same object. Nonetheless, fuzzy metrics show some differences compared with classical ones which still make their study of interest nowadays. On the one hand, they differ in some purely metric topics as completeness or fixed point theory, which are two active topics of research in the literature (see, for instance, the following recent references [5–11]). On the other hand, fuzzy metrics have been successfully used, compared with their classical counterparts, in engineering problems such as model estimation, modelling multi-agent systems or image filtering (see, for instance, [12–17] and references therein). With the aim of obtaining a fuzzy version of the celebrated Banach fixed point theorem, Ref. [18] introduced a notion of Cauchy sequence and completeness for fuzzy metric spaces in the sense of Kramosil and Michalek. However, Ref. [3] pointed out some drawbacks of the assumptions of [18] and defined the Cauchy sequences and completeness of fuzzy metric spaces in a new sense. Coming back to the fixed point theorem established in [18], the contractive condition used in it is directly associated with the parameter with respect to which the fuzzy distance is measured. In contrast with [18,19] (see also, [20]) where a more natural contractive condition is used, which is different in nature, to prove fixed point results. In this sequel, Refs. [21–24] and several others introduced various generalized contractive conditions. Recently, Ref. [25] introduced the notion of complex-valued fuzzy metric spaces and extended the idea of t-norm and fuzzy metric from the set $[0,1]$ to a subset of complex numbers. They also proved some fixed point theorems and discussed some possible applications of such spaces.

On the other hand, the fixed point theory in the vector-valued metric spaces was initiated by [26]. In such spaces, the metric function can take values not only in the set $\mathbb{R}$ of real numbers, but in a cone associated with a Banach space. This interesting generalization of metric attracted several researchers. An important improvement to the notion due to [26] was given by [27]. They introduced the cone metric spaces over Banach algebras and showed the extensive nature of contractive conditions and fixed point theorems in such spaces.

Topological algebras consist of a very useful subclass known as Banach algebras. Banach algebras have a norm structure and hence have applications in various branches of pure and applied mathematics as well as in other branches of sciences, e.g., in solving nonlinear integral equations, functional integral equations, in the study of Fourier series, representation theory, harmonic analysis and other significant areas of sciences. Nowadays, the class of Banach algebras is an interesting and vast discipline with a variety of specializations and applications (see, e.g., [28–32]). Ref. [27] introduced a kind of space in which metric function attains values in the form of vectors in a Banach algebra and utilized the properties of Banach algebras to show the superiority of such vector-valued metric functions over the usual ones. The space of the complex numbers is a particular type of Banach algebra and has great significance in the study of mathematical and physical systems. Ref. [25] utilized the space of complex numbers to introduce the complex-valued fuzzy sets and complex-valued fuzzy metric spaces and proved some fixed point results in complex-valued fuzzy metric spaces. Ref. [25] discussed significance and applications of such spaces and their fixed point results. The motivation for this work comes from the question: "can fuzzy sets (fuzzy metrics) be extended to a vector-valued version in such a way that this notion generalizes and unifies to both the usual fuzzy sets (usual fuzzy metrics) and complex-valued fuzzy sets (complex-valued fuzzy metrics) and are the previously mentioned fixed point results provable in this new generalized setting"?

In this paper, we introduce the notion of vector-valued t-norm and vector-valued fuzzy metric space and prove some fixed point theorems for contractive mappings in such spaces. The notion of vector-valued fuzzy metric space generalizes, extends and unifies the usual fuzzy metric spaces and the complex-valued fuzzy metric spaces. A new class of mappings in this new setting called the class of generalized ξ-contractions is introduced. The fixed point theorems for the mappings of such class generalize and extend some results established in the literature. Concretely, Theorems 3, 5 and 6 generalize and extend the main results provided in [23,25], due to their being established in a more general framework. Some topological properties of such spaces are also discussed. We present suitable examples which illustrate the new notions and justify our claims.

2. Preliminaries

We first state some basic notions and definitions about the fuzzy metric and complex-valued fuzzy metric spaces which will be useful throughout the paper.

Definition 1. *A triangular norm (briefly, t-norm) is a binary operation $*$ on $[0,1]$ such that, for all $x, y, z \in [0,1]$, the following axioms are satisfied:*

(T1) $x * y = y * x$; *(Commutativity)*

(T2) $x * (y * z) = (x * y) * z$; *(Associativity)*

(T3) $x * y \geq x * z$, *whenever* $y \geq z$; *(Monotonicity)*
(T4) $x * 1 = x$. *(Boundary Condition)*

Definition 2 ([3])**.** *A triple* $(X, M, *)$ *is called a fuzzy metric space if* X *is a nonempty set,* $*$ *is a continuous t-norm and* $M : X \times X \times (0, \infty) \to [0, 1]$ *is a fuzzy set satisfying the following conditions:*

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is a continuous mapping;*

for all $x, y, z \in X$ *and* $s, t > 0$*. Then, the triplet* $(X, M, *)$ *is called a fuzzy metric space and* M *is called a fuzzy metric on* X*. For various properties of a fuzzy metric space the reader is referred to [3].*

Let $P = \{(a, b) : 0 \leq a < \infty, 0 \leq b < \infty\} \subset \mathbb{C}$ and denote by θ and ℓ the elements $(0, 0)$, $(1, 1) \in P$, respectively (see [25]). Then, the relation $\preceq$ on $\mathbb{C}$ such that $c_1 \preceq c_2$ (or, equivalently, $c_2 \succeq c_1$) if and only if $c_2 - c_1 \in P$ defines a partial ordering on $\mathbb{C}$. $c_1 \prec c_2$ (or, equivalently, $c_2 \succ c_1$) indicates that $\mathrm{Re}(c_1) < \mathrm{Re}(c_2)$ and $\mathrm{Im}(c_1) < \mathrm{Im}(c_2)$. Define $I = \{(a, b) : 0 \leq a \leq 1, 0 \leq b \leq 1\}$, $I_O = \{(a, b) : 0 < a < 1, 0 < b < 1\}$ and $P_\theta = \{(a, b) : 0 < a < \infty, 0 < b < \infty\}$.

Definition 3. *Let* X *be a nonempty set. A complex fuzzy set* M *is characterized by a mapping with domain* X *and values in complex interval* I*.*

Definition 4 ([25])**.** *A binary operation* $* : I \times I \to I$ *is called a complex-valued t-norm if (i)* $c_1 * c_2 = c_2 * c_1$*; (ii)* $c_1 * c_2 \preceq c_3 * c_4$ *whenever* $c_1 \preceq c_3, c_2 \preceq c_4$*; (iii)* $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$*; (iv)* $c * \theta = \theta, c * \ell = c$*; for all* $c, c_1, c_2, c_3, c_4 \in I$*.*

Definition 5 ([25])**.** *Let* X *be a nonempty set,* $*$ *a continuous complex-valued t-norm and* M *a complex fuzzy set on* $X \times X \times P_\theta$ *satisfying the following conditions:*

(CV1) $\theta \prec M(x, y, c)$;
(CV2) $M(x, y, c) = \ell$ *if and only if* $x = y$;
(CV3) $M(x, y, c) = M(y, x, c)$;
(CV4) $M(x, y, c) * M(y, z, c') \preceq M(x, z, c + c')$;
(CV5) $M(x, y, \cdot) : P_\theta \to I$ *is continuous;*

for all $x, y, z \in X$ *and* $c, c' \in P_\theta$*. Then, the triplet* $(X, M, *)$ *is called a complex-valued fuzzy metric space and* M *is called a complex-valued fuzzy metric on* X*. A complex-valued fuzzy metric can be thought of as the degree of nearness between two points of* X *with respect to a complex parameter* $c \in P_\theta$*. For the examples of complex-valued t-norms and complex-valued fuzzy metric spaces, the reader is referred to [25].*

In the next section, we introduce the vector-valued fuzzy sets and vector-valued fuzzy metric spaces and study their properties.

3. Vector-Valued Fuzzy Sets and Vector-Valued Fuzzy Metric Spaces

Let $\mathcal{A}$ always be a real Banach algebra with zero vector θ and multiplicative unit e. A subset $\mathcal{P}$ of $\mathcal{A}$ is called a cone if:

(C1) $\mathcal{P}$ is nonempty closed and $\{\theta, e\} \subset \mathcal{P}$;
(C2) $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$ for all nonnegative real numbers α, β;
(C3) $\mathcal{P}^2 = \mathcal{P}\mathcal{P} \subset \mathcal{P}$;

(C4) $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$.

Given a cone $\mathcal{P} \subset \mathcal{A}$, we define a partial ordering $\preceq$ in $\mathcal{A}$ with respect to $\mathcal{P}$ by $x \preceq y$ (or equivalently $y \succeq x$) if and only if $y - x \in \mathcal{P}$. We shall write $x \prec y$ (or equivalently $y \succ x$) to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in \mathcal{P}^\circ$, where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

A cone $\mathcal{P}$ is said to be solid if $\mathcal{P}^\circ \neq \emptyset$. For $a \in \mathcal{A}$, the spectral radius of a is denoted by $\rho(a)$. By an ordered interval in $\mathcal{A}$ with the end points $a, b \in \mathcal{A}, a \preceq b$, we mean a set which contains all vectors c such that $a \preceq c \preceq b$ and it is denoted by $[a, b]$, i.e.,

$$[a, b] = \{c \in \mathcal{A} \colon a \preceq c \preceq b\}.$$

We denote by I_e a particular ordered interval $[\theta, e]$, i.e.,

$$I_e = [\theta, e] = \{a \in \mathcal{A} \colon \theta \preceq a \preceq e\}.$$

We call I_e the unit vector interval.

