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Computing parameter planes of iterative root-finding methods with several free critical points

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ABSTRACT

In this paper we present an algorithm to obtain the parameter planes of families of root-finding methods with several free critical points. The parameter planes show the joint behaviour of all critical points. This algorithm avoids the inconsistencies arising from the relationship between the different critical points as well as the indeterminacy caused by the square roots involved in their computation.

We analyse the suitability of this algorithm by drawing the parameter planes of different Newton-like methods with two and three critical points. We also present some results of the expressions of the Newton-like operators and their derivatives in terms of palindromic polynomials, and we show how to obtain the expression of the critical points of a Newton-like method with real coefficients.

1. Introduction

Iterative root-finding methods are used to solve equations whose solutions cannot be obtained by means of algebraic procedures. The development of new root-finding methods has become a very active area of research: it is sought to find new methods which increase the order of convergence to the solutions of the equation and have better computational efficiency. However, the radii of convergence may decrease as the order of the methods increases. At this point, a dynamical study can provide valuable information on the behaviour of these methods in a qualitative way.

When we apply an iterative root-finding method to solve the non-linear equation $f(z) = 0$ we obtain an operator O_f. If z^* is a solution of the equation and an initial guess z_0 is close enough to z^*, the operator provides a sequence

$$z_{k+1} = O_f(z_k), \quad k \geq 0,$$

that converges to z^*. The sequence $\{z_k\}_{k\geq 0}$ defines the orbit of the point z_0. Nonetheless, in general we cannot know a priori if an initial guess z_0 is close enough to a root. A dynamical study can provide information on the kind of asymptotic behaviour presented by the orbits of an initial condition z_0. In particular, it can detect whether there are attracting cycles or other stable behaviour not corresponding to the basins of attraction of the solutions of $f(z) = 0$. Such stable behaviours would provide open sets of initial conditions which do not converge to any of the roots, which is an important drawback when applying the algorithm. In many cases, if f is a rational map, then O_f is also a rational map and we can use the theory of complex dynamics (see Section 2.1) to study the existence of such stable domains. Indeed, all stable behaviours are related to a critical point, a point c such that $O'_f(c) = 0$, so it is enough to study the orbits of all critical points of f.

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If we study a family of root-finding algorithms depending on parameters (or if f depends on parameters), the operator O_f will also depend on parameters. In this scenario, parameter planes play an important role on helping us understand the family. For each parameter, we can use the orbit of critical points to determine if there are stable domains other than the basins of attraction of the roots and then plot the parameter accordingly. Using the parameter plane we can find the members of the family with better behaviour.

When the operator has a single free critical point (a critical point which is not fixed under the dynamics of O_f), each colour of the parameter plane explains the asymptotic behaviour of that critical point (see [1,2], for example). However, as the order of convergence of the algorithm increases, the number of free critical points also increases, and drawing parameter planes becomes challenging (see, for instance, [3–13] and references therein). A usual approach to tackle this problem is to produce a different parameter plane for each different free critical point (see [14]). When considered simultaneously, these parameter planes provide complete information of the asymptotic behaviour of all critical orbits. However, when regarded separately these planes may contain inconsistencies which are usually due to changes of determination of roots which appear in the definition of the critical points and the dynamical relations amongst them (compare with Section 4.1). Moreover, some bifurcations may be difficult to understand when plotting parameter planes separately. For instance, if one critical point is captured by an attracting cycle controlled by another critical point (a capture parameter), the parameter planes may seem unusual when regarded separately. An alternative approach to this problem is to plot a single parameter plane which considers the dynamics of these critical points simultaneously, plotting the parameter in black if any of the critical orbits does not converge to the roots. Those black parameters correspond to operators for which the root-finding algorithm may not behave appropriately: there can be stable behaviours other than the roots. This approach is followed, for instance, in [15,16]. However, those parameter planes may have the disadvantage of losing the information of how many critical orbits fail to converge to the roots. For instance, in [15, Figure 5] there are black parameters for which only one critical orbit fails to converge to the roots and other parameters for which no free critical orbit converges to the roots. This information is relevant since we can have as many attracting cycles not coming from the basins of attraction of the roots as critical orbits failing to converge to the roots.

The goal of this paper is to present an algorithm to draw parameter planes taking into account all critical points simultaneously in a single plane and not losing any information. This algorithm is presented in Section 3, where we also provide the corresponding pseudocode (furthermore, an implementation in C of the algorithm can be found in the GitHub repository [17]). It is based on the escaping algorithm and can be used with no modification for any number of free critical points. The idea of the algorithm is the following. If all critical points converge to the roots, then we use a scaling of colours which indicates the slowest time of convergence to a root amongst all critical points. This criterion avoids analysing to which roots the critical orbits converge, since changes on the determination of the roots could lead to lines in the parameter planes which do not actually correspond to bifurcations, similar to what can be observed in the parameter planes in Section 4.1. If any of the critical orbits does not converge to a root, then we plot the parameter with a different colour depending on the number of critical points which do not escape (see Fig. 1 for a short explanation of the colours used and Fig. 2 for a first example of how we avoid inconsistencies using the algorithm). Along the paper we explain how using this algorithm can help to better interpret the bifurcations in parameter plane. In Section 3.1 we also explain different modifications of the algorithm that can help us get extra information (compare with Sections 4.2 and 4.3).

Even though the algorithm presented can be adapted to plot the parameter plane of any family of root-finding algorithms, the implementation that we present is done keeping in mind the so called *Newton-like methods*. These methods are variations of Newton's method and are analysed in [18]. Many of the root-finding algorithms in the literature are Newton-like methods ([2,5,10,13,19–26], for example). When Newton-like algorithms are applied on quadratic polynomials $p(z) = z^2 - c$, an intrinsic symmetry appears in the operator obtained. We prove that, after applying a conjugacy that sends the roots to $z = 0$ and $z = \infty$, such operators have the following generic expression:

$$O(z) = z^n \frac{a_k + a_{k-1} z + \cdots + a_1 z^{k-1} + z^k}{1 + a_1 z + \cdots + a_{k-1} z^{k-1} + a_k z^k} = z^n \prod_{i=1}^{k} \frac{(z - r_i)}{(1 - r_i z)}.$$

Actually, in [18] we study such methods when applied on polynomials $p(z) = z^d - c$ and we show that the maps obtained are symmetric with respect to a rotation by a dth root of the unit. The previous operator is obtained when restricting to $d = 2$ and applying the conjugacy, regardless of the method used. After applying the conjugacy, the operator $O(z)$ obtained is symmetric with respect to the map $z \to 1/z$. This symmetry is taken into consideration when implementing the algorithm. Indeed, in order to avoid inconsistencies in the colour scheme used when all critical orbits converge to the roots, we need to implement stop conditions for the convergence to $z = 0$ and $z = \infty$ which are preserved by $z \to 1/z$. Moreover, if c is a critical point of $O(z)$, by symmetry, then $\tilde{c} = 1/c$ is also a critical point of $O(z)$ and their orbits have symmetric asymptotic behaviour. Therefore, we count each pair $\{c, 1/c\}$ as a single free critical point and only iterate one of them when drawing parameter planes.

Up to this moment we have not talked about another crucial procedure to draw parameter planes. We need to actually compute the expressions of all critical points of the operator $O(z)$. As the degree of $O(z)$ increases, the number of critical points also increases (a rational map of degree d has $2d - 2$ critical points counting multiplicity), so obtaining expressions of all the critical points can be challenging. However, the operators $O(z)$ coming from Newton-like methods satisfy certain properties which may help us find all of their critical points. In Section 2.2 we prove that the derivative of these operators gives rise to palindromic polynomials, which allows us to halve the degree of the polynomial we need to solve. In Proposition 2.10 we show that the free critical points from the operator $O(z)$ with real coefficients satisfy that they are either pairs of inverse real roots, or complex conjugates lying on the unit circle or a set of four related roots of a quartic palindromic polynomial. Even though Section 2.2 is not strictly related to the development of the algorithm, the methods introduced in this section are later used in Section 4 in order to obtain all critical points of different Newton-like root-finding algorithm's for which we later plot the parameter planes.

This paper is organized as follows. Section 2 contains the preliminary results which are used for the implementation of the algorithm: in Section 2.1 we present the basic concepts of complex dynamics and in Section 2.2 we introduce palindromic polynomials and their relation with Newton-like root-finding algorithms. In Section 3 we explain in detail the algorithm for drawing parameter planes and we provide the corresponding pseudocode as well as the pseudocode for two variants of the algorithm. Next, in Section 4 we illustrate the convenience of using the algorithm by drawing the parameter planes of different Newton-like methods with two and three critical points. We also plot several dynamical planes corresponding to each of the examples for a better understanding of the colours in the parameter plane. Moreover, we explain different modifications to the algorithm that can be implemented to obtain a better understanding of the parameter planes. Finally, in Section 5 we provide the conclusions of the paper.

2. Preliminary results

In this section we present the main theoretical results used to develop the algorithm. First, in Section 2.1 we recall basic concepts of holomorphic dynamics, such as critical points and their importance. Next, in Section 2.2 we introduce the concept of palindromic polynomials. For Newton-like methods applied to quadratic maps, we prove that the critical points of the operator are solutions of palindromic polynomials. Afterwards, we provide several results on how to find the zeros of such polynomials. This results are used to find the critical points of the methods studied as examples in this paper.

2.1. Introduction to complex dynamics

For a better understanding of the exhibited results, we recall some basic concepts of complex dynamics. For a more detailed introduction to the topic we refer to [27,28].

Given a rational map $Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere, we consider the dynamical system provided by the iterates of Q. A point z_0 is called fixed if $Q(z_0) = z_0$. A point z_0 is called periodic of period $p \geq 1$ if $Q^p(z_0) = z_0$ and $Q^\ell(z_0) \neq z_0$ for all $\ell < p$. In the later case we denote by $\langle z_0 \rangle = \{z_0, z_1, \dots, z_{p-1}\}$, where $z_\ell = Q(z_{\ell-1})$, the cycle of period p generated by z_0. The multiplier of a fixed point is given by $\lambda(z_0) = Q'(z_0)$. Similarly, the multiplier of a periodic point z_0 of period p is given by $\lambda\left(z_0\right) = (Q^p)'\left(z_0\right) = Q'(z_0) \cdot \ldots \cdot Q'(z_{p-1})$. A periodic or fixed point z_0 is called *attracting* (resp. *superattracting*) if $|\lambda(z_0)| < 1$ (resp. $\lambda(z_0) = 0$), *repelling* if $|\lambda(z_0)| > 1$, and *indifferent* if $|\lambda(z_0)| = 1$. An indifferent point z_0 is called *parabolic* (or *rationally indifferent*) if $\lambda(z_0) = e^{2\pi i r/s}$ with $r/s \in \mathbb{Q}$. If $\lambda(z_0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the point z_0 is called *irrationally indifferent*. Attracting fixed (or periodic) points z_0 have associated a basin of attraction $\mathcal{A}(z_0)$ associated to them, which consists of the set of points that converge to z_0 (or the cycle $\langle z_0 \rangle$) under iteration of Q. Similarly, the basin of attraction $\mathcal{A}(z_0)$ of an a parabolic fixed (or periodic) point is defined as the set of points which converge to z_0 (or $\langle z_0 \rangle$). Unlike in the attracting case, a parabolic point z_0 belongs to the boundary of $\mathcal{A}(z_0)$, $z_0 \notin \mathcal{A}(z_0)$. With respect to the irrationally indifferent point z_0, if the map Q (Q^p in the periodic case) is conjugate to the rigid rotation $z \to \theta \cdot z$ in some neighbourhood of z_0 we say that z_0 is a *Siegel point* and the maximal domain of the conjugation is called *Siegel disk*. Otherwise we say that z_0 is a *Cremer point*.

