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On Modular b-Metrics

Salvador Romaguera

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain; sromague@mat.upv.es

Abstract: The notions of modular b-metric and modular b-metric space were introduced by Ege and Alaca as natural generalizations of the well-known and featured concepts of modular metric and modular metric space presented and discussed by Chistyakov. In particular, they stated generalized forms of Banach's contraction principle for this new class of spaces thus initiating the study of the fixed point theory for these structures, where other authors have also made extensive contributions. In this paper we endow the modular b-metrics with a metrizable topology that supplies a firm endorsement of the idea of convergence proposed by Ege and Alaca in their article. Moreover, for a large class of modular b-metric spaces, we formulate this topology in terms of an explicitly defined b-metric, which extends both an important metrization theorem due to Chistyakov as well as the so-called topology of metric convergence. This approach allows us to characterize the completeness for this class of modular b-metric spaces that may be viewed as an offsetting of the celebrated Caristi–Kirk theorem to our context. We also include some examples that endorse our results.

Keywords: modular b-metric; uniformity; metrizable; complete; modular b-Caristi mapping; Caristi–Kirk's theorem

MSC: 54E35; 54E50; 54H25

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Citation: Romaguera, S. On Modular b-Metrics. *Symmetry* **2024**, *16*, 1333. https://doi.org/10.3390/sym16101333

Academic Editors: Qing-Wen Wang, Ghasem Soleimani Rad and Stojan Radenovic

Received: 19 September 2024
Revised: 5 October 2024
Accepted: 7 October 2024
Published: 9 October 2024

1. Introduction and Preliminaries

We start by pointing out the following issues: for notions and properties on general topology, including uniformities and uniform spaces, we refer the reader to the valuables classical texts [1,2].

If $\mathcal{O}$ is an open set in a topological space $(\mathcal{S}, \Gamma)$, we will say that it is Γ-open (recall that the empty-set is open for every topological space), and if $\{x_n\}_{n\geq 1}$ is a sequence in $\mathcal{S}$ that converges in $(\mathcal{S}, \Gamma)$, we will say that it is Γ-convergent or that Γ-converges. As usual, a topological space $(\mathcal{S}, \Gamma)$ is called metrizable if there is a metric ϱ on $\mathcal{S}$ whose induced topology Γ_ϱ agrees with Γ. In such a case, we say that the topology Γ is metrizable.

The idea of exploring a notion of distance that simultaneously generalizes the concept of a metric and the notion of a quasi-norm in the classical sense of functional analysis [3–7] was independently discussed by several authors under different perspectives and denominations [8–13]. Here, we will use Czerwik's terminology [10,11] as follows.

Definition 1 ([10,11])**.** *A b-metric on a set $\mathcal{S}$ is a pair $(\mathcal{D}, K)$ such that $\mathcal{D}$ is a function from $\mathcal{S}\times\mathcal{S}$ to $[0,\infty)$ and K is a real constant with $K\geq 1$ satisfying the next conditions for every $x, y, z \in \mathcal{S}$:*

(bm0) $x = y$ *if and only if* $\mathcal{D}(x, y) = 0$;
(bm1) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$;
(bm2) $\mathcal{D}(x, y) \leq K[\mathcal{D}(x, z) + \mathcal{D}(z, y)]$.

Then, the triple $(\mathcal{S}, \mathcal{D}, K)$ is called a b-metric space.

Note that if $K = 1$, we get the notions of a metric and a metric space, respectively.

Several examples of b-metric spaces can be meet in [14–16]. The following typical instance will be utilized later (see, e.g., [14] (Example 2.2) and [16] (Example 12.2)).

Example 1. *Let $(\mathcal{S}, \varrho)$ be a metric space, and c and C be constants with $c > 0$ and $C > 1$. Then, the pair $(\mathcal{D}_\varrho, 2^{C-1})$ is a b-metric on $\mathcal{S}$, with $\mathcal{D}_\varrho$ given by $\mathcal{D}_\varrho(x,y) = c(\varrho(x,y))^C$ for every $x, y \in \mathcal{S}$.*

The study of the topological aspects of b-metric spaces has received the attention of several authors, while the problem of obtaining relevant fixed point theorems in the setting of these spaces has been the subject of considerable research (see, e.g., [17–23] and the references therein). In this regard, Chapter 5 of the recent book by Karapinar and Agarwal [24] offers a complete treatment of fixed point theory for b-metric spaces.

In the sequel, we recall some well-known and pertinent properties of b-metric spaces (see, e.g., [23] (Section 2), [16] (Chapter 12)).

Each b-metric $(\mathcal{D}, K)$ on a set $\mathcal{S}$ induces a metrizable topology $\Gamma_{(\mathcal{D},K)}$ on $\mathcal{S}$ given by
$$\Gamma_{(\mathcal{D},K)} = \{\mathcal{O} \subseteq \mathcal{S} : \text{for each } x \in \mathcal{O}, \text{ there is } \varepsilon > 0 \text{ such that } B_{(\mathcal{D},K)}(x,\varepsilon) \subseteq \mathcal{O}\},$$
where $B_{(\mathcal{D},K)}(x,\varepsilon) = \{y \in \mathcal{S} : \mathcal{D}(x,y) < \varepsilon\}$ for every $x \in \mathcal{S}$ and $\varepsilon > 0$.