Remark 1 ([33]). *Let $\mathcal{P}$ be a cone in a Banach space $\mathcal{A}$ and $a, b, c \in \mathcal{P}$.*

(i) *If $a \preceq b$ and $b \ll c$ then $a \ll c$.*
(ii) *If $a \ll b$ and $b \ll c$ then $a \ll c$.*
(iii) *If $\theta \preceq u \ll c$ for every $c \in \mathcal{P}^\circ$ then $u = \theta$.*

Definition 6. *Let X be a nonempty set, $\mathcal{A}$ be a Banach algebra with unit e and $\mathcal{P}$ be a cone in $\mathcal{A}$. A vector-valued or $\mathcal{P}$-valued fuzzy set on X is a function $F \colon X \to I_e$.*

Note that for $\mathcal{A} = \mathbb{R}$ with usual norm, ordinary multiplication and $\mathcal{P} = [0, \infty)$, we have $I_e = [0, 1]$ the real unit interval; the above definition reduces into the definition of usual fuzzy sets given by [1].

Definition 7. *A mapping $* \colon I_e \times I_e \to I_e$ is called a $\mathcal{P}$-valued (or vector-valued) triangular norm ($t_{\mathcal{P}}$-norm for short) if $*$ satisfies the following conditions:*

(i) *$*$ is commutative and associative, i.e., $a * b = b * a$ and $a * (b * c) = (a * b) * c$, for all $a, b, c \in I_e$;*
(ii) *$e * a = a$, for all $a \in I_e$;*
(iii) *$a * b \preceq c * d$, whenever $a \preceq c$ and $b \preceq d$, with $a, b, c, d \in I_e$.*

Note that for $\mathcal{A} = \mathbb{R}$ the Banach algebra of real numbers with usual norm, ordinary multiplication and $\mathcal{P} = [0, \infty)$ and $I_e = [0, 1]$ the real unit interval, the above definition reduces into the definition of t-norm given by [34]. While, for $\mathcal{A} = \mathbb{C} = \{(x, y) \colon x, y \in \mathbb{R}\}$ the Banach algebra of complex numbers with coordinate-wise multiplication, maximum norm $\|(x, y)\| = \max\{|x|, |y|\}$, $e = (1, 1)$, $\mathcal{P} = \{(x, y) \in \mathbb{C} \colon x, y \geq 0\}$ and $I_e = \{(x, y) \in \mathbb{C} \colon 0 \leq x, y \leq 1\}$, the above definition reduces into the definition of complex-valued t-norm given by [25]. Therefore, all the examples of complex-valued t-norms given in [25] are the examples of $t_{\mathcal{P}}$-norms.

In the next example, part (A) shows that for every given t-norm one can construct a $t_{\mathcal{P}}$-norm. While part (B) defines a $t_{\mathcal{P}}$-norm with a structure disimilar to part (A). Thus, these parts show the novelty and generalized nature of $t_{\mathcal{P}}$-norms.

Example 1. *Let $\mathcal{A} = \mathbb{R}^n$ be the Banach algebra with multiplication defined by $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$, the norm defined by $\|(x_1, \ldots, x_n)\| = \max\{|x_i| \colon 1 \leq i \leq n\}$ and with unit $e = (1, \ldots, 1)$. Define $\mathcal{P} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_i \geq 0, 1 \leq i \leq n\}$, then $I_e = \{(a_1, \ldots, a_n) \colon 0 \leq a_i \leq 1, 1 \leq i \leq n\}$. Then:*

(A) *For each given usual t-norm* $\star_i\colon [0,1]\times[0,1]\to[0,1]$, $1\le i\le n$ *the mapping* $*\colon I_e\times I_e\to I_e$ *defined by:*

$$(a_1,\ldots,a_n)*(b_1,\ldots,b_n)=(a_1\star_1 b_1,\ldots,a_n\star_n b_n)\text{ for all } a_i,b_i\in[0,1]$$

is a $t_{\mathcal{P}}$*-norm.*

(B) *The mapping* $*\colon I_e\times I_e\to I_e$ *defined by:*

$$(a_1,\ldots,a_n)*(b_1,\ldots,b_n)=\begin{cases}(a_1,\ldots,a_n), & \text{if } b_i=1 \text{ for } 1\le i\le n;\\ (b_1,\ldots,b_n), & \text{if } a_i=1 \text{ for } 1\le i\le n;\\ \theta, & \text{otherwise.}\end{cases}$$

is a $t_{\mathcal{P}}$*-norm which cannot be expressed as* $(a_1\star_1 b_1,\ldots,a_n\star_n b_n)$, *where each* $\star_i$ *is a usual t-norm.*

Example 2. *Suppose* $\mathcal{A}=\{M_r\colon r\in\mathbb{R}\}$, *where* $M_r=\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}$, $r\in\mathbb{R}$. *Then,* $\mathcal{A}$ *is a Banach algebra with usual matrix addition and multiplication, with unit* $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *as unit vector and the norm defined by* $\|M_r\|=|r|$ *for all* $r\in\mathbb{R}$. *Then,* $\mathcal{P}=\{M_r\colon r\ge 0\}$ *is a cone in* $\mathcal{A}$ *and* $I_e=\{M_r\in\mathcal{A}\colon 0\le r\le 1\}$. *For every given usual t-norm* $\star\colon[0,1]\times[0,1]\to[0,1]$, *the mapping* $*$ *defined by* $M_r*M_s=M_{r\star s}$ *for all* $M_r,M_s\in I_e$ *is a* $t_{\mathcal{P}}$*-norm.*

Example 3. *Let* $\mathcal{A}=C^1_{\mathbb{R}}[0,1]$ *with pointwise multiplication and norm defined by* $\|a(t)\|=\|a(t)\|_\infty+\|a'(t)\|_\infty$. *Then,* $\mathcal{A}$ *is a Banach algebra with* $e=1$. *Define* $\mathcal{P}=\{a\in\mathcal{A}\colon a(t)\ge 0 \text{ for all } t\in[0,1]\}$; *then*

$$I_e=\{a\in\mathcal{P}\colon a(t)\le 1 \text{ for all } t\in[0,1]\}.$$

Then, the mapping $*\colon I_e\times I_e\to I_e$ *defined by* $\psi(t)*\varphi(t)=\psi(t)\varphi(t)$ *for all* $t\in[0,1]$, *i.e., the pointwise multiplication, is a* $t_{\mathcal{P}}$*-norm.*

In what follows, for $a,b\in\mathcal{A}$, by (a,b) we denote the set $\{c\in\mathcal{A}\colon a\ll c\ll b\}$ and by $(a,b]$ we denote the set $\{c\in\mathcal{A}\colon a\ll c\preceq b\}$.

Definition 8. *Let X be a nonempty set and* $\mathcal{A}$ *be a Banach algebra with cone* $\mathcal{P}$. *Then, a vector-valued fuzzy metric space over* $\mathcal{A}$ *is a triplet* $(X,M,*)$ *such that* $*$ *is a continuous* $t_{\mathcal{P}}$*-norm and M is a* $\mathcal{P}$*-valued fuzzy set on* $X\times X\times\mathcal{P}^\circ$ *satisfying the following conditions:*

(i) $\theta\ll M(x,y,c)$;
(ii) $M(x,y,c)=e$ *if and only if* $x=y$;
(iii) $M(x,y,c)=M(y,x,c)$;
(iv) $M(x,y,c)*M(y,z,c')\preceq M(x,z,c+c')$;
(v) $M(x,y,\cdot)\colon\mathcal{P}^\circ\to I_e$ *is continuous*

for all $x,y,z\in X$ *and* $c,c'\in\mathcal{P}^\circ$. *In this case, M is said to be a vector-valued fuzzy metric.*

Remark 2. *Condition (ii) of the above definition is equivalent to the following:*

$$M(x,x,c)=e \text{ for all } x\in X, c\in\mathcal{P}^\circ; \text{ and } \theta\ll M(x,y,c)\prec e \text{ for all } x\ne y, c\in\mathcal{P}^\circ.$$

While, condition (iv) shows that if $\theta\ll c_1\ll c_2$, *then we have* $M(x,y,c_1)\preceq M(x,y,c_2)$.

Remark 3. *Note that for* $\mathcal{A}=\mathbb{R}$ *the Banach algebra of real numbers with usual norm, ordinary multiplication,* $\mathcal{P}=[0,\infty)$ *and* $I_e=[0,1]$ *the real unit interval, the above definition reduces into the definition of fuzzy metric spaces given by [3]. On the other hand, if* $(X,M,*)$ *is a complex-valued fuzzy metric space (in the sense of [25]), then* $(X,M,*)$ *is a vector-valued fuzzy metric space with* $\mathcal{A}=\mathbb{C}=\{(x,y)\colon x,y\in\mathbb{R}\}$ *the Banach algebra of complex numbers with coordinate-*

wise multiplication and with maximum norm $\|(x,y)\| = \max\{|x|,|y|\}$, *in which* $e = (1,1)$, $\mathcal{P} = \{(x,y) \in \mathbb{C} \colon x, y \geq 0\}$, $I_e = \{(x,y) \in \mathbb{C} \colon 0 \leq x, y \leq 1\}$.

Thus, the notion of vector-valued fuzzy metric spaces generalizes and unifies the classical fuzzy metric spaces and the notion of complex-valued fuzzy metric spaces (in the sense of [25]).

The following two examples show that for every fuzzy metric space (in the sense of [3]) there exists a vector-valued fuzzy metric space.

Example 4. *Let* $(X, f, \star)$ *be a fuzzy metric space (in the sense of [3]). Let* $\mathcal{A}, \mathcal{P}, e, I_e$ *and* $*$ *be taken from Example 2, then* $(X, M, *)$ *is a vector-valued fuzzy metric space over* $\mathcal{A}$*, where* M *is defined by:*

$$M(x,y,c) = f(x,y,\det(c))e \text{ for all } x,y \in X, c \in \mathcal{P}^\circ.$$

The following proposition shows that the vector-valued fuzzy metric spaces can be constructed in a more general way.

Proposition 1. *Let* (X, d) *be a cone metric space (see [27]) over a Banach algebra* $\mathcal{A}$ *with cone* $\mathcal{P}$ *such that every pair of points in the ordered interval* $[-e, e]$ *is comparable and* $d(x,y) \ll e$ *for all* $x, y \in X$*. Then* $(X, M, *)$ *is a vector-valued fuzzy metric space over* $\mathcal{A}$*, where* $a * b = \max\{a + b - e, \theta\}$ *and*

$$M(x,y,c) = e - d(x,y) \text{ for all } x,y \in X, c \in \mathcal{P}^\circ.$$

Indeed, it is sufficient to assume that every pair of points in $[-e, e]$ *has a supremum (instead pair is comparable) and to take* $a * b = \sup\{a + b - e, \theta\}$.