The iteration of Q defines a completely invariant partition of $\widehat{\mathbb{C}}$. The *Fatou set* $\mathcal{F}(Q)$ is defined as the set of points $z \in \widehat{\mathbb{C}}$ such that the family of iterates of Q is normal in some open neighbourhood of z. Its complement, the *Julia set* $\mathcal{J}(Q) = \widehat{\mathbb{C}} \setminus \mathcal{F}(Q)$, is closed and corresponds to the set of points with chaotic behaviour. The connected components of $\mathcal{F}(Q)$ are called Fatou components and are mapped under iteration of Q amongst themselves. It follows from Sullivan's No Wandering Theorem [29] that all Fatou components of a rational map are either periodic or preperiodic. All periodic Fatou components of a rational map are either basins of attraction of attracting or parabolic cycles, or simply connected rotation domains (Siegel disks) or doubly connected rotation domains (Herman rings). Moreover, all these periodic Fatou components are related to a critical point, i.e. a point $c \in \widehat{\mathbb{C}}$ such that $Q'(c) = 0$. Indeed, all attracting and parabolic basins of attraction contain, at least, a critical point. Furthermore, given any Siegel disc or Herman ring U there is a critical point (two in the case of Herman rings) whose orbit accumulates on ∂U. If a critical point is not a fixed point of Q it is called free critical point. Two or more critical points can satisfy relations among them that imply a symmetry in their dynamics; therefore, in order to detect all stable behaviours of the map Q it is enough to study the asymptotic behaviour of all free critical points of Q up to symmetry.

2.2. Critical points of Newton-like methods

In this section we study properties of the operators obtained when applying Newton-like methods to polynomials of degree two $z^2 - c$, that allow us to obtain the expressions of all critical points in terms of the parameter. As proved in [18], after a conjugation via a Möbius map that sends the roots $\pm\sqrt{c}$ to 0 and ∞, these operators have the generic expression:

$$O(z) = z^n \frac{a_k + a_{k-1}z + \cdots + a_1 z^{k-1} + z^k}{1 + a_1 z + \cdots + a_{k-1} z^{k-1} + a_k z^k}, \qquad \text{where } a_k \neq 0. \tag{1}$$

The fixed points of $O(z)$ different from 0 and ∞ are called *strange fixed points*.

We can observe that the polynomials in the numerator and denominator of this expression have the same coefficients in reciprocal order.

Definition 2.1. Two degree n polynomials $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ and $q(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + b_n z^n$ are reciprocal if $b_i = a_{n-i}$, for $i \in \{0, 1, \ldots, n\}$.

In the following, we study some results involving reciprocal polynomials.

Lemma 2.2. *The quotient of two reciprocal polynomials p and $\widehat{p}$ satisfies the symmetry property*

$$\frac{p(z)}{\widehat{p}(z)} = \frac{1}{\frac{p(1/z)}{\widehat{p}(1/z)}}.$$

Proof. Let us consider a degree n polynomial $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$. The reciprocal polynomial of $p(z)$ is $\widehat{p}(z) = a_n + a_{n-1} z + \cdots + a_1 z^{n-1} + a_0 z^n$, that can be written as $\widehat{p}(z) = z^n p(1/z)$. Then,

$$\frac{p(1/z)}{\widehat{p}(1/z)} = \frac{z^{-n}\widehat{p}(z)}{(1/z)^n p(z)} = \frac{\widehat{p}(z)}{p(z)}. \quad \square$$

Remark 2.3. The result of the previous lemma implies that the operator (1) satisfies $O(z) = \frac{1}{O(1/z)}$. So, the strange fixed points different from $z = 1$ and $z = -1$ and the critical points different from $z = 0$ of the operator $O(z)$ come in inverse pairs.

The fact that reciprocal polynomials appear in the expression of the operator $O(z)$ leads to a special type of polynomials in the expression of $O'(z)$, the so-called palindromic polynomials.

Definition 2.4. A polynomial $P(z) = A_0 + A_1 z + \cdots + A_{n-1} z^{n-1} + A_n z^n$ is called palindromic if $A_i = A_{n-i}$, $i \in \{0, 1, \ldots, n\}$.

Definition 2.5. A polynomial $P(z) = A_0 + A_1 z + \cdots + A_{n-1} z^{n-1} + A_n z^n$ is called antipalindromic if $A_i = -A_{n-i}$, $i \in \{0, 1, \ldots, n\}$.

Some properties of this type of polynomials are given in the following proposition (see, for example, [30] for a proof of the result):

Proposition 2.6. *Palindromic and antipalindromic polynomials satisfy the following properties:*

(a) If α is a root of a polynomial that is either palindromic or antipalindromic, then $1/\alpha$ is also a root and has the same multiplicity.
(b) The converse is true: if a polynomial is such that if α is a root then $1/\alpha$ is also a root of the same multiplicity, then the polynomial is either palindromic or antipalindromic.
(c) The sum of two palindromic (antipalindromic) polynomials is a palindromic (antipalindromic) polynomial.
(d) The product of a constant by a palindromic (antipalindromic) polynomial is a palindromic (antipalindromic) polynomial.
(e) The product of two palindromic or two antipalindromic polynomials is palindromic.
(f) A palindromic polynomial $P(z)$ of odd degree is a multiple of $z+1$ (it has -1 as a root) and its quotient by $z+1$ is also palindromic.
(g) An antipalindromic polynomial $Q(z)$ is a multiple of $z-1$ (it has 1 as a root) and its quotient by $z-1$ is palindromic.

Next we prove that the critical points of a Newton-like method applied on degree 2 polynomials are the roots of a palindromic polynomial.

Lemma 2.7. *The product of two reciprocal polynomials is a palindromic polynomial.*

Proof. Let us consider a polynomial $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ of degree n. The reciprocal polynomial of $p(z)$ is $\widehat{p}(z) = a_n + a_{n-1} z + \cdots + a_1 z^{n-1} + a_0 z^n$, that can be written as $\widehat{p}(z) = z^n p(1/z)$. Then,

$$p(z)\widehat{p}(z) = \sum_{i=0}^{n} a_i z^i \cdot z^n \sum_{j=0}^{n} a_j (\frac{1}{z})^j = \sum_{i,j=0}^{n} a_i a_j z^{n+i-j} = \sum_{k=0}^{2n} b_k z^k$$

where:

$$b_k = \sum_{i=0}^{k} a_i a_{n-k+i}, \quad k \in \{0, 1, \ldots, n\}$$
$$b_k = \sum_{i=0}^{2n-k} a_{k-n+i} a_i, \quad k \in \{n, n+1, \ldots, 2n\}.$$

Let us prove that $b_k = b_{2n-k}$, for $k \in \{0, 1, \ldots, n\}$:

$$b_{2n-k} = \sum_{i=0}^{2n-(2n-k)} a_{(2n-k)-n+i} a_i = \sum_{i=0}^{k} a_{n-k+i} a_i = b_k.$$

We can conclude that the product $p(z)\widehat{p}(z)$ is a palindromic polynomial. $\square$

In [18] we deduce that $z = 1$ is a fixed point of the operator $O(z)$. Moreover, if $n + k$ is odd, then $z = -1$ is also a fixed point. As we have remarked above, from Lemma 2.2 it is easy to prove that the strange fixed points of the operator $O(z)$, different from $z = 1$ and $z = -1$, come in inverse pairs. The same occurs for the critical points different from $z = 0$. In fact, let us see that these critical points are the roots of a palindromic polynomial.

Lemma 2.8. *The polynomial in the numerator of $O'(z)$ is palindromic.*

Proof. Let us consider the operator $O(z)$ given in (1) written as

$$O(z) = z^n \frac{p(z)}{\widehat{p}(z)}.$$

Then, we have that:

$$O'(z) = z^{n-1} \frac{n\widehat{p}(z)p(z) + z\left(\widehat{p}'(z)p(z) - \widehat{p}(z)p'(z)\right)}{p^2(z)}$$

and the critical points are $z = 0, \infty$ and the roots of the polynomial $P(z) = n\widehat{p}(z)p(z) + z\left(\widehat{p}'(z)p(z) - \widehat{p}(z)p'(z)\right)$ that appears in the numerator of $O'(z)$. Let us see that $P(z)$ is a palindromic polynomial.

From the expressions $p(z)$ and $\widehat{p}(z)$ we obtain $p'(z)$ and $\widehat{p'}(z)$:

$$p(z) = \sum_{i=0}^{n} a_i z^i \Rightarrow p'(z) = \sum_{i=0}^{n} i a_i z^{i-1}$$
$$\widehat{p}(z) = \sum_{j=0}^{n} a_j z^{n-j} \Rightarrow \widehat{p}'(z) = \sum_{j=0}^{n} (n-j) a_j z^{n-j-1},$$

and we can write:

$$\begin{aligned} z\left(p(z)\widehat{p}'(z) - p'(z)\widehat{p}(z)\right) &= z\left(\sum_{i=0}^{n} i a_i z^{i-1} \sum_{j=0}^{n} a_j z^{n-j}\right) - z\left(\sum_{i=0}^{n} a_i z^i \sum_{j=0}^{n} (n-j) a_j z^{n-j-1}\right) \\ &= \sum_{i,j=0}^{n} (i-n+j) a_i a_j z^{i+n-j} = \sum_{k=0}^{2n} b_k z^k, \end{aligned}$$

where:

$$\begin{aligned} b_k &= \sum_{i=0}^{k} (k-2i) a_i a_{n+i-k}, \quad k \in \{0, 1, \dots, n\} \\ b_k &= \sum_{i=0}^{2n-k} (2n-2i-k) a_{i+k-n} a_i, \quad k \in \{n, n+1, \dots, 2n\}. \end{aligned}$$

We can check that the coefficients verify $b_k = b_{2n-k}$, for $k \in \{0, 1, \dots, n\}$:

$$b_{2n-k} = \sum_{i=0}^{2n-(2n-k)} (2n-2i-(2n-k)) a_{i+(2n-k)-n} a_i = \sum_{i=0}^{k} (k-2i) a_{n+i-k} a_i = b_k.$$

Then, the expression $z\left(p(z)\widehat{p}'(z) - p'(z)\widehat{p}(z)\right)$ is a palindromic polynomial. By Lemma 2.7, we have that $\widehat{p}(z)p(z)$ is also palindromic. By applying properties of palindromic polynomials (Proposition 2.6), we conclude that $P(z)$ is a palindromic polynomial. □

From the previous lemmas we see that the critical points different from $z = 0$ and $z = \infty$ are the roots of a palindromic polynomial. Moreover, if the coefficients of the palindromic polynomial in the numerator of $O'(z)$ are real, it can be decomposed as a finite product of polynomials of degree at most four. To achieve this goal, we rely on the following theorem concerning palindromic polynomials (see [31], for example):

Theorem 2.9. *For a polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ with coefficients in $\mathbb{C}$ and degree n, the following conditions are equivalent:*

- *the polynomial has palindromic coefficients: $a_k = a_{n-k}$ for all k,*
- *$z^n P(1/z) = P(z)$,*
- *(if $n = 2m$) $P(z) = z^m q(z + 1/z)$ for a polynomial q with coefficients in $\mathbb{C}$ and degree m.*

Given that an odd degree palindromic polynomial can be written as $(z + 1)$ multiplied by a palindromic polynomial of even degree, we restrict our study to polynomials of even degree. Moreover, let us notice that the change of variable $x = z + \frac{1}{z}$ transforms a palindromic polynomial $p(z)$ of degree $2m$ into a polynomial $q(x)$ of degree m.

In the following result we show how to find all the critical points of the operator (1) when the coefficients of the rational function are real.

Proposition 2.10. *The free critical points of the rational function given in (1) with real coefficients satisfy that they are either pairs of inverse real roots, or complex conjugates and lie on the unit circle or they are a set of four related roots of a quartic palindromic polynomial.*

Proof. By applying the fundamental theorem of algebra, a polynomial $q(x)$ with real coefficients can be decomposed as a product of monomials (corresponding to their real roots) and quadratic polynomials (corresponding to their complex conjugate roots).

From the above results, it is obtained that every palindromic polynomial $P(z)$ with real coefficients can be factorized into a product of palindromic polynomials of order two and four:

$$P(z) = K\prod_i (z^2 + a_i z + 1)\prod_j (z^4 + b_j z^3 + c_j z^2 + b_j z + 1).$$

This statement is easy to see since, from the previous theorem, if P has $n = 2m$ degree, it can be written as $P(z) = z^m q(z + 1/z)$ and, by applying the fundamental theorem of algebra on $q(z + 1/z)$ we obtain:

$$\begin{aligned} q(z+1/z) &= K\prod_{i=1}^{m_1}\left((z+1/z)+A_i\right)\prod_{j=1}^{m_2}\left((z+1/z)^2 + B_j(z+1/z) + C_j\right) \\ &= K\prod_{i=1}^{m_1}\frac{1}{z}\left(z^2 + A_i z + 1\right)\prod_{j=1}^{m_2}\frac{1}{z^2}\left(z^4 + B_j z^3 + (2+C_j)z^2 + B_j z + 1\right) \end{aligned}$$

where $m = m_1 + 2m_2$.

When studying the solutions of the polynomials P and q we want to highlight the following considerations:

- Polynomials $(z^2 + A_i z + 1)$ can be decomposed as a product of two monomials when $|A_i| \geq 2$; so, the corresponding roots of $P(z)$ are real and inverse. If $|A_i| < 2$, the corresponding roots of $P(z)$ are complex conjugate and they are on the unit circle.
- The roots of the polynomial $\left((z+1/z)^2 + B_j(z+1/z) + C_j\right)$ for $(z + 1/z)$ are real for $B_j^2 - 4C_j \geq 0$; so, the corresponding roots of $P(z)$ are complex conjugate and they are on the unit circle. When $B_j^2 - 4C_j < 0$, the inverse of a root of $P(z)$ is not its conjugate, then it must be one of a set of four related roots that satisfy a quartic palindromic polynomial.
- Moreover, it is easy to check that the roots of P on the unit circle, considered as pairs of reciprocals, correspond to the roots of q in the interval $[-2, 2]$. □

From the above results, it follows that if we are able to obtain the corresponding decomposition, we can always find all the critical points of a rational function of the type given in (1).

3. The algorithm

When a family of rational maps has more than one free critical point, understanding the parameter plane can be tricky. In this case, a usual procedure is to plot the parameter plane of every critical point separately. However, this poses two problems. First, it might be challenging to understand the whole bifurcation locus by observing the different plots separately (see Fig. 2, upper plots). Indeed, many times the changes in the asymptotic behaviour of one critical orbit may be determined by changes on other orbits (for instance, when one critical orbit is captured by an attracting cycle "controlled" by another critical orbit), but it is difficult to understand such behaviour by observing the plots obtained by iterating each free critical point separately. Moreover, changes in the determination of the roots may lead to non-continuous parametrizations of the critical points (when the determination changes the different critical points "permute" amongst one other). These phenomenon leads to curves in the plots which may be confused with bifurcations (see Figs. 2 and 9).

In order to avoid these problems we plot the parameter plane by studying the orbits of all free critical points simultaneously. In this section we explain how the proposed algorithm works and provide a pseudocode of the main part of the procedure where the iteration of the critical points takes place and colours are assigned. We also explain two different variants. Moreover, in the GitHub repository [17] we provide a program written in C which applies the algorithm.

The procedure used for the algorithm is based on the fact that every Fatou component of a rational map is related to, at least, a critical point (see Section 2.1). Therefore, in order to know if there can be stable behaviour other than the basins of attraction of the roots it is enough to study the asymptotic behaviour of all free critical points (up to symmetry). To do so, all free critical points are iterated a given amount of times. Then, the amount of critical orbits which do not converge to a root after the given amount of iterates tells us how many cycles of Fatou components, other than the roots, there can be.

The algorithm works as follows. First we create a grid of points. Each point of the grid is associated to a parameter in the region of the parameter plane that we want to draw. Then, for each of these parameters we compute all different free critical points. Once all critical points are defined, each of them is iterated up to a given maximum number of iterates *maxit* (for our implementation we use $maxit = 150$). Upon each iterate we verify if the orbit has converged to any of the roots. Since for this paper we consider root-finding algorithms applied to quadratic polynomials and we conjugate the operator obtained so that the roots are placed at 0 and ∞, in order to verify if we have convergence, we check if the iterate z satisfies $|z| < eps$ (convergence to 0) or if $|z| > esc$ (convergence to ∞). We use $esc = 10^4$ and $eps = 1/esc$. If the critical orbit converges to one of the roots before that, we stop the process and iterate the next critical point. Moreover, we store the information of the amount of iterates needed by the "slowest" critical point to converge to the roots in *speedconv*.

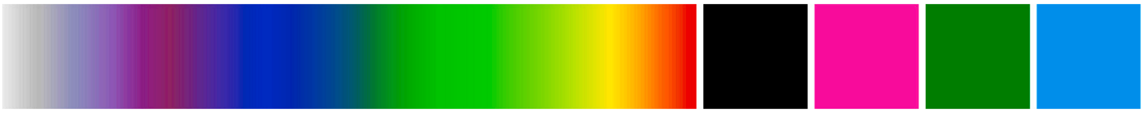

Fig. 1. Colours used in the drawings of the paper. The scaling of colours on the left goes from slow convergence (left) up to fast convergence (right) to the roots. On the right we show, in order, the other colours used in figures: black, pink, dark green, and pale blue. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

If at the end of the process all free critical orbits converge to the roots, we plot the pixel using a scaling of colours which goes from red (speedconv is close to 0), to yellow, to pale green, to blue, and up to grey (speedconv is close to maxit). This scaling of colours is shown in Fig. 1. If any of the critical orbits does not converge to the roots then we plot the pixel with a different colour depending on the number of orbits which do not converge to the roots (black if no free critical points converges to the roots, pink if only one critical point converges to the roots, dark green if two converge to the roots, etc.). The number of critical orbits converging to the roots is stored at *controlescape*. In Algorithm 1 we show a pseudocode of the main part of the algorithm. Given a parameter a, this pseudocode shows how the colour is chosen when iterating the function $f(z) := f_a(z)$.

Algorithm 1 Main algorithm

```
1:  ncrit ← amount of critical points
2:  maxit ← maximum number of iterations
3:  eps ← control of convergence to 0
4:  esc ← 1/eps
5:  procedure ITERATION OF THE CRITICAL POINTS
6:      controlescape ← 0
7:      speedconv ← 0
8:      for each k from 1 to ncrit do
9:          z ← k-th critical point
10:         for each i from 1 to maxit do
11:             z ← f(z)
12:             if |z| > esc or |z| < eps then
13:                 controlescape ← controlescape + 1
14:                 if speedconv < i then speedconv ← i
15:                 end if
16:                 break
17:             end if
18:         end for
19:     end for
20: end procedure
21: procedure ASSIGNATION OF COLOURS
22:     if controlescape = ncrit then
23:         colour ← colour proportional to speedconv
24:     else
25:         colour ← colour corresponding to ncrit − controlescape
26:     end if
27: end procedure
```

Even though in this paper we only apply the algorithm to families with up to three free critical orbits, the program is designed to handle without modification any number of free critical orbits.

We want to make a remark on how the parameters esc and eps that are used to determine convergence to the roots are chosen. In this paper we work with Newton-like families applied to quadratic polynomials and, hence, the operators obtained are symmetric with respect to the map $z \to 1/z$ (see [18]). It follows that if c is a critical point then $1/c$ is also a critical point and their orbits are symmetric. In the program we only iterate one of each pair of critical points since both provide the same information. However, in order to guarantee that the information of how fast a critical point converges to the roots does not depend on the critical point chosen, the stop criterium needs to respect the symmetry $z \to 1/z$. This is why we choose $eps = 1/esc$.

Remark 3.1. The procedure presented in Algorithm 1 can be adapted to work with any family of rational maps depending on a complex parameter or two real parameters provided that the escaping algorithm can be applied, i.e. there is (at least) a persistent (super)attracting cycle towards which the free critical orbits can converge. In that case, the only modification in Algorithm 1 would consist on replacing the escaping conditions (line 12 of the pseudocode) by suitable conditions of convergence to the persistent attracting cycle(s) of the family. The algorithm may also be applied in the general case of rational families depending on 2 or more complex parameters. However, in order to apply the procedure in this case we should first restrict ourselves to slices of parameters of real dimension 2. The main difficulty in this case is, therefore, to identify which slices provide the most relevant dynamical information.

3.1. Variants of the algorithm

In the following we present two variants of the algorithm. The first of them (see Algorithm 2) is a modification to increase the amount of information provided by the algorithm in the case there are two free critical points (the modification can be adapted to any amount of critical points). In case that no critical orbit converges to a root, the original algorithm plots the parameter in black. However, this does not distinguish the scenarios where both critical orbits converge to the same attracting cycle or to two different ones (up to symmetry). The goal of the modification is to detect whether there can be two cycles of stable behaviour (attracting or parabolic cycles, Siegel disks, or Herman rings) other than the basins of attraction of the roots.

If no critical orbit converges to the roots, the algorithm compares the critical orbits. The only difference with the original one is how the colour is assigned when no critical orbit converges to a root ($controlescape = 0$): we keep assigning the colour black if both critical orbits converge to the same cycle but we assign the colour pale blue if they converge to different cycles. To check their convergence we compare their asymptotic behaviour. If after a given number of iterates ($maxit$) none of the critical orbits has converged to a root, we iterate both of the critical orbits further (by $extraiterates$) to obtain better approximations of points in the respective limit sets of their orbits (we call them $asymptc1$ and $asymptc2$). Next, we verify whether these two points belong to the same cycle (up to symmetry). Given a maximum period ($maxPeriod$), we compare $asymptc1$ and $1/asymptc1$ with the next $maxPeriod$ iterates of $asymptc2$. If any of the comparisons succeeds (the iterate is closer to $asymptc1$ or its inverse than $eps1$), we conclude that both of the orbits belong to the same cycle (or to symmetric cycles). We show the pseudocode of this procedure in Algorithm 2.

The second variant of the algorithm (see Algorithm 3) deals with the case where there is a persistent cycle of Fatou components (attracting, parabolic, Siegel disk, or Herman ring) which is not tied to a given critical point. In this case, at least one free critical orbit must be tied to such cycle and there are at most $ncrit - 1$ other cycles of stable behaviour. This variant does not modify the procedure of iteration of the critical points of Algorithm 1. Instead, it only modifies how colours are assigned. Since at most $ncrit-1$ critical orbits may converge to the roots, then $constrolescape$ can be at most $ncrit - 1$. Therefore, we consider the number of critical orbits is $ncrit - 1$, for the purpose of assigning the colours: if $constrolescape = ncrit - 1$ we use the scaling of colours depending on $speedconv$ and if $constrolescape < ncrit - 1$ we change to colours accordingly. The pseudocode for this modification is shown in Algorithm 3.