Example 5. *Let* $X = [0, b]$, $0 < b < 1$ *and* $\mathcal{A} = \mathbb{R}^2$ *be the Banach algebra with the norm* $\|(x_1, x_2)\| = |x_1| + |x_2|$*, with the multiplication "*$\cdot$*" defined by* $(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2)$ *and the unit* $e = (1,1)$*. Let* $\mathcal{P} = \{(x_1, x_2) \colon x_1, x_2 \geq 0\}$*, then* $I_e = \{(x,y) \in \mathcal{A} \colon 0 \leq x, y \leq 1\}$*. Define a cone metric* $d \colon X \times X \to \mathcal{P}$ *by* $d(x,y) = |x - y|e$ *for all* $x, y \in X$*. Then,* $d(x,y) \ll e$ *for all* $x, y \in X$ *and* $(X, M, *)$ *is a vector-valued fuzzy metric space over* $\mathcal{A}$*, where* $x * y = \sup\{x + y - e, \theta\}$ *for all* $x, y \in I_e$ *and*

$$M(x,y,c) = e - d(x,y) \text{ for all } x,y \in X, c \in \mathcal{P}^\circ.$$

Example 6. *Let* $X = \mathbb{R}$, $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ *with pointwise multiplication "*$\cdot$*" and supremum norm. Let* $\mathcal{P} = \{\psi(t) \in \mathcal{A} \colon \psi(t) \geq 0 \text{ for all } t \in [0,1]\}$*, then* $\mathcal{A}$ *is a Banach algebra with* $e = 1$ *and* $I_e = \{\psi(t) \in \mathcal{A} \colon 0 \leq \psi(t) \leq 1 \text{ for all } t \in [0,1]\}$*. Define a cone metric* $d \colon X \times X \to \mathcal{P}$ *by* $d(x,y) = |x - y|e^t$ *for all* $x, y \in X$*. Then, for every* $c \in \mathcal{P}^\circ$ *the vector* $c + d(x,y)$ *is invertible; indeed, the pointwise multiplicative inverse with respect to ordinary multiplication is the inverse of* $c(t) + d(x,y)(t)$ *in* $\mathcal{A}$*. Then* $(X, M, *)$ *is a vector-valued fuzzy metric space over* $\mathcal{A}$*, where* $*$ *is the pointwise multiplication and*

$$M(x,y,c)(t) = c(t)\left[c(t) + |x-y|e^t\right]^{-1} \text{ for all } x,y \in X, c(t) \in \mathcal{P}^\circ, t \in [0,1].$$

Definition 9. *A sequence* $\{u_n\}$ *in* I_e *is said to be an e-sequence if, for each* $\varepsilon \in (\theta, e)$*, there exists* $n_0 \in \mathbb{N}$ *such that* $u_n \gg e - \varepsilon$ *for all* $n > n_0$.

Proposition 2. *In a Banach algebra* $\mathcal{A}$ *with cone* $\mathcal{P}$*:*

(A) *If* $\varepsilon_1, \varepsilon_2 \in (\theta, e)$ *are such that* $\varepsilon_1 \gg \varepsilon_2$*, then we can find an* $\varepsilon_3 \in (\theta, e)$ *such that* $\varepsilon_1 * \varepsilon_3 \gg \varepsilon_2$*;*
(B) *For every* $\varepsilon_1 \in (\theta, e)$ *we can find* $\varepsilon_2, \varepsilon_3 \in (\theta, e)$ *such that* $\varepsilon_2 * \varepsilon_2 \gg \varepsilon_1$ *and* $\varepsilon_1 \gg \varepsilon_3 * \varepsilon_3$.

Proof. (A) Suppose, $\varepsilon_1, \varepsilon_2 \in (\theta, e)$ are such that $\varepsilon_1 \gg \varepsilon_2$ and

$$\varepsilon_1 * \varepsilon \not\gg \varepsilon_2 \text{ for all } \varepsilon \in (\theta, e).$$

Then, we can choose a sequence $\{\varepsilon_n\}$ in (θ, e) such that $\varepsilon_n \to e$ as $n \to \infty$ and $\varepsilon_1 * \varepsilon_n \not\gg \varepsilon_2$ for all $n \in \mathbb{N}$, i.e.,

$$\varepsilon_1 * \varepsilon_n - \varepsilon_2 \in \mathcal{A} \setminus \mathcal{P}^\circ \text{ for all } n \in \mathbb{N}.$$

Since $\mathcal{P}^\circ$ is open, $*$ is continuous and $\varepsilon_n \to e$ as $n \to \infty$; the above inclusion yields

$$\varepsilon_1 - \varepsilon_2 = \varepsilon_1 * e - \varepsilon_2 \notin \mathcal{P}^\circ.$$

This shows that $\varepsilon_1 \not\gg \varepsilon_2$ (a contradiction) and proves the result.

(B) It can be proved by following a process similar to the one used in part (A). □

Proposition 3. *In a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If $\{a_n\}$ and $\{b_n\}$ are two e-sequences in I_e, then $\{a_n * b_n\}$ is an e-sequence in I_e.*

Proof. Let $\varepsilon \in (\theta, e)$ be given; then, we have $e - \varepsilon \in (\theta, e)$. Then, by Proposition 2, there exists $\delta \in (\theta, e)$ such that $\delta * \delta \gg e - \varepsilon$. Again, since $e - \delta \in (\theta, e)$ and $\{a_n\}$ and $\{b_n\}$ are e-sequences in I_e there exists $n_0 \in \mathbb{N}$ such that

$$a_n \gg e - (e - \delta) = \delta,\ b_n \gg e - (e - \delta) = \delta \text{ for all } n > n_0.$$

As $*$ is nondecreasing, it yields

$$a_n * b_n \succeq \delta * \delta \gg e - \varepsilon.$$

Hence, $\{a_n * b_n\}$ is an e-sequence. □

Definition 10. *Let $(X, M, *)$ be a vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Cauchy sequence if for each $\varepsilon \in (\theta, e)$ and each $c \in \mathcal{P}^\circ$, there is $n_0 \in \mathbb{N}$ such that $e - \varepsilon \ll M(x_n, x_m, c)$, for all $n, m > n_0$.*

On the other hand, $\{x_n\}$ is called a weak Cauchy sequence if for each $\varepsilon \in (\theta, e)$ and each $c \in \mathcal{P}^\circ$, there is $n_0 \in \mathbb{N}$ such that $e - \varepsilon \ll M(x_n, x_{n+1}, c)$, for all $n > n_0$. Or equivalently, $\{x_n\}$ is called a weak Cauchy sequence if $\{M(x_n, x_{n+1}, c)\}$ is an e-sequence for all $c \in \mathcal{P}^\circ$.

It is easy to see that every Cauchy sequence is a weak Cauchy sequence, but the converse is not necessarily true.

Sequence $\{x_n\}$ is called convergent and converges to $x \in X$ if, for each $\varepsilon \in (\theta, e)$ and each $c \in \mathcal{P}^\circ$, there exists $n_0 \in \mathbb{N}$ such that $e - \varepsilon \ll M(x_n, x, c)$, for all $n > n_0$. Or equivalently, $\{x_n\}$ is called a convergent and converges to $x \in X$ if $\{M(x_n, x, c)\}$ is an e-sequence for all $c \in \mathcal{P}^\circ$.

Remark 4. *In a fuzzy metric space $(X, M, *)$, for the convergence of a sequence $\{x_n\}$ to a point $x \in X$ the necessary and sufficient condition is that the limit $\lim_{n\to\infty} M(x_n, x, t)$ must exist and be equal to 1 for all $t > 0$. The same is true for convergent sequences in complex-valued fuzzy metric spaces (with the mentioned limit equal to ℓ). On the other hand, in cases of vector-valued fuzzy metric spaces this condition is much weaker. Indeed, in a vector-valued fuzzy metric space the limit $\lim_{n\to\infty} M(x_n, x, c)$ may not exist for some $c > 0$ and for all $x \in X$, although the sequence $\{x_n\}$ may converge to some $x \in X$. For instance, let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ with pointwise multiplication "$\cdot$" and the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and let $\mathcal{P} = \{x(t) \in \mathcal{A} \colon x(t) \geq 0 \text{ for all } t \in [0,1]\}$, then $\mathcal{A}$ is a Banach algebra with $\theta = 0$, $e = 1$, $I_e = \{\psi(t) \in \mathcal{A} \colon 0 \leq \psi(t) \leq 1 \text{ for all } t \in [0,1]\}$. Suppose that $X = \mathcal{P}$, then $(X, M, *)$ is a vector-valued fuzzy metric space over $\mathcal{A}$, where $*$ is the pointwise multiplication and*

$$M(x, y, c)(t) = [1 + x(t) + y(t)]^{-1}, x \neq y, t \in [0,1] \text{ and } M(x, x, c) = e$$

for all $x, y \in X, c \in \mathcal{P}^\circ$. Now consider the sequence $\{x_n(t)\}$ in X, where $x_n(t) = \frac{t^n}{n}$. Note that $\lim_{n\to\infty} M(x_n, x, c)$ *does not exist for all $x \in X$ (otherwise, $\lim\limits_{n\to\infty} x_n = 0$ in $\mathcal{A}$ which is not true). This sequence is convergent and converges to $\theta = 0$ in X. Indeed, for each $\varepsilon \in (\theta, e)$ and each $c \in \mathcal{P}^\circ$, we can find $n_0 \in \mathbb{N}$ such that $e - \varepsilon \ll M(x_n, \theta, c)$ for all $n_0 \in \mathbb{N}$. The above example reflects the fact that the concepts introduced here are not trivial analogues of the usual versions, it also reflects the case when existing concepts cannot be applied but new concepts can be applied.*