4. Newton-like methods with more than one free critical point

In this section we apply the program to plot parameter planes of different operators with more than one free critical orbit obtained from applying Newton-like methods on degree two polynomials. Examples with more than one free critical point appear with some frequency in the literature, especially when high-order numerical methods are studied. We consider some cases that are representative of the type of dynamics that they give rise to.

As proved in [18], $z = 1$ is always a strange fixed point of this type of methods; $z = -1$ is also a fixed point when $n + k$ is odd and it is a preimage of $z = 1$ when $n + k$ is even. We check these statements in the methods that we study.

As the operators have two or more critical points, if we plot separately the parameter plane of each of them there appear inconsistencies produced by the indeterminacies of the square roots in the expression of the critical points, as we can see in the figures that appear in each subsection. However, this problem does not appear when using the program since the parameter plane that it plots takes into account simultaneously the behaviour of all free critical orbits.

The first three subsections correspond to Newton-like systems with operators satisfying that $n \geq k$. The family studied in Section 4.3 corresponds to a limit case of this type of numerical methods, so it deserves a more detailed study. Although Ermakov–Kalitkin family has two free critical points, one of them is necessarily in the basin of attraction of the point $z = -1$, which is a parabolic point located on the boundary of two attractor petals. So, it can be considered as a family with a single free critical point.

Finally, in Section 4.4 we consider an example where we use our algorithm to obtain the parameter plane of a family with three free critical points.

4.1. Fourth-order methods derived from the Kim family

In [14], the authors study a parametric family of fourth-order methods coming from the Kim family. After applying it on quadratic polynomials they obtain the following operator:

$$O_a(z) = \frac{z^4(1 - a + 4z + 6z^2 + 4z^3 + z^4)}{1 + 4z + 6z^2 + 4z^3 + (1 - a)z^4} \tag{2}$$

whose derivative is:

$$O'_a(z) = \frac{4z^3(1 + z)^4(-1 + a + (-4 - a)z + (-6 + a)z^2 + (-4 - a)z^3 + (-1 + a)z^4)}{(1 + 4z + 6z^2 + 4z^3 + (1 - a)z^4)^2}. \tag{3}$$

Algorithm 2 Variation to compare attractors

```
ncrit ← amount of critical points (2)
maxit ← maximum number of iterations
eps ← control of convergence to 0
esc ← 1/eps
eps1 ← tolerance to compare orbits
extraiterates ← number of iterations to refine asymptotic orbits
maxPeriod ← maximum period to compare orbits
procedure ITERATION OF THE CRITICAL POINTS
    controlescape ← 0
    speedconv ← 0
    for each k from 1 to ncrit do
        z ← k-th critical point
        for each i from 1 to maxit do
            z ← f(z)
            if |z| > esc or |z| < eps then
                controlescape ← controlescape + 1
                if speedconv < i then speedconv ← i
                end if
                break
            end if
        end for
        if i = maxit then
            if k = 1 then asymptc1 ← z
            end if
            if k = 2 and controlescape = 0 then
                flag2attr ← 1
                asymptc2 ← z
                for each j from 1 to extraiterates do
                    asymptc1 ← f(asymptc1)
                    asymptc2 ← f(asymptc2)
                end for
                asymptc1inv ← 1/asymptc1
                for each j from 1 to maxPeriod do
                    asymptc2 ← f(asymptc2)
                    if |asymptc1inv − asymptc2| < eps1 or |asymptc1 − asymptc2| < eps1 then
                        flag2attr ← 0
                        break
                    end if
                end for
            end if
        end if
    end for
end procedure
procedure ASSIGNATION OF COLOURS
    if controlescape = 2 then colour ← colour proportional to speedconv
    end if
    if controlescape = 1 then colour ← colour corresponding to ncrit − controlescape(= 1)
    end if
    if controlescape = 0 then
        if flag2attr = 0 then colour ← black
        else colour ← pale blue
        end if
    end if
end procedure
```

The fixed points of $O_a(z)$ are $z = 0$, $z = \infty$ and the solutions of the equation:

$$(z-1)(1+5z+11z^2+(14+a)z^3+11z^4+5z^5+z^6)=0.$$

Algorithm 3 Variation when there is an extra persistent cycle of Fatou components

```
procedure ASSIGNATION OF COLOURS
    if controlescape = ncrit − 1 then
        colour ← colour proportional to speedconv
    else
        colour ← colour corresponding to ncrit − 1 − controlescape
    end if
end procedure
```

So, there exist seven strange fixed points, $z = 1$ and the six roots of the polynomial of degree six above. As proved in [18], the fixed point $z = 1$ is attractive outside the circle

$$|a - 16| = 64$$

and the point $z = -1$ is a preimage of $z = 1$.

The free critical points are the roots of the four-degree polynomial in the numerator of (3). With the change $x = z + \frac{1}{z}$, the problem is reduced to find the solutions of the equation

$$(1 - a)x^2 + (4 + a)x + 4 + a = 0.$$

By undoing this change, the four free critical points are:

$$c_1(a) = \frac{4 + a - \sqrt{5a(4 + a)} - \sqrt{10a(6 - a) - 2(4 + a)\sqrt{5a(4 + a)}}}{4(a - 1)},$$
$$c_2(a) = \frac{4 + a - \sqrt{5a(4 + a)} + \sqrt{10a(6 - a) - 2(4 + a)\sqrt{5a(4 + a)}}}{4(a - 1)},$$
$$c_3(a) = \frac{4 + a + \sqrt{5a(4 + a)} - \sqrt{10a(6 - a) + 2(4 + a)\sqrt{5a(4 + a)}}}{4(a - 1)},$$
$$c_4(a) = \frac{4 + a + \sqrt{5a(4 + a)} + \sqrt{10a(6 - a) + 2(4 + a)\sqrt{5a(4 + a)}}}{4(a - 1)}.$$

As $c_2(a) = \frac{1}{c_1(a)}$ and $c_4(a) = \frac{1}{c_3(a)}$, it is enough to study the behaviour of c_1 and c_3. If we draw the parameter plane of each of them (see upper plots in Fig. 2), we observe some inconsistencies due to the indeterminacy generated by the square roots appearing in the critical points.

These inconsistencies disappear when drawing a parameter plane that takes into account both free critical points, as it can be observed in the lower plot in Fig. 2. Recall that, when plotting the parameter plane using two critical points, black indicates that no critical orbit converges to the roots while pink indicates that only one critical orbit converges to the roots. If both critical orbits converge to the roots we use a scaling of colours depending on the slower time of convergence (see Fig. 1).

In order to illustrate the different situations in the parameter plane, we finish this subsection by showing some dynamical planes of this operator (see Fig. 3). For these dynamical planes, we use the same scaling of colours used in the parameter planes to indicate convergence to the roots $z = 0$ and $z = \infty$ (from red (fast convergence to the roots), to yellow, to pale green, to blue and up to grey (slow convergence, (compare with Fig. 1), we use dark green if the point converges to $z = 1$ (in case that $z = 1$ is attracting), and we use black if the point does neither converge to the roots 0 and ∞ nor to the fixed point $z = 1$. We also plot using white squares the location of the critical points. Notice that the family has four different free critical points (two modulo symmetry). Therefore, we will always have two critical points with the same dynamical behaviour.

The parameters are chosen as follows. The value $a = -30$ belongs to a black parameter at the bulb on the left. Since the parameter is black, no critical point converges to the roots. Indeed, the two critical points located more to the right lie in the immediate basin of attraction of two different attracting fixed points (which are symmetric). The other two critical points belong the preimages of those basins of attraction. For $a = -30 + 8i$ the parameter is pink, so two critical points belong to the basins of attractions of the root. Indeed, this parameter belongs to the same bulb as $a = 30$. In this case the two critical points to the left have moved out the basins of attraction of the fixed points, while the critical points to the right belong to immediate basins of attraction of the continuation of the attracting fixed points of $a = -30$. For $a = -2$ the parameter is red, so the four critical points are in the basin of attraction of the points 0 and ∞, which correspond to the basins of attraction of the roots. Finally, the parameter $a = 84$ lies on the unbounded black disk of parameters for which the point $z = 1$ is attracting (and all free critical orbits converge to it).

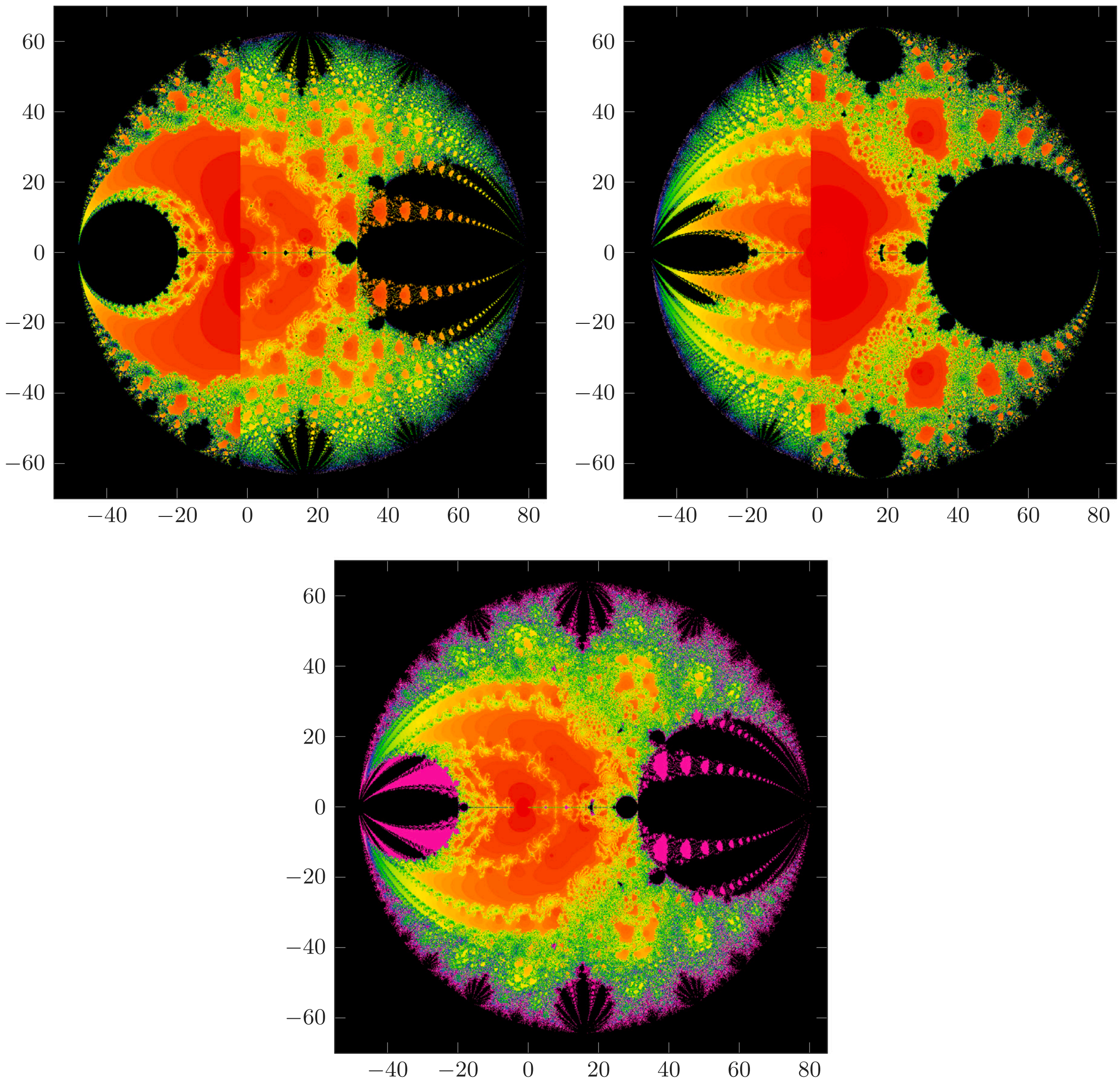

Fig. 2. Upper figures show the parameter planes of the operator (2) using the critical points $c_1(a)$ (left) and $c_3(a)$ (right) separately. Lower figure shows the parameter plane obtained when using both critical points simultaneously.