Proposition 4. *If $(X, M, *)$ is a vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$, then every convergent sequence in X is a Cauchy (therefore weak Cauchy) sequence in X.*

Proof. Suppose that the sequence $\{x_n\}$ converges to some $x \in X$; i.e., $\{M(x_n, x, c)\}$ is an e-sequence for all $c \in \mathcal{P}^\circ$. Since $\{M(x_n, x, c)\}$ is an e-sequence for all $c \in \mathcal{P}^\circ$, by Proposition 3 $M(x_n, x, c/2) * M(x, x_m, c/2)$ is an e-sequence. Hence, for each $\varepsilon \in (\theta, e)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, c/2) * M(x, x_m, c/2) \gg e - \varepsilon$ for all $n, m > n_0$. Since

$$M(x_n, x_m, c) \succeq M(x_n, x, c/2) * M(x, x_m, c/2)$$

for all $c \in \mathcal{P}^\circ$, we must have $M(x_n, x_m, c) \gg e - \varepsilon$ for all $n, m > n_0$. Hence, $\{x_n\}$ is a Cauchy sequence. □

It is easy to see that the converse of the above proposition is not necessarily true; hence, we define the following:

Definition 11. *We say that the space $(X, M, *)$ is complete (respectively, strong-complete) if every Cauchy (respectively, weak-Cauchy) sequence in X converges to some $x \in X$.*

Remark 5. *The types of the Cauchy sequences (respectively, weak Cauchy sequences) and the completeness (respectively, strong completeness) defined here are a vector-valued analogue of M-Cauchy sequences defined by [3] (respectively, G-Cauchy sequences defined by [18]) and M-completeness (respectively, G-completeness), respectively, in classical fuzzy metric spaces. Hence, every G-complete (respectively, M-complete) fuzzy metric space is a particular case of strong complete (respectively, complete) vector-valued fuzzy metric spaces.*

Remark 6. *In a Banach algebra, we observe that the set (θ, e) may be empty, e.g., let $\mathcal{A} = \mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, with the multiplication "$\cdot$" defined by $(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_1 + x_1 y_2)$, cone $\mathcal{P} = \{(x_1, x_2) \colon x_1, x_2 \geq 0\}$ and the unit $e = (1, 0)$. Then, it is easy to see that $(\theta, e) = \emptyset$. Therefore, in a vector-valued fuzzy metric space with such Banach algebra and cone, every sequence is a Cauchy sequence as well as convergent and so such spaces are trivially complete (strong-complete). Hence, we can say that the presented concepts of Cauchyness and completeness are new and quite different from the Cauchyness and completeness of ordinary fuzzy metric spaces and complex-valued fuzzy metric spaces.*

Note that (θ, e) is nonempty if and only if $e \in \mathcal{P}^\circ$. Therefore, throughout the paper, we assume that $e \in \mathcal{P}^\circ$.

Proposition 5. *In a vector-valued fuzzy metric space $(X, M, *)$ over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$, if $x, y \in X, c \in \mathcal{P}^\circ, \varepsilon \in (\theta, e)$ are such that $M(x, y, c) \gg e - \varepsilon$, then we can find $c_0 \in (\theta, c)$ such that $M(x, y, c_0) \gg e - \varepsilon$.*

Proof. We prove the result by contradiction. Then, suppose that $x, y \in X, c \in \mathcal{P}^\circ, \varepsilon \in (\theta, e)$ are such that $M(x, y, c) \gg e - \varepsilon$ and

$$M(x, y, c') \not\gg e - \varepsilon \text{ for all } c' \in (\theta, c).$$

Since $\alpha\mathcal{P}^\circ \subset \mathcal{P}^\circ$ for all $\alpha > 0$, it follows from the above that $M\left(x, y, c - \frac{c}{n+1}\right) - e + \varepsilon \notin \mathcal{P}^\circ$ for all $n \in \mathbb{N}$, i.e.,

$$M\left(x, y, c - \frac{c}{n+1}\right) - e + \varepsilon \in \mathcal{A} \setminus \mathcal{P}^\circ \text{ for all } n \in \mathbb{N}.$$

Since $M(x, y, \cdot)$ is continuous and $\mathcal{P}^\circ$ is open, the above inclusion yields

$$M(x, y, c) - e + \varepsilon \in \mathcal{A} \setminus \mathcal{P}^\circ.$$

This shows that $M(x, y, c) \not\gg e - \varepsilon$ (a contradiction) and proves the result. □

Definition 12. *Let $(X, M, *)$ be a vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$, $x \in X$, $c \in \mathcal{P}^\circ$ and $\varepsilon \in (\theta, e)$. Then, the open ball with center x and radius ε is denoted by $B(x, \varepsilon, c)$ and it is defined by:*

$$B(x, \varepsilon, c) = \{y \in X \colon M(x, y, c) \gg e - \varepsilon\}.$$

A subset $S \subseteq X$ is called open if for every $x \in S$, there exist $c \in \mathcal{P}^\circ$ and $\varepsilon \in (\theta, e)$ such that $B(x, \varepsilon, c) \subseteq S$.

Theorem 1. *In a vector-valued fuzzy metric space every open ball is an open set.*

Proof. Consider an open ball $B(x, \varepsilon, c)$ and suppose $y \in B(x, \varepsilon, c)$. Then we have:

$$M(x, y, c) \gg e - \varepsilon.$$

Therefore, by Proposition 5, there exists $c_0 \in (\theta, c)$ such that $M(x, y, c_0) \gg e - \varepsilon$. Let $\varepsilon_0 = M(x, y, c_0) \gg e - \varepsilon$. As $\varepsilon_0 \gg e - \varepsilon$, there exists $\delta \in (\theta, e)$ such that $\varepsilon_0 \gg e - \delta \gg e - \varepsilon$. Therefore, by Proposition 2 there exists $\varepsilon_1 \in (\theta, e)$ such that $e - \delta \ll \varepsilon_0 * \varepsilon_1$. Now consider the open ball $B(y, e - \varepsilon_1, c - c_0)$. If $z \in B(y, e - \varepsilon_1, c - c_0)$, then we have $M(y, z, c - c_0) \gg \varepsilon_1$. Therefore:

$$\begin{aligned} M(x, z, c) &\succeq M(x, y, c_0) * M(y, z, c - c_0) \\ &\succeq \varepsilon_0 * \varepsilon_1 \\ &\gg e - \delta \\ &\gg e - \varepsilon. \end{aligned}$$

Therefore, $z \in B(x, \varepsilon, c)$ and hence $B(y, e - \varepsilon_1, c - c_0) \subset B(x, \varepsilon, c)$. This shows that $B(x, \varepsilon, c)$ is an open set. □

It is obvious that, if $(X, M, *)$ is a vector-valued fuzzy metric space, then the collection

$$\mathfrak{T} = \{S \subseteq X \colon \text{ for all } x \in S \text{ there exist } c \in \mathcal{P}^\circ, \varepsilon \in (\theta, e) \text{ such that } B(x, \varepsilon, c) \subseteq S\}$$

is a topology on X. Also, by properties of a cone in a Banach algebra, for given $c \in \mathcal{P}^\circ$ and $\varepsilon \in (\theta, e)$ we can always find $N \in \mathbb{N}$ such that $c - \frac{e}{N}, \varepsilon - \frac{e}{N} \in \mathcal{P}^\circ$; hence, Remark 2 yields that for every $x \in X$ the collection:

$$\mathcal{B} = \left\{B\left(x, \frac{e}{n}, \frac{e}{n}\right) \colon n \in \mathbb{N}\right\}$$

is a local base at $x \in X$. Therefore, the topology $\mathfrak{T}$ is first countable.

We next show that this topology is Hausdorff.

Theorem 2. *The topology $\mathfrak{T}$ is Hausdorff.*

Proof. Let $(X, M, *)$ be the given vector-valued fuzzy metric space and $x, y \in X$ with $x \neq y$. Then, by definition we have

$$\theta \ll M(x,y,c) \prec e \text{ for all } c \in \mathcal{P}^\circ.$$

Therefore, for any fixed $c \in \mathcal{P}^\circ$ we have $\theta \prec e - M(x,y,c)$. Therefore, by (iii) of Remark 1, there exists $\varepsilon_0 \in \mathcal{P}^\circ$ such that $e - M(x,y,c) \not\ll \varepsilon_0$. Without loss of generality, we can assume that $\varepsilon_0 \in (\theta, e)$.

Since $\varepsilon_0 \ll e$, by Proposition 2, there exists $\varepsilon_1 \in (\theta, e)$ such that $\varepsilon_1 * \varepsilon_1 \succeq e - \varepsilon_0$. We claim that

$$B\Big(x, e - \varepsilon_1, \frac{c}{2}\Big) \cap B\Big(y, e - \varepsilon_1, \frac{c}{2}\Big) = \emptyset.$$

If there exists

$$z \in B\Big(x, e - \varepsilon_1, \frac{c}{2}\Big) \cap B\Big(y, e - \varepsilon_1, \frac{c}{2}\Big)$$

then $M\Big(x, z, \frac{c}{2}\Big) \gg \varepsilon_1$ and $M\Big(z, y, \frac{c}{2}\Big) \gg \varepsilon_1$ and hence

$$\begin{aligned} M(x,y,c) &\succeq M\Big(x,z,\frac{c}{2}\Big) * M\Big(z,y,\frac{c}{2}\Big) \\ &\gg \varepsilon_1 * \varepsilon_1 \\ &\succeq e - \varepsilon_0 \end{aligned}$$

i.e., $e - M(x,y,c) \ll \varepsilon_0$. This contradiction proves the result. □

In the next section, we prove some fixed point results in vector-valued fuzzy metric spaces.

4. Fixed Point Theorems

Let $(X, M, *)$ be a vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. We say that the mapping $T\colon X \to X$ is a generalized Tirado contraction or a generalized fuzzy Banach contraction with contractive vector k if the following condition is satisfied: $k \in \mathcal{P}$ such that $\rho(k) < 1$ and

$$e - M(Tx, Ty, c) \preceq k(e - M(x,y,c))$$

for all $x, y \in X$ and $c \in \mathcal{P}^\circ$.

Note that for $\mathcal{A} = \mathbb{R}$ the Banach algebra of real numbers with ordinary multiplication, usual norm, $\mathcal{P} = [0, \infty)$ and $I_e = [0,1]$ the real unit interval, the generalized Tirado contractions reduce into the contractions considered by [19] (see also [20]). Hence, Tirado's contraction is a particular case of generalized Tirado contractions.

Let $\Xi_{\mathcal{A}}$ denote the family of all functions $\xi\colon I_e \to I_e$ satisfying the following properties:

(ξ_1) ξ is nondecreasing;
(ξ_2) if $\{c_n\}$ is an e-sequence, then $\{\xi(c_n)\}$ is an e-sequence, where $\theta \ll c_n \preceq e$;
(ξ_3) $\{\xi^n(c)\}$ is an e-sequence for all $\theta \ll c \preceq e$.

Example 7. *Consider the Banach algebra $C^1_{\mathbb{R}}[0,1]$ with pointwise multiplication and norm defined by $\|a(t)\| = \|a(t)\|_\infty + \|a'(t)\|_\infty$ and with unit $e = 1$. Let $\mathcal{P} = \{a \in C^1_{\mathbb{R}}[0,1]\colon a(t) \geq 0 \text{ for all } t \in [0,1]\}$, then $I_e = \{a \in \mathcal{P}\colon a(t) \leq 1 \text{ for all } t \in [0,1]\}$. If we define $\xi\colon I_e \to I_e$ by $\xi(a(t)) = \sqrt{a(t)}$ for all $a \in I_e$ and $t \in [0,1]$, then it is easy to see that $\xi \in \Xi_{C^1_{\mathbb{R}}[0,1]}$.*

By Ψ we denote the family of all functions $\psi\colon [0,1] \to [0,1]$ such that ψ is continuous, nondecreasing and $\psi(t) > t$ (or equivalently, $\lim_{n\to\infty} \psi^n(t) = 1$) for all $t \in (0,1)$ [23].

Example 8. *Let $\mathcal{A} = \mathbb{R}^n$ with Euclidian norm, coordinate-wise multiplication and $\mathcal{P} = \{(a_1, \ldots, a_n) \in \mathbb{R} \colon a_i \geq 0, 1 \leq i \leq n\}$. Then $e = (1, \ldots, 1)$. If $\psi_i \in \Psi$ for $1 \leq i \leq n$, then the function $\xi \colon I_e \to I_e$, where*

$$\xi(c_1, \ldots, c_n) = (\psi_1(c_1), \ldots, \psi_n(c_n)) \text{ for all } (c_1, \ldots, c_n) \in I_e$$

is a member of $\Xi_{\mathbb{R}^n}$.

Note that every member of the family Ψ is a member of $\Xi_{\mathbb{R}^n}$ for $n = 1$.

Remark 7. *Consider the family $\mathcal{B}$ of nondecreasing functions $\beta \colon [0,1] \to [0,1]$ such that $\lim_{n\to\infty} \beta^n(t) = 1$ for all $t \in (0,1)$ (see [35]). In the above example, all ψ_i are assumed to be continuous, but note that even if we omit the continuity of these functions the conclusion remains same. More precisely, we have the proper inclusion:*

$$\Psi \subset \mathcal{B} = \Xi_{\mathbb{R}}.$$

Definition 13. *Let $(X, M, *)$ be a vector-valued fuzzy metric space over Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$ and $T \colon X \to X$ be a mapping. Then, T is called a generalized ξ-contraction if there exists a $\xi \in \Xi_{\mathcal{A}}$ such that the following condition is satisfied:*

$$M(Tx, Ty, c) \succeq \xi(M(x, y, c))$$

for all $x, y \in X$ and $c \in \mathcal{P}^\circ$.

Example 9. *If $e - k \in \mathcal{P}$ and $k \neq e$ (i.e., $k \prec e$), then every generalized Tirado contraction with a contractive vector k is a generalized ξ-contraction with $\xi(c) = e - k + kc$ for all $c \in I_e$.*

Remark 8. *In the above example, we assume that $e - k \in \mathcal{P}$. We point out that this assumption is necessary. In fact, if we discard this assumption, the conclusion of the above example may not be valid. Indeed, if $c = \theta$, then $\xi(c) = \xi(\theta) = e - k$, so, if $e - k \notin \mathcal{P}$, then we have $\xi(c) \notin \mathcal{P}$; i.e., we obtain $c \in I_e$ such that $\xi(c) \notin I_e$. Hence, $\xi \notin \Xi_{\mathcal{A}}$.*

Remark 9. *On a fuzzy metric space $(X, M, *)$ (in the sense of [3]), we say that a mapping $T \colon X \to X$ is a ψ-contraction ([23]) if there exists $\psi \in \Psi$ such that the following condition is satisfied:*

$$M(Tx, Ty, t) \geq \psi(M(x, y, t)) \text{ for all } x, y \in X \text{ and } t > 0.$$

In view of Remark 3, it is easy to see that every ψ-contraction is a generalized ξ-contraction with $\psi(t) = \xi(t)$ for all $t \in [0,1]$. As the contractions considered by [21,24] (see [36]) are included in the class of ψ-contractions, these types of contractions are also members of the family $\Xi_{\mathbb{R}}$.

On the other hand, in view of Remark 3, it is easy to see that every fuzzy Banach contraction (considered by [25]) with fuzzy contractive constant $k \in (0,1)$ is a generalized Tirado contraction with contractive vector (k, k). Hence, the contractions considered by [25] are also a particular type of generalized ξ-contractions.

We next prove some fixed point theorems for generalized ξ-contractions in strong complete and complete vector-valued fuzzy metric spaces.

Theorem 3. *Let $(X, M, *)$ be a strong complete vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If T is a generalized ξ-contraction, then T has a unique fixed point.*

Proof. We start with an arbitrary $x_0 \in X$ and define an iterative sequence $\{x_n\}$ by:

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

We shall show that this sequence is a weak Cauchy sequence. Then, for every $n \in \mathbb{N}$, $c \in \mathcal{P}^\circ$ we have:

$$\begin{array}{rcl} M(x_n, x_{n-1}, c) & = & M(Tx_{n-1}, Tx_{n-2}, c) \\ & \succeq & \xi(M(x_{n-1}, x_{n-2}, c)). \end{array}$$

Replacing n by $n-1$ in the above inequality, we obtain

$$M(x_{n-1}, x_{n-2}, c) \succeq \xi(M(x_{n-2}, x_{n-3}, c)).$$

Since ξ is nondecreasing, the above inequality yields

$$\xi(M(x_{n-1}, x_{n-2}, c)) \succeq \xi(\xi(M(x_{n-2}, x_{n-3}, c))) = \xi^2(M(x_{n-2}, x_{n-3}, c)).$$

Therefore

$$M(x_n, x_{n-1}, c) \succeq \xi^2(M(x_{n-2}, x_{n-3}, c)).$$

Repeating this process, we obtain:

$$M(x_n, x_{n-1}, c) \succeq \xi^n(M(x_0, x_1, c)). \tag{1}$$

By (ξ_3) we have $\{\xi^n(M(x_0, x_1, c))\}$ is an e-sequence. So, for every $\varepsilon \in (\theta, e)$ there exists $n_0 \in \mathbb{N}$ such that

$$\xi^n(M(x_0, x_1, c)) \gg e - \varepsilon \text{ for all } n > n_0.$$

It follows from the above inequality and (1) that

$$M(x_n, x_{n-1}, c) \gg e - \varepsilon \text{ for all } n > n_0.$$

This shows that $\{x_n\}$ is a weak Cauchy sequence.

Since $(X, M, *)$ is strong complete, $\{x_n\}$ converges to some $u \in X$. We now show that u is a fixed point of T.

Fix an arbitrary $c \in \mathcal{P}^\circ$. As $\{x_n\}$ converges to u, the sequence $\{M(x_n, u, c)\}$ is an e-sequence; hence, by (ξ_2) we have $\{\xi(M(x_n, u, c/2))\}$ is an e-sequence. By Proposition 3, the sequence $\{\xi(M(x_n, u, c/2)) * M(x_{n+1}, u, c/2)\}$ is an e-sequence. This shows that for every $\varepsilon \in (\theta, e)$ there exists $n_1 \in \mathbb{N}$ such that

$$\xi(M(x_n, u, c/2)) * M(x_{n+1}, u, c/2) \gg e - \varepsilon \text{ for all } n > n_1. \tag{2}$$

Using (2), we obtain the following: for all $n > n_1$

$$\begin{array}{rcl} M(Tu, u, c) & \succeq & M(Tu, x_{n+1}, c/2) * M(x_{n+1}, u, c/2) \\ & = & M(Tu, Tx_n, c/2) * M(x_{n+1}, u, c/2) \\ & \succeq & \xi(M(x_n, u, c/2)) * M(x_{n+1}, u, c/2) \\ & \gg & e - \varepsilon. \end{array}$$

This shows that $M(Tu, u, c) - e + \varepsilon \in \mathcal{P}^\circ$. As $\varepsilon \in (\theta, e)$ is arbitrary, $*$ is continuous and $\mathcal{P}$ is closed, we must have $M(Tu, u, c) - e \in \mathcal{P}$. Because $e - M(Tu, u, c) \in \mathcal{P}$ by definition of $\mathcal{P}$ we obtain $M(Tu, u, c) = e$. Taking into account that $c \in \mathcal{P}^\circ$ is arbitrary, we conclude $Tu = u$.