4.2. A multipoint variant of Chebyshev's method

In [8], the authors study the dynamics of a multipoint variant of Chebyshev's method. After applying it on quadratic polynomials they obtain the following operator:

$$O_a(z) = \frac{z^3(2-4a+(5-8a+4a^2)z+(4-4a)z^2+z^3)}{(1+(4-4a)z+(5-8a+4a^2)z^2+(2-4a)z^3)}. \tag{4}$$

The fixed points of $O_a(z)$ are $z=0$, $z=\infty$ and the solutions of the equation:

$$(z-1)(1+(5-4a)z+4(2-2a+a^2)z^2+(5-4a)z^3+z^4)=0.$$

So, there are four strange fixed points in addition to $z=1$ that satisfy $z_1(a)=\frac{1}{z_2(a)}$ and $z_3(a)=\frac{1}{z_4(a)}$.

The derivative of operator (4) is:

$$O_a'(z) = \frac{-2z^2(1+(2-2a)z+z^2)P(z,a)}{(-1+(2a-1)z)^2(1+(3-2a)z+2z^2)^2}, \tag{5}$$

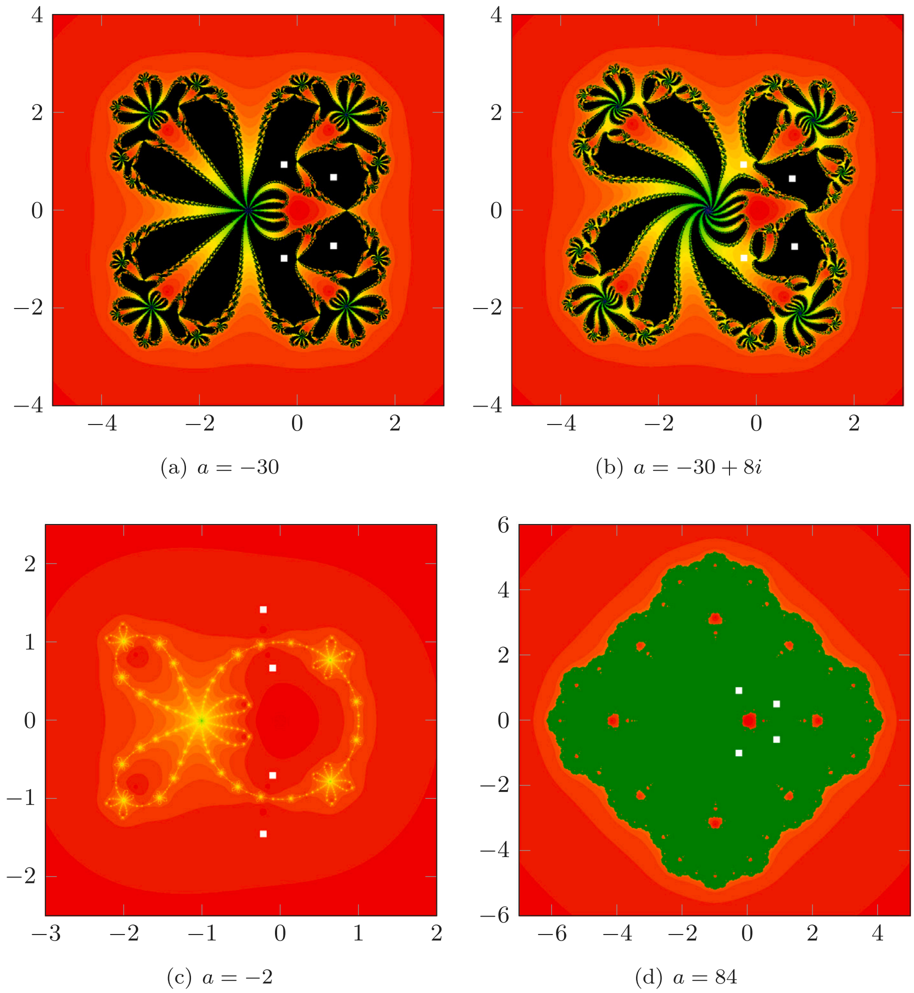

Fig. 3. Dynamical planes of the operator (2) for different values of the parameter a.

where

$$P(z,a) = 6a - 3 + (-12 + 22a - 12a^2)z + (-18 + 32a - 24a^2 + 8a^3)z^2 + (-12 + 22a - 12a^2)z^3 + (6a - 3)z^4.$$

The fixed point $z = 1$ is attractive inside the curve

$$55 - 24\alpha^3 + 3\alpha^4 + 22\beta^2 + 3\beta^4 - 8\alpha(13 + 3\beta^2) + \alpha^2(74 + 6\beta^2) = 0,$$

being $a = \alpha + i\beta$. The point $z = -1$ is a preimage of $z = 1$. We can also see in Fig. 4 the stability curves delimiting the regions where $z = 1$ and the inverse pair z_1 and $z_2 = 1/z_1$ are attractive (coloured respectively in green and red). Notice that, by symmetry, the strange fixed points z_1 and z_2 are attractive for the same set of parameters.

From Eq. (5), the points that satisfy $1 + (2 - 2a)z + z^2 = 0$ are preimages of $z = 1$. So, the free critical points are the solutions of $P(z,a) = 0$. As in the previous subsection, the degree of the equation is reduced to the half by means of the change $z + \frac{1}{z} = x$. After solving the equation and undoing the change we have the four critical points:

$$c_1(a) = \frac{6 - 11a + 6a^2 - a\sqrt{1 + 36a - 12a^2} - \sqrt{2a\left(6 + 25a - 48a^2 + 12a^3 - (6 - 11a + 6a^2)\sqrt{1 + 36a - 12a^2}\right)}}{6(-1 + 2a)},$$

$$c_2(a) = \frac{6 - 11a + 6a^2 - a\sqrt{1 + 36a - 12a^2} + \sqrt{2a\left(6 + 25a - 48a^2 + 12a^3 - (6 - 11a + 6a^2)\sqrt{1 + 36a - 12a^2}\right)}}{6(-1 + 2a)},$$

$$c_3(a) = \frac{6 - 11a + 6a^2 + a\sqrt{1 + 36a - 12a^2} - \sqrt{2a\left(6 + 25a - 48a^2 + 12a^3 + (6 - 11a + 6a^2)\sqrt{1 + 36a - 12a^2}\right)}}{6(-1 + 2a)},$$

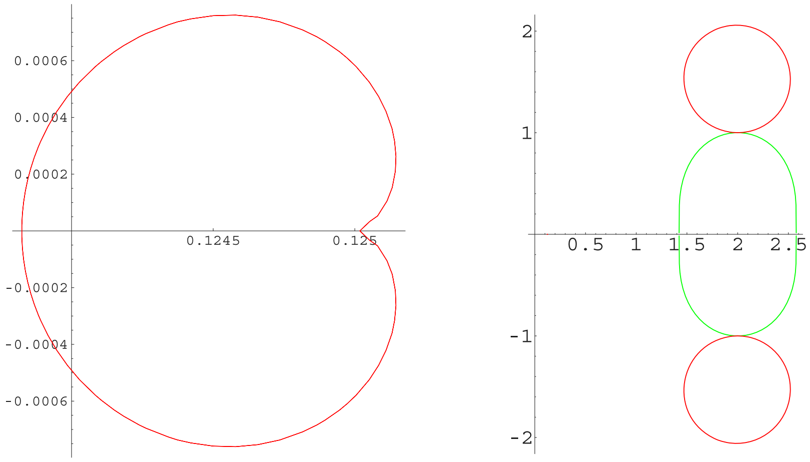

Fig. 4. Stability regions for $z = 1$, z_1 and z_2 under the operator (4). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

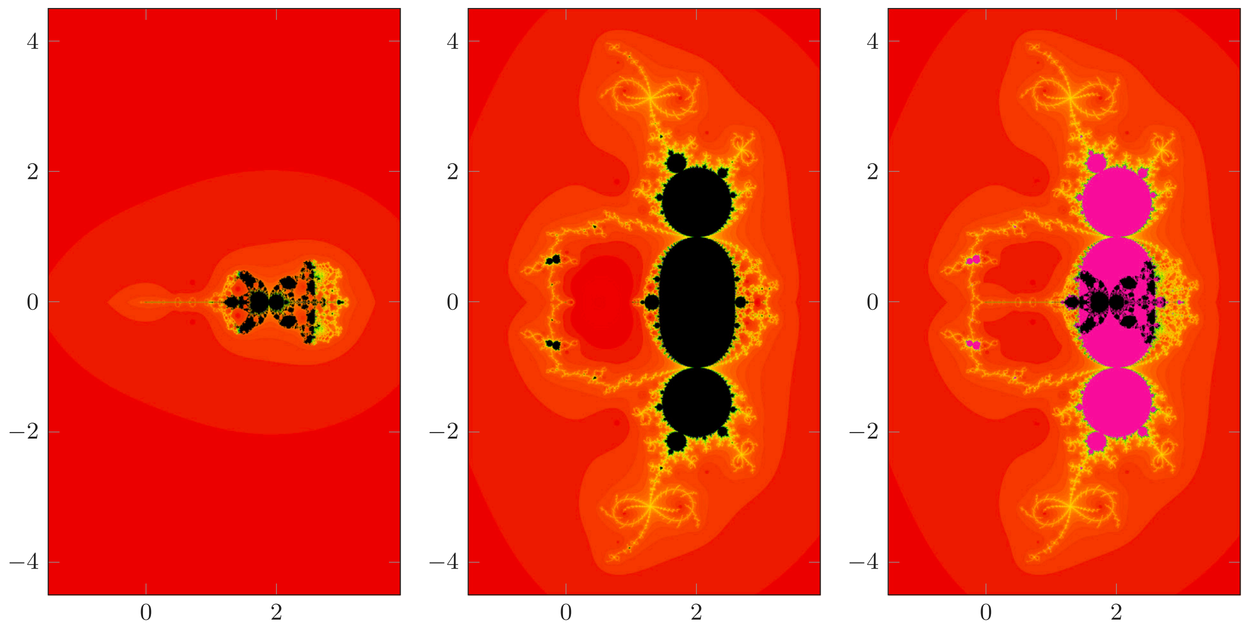

Fig. 5. Parameter planes of the operator (4) obtained using the critical points $c_1(a)$ (left), the critical point $c_3(a)$ (centre), and both critical points simultaneously (right).

$$c_4(a) = \frac{6 - 11a + 6a^2 + a\sqrt{1+36a-12a^2} + \sqrt{2a\left(6+25a-48a^2+12a^3+(6-11a+6a^2)\sqrt{1+36a-12a^2}\right)}}{6(-1+2a)}.$$

As $c_2(a) = \frac{1}{c_1(a)}$ and $c_4(a) = \frac{1}{c_3(a)}$, it is enough to study the behaviour of c_1 and c_3. In Fig. 5 left and centre we show the parameter planes obtained using c_1 and c_3, respectively. In this case we do not observe any incongruence coming from changes of determination of the critical points. However, by obtaining a parameter plane using both critical points (see Fig. 5 right), we do obtain a better understanding of the bifurcation set.