For uniqueness of fixed point, in contrast, suppose that $v \in X$ is another fixed point of T and $u \neq v$. Let $c \in \mathcal{P}^\circ$ be arbitrary; since T is a generalized ξ-contraction, we have:

$$M(u, v, c) = M(Tu, Tv, c) \succeq \xi(M(u, v, c)).$$

Since ξ is nondecreasing, it follows from the above inequality that

$$M(u,v,c) \succeq \xi^n(M(u,v,c)) \text{ for all } n \in \mathbb{N}. \tag{3}$$

By (ξ_3), the sequence $\{\xi^n(M(u,v,c))\}$ is an e-sequence; hence, for every $\varepsilon \in (\theta, e)$ there exists $n_2 \in \mathbb{N}$ such that

$$\xi^n(M(u,v,c)) \gg e - \varepsilon \text{ for all } n > n_2.$$

We obtain from (3) and the above that $M(u,v,c) \gg e - \varepsilon$; i.e., $M(u,v,c) - e \in \mathcal{P}^\circ \subset \mathcal{P}$. So, we obtain $M(u,v,c) = e$ and due to c being arbitrary we conclude that $u = v$, a contradiction. □

Example 10. *Consider the Banach algebra $\mathbb{R}^2$ with coordinate-wise multiplication, the Euclidean norm, zero vector $\theta = (0,0)$ and unit $e = (1,1)$. Consider the cone $\mathcal{P} = \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1, x_2 \geq 0\}$ in $\mathbb{R}^2$, then $I_e = \{(x_1,x_2) \in \mathbb{R}^2 \colon 0 \leq x_1, x_2 \leq 1\}$. Note that the partial ordering "$\preceq$" induced by $\mathcal{P}$ on $\mathbb{R}^2$ is not a linear order; nevertheless, we can always find the infimum (supremum) of any pair of elements of $\mathbb{R}^2$; i.e., the pair $(\mathbb{R}^2, \preceq)$ is a lattice. Indeed, $\inf\{(x_1,x_2),(y_1,y_2)\} = (\min\{x_1,y_1\}, \min\{x_2,y_2\})$. Hence, we define $* \colon I_e \times I_e \to I_e$ by:*

$$x * y = \inf\{x,y\} \text{ for all } x,y \in I_e.$$

Let $X = (0,1] \times (0,1]$ and consider a $\mathcal{P}$-valued fuzzy set M on $X \times X \times \mathcal{P}^\circ$ defined by:

$$M(x,y,c) = \begin{cases} \inf\{x,y\}, & \text{if } x \neq y; \\ e, & \text{if } x = y. \end{cases}$$

for all $x,y \in X$ and $c \in \mathcal{P}^\circ$. Then $(X,M,*)$ is a vector-valued fuzzy metric space. It is easy to see that if a sequence is weak Cauchy in X, then it must be convergent to e. Hence, $(X,M,*)$ is a strong complete vector-valued fuzzy metric space. Let $a_i, b_i \in \mathbb{R}$, $i = 1,2$ be such that $a_i > 1, 0 < b_i < 1, a_i b_i < 1$ and let $f_i \colon (0,b_i] \to (0,1]$, $i = 1,2$ be such that $f_i(r) \geq a_i r$ for all $r \in (0,b_i]$.

Let $T : X \to X$ be a mapping defined by

$$T(x) = T(x_1,x_2) = \begin{cases} (f_1(x_1), f_2(x_2)), & \text{if } 0 \leq x_1 \leq b_1, 0 \leq x_2 \leq b_2; \\ (1,1), & \text{otherwise.} \end{cases}$$

Now consider the mapping $\xi \colon I_e \to I_e$ defined by

$$\xi(x_1,x_2) = \begin{cases} (a_1 x_1, a_2 x_2), & \text{if } 0 \leq x_1 \leq b_1, 0 \leq x_2 \leq b_2; \\ (1,1), & \text{otherwise.} \end{cases}$$

Then, one can verify easily that $\xi \in \Xi_{\mathbb{R}^2}$ and T is a generalized ξ-contraction. Thus, all the conditions of Theorem 3 are satisfied. Hence, by Theorem 3 we can conclude the existence and uniqueness of the fixed point of the mapping T. Indeed, $(1,1) \in X$ is the unique fixed point of T.

Remark 10. *Theorems 3.1 and 3.7 of [25] are the main results of them. We show that the results from [25] are not applicable to the above example.*

It is obvious that the vector-valued fuzzy metric space $(X,M,*)$ in the above example is indeed a complex-valued fuzzy metric space. Set $a_1 = a_2 = 2, b_1 = b_2 = 0.2$ and $f_1(r) = f_2(r) = 2r$ for all $r \in (0, 0.2]$. We first show that there exists no $k \in [0,1)$ such that

$$\ell - M(Tx,Ty,c) \preceq k[\ell - M(x,y,c)] \tag{4}$$

for all $x, y \in X$ and for all $c \in P_\theta$. In contrast, suppose that such a k exists. Then, taking $x = (1,1), y = (y_1, y_2)$ and $y_1, y_2 \le 0.2$, we have

$$M(Tx, Ty, c) = \inf\{(1,1), (2y_1, 2y_2)\} = (2y_1, 2y_2) \text{ and}$$

$$M(x, y, c) = \inf\{(1,1), (y_1, y_2)\} = (y_1, y_2).$$

If $k < 0.6$, take $y_1 = y_2 = 0.05$; then, (4) yields a contradiction. If $k \ge 0.6$, then again take $y_1 = y_2 = \frac{1-k}{2}$ and (4) yields a contradiction. Hence, Theorem 3.1 of [25] cannot be used here.

Next, we observe that the value of $M(x, y, c)$ is independent of c; hence, the condition "$\lim_{n\to\infty} \inf_{y \in X} M(x, y, c_n) = \ell$ for all $x \in X$, whenever $\lim_{n\to\infty} c_n = \infty$" (i.e., a necessary condition in Theorem 3.7 of [25]) is not satisfied. Hence, Theorem 3.7 of [25] cannot be used here.

The following corollary is an extension and a generalization of the result of [23] in strong complete vector-valued fuzzy metric spaces. Note that the assumption of continuity (or even of semi-continuity) of function ψ is not needed.

Corollary 1. *Let $(X, M, *)$ be a strong complete vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If there exists a nondecreasing continuous function $\psi \colon I_e \to I_e$ such that $\psi(\varepsilon) \gg \varepsilon$ and $\lim_{n\to\infty} \psi^n(\varepsilon) = e$ for all $\varepsilon \in (\theta, e]$ and*

$$M(Tx, Ty, c) \succeq \psi(M(x, y, c))$$

for all $c \in \mathcal{P}^\circ$ and $x, y \in X$, then T has a unique fixed point.

In view of Example 9, we obtain the following corollaries which generalize and extend the fixed point result of [19] in strong complete vector-valued fuzzy metric space.

Corollary 2. *Let $(X, M, *)$ be a strong complete vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If T is a generalized Tirado contraction with contractive vector k such that $e - k \in \mathcal{P}$ and $k \neq e$, then T has a unique fixed point.*

Different authors have pointed out the drawbacks of the notion of strong completeness used in the preceding results (see [3,37,38]). So we are interested in proving a fixed point result for generalized ξ-contraction in a complete vector-valued fuzzy metric space. However, it will be established only for a class of vector-valued fuzzy metric spaces, which are defined below.

Definition 14. *Let $(X, M, *)$ be a vector-valued fuzzy metric space over Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. We will say that $(X, M, *)$ (or simply M) is a non-Archimedean vector-valued fuzzy metric space if (in addition), for each $x, y, z \in X$ and $c \in \mathcal{P}^\circ$, the following inequality is satisfied:*

$$M(x, y, c) * M(y, z, c) \preceq M(x, z, c). \tag{5}$$

Theorem 4. *Let $(X, M, *)$ be a complete non-Archimedean vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$ such that $a * a = a$ for all $a \in I_e$. If $T \colon X \to X$ is a generalized ξ-contraction, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and define an iterative sequence $\{x_n\}$ by:

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the existence is proved. In contrast, suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. It is not hard to check by induction on n that, for every $c \in \mathcal{P}^\circ$, we have $M(x_n, x_{n+1}, c) \succeq \xi^n(M(x_0, x_1, c))$ for all $n \in \mathbb{N}$. In addition, by (ξ_3)

we have $\{\xi^n(M(x_0, x_1, c))\}$ is an e-sequence for every $c \in \mathcal{P}^\circ$. Therefore, $\{M(x_n, x_{n+1}, c)\}$ is an e-sequence, for every $c \in \mathcal{P}^\circ$.

Next we will show that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon \in (\theta, e)$ and $c \in \mathcal{P}^\circ$. Taking into account that $\{M(x_n, x_{n+1}, c)\}$ is an e-sequence, given $\varepsilon' \in (\theta, \varepsilon)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+1}, c) \gg e - \varepsilon'$ for all $n > n_0$. Now consider $m, n \in \mathbb{N}$ such that $m \geq n > n_0$. Then,

$$\begin{aligned} M(x_n, x_m, c) &\succeq M(x_n, x_{n+1}, c) * M(x_{n+1}, x_{n+2}, c) * \cdots * M(x_{m-1}, x_m, c) \\ &\succeq (e - \varepsilon') * (e - \varepsilon') * \cdots * (e - \varepsilon') = e - \varepsilon' \gg e - \varepsilon. \end{aligned} \tag{6}$$

This shows that $\{x_n\}$ is a Cauchy sequence.

Since $(X, M, *)$ is complete, $\{x_n\}$ converges to some $u \in X$. We now show that u is a fixed point of T. Fix an arbitrary $c \in \mathcal{P}^\circ$, then:

$$M(Tu, u, c) \succeq M(Tu, x_{n+1}, c) * M(x_{n+1}, u, c). \tag{7}$$

As $\{x_n\}$ converges to some $u \in X$, the sequence $\{M(x_n, u, c)\}$ is an e-sequence. Also, for every $n \in \mathbb{N}$ we have

$$M(Tu, x_{n+1}, c) = M(Tu, Tx_n, c) \succeq \xi(M(u, x_n, c)).$$

As $\{M(x_n, u, c)\}$ is an e-sequence, the above inequality and (ξ_2) imply that $\{M(Tu, x_{n+1}, c)\}$ is an e-sequence. Hence, by Proposition 3 the sequence $\{M(Tu, x_{n+1}, c) * M(x_{n+1}, u, c)\}$ is an e-sequence. Therefore, for each $\varepsilon \in (\theta, e)$ there exists $n_1 \in \mathbb{N}$ such that

$$M(Tu, x_{n+1}, c) * M(x_{n+1}, u, c) \gg e - \varepsilon \text{ for all } n > n_1, c \in \mathcal{P}^\circ.$$

Using the above inequality in (7) we obtain $M(Tu, u, c) \gg e - \varepsilon$ for all $n > n_1$, i.e., $\theta \preceq e - M(Tu, u, c) \ll \varepsilon$ for all $n > n_1$. This inequality with Remark 1 yields that $e - M(Tu, u, c) = \theta$, i.e., $M(Tu, u, c) = e$. Due to $c \in \mathcal{P}^\circ$ being arbitrary, we conclude $Tu = u$.

For uniqueness of fixed point, in contrast, suppose that there exists a fixed point $v \in X$ of T and $u \neq v$.

Fix an arbitrary $c \in \mathcal{P}^\circ$. Then, since T is a generalized ξ-contraction, we have:

$$M(u, v, c) = M(Tu, Tv, c) \succeq \xi(M(u, v, c)).$$

Since ξ is nondecreasing, it follows from the above inequality that

$$M(u, v, c) \succeq \xi^n(M(u, v, c)) \text{ for all } n \in \mathbb{N}. \tag{8}$$

By (ξ_3), the sequence $\{\xi^n(M(u, v, c))\}$ is an e-sequence; hence, for each $\varepsilon \in (\theta, e)$ there exists $n_2 \in \mathbb{N}$ such that

$$\xi^n(M(u, v, c)) \gg e - \varepsilon \text{ for all } n > n_2.$$

We obtain from (8) and the above inequality that $M(u, v, c) \gg e - \varepsilon$, i.e., $\theta \preceq e - M(u, v, c) \ll \varepsilon$. By Remark 1, we have $M(u, v, c) = e$ and, since $c \in \mathcal{P}^\circ$ is arbitrary, we conclude $u = v$. This contradiction proves the uniqueness. □

Example 11. *Consider the Banach algebra $C^1_{\mathbb{R}}[0,1]$ with pointwise multiplication, norm defined by $\|a(t)\| = \|a(t)\|_\infty + \|a'(t)\|_\infty$, zero vector $\theta = 0$ and with unit $e = 1$. Let $\mathcal{P} = \{a \in C^1_{\mathbb{R}}[0,1] \colon a(t) \geq 0$ for all $t \in [0,1]\}$, then $I_e = \{a \in \mathcal{P} \colon a(t) \leq 1$ for all $t \in [0,1]\}$. Define $* \colon I_e \times I_e \to I_e$ by pointwise minimum: for each $t \in [0,1]$*

$$(a * b)(t) = \min\{a(t), b(t)\}.$$

for all $a, b \in I_e$. Then, $(a * a)(t) = \min\{a(t), a(t)\} = a(t)$ for all $t \in [0,1]$. Hence, $a * a = a$ for all $a \in I_e$. Let $X = (\theta, e]$ and consider a $\mathcal{P}$-valued fuzzy set M on $X \times X \times \mathcal{P}^\circ$ defined by the following: for each $t \in [0,1]$

$$M(x,y,c)(t) = \begin{cases} e, & \text{if } x = y; \\ (x * y)(t), & \text{otherwise.} \end{cases}$$

for all $x, y \in X$ and $c \in \mathcal{P}^\circ$. Then, it is easy to verify that $(X, M, *)$ is a vector-valued fuzzy metric space. Also, since every Cauchy sequence in X must converge to 1, $(X, M, *)$ is complete. Let $T\colon X \to X$ be a mapping defined by $T(x)(t) = \sqrt{x(t)}$ for all $t \in [0,1], x \in X$. Consider $\xi\colon I_e \to I_e$ defined by $\xi(a(t)) = \sqrt{a(t)}$ for all $a \in I_e$. Then T is a generalized ξ-contraction. Thus, all the conditions of Theorem 4 are satisfied. Hence, by Theorem 4 we can conclude the existence and uniqueness of fixed point of the mapping T. Indeed, $1 \in X$ is the unique fixed point of T.

The following result is an extension of the result of [23] in a complete non-Archimedean vector-valued fuzzy metric space which establishes the existence and uniqueness of fixed point. Note that conditions demanded on ψ are more restrictive than those fulfilled by the functions included in $\Xi_{\mathcal{A}}$. So, the hypothesis imposed on the $t_{\mathcal{P}}$-norm can be deleted.

Theorem 5. *Let $(X, M, *)$ be a complete non-Archimedean vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If there exists a nondecreasing function $\psi\colon I_e \to I_e$ such that $\psi(\varepsilon) \gg \varepsilon$ and $\lim_{n\to\infty} \psi^n(\varepsilon) = e$ for all $\varepsilon \in (\theta, e]$ and*

$$M(Tx, Ty, c) \succeq \psi(M(x,y,c))$$

for all $c \in \mathcal{P}^\circ$ and $x, y \in X$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define an iterative sequence $\{x_n\}$ by:

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

Following the same arguments as those used in the proof of Theorem 4, we obtain that $\{M(x_n, x_{n+1}, c)\}$ is an e-sequence for every $c \in \mathcal{P}^\circ$. Next, we will show that $\{x_n\}$ is a Cauchy sequence by contradiction.

So assume that $\{x_n\}$ is not Cauchy. Then, there exists $\varepsilon \in (\theta, e)$ and $c \in \mathcal{P}^\circ$ such that for all $n \in \mathbb{N}$ we can find $m > n$ such that $M(x_n, x_m, c) \not\gg e - \varepsilon$. Under such an assumption, we construct a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ as follows.

First of all, $\{M(x_n, x_{n+1}, c)\}$ is an e-sequence, so there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+1}, c) \gg e - \varepsilon$ for all $n \geq n_0$. Let $n(1) = n_0$. Now, we take, for all $k \in \mathbb{N}$, $n(k+1)$ as the (unique) integer greater than $n(k)$ such that $M(x_{n(k)}, x_m, c) \gg e - \varepsilon$, for all $n(k) \leq m < n(k+1)$ and $M(x_{n(k)}, x_{n(k+1)}, c) \not\gg e - \varepsilon$. Observe that our assumption allows us to obtain such a construction. In addition, for each $k \in \mathbb{N}$ we have that $n(k+1) > n(k) + 1$, since for each $n \geq n(1)$ satisfied $M(x_n, x_{n+1}, c) \gg e - \varepsilon$. Then, we have, for all $k \in \mathbb{N}$, the following

$$\begin{aligned} M(x_{n(k)}, x_{n(k+1)}, c) &\succeq M(x_{n(k)}, x_{n(k)+1}, c) * M(x_{n(k)+1}, x_{n(k+1)}, c) \\ &= M(x_{n(k)}, x_{n(k)+1}, c) * M(Tx_{n(k)}, Tx_{n(k+1)-1}, c) \\ &\succeq M(x_{n(k)}, x_{n(k)+1}, c) * \psi(M(x_{n(k)}, x_{n(k+1)-1}, c)) \\ &\succeq M(x_{n(k)}, x_{n(k)+1}, c) * \psi(e - \varepsilon). \end{aligned} \tag{9}$$

By (A) in Proposition 2, there exists $\delta \in (\theta, e)$ such that $(e - \delta) * \psi(e - \varepsilon) \gg e - \varepsilon$. In addition, since $\left\{M(x_{n(k)}, x_{n(k)+1}, c)\right\}$ is an e-sequence we can find $k_0 \in \mathbb{N}$ such that $M(x_{n(k)}, x_{n(k)+1}, c)$

$\gg e - \delta$ for every $k \geq k_0$. So, on account of inequality (9) we have, for every $k \geq k_0$, the following:

$$M(x_{n(k)}, x_{n(k+1)}, c) \quad \succeq \quad (e - \delta) * \psi(e - \varepsilon) \gg e - \varepsilon,$$

a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Then, the remainder of the proof follows the same argumentation used in the proof of Theorem 4. $\square$

The following corollary extends the fixed point result of [19] in complete vector-valued fuzzy metric spaces.

Corollary 3. *Let $(X, M, *)$ be a complete non-Archimedean vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with cone $\mathcal{P}$. If T is a generalized Tirado contraction with contractive vector k such that $e - k \in \mathcal{P}$ and $k \neq e$, then T has a unique fixed point.*

The above corollary is established for non-Archimedean vector-valued fuzzy metric spaces; therefore, the above corollary cannot be treated as a proper generalization of the main fixed point result of [25] (because in their main fixed point result no such assumption was imposed). In the next theorem, we establish a fixed point result by omitting such a requirement and applying a strong condition on cone $\mathcal{P}$ in such a way that the fixed point result of [25] is generalized in vector-valued fuzzy metric spaces. Before stating the theorem, we recall some definitions.

The following definitions are well known (see, e.g., [39]):

Definition 15. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{P}$ a cone in $\mathcal{A}$. Then:*

(a) *$\mathcal{P}$ is called normal if there is a number $K \geq 1$ such that for all $a, b \in \mathcal{A}$, $\theta \preceq a \preceq b$ implies that $\|a\| \leq K\|b\|$.*

(b) *$\mathcal{P}$ is called regular if every monotonic nondecreasing sequence which is bounded from above is convergent.*

(c) *$\mathcal{P}$ is called minihedral if $\sup\{x, y\}$ exists for all $x, y \in \mathcal{A}$ and strongly minihedral if every subset of $\mathcal{A}$ which is bounded from above has a supremum or equivalently, if every subset of $\mathcal{A}$ which is bounded from below has an infimum.*

Lemma 1 ([40])**.** *In a Banach space:*

(a) *Every strongly minihedral (closed) cone is normal;*

(b) *Every strongly minihedral normal (not necessarily closed) cone is regular.*

Theorem 6. *Let $(X, M, *)$ be a complete vector-valued fuzzy metric space over a Banach algebra $\mathcal{A}$ with strongly minihedral cone $\mathcal{P}$. If T is a generalized Tirado contraction with contractive vector k such that $e - k \in \mathcal{P}$ and $k \neq e$, then T has a unique fixed point.*

Proof. For any $x_0 \in X$, we define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We can assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$; otherwise, T will have a fixed point. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. We shall show that $\{x_n\}$ is a Cauchy sequence.