Recall that in Fig. 5 right, colour black indicates that no free critical orbit converges to the roots while pink indicates that only one free critical orbit converges to the root. Therefore, if the parameter is black there may be up to four basins of attraction other than the roots (two up to symmetry). It is then relevant to know if the critical points c_1 and c_3 converge to two different attracting cycles (not related by symmetry), since those parameters would be particularly inconvenient.

In Fig. 6 we show the result of applying the variant of the algorithm described in Section 3.1 that compares the asymptotic behaviour of two non-escaping critical orbits (see Algorithm 2) to detect if there can be 2 cycles of stable behaviour other than

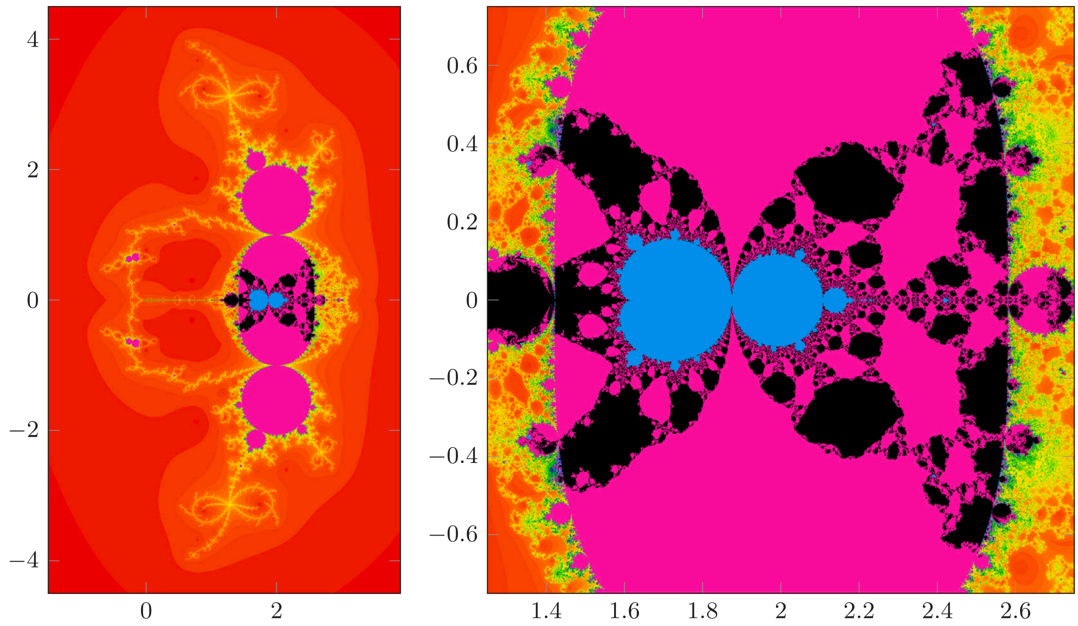

Fig. 6. Parameter plane of the operator (4) obtained using the critical points c_1 and c_3. We use colour pink if only one free critical orbit converges to the roots, blue if the orbits od c_1 and c_3 converge to two different non-symmetric attracting cycles, and black if both free critical orbits converge to symmetric attracting cycles. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the basins of attraction of the roots. As before, pink indicates that only one critical value converges to the roots. Black indicates that both c_1 and c_3 converge to the same cycle (up to symmetry). Pale blue indicates that c_1 and c_3 converge to different cycles. Parameters corresponding to black points, for which there is more than one free critical orbit within the same basin of attraction, are usually called capture parameters. On the other hand, if c_1 and c_3 converge to different cycles we call the parameter disjoint. We would like to point out that disjoint parameters can usually be recognized without having to use this colouring. Indeed, disjoint parameters usually lead to Mandelbrot-like structures in the parameter plane which are easily recognizable (see Fig. 6). We would like to point out that the black components in Fig. 2 are mostly capture components (disjoint hyperbolic components of Operator (2) are very small).

We end this subsection with some dynamical planes that illustrate the different colours in the parameter plane of Operator (4) (see Fig. 7). The colours are as in the dynamical planes in Section 4.1. The value $a = 1.5$ is located in a black region inside the main bulb of Fig. 6. All four critical points (two up to symmetry) belong to the basin of attraction of $z = 1$ (in green). The parameter $a = 1.7$ lies in the biggest blue cardioid of Fig. 6. The basin of attraction of $z = 1$ (in green) contains two critical points (one up to symmetry). The other two free critical points lie in the basin of attraction of an attracting cycle of period two (in black). The parameter $a = 2+1.5i$ belongs to a pink bulb where the fixed points z_1 and z_2 are attracting (compare with Fig. 4 (right)). In black we observe the basins of attraction of these two points, each of them containing a free critical point. The other two free critical points belong to the basins of attraction of the roots $z = 0$ and $z = \infty$. Finally, the parameter $a = 2i$ (which appears as a red parameter) is chosen so that all critical points belong to the basins of attraction of the roots.

4.3. The Ermakov–Kalitkin family

The next example of application of the program corresponds to a family with two free critical orbits (up to symmetry) from which only one orbit is actually free. In [10] the authors study the dynamical behaviour of the family of Ermakov–Kalitkin type methods applied to quadratic polynomials. In this case, the strange fixed point $z = -1$ is a parabolic point of multiplicity three and it is located on the boundary of two parabolic basins. It follows that each of these parabolic basins must contain a critical point.

After applying the method on quadratic polynomials the following operator is obtained:

$$O_a(z) = \frac{z^3(2(a-1)+a^2+4(a-1)z+2(a-1)z^2)}{2(a-1)+4(a-1)z+(a^2+2a-2)z^2}, \tag{6}$$

whose derivative is:

$$O'_a(z) = \frac{z^2 P(a,z)}{(2(a-1)+4(a-1)z+(a^2+2a-2)z^2)^2}, \tag{7}$$

being

$$P(a,z) = 6(a-1)(-2+2a+a^2)+8(a-1)(-6+6a+a^2)z+(72-144a+68a^2+4a^3+a^4)z^2+$$

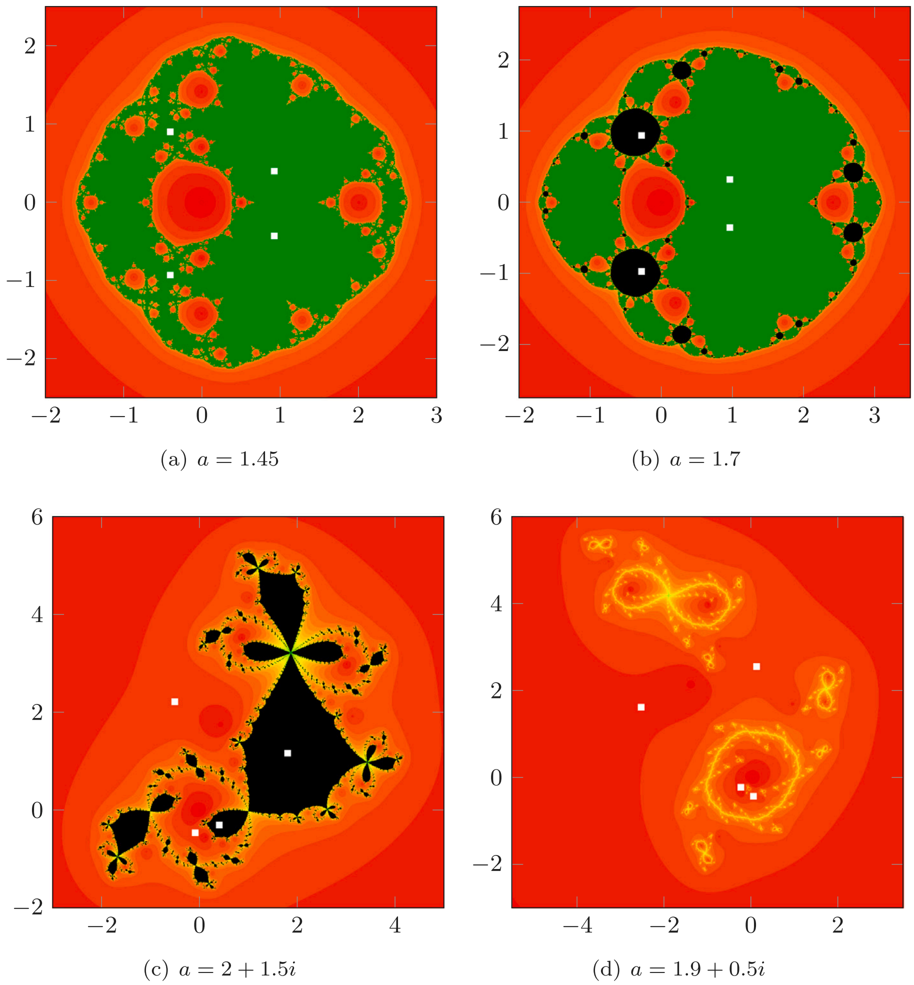

Fig. 7. Dynamical planes of the operator (4) for different values of the parameter a.

$$+\ 8(a-1)(-6+6a+a^2)z^3 + 6(a-1)(-2+2a+a^2)z^4.$$

In this case, the strange fixed points are only $z = 1$ and $z = -1$, since $O_a(z) = z$ implies:

$$2(-1+a)(-1+z)(1+z)^3 = 0.$$

The point $z = 1$ is attractive inside the curve defined by

$$16 - 32\alpha + 15\alpha^2 + \alpha^3 + 17\beta^2 + \alpha\beta^2 = 0,$$

being $a = \alpha + i\beta$ (see Fig. 8).

The fixed point $z = -1$ is a parabolic point with multiplicity 3, as it is a triple solution of $O_a(z) = z$. Therefore, $z = -1$ lies on the boundary of two attractive parabolic basins, each of them containing a critical point (see [28, Section 10], for instance).

As in the previous examples, we can obtain the four roots of the polynomial $P(a, z)$ in . They correspond to the four free critical points:

$$c_1(a) = \frac{4(1-a)(-6+6a+a^2) - a^2\sqrt{2(1-a)(26-26a+3a^2)}}{12(a-1)(-2+2a+a^2)} - \frac{a\sqrt{2(1-a)\left(192-384a+154a^2+38a^3+3a^4-4(-6+6a+a^2)\sqrt{2(1-a)(26-26a+3a^2)}\right)}}{12(a-1)(-2+2a+a^2)},$$

$$c_2(a) = \frac{4(1-a)(-6+6a+a^2) - a^2\sqrt{2(1-a)(26-26a+3a^2)}}{12(a-1)(-2+2a+a^2)} +$$

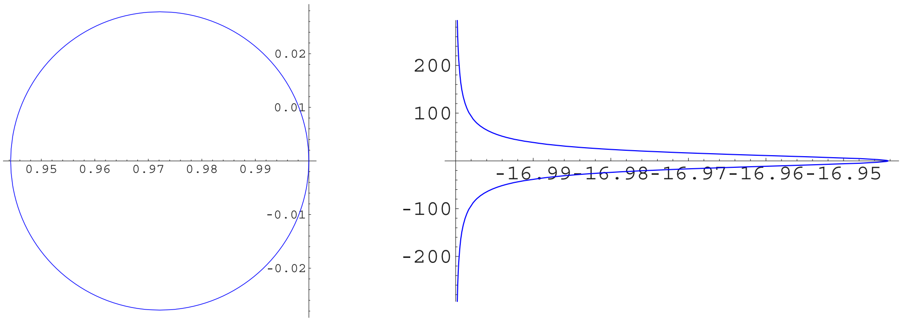

Fig. 8. Stability regions for $z = 1$.

$$+\frac{a\sqrt{2(1-a)\left(192-384a+154a^2+38a^3+3a^4-4(-6+6a+a^2)\sqrt{2(1-a)(26-26a+3a^2)}\right)}}{12(a-1)(-2+2a+a^2)},$$

$$c_3(a) = \frac{4(1-a)(-6+6a+a^2)+a^2\sqrt{2(1-a)(26-26a+3a^2)}}{12(a-1)(-2+a+a^2)} -$$
$$-\frac{a\sqrt{2(1-a)\left(192-384a+154a^2+38a^3+3a^4+4(-6+6a+a^2)\sqrt{2(1-a)(26-26a+3a^2)}\right)}}{12(a-1)(-2+2a+a^2)},$$

$$c_4(a) = \frac{4(1-a)(-6+6a+a^2)+a^2\sqrt{2(1-a)(26-26a+3a^2)}}{12(a-1)(-2+2a+a^2)} +$$
$$+\frac{a\sqrt{2(1-a)\left(192-384a+154a^2+38a^3+3a^4+4(-6+6a+a^2)\sqrt{2(1-a)(26-26a+3a^2)}\right)}}{12(a-1)(-2+2a+a^2)}.$$

As $c_2(a) = \frac{1}{c_1(a)}$ and $c_4(a) = \frac{1}{c_3(a)}$, it is enough to study the behaviour of c_1 and c_3. If we plot the parameter plane of each of them (see Fig. 9 upper) we observe the inconsistencies due to the interaction between the two critical points.

Since $z = -1$ is a parabolic point of multiplicity three and therefore, it is located at the boundary two parabolic basins. One critical point must lie in each of these parabolic basins. However this may take only one free critical orbit up to symmetry, since it is enough that one of the critical points and its inverse lie in each of these basins. It follows that at most one of the critical points (up to symmetry) may lead to new stable dynamics (other than the basins of attraction of the roots and $z = -1$).

The parameter plane obtained using the two free critical points c_1 and c_3 can be observed in Fig. 9 (lower). However, since only one of the orbits may actually be free (one of the critical orbits has to converge to the parabolic point $z = -1$), it is convenient to apply the variant of the algorithm described in Section 3.1 devoted to the cases where there is a persistent cycle of Fatou components which is not related to a specific critical point (compare with Algorithm 3). Recall that, for this variant, if no critical orbit converges to the roots we use black and we use the scaling of colours that we usually use when all critical orbits converge to the roots when one critical orbit converges to the roots (that is the maximum amount critical orbits that can escape to the roots). If we did not apply this variant of the algorithm then regions with scaling of red would appear as pink (which would still be correct but would make more difficult to recognize the thinner regions of bifurcation parameters).

In Fig. 10 we show some dynamical planes for different values of the parameters. We use the same colours as in the dynamical planes of Section 4.1. The parameter $a = -20$ lies in the black region which is unbounded to the left for which $z = 1$ is attracting. Two free critical points (one up to symmetry) lie in the two petals of the basin of attraction of $z = -1$ (in black) and the other two critical points lie in the basin of attraction of $z = 1$ (in green). Similarly, $a = 0.97$ is chosen in the disk described in Fig. 8 for which $z = 1$ is also attracting. The parameter $a = -7$ is chosen on the central red strip of parameters for which the two critical points that do not converge to -1 converge to the roots. The parameter $a = 9$ lies in the big black component to the right of the parameter plane (see Fig. 9 lower), which corresponds to sets of parameters for which all critical orbits lie in the immediate basin of attraction of $z = -1$.

4.4. A sixth order iterative scheme

The last example of application of the program we show is a family with three free critical orbits (up to symmetry). In [6] the authors study the dynamics of a bi-parametric sixth order family of iterative methods for solving non-linear equations.

After applying it on quadratic polynomials, for a fixed value of one of the parameters, they obtain the following operator:

$$O_a(z) = \frac{z^6(10-4a+(48-28a+4a^2)z+(69-40a+6a^2)z^2+(56-30a+4a^2)z^3+(28-12a+a^2)z^4+(8-2a)z^5+z^6)}{1+(8-2a)z+(28-12a+a^2)z^2+(56-30a+4a^2)z^3+(69-40a+6a^2)z^4+(48-28a+4a^2)z^5+(10-4a)z^6},$$

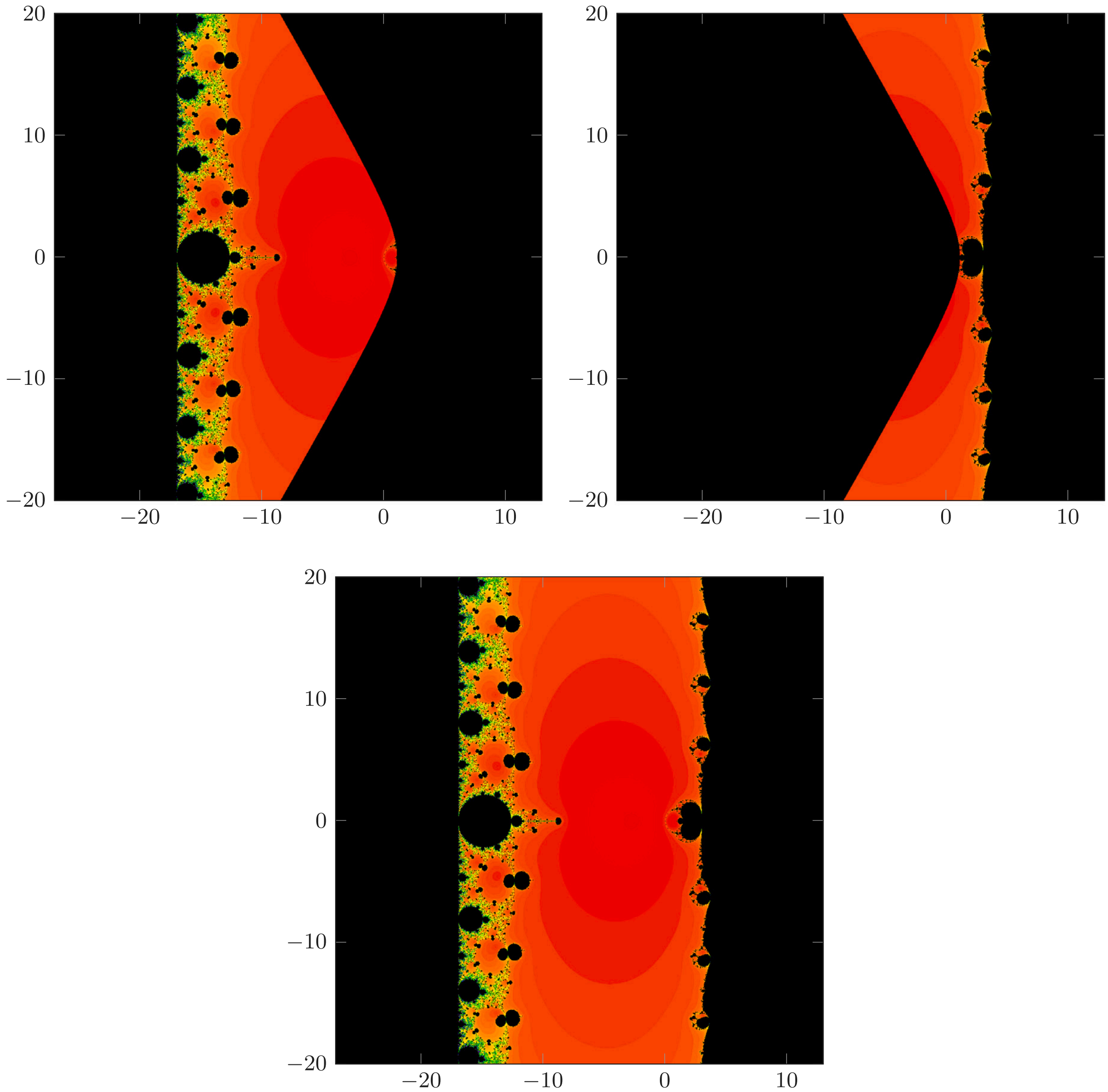

Fig. 9. Parameter planes of the operator (6) obtained using the critical points $c_1(a)$ (upper left) and $c_3(a)$ (upper right) separately and using both critical points simultaneously (lower).

whose derivative is:

$$O'_a(z) = \frac{4z^5(1+z)^4(-1+(-2+a)z-z^2)P(a,z)}{(1+(8-2a)z+(28-12a+a^2)z^2+(56-30a+4a^2)z^3+(69-40a+6a^2)z^4+(48-28a+4a^2)z^5+(10-4a)z^6)^2}$$

where

$$\begin{aligned} P(a,z) \;=\; & -15+6a+(-94+63a-11a^2)z+(-205+170a-49a^2+5a^3)z^2+(-252+206a-56a^2+5a^3)z^3+ \\ & +\;(-205+170a-49a^2+5a^3)z^4+(-94+63a-11a^2)z^5+(-15+6a)z^6. \end{aligned}$$

The solutions of equation $-1+(-2+a)z-z^2=0$ are preimages of $z=1$ and the point $z=-1$ is also a preimage of $z=1$. So, there are six free critical points, that are the solutions of the palindromic polynomial of degree six $P(a,z)$. As before, the roots of this polynomial are obtained by doing the change $x = z + \frac{1}{z}$ and solving the polynomial equation of degree three:

$$-64+80a-34a^2+5a^3+(-160+152a-49a^2+5a^3)x+(-94+63a-11a^2)x^2+(-15+6a)x^3=0.$$

We obtain six different free critical points, only three up to the symmetry of the operator given by $z \to 1/z$. We plot the parameter plane using simultaneously the three free critical orbits with independent dynamics (see Fig. 11). In this case, we use black if no