Fix an arbitrary $c \in \mathcal{P}^\circ$. Since $\mathcal{P}$ is strongly minihedral, $\theta \prec M(x_n, x_m, c) \preceq e$ for all $n, m \in \mathbb{N}$; therefore, for every $n \in \mathbb{N}$ the infimum $\inf_{m>n} M(x_n, x_m, c)$ exists. Let

$$\alpha_n = \inf_{m>n} M(x_n, x_m, c).$$

Obviously, $\theta \preceq \alpha_n \preceq e$ for all $n \in \mathbb{N}$. Since T is a generalized Tirado contraction with contractive vector k, for every $n \in \mathbb{N}$ and $m > n$ we have

$$e - M(x_{n+1}, x_{m+1}, c) = e - M(Tx_n, Tx_m, c) \preceq k[e - M(x_n, x_m, c)]. \tag{10}$$

Since $e - k \in \mathcal{P}$ and $k \neq e$, the above inequality implies that

$$M(x_n, x_m, c) \preceq M(x_{n+1}, x_{m+1}, c) \quad \text{for all} \quad n \in \mathbb{N},\ m > n.$$

As $\mathcal{P}$ is strongly minihedral, taking infimum over $m > n$ and using the definition of α_n in the above inequality we obtain

$$\theta \preceq \alpha_n \preceq \alpha_{n+1} \preceq e \quad \text{for all} \quad n \in \mathbb{N}. \tag{11}$$

Thus, $\{\alpha_n\}$ is a monotonic nondecreasing sequence in $\mathcal{P}$ which is strongly minihedral (therefore regular); hence, there exists $e_1 \in \mathcal{P}$ such that $\lim_{n\to\infty} \alpha_n = e_1 \preceq e$.

Inequality (10) implies that

$$e - k + kM(x_n, x_m, c) \preceq M(x_{n+1}, x_{m+1}, c)$$

for all $m > n$. Again, since $\mathcal{P}$ is strongly minihedral, taking infimum over $m > n$ and using the definition of α_n in the above inequality we obtain $e - k + k\alpha_n \preceq \alpha_{n+1}$, for every $n \in \mathbb{N}$. Hence, by the closedness of $\mathcal{P}$ we obtain $e - k \preceq (e - k)e_1 \preceq e - k$. So, $e_1 = e$ and

$$\lim_{n\to\infty} \alpha_n = e.$$

Therefore, for every given $\varepsilon \in (\theta, e)$ there exists $n_0 \in \mathbb{N}$ such that

$$\alpha_n \gg e - \varepsilon \text{ for all } n > n_0.$$

By the definition of α_n, we have $M(x_n, x_m, c) \succeq \alpha_n$ for all $m > n \in \mathbb{N}$; hence, it follows from the above inequality that

$$M(x_n, x_m, c) \gg e - \varepsilon \text{ for all } m > n > n_0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Hence, by the completeness of X there exists $u \in X$ such that the sequence $\{M(x_n, u, c)\}$ is an e-sequence for all $c \in \mathcal{P}^\circ$.

We shall show that u is the fixed point of T. Since T is a generalized Tirado contraction we have $e - M(Tx_n, Tu, c) \preceq k[e - M(x_n, u, c)]$, i.e.,

$$e - k + kM(x_n, u, c) \preceq M(Tx_n, Tu, c). \tag{12}$$

Since $\{M(x_n, u, c)\}$ is an e-sequence for every $c \in \mathcal{P}^\circ$, for given $\varepsilon \in (\theta, e)$ there exists $n_1 \in \mathbb{N}$ such that

$$M(x_n, u, c) \gg e - \varepsilon \text{ for all } n > n_1, c \in \mathcal{P}^\circ. \tag{13}$$

By the use of (12) and (13), we obtain the following: for every $c \in \mathcal{P}^\circ$ and for all $n > n_0$

$$\begin{aligned} M(u, Tu, c) &\succeq M(u, x_{n+1}, c/2) * M(x_{n+1}, Tu, c/2) \\ &= M(u, x_{n+1}, c/2) * M(Tx_n, Tu, c/2) \\ &\gg (e - \varepsilon) * (e - k\varepsilon). \end{aligned}$$

Since $*$ and multiplication in $\mathcal{A}$ both are continuous, the above inequality shows that $M(u, Tu, c) \succeq e$, i.e., $M(u, Tu, c) = e$ for all $c \in \mathcal{P}^\circ$. Thus, $Tu = u$; i.e., u is a fixed point of T.

If $v \in X$ is another fixed point of T, then we must have:

$$e - M(u, v, c) = e - M(Tu, Tv, c) \preceq k[e - M(u, v, c)] \text{ for all } c \in \mathcal{P}^\circ.$$

Since $\rho(k) < 1$, the above inequality yields $e - M(u, v, c) = \theta$; i.e., $u = v$. Thus, u is the unique fixed point of T. □

Corollary 4 ([25])**.** *Let $(X, M, *)$ be a complete complex-valued fuzzy metric space. If T is a fuzzy Banach contraction with contractive constant k, then T has a unique fixed point in X.*

Proof. In view of Remark 3, $(X, M, *)$ is a complex-valued fuzzy metric space over $\mathcal{A} = \mathbb{C} = \{(x,y) \colon x, y \in \mathbb{R}\}$ the Banach algebra of complex numbers with coordinate-wise multiplication, maximum norm $\|(x,y)\| = \max\{|x|, |y|\}$, $e = (1,1)$, cone $\mathcal{P} = \{(x,y) \in \mathbb{C} \colon x, y \geq 0\}$ and $I_e = \{(x,y) \in \mathbb{C} \colon 0 \leq x, y \leq 1\}$. We notice that this cone $\mathcal{P}$ is strongly minihedral. Also, as T is a fuzzy Banach contraction with contractive constant k, it is a generalized Tirado contraction with contractive vector $(k,k) \in \mathcal{P}$. Hence, the existence and uniqueness of the fixed point of T follows from Theorem 6. □

5. Conclusions

The abstract spaces have several applications in various branches of science, e.g., in the theory of relativity, quantum mechanics and in engineering problems where the use of vector variables and functions makes procedures of calculations and finding solutions much simpler (see [41–43]). The establishment of most of the scientific processes and problems are influenced directly by some mathematical structures in the form of distance spaces with some particular properties, e.g., complexification of Minkowski spacetime and quantum decoherence involves the role of complex spaces (see, e.g., [44–46]). From a mathematical point of view, the solutions of various problems involve the uses of Banach spaces, Hilbert spaces and several other generalized spaces. The use of generalized spaces permits us to deal with the systems more effectively and simply. This shows the preponderance of generalized spaces over the spaces without generalized structure.

On the other hand, the concept of distance plays an important role in the analysis of systems and processes that frequently occur in practical problems. The deportment of systems having an uncertain nature is successfully analyzed through the concept of fuzzyness. In contrast with the classical notion of distances (metrics), the fuzzy distances have a larger domain of applicability and can be used for the analysis of such systems. In [47], authors showed how the use of fuzzy metrics makes the filtering computationally simpler and more efficient than the usual distances. In [48], the proximity of two pixels in a color image in image filtering and processing is analyzed with the fuzzy metrics.

Here, we have presented a new type of fuzzy metric space in which the fuzzy metric and t-norm can attain the values in generalized spaces (Banach algebras, e.g., in $\mathbb{R}^n$) instead of the real numbers. This approach can be used for the systems in which the proximity of objects is desirable with respect to various components of a vector parameter, e.g., the proximity of two objects is measured with respect to some parameter (variable) associated with the t-norm; the new approach can make us able to consider the proximity of objects with respect to not only one, but more than one (n) variable simultaneously through the n-dimensional vectors. For instance, when comparing two pixels we can use a vector valued fuzzy metric that attains values in $[0,1]^3$ which could provide a degree of similarity of red, green and blue with respect to a parameter (not necessarily the same), respectively, in each component. Moreover, theoretically this is not limited to the finite dimensional cases; therefore, this concept can be used for infinite dimensional cases as well. The fixed point results in fuzzy metric spaces can be applied on the problems associated with recursive algorithms (see [49–51]). In the papers [52,53], authors applied the fixed point results of contractive mappings in fuzzy metric spaces to the analysis of quicksort algorithms. Here, we have extended the fixed point results of ordinary fuzzy metric spaces into the vector-valued fuzzy metric spaces, so the new results can be applied on a larger domain of problems.

From a theoretical point of view, we have extended the ordinary t-norm and ordinary fuzzy metrics to their vector-valued versions and proved the fixed point results for contractive type mappings in both strong complete and complete vector-valued fuzzy metric spaces. We use the class Ξ of the functions in contractive condition which is a generalized and extended form of the class ψ introduced by [23,54] and so several fixed point results in ordinary fuzzy metric spaces have been extended and generalized to this new type of fuzzy metric space.

Thus, we have extended the notion of t-norms to their vector-valued version, and unified and generalized the notions of ordinary and complex-valued fuzzy metric spaces and corresponding fixed point theorems in vector-valued fuzzy metric spaces. An attempt to generalize some existing fixed point results in new generalized settings was made. In our fixed point result on vector-valued fuzzy metric space we have used a strong condition on the underlying cone, i.e., the cone is strongly minihedral. As we know, there are several examples of important cones which are not minihedral. Due to this fact, the applicability of our fixed point result for generalized Tirado contractions is limited and therefore we conclude this paper with the following open problem:

Can we prove Theorem 6 for vector-valued fuzzy metric spaces with cones not necessarily miniheral?

Furthermore, following the ideas of the presented work, it may be investigated whether some important generalized notions, such as fuzzy partial metric space [55] and fuzzy metric-like spaces [56], can be extended to their vector-valued versions. Apart from this, the fixed point result for generalized contractions, e.g., for $\mathcal{Z}$-contractions [57] in the new setting and their applications (e.g., to integral differential systems and dynamic systems) can be investigated.

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