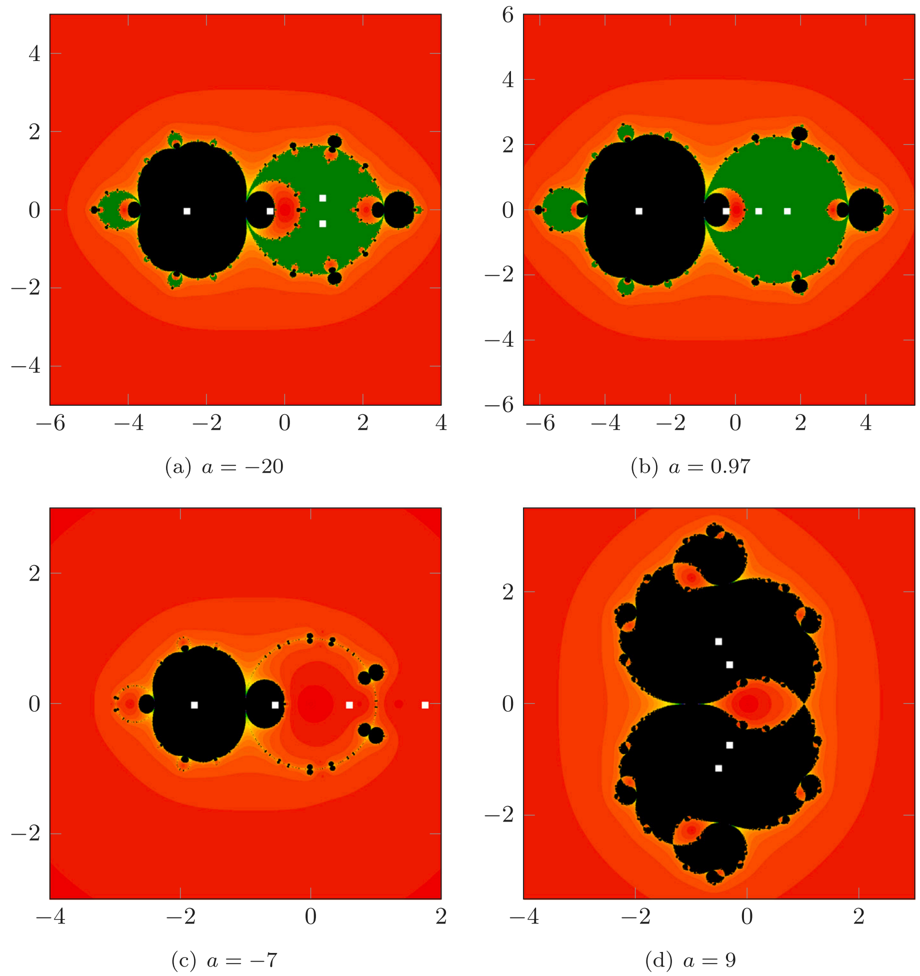

Fig. 10. Dynamical planes of the operator (6) for different values of a.

free critical point converges to the roots, pink if one critical orbit converges to a root, green if two critical orbits escape to any of the roots, and a scaling of colours if all critical orbits converge to the roots.

By checking the values of the parameter where $|O'_a(1)| = 1$, we have that the fixed point $z = 1$ is attractive inside the curve

$$1000176 - 997536\alpha + 373160\alpha^2 - 62056\alpha^3 + 3871\alpha^4 + (124216 - 62056\alpha + 7742\alpha^2)\beta^2 + 3871\beta^4 = 0,$$

being $a = \alpha + i\beta$. This curve can be observed in Fig. 12 and corresponds to the biggest green oval (with decorations inside) of Fig. 11.

In Fig. 13 we show some dynamical planes for different values of the parameters. We use the same colours as in the dynamical planes of Section 4.1. The first three parameters are chosen within the region for which $z = 1$ is attracting. The parameter $a = 3.9$ (which appears in black in Fig. 11) is such that no free critical orbit converges to the roots: four critical points belong to the basin of attraction of $z = 1$ (in green) and the other two critical points belong to the basins of attraction of two different attracting fixed points. The parameter $a = 3.9+0.04i$ (which appears in pink in Fig. 11) is such that two critical orbits belong to the basin of attraction of $z = 1$, two critical orbits converge to the roots, and the remaining two critical points converge to the basins of attraction of two different attracting fixed points. The parameter $a = 4.1 + 0.2i$ (which appears in green in Fig. 11) is such that two critical points belong to the basin of attraction of $z = 1$ and the remaining four critical points belong to the basins of attraction of the roots. The last parameter, $a = 4.4 + 0.4i$, is chosen so that all six critical points converge to the roots.

5. Conclusions

In this section we summarize the main conclusions of this paper.

We have provided an algorithm to draw parameter planes taking into account all critical points simultaneously in a single plane and not losing any information. This algorithm avoids the inconsistencies arising from the relationship between the different critical points as well as the indeterminacy caused by the square roots involved in their computation. We have also explained how

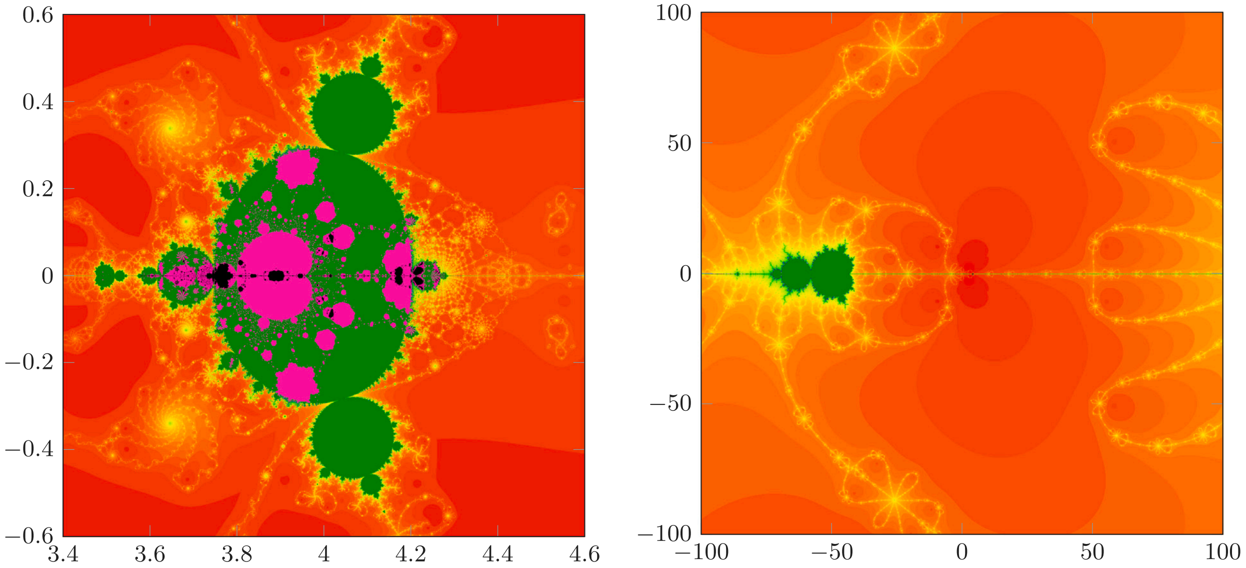

Fig. 11. Parameter planes of the operator 4.4 obtained using all three free critical orbits simultaneously. Black, pink and green indicate, respectively, that zero, one and two critical orbits have converged to the roots. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

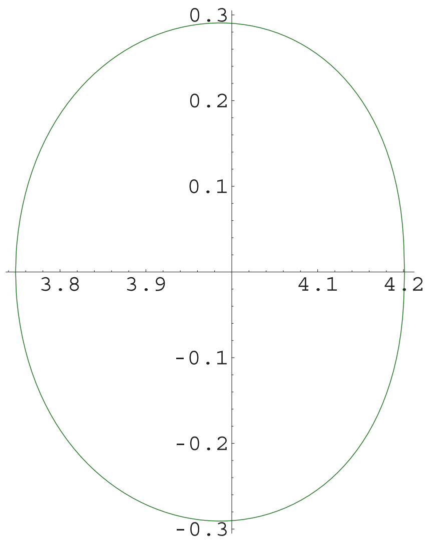

Fig. 12. Stability region for $z = 1$.

this algorithm can be modified in order to obtain more information. These modifications focus on how to determine whether two critical points that do not belong to the basins of attraction of the roots can lead to two different cycles of Fatou components and on how to study the case where there is a persistent Fatou component which is not related to a given critical point. Along the paper we have provided several examples which show that this algorithm can be used to illustrate the bifurcation locus of operators coming from Newton-like root-finding algorithms in a single dynamical plane without losing any information and avoiding inconsistencies arising from relationships amongst critical points and indeterminacies caused by squared roots. We consider that this algorithm will be a powerful tool for future studies of rational maps arising from root-finding algorithms or, more generally, modelling physical or natural events.

The main difficulty to apply this algorithm is to find all critical points. In Section 2.2 we prove that in order to find the critical points of operators obtained from Newton-like root-finding methods applied on quadratic polynomials we need to solve a palindromic polynomial. Following this result, in this section of the preliminaries we provide a short discussion on palindromic polynomials and how to find their roots. Even though this preliminary section is not strictly related with the development of the

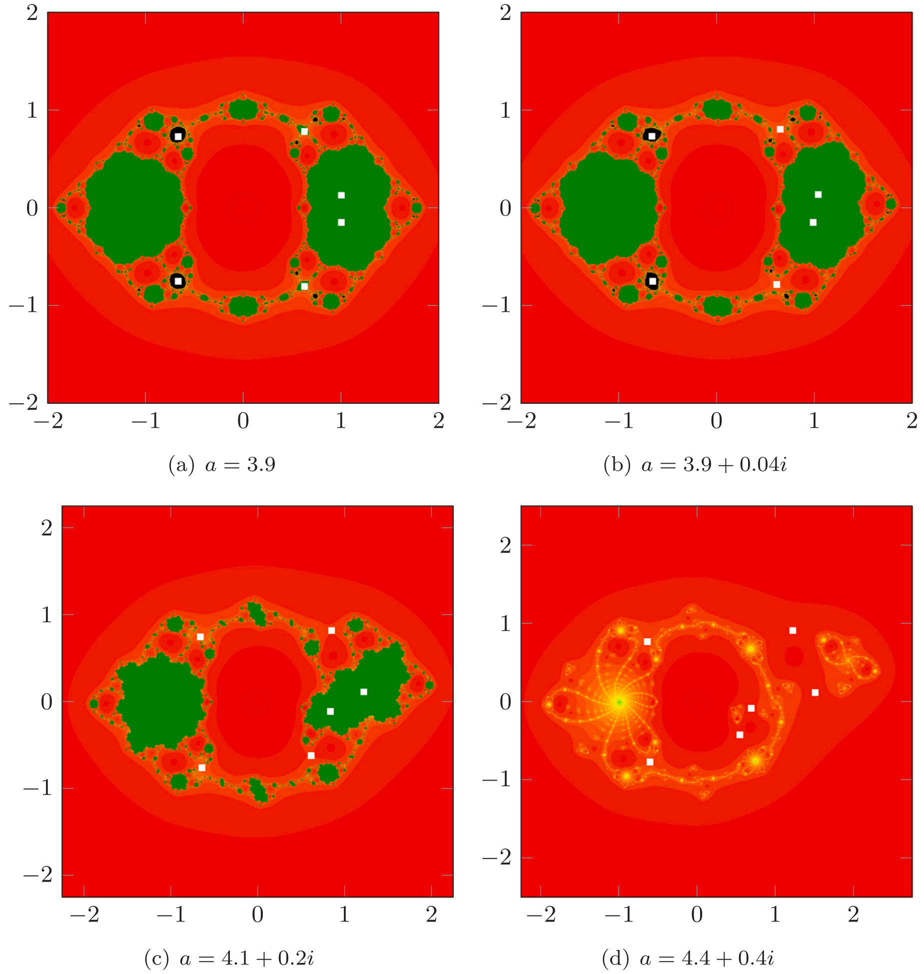

Fig. 13. Dynamical planes of the operator (6) for different values of a.

algorithm, the techniques discussed there can be particularly useful when implementing the algorithm in operators obtained from Newton-like root-finding methods.

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Consent for publication

All authors give their consent for publication.

CRediT authorship contribution statement

Beatriz Campos: Investigation. **Jordi Canela:** Conceptualization. **Alberto Rodríguez-Arenas:** Conceptualization. **Pura Vindel:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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