Remark 1. *Since the definition of the balls $B_{(\mathcal{D},K)}(x,\varepsilon)$ does not depend from the constant K, we infer that if $(\mathcal{D}, K)$ and $(\mathcal{D}, K')$ are b-metrics on a set $\mathcal{S}$, we have $B_{(\mathcal{D},K)}(x,\varepsilon) = B_{(\mathcal{D},K')}(x,\varepsilon)$ for every $x \in \mathcal{S}$ and $\varepsilon > 0$, which implies the well-known fact that the topologies $\Gamma_{(\mathcal{D},K)}$ and $\Gamma_{(\mathcal{D},K')}$ agree on $\mathcal{S}$. For this reason, in the rest of the paper, we will write $\Gamma_\mathcal{D}$ and $B_\mathcal{D}(x,\varepsilon)$ instead of $\Gamma_{(\mathcal{D},K)}$ and $B_{(\mathcal{D},K)}(x,\varepsilon)$, respectively, if there is no possibility of ambiguity.*

The following fundamental property will be employed in the proof of Theorem 2 in Section 2: a sequence $\{x_n\}_{n\geq 1}$ in a b-metric space $(\mathcal{S}, \mathcal{D}, K)$ is $\Gamma_\mathcal{D}$-convergent to $x \in \mathcal{S}$ if and only if $\mathcal{D}(x, x_n) \to 0$ as $n \to \infty$.

We emphasize that, in contrast to the standard metric case, the balls $B_\mathcal{D}(x,\varepsilon)$ are not necessarily $\Gamma_\mathcal{D}$-open sets (see, e.g., [18] (Example 3.9)).

We also remind the reader that the notions of Cauchy sequence and completeness in the context of b-metric spaces are defined in the same way as in the metric case.

In the next section we will need the following slight generalization of the notion of a b-metric.

Definition 2. *An enlarged b-metric on a set $\mathcal{S}$ is a pair $(\mathcal{E}, K)$ such that $\mathcal{E}$ is a function from $\mathcal{S} \times \mathcal{S}$ to $[0, \infty]$ and K is a real constant with $K \geq 1$ satisfying conditions (bm0), (bm1) and (bm2) in Definition 1. (As usual, we adopt the convention that $\infty = \infty + \infty = \infty + x$ for every real number x).*

Encouraged in part by research concerning modulars on vector spaces [25–28], Chistyakov introduced and deeply studied [29–31] the notions of modular metric and modular metric space. In [32], Ege and Alaca introduced the notions of modular b-metric and modular b-metric space as natural generalizations of Chistyakov's concepts from a b-metric point of view. In particular, they stated generalized forms of Banach's contraction principle for this new class of spaces, thus initiating the study of the fixed point theory for these structures, where several authors have also made extensive contributions [33–37].

In this paper, we assign a topology upon modular b-metrics that robustly supports the convergence concept introduced by Ege and Alaca in [32]. We authenticate that this topology is susceptible to metrization, achieved through the construction of a compatible uniformity possessing a countable base. This formulation, applicable across a broad spectrum of modular b-metric spaces is articulated via a specifically defined b-metric. Thus, we amplify an important metrization theorem due to Chistyakov, alongside the extension of the so-called topology of metric convergence. Our methodology further facilitates the characterization of the completeness of this subclass of modular b-metric spaces, an analysis paralleling the characterization of complete metric spaces provided by the renowned Caristi–Kirk theorem. We also present some instances (see Examples 4–8 below) that support our findings.

2. Properties of Modular b-Metrics

We start this section by recalling the aforementioned concepts due to Ege and Alaca.

Definition 3 ([32]). *A modular b-metric on a set $\mathcal{S}$ is a pair $(\mathcal{M}, K)$ such that $\mathcal{M}$ is a function from $(0,\infty) \times \mathcal{S} \times \mathcal{S}$ to $[0,\infty]$ and K is a real constant, with $K \geq 1$ satisfying the next conditions for every $x, y, z \in \mathcal{S}$:*

(Mbm0) $x = y$ *if and only if* $\mathcal{M}(t,x,y) = 0$ *for every* $t > 0$;
(Mbm1) $\mathcal{M}(t,x,y) = \mathcal{M}(t,y,x)$ *for every* $t > 0$;
(Mbm2) $\mathcal{M}(t+s,x,y) \leq K[\mathcal{M}(t,x,z) + \mathcal{M}(s,z,y)]$ *for every* $t, s > 0$.

Then, the triple $(\mathcal{S}, \mathcal{M}, K)$ is called a modular b-metric space.

Note that if $K = 1$, we get the notions of a modular metric and a modular metric space due to Chistyakov [29].

Remark 2. *It is a well-known consequence of condition (Mbm2) that, for $x, y \in \mathcal{S}$ fixed, we have $\mathcal{M}(t,x,y) \leq K\mathcal{M}(s,x,y)$ whenever $t > s > 0$.*

Remark 3. *As a consequence of Remark 2, we obtain the following useful modification of condition (Mbm0): Let $(\mathcal{S}, \mathcal{M}, K)$ be a modular b-metric space and let $x, y \in \mathcal{S}$. Then, $x = y$ if and only if $\mathcal{M}(t,x,y) \leq Kt$ for every $t > 0$.*

To find various examples of modular b-metric spaces see, e.g., [32,33]. We recall the following representative one.

Example 2. *Let $(\mathcal{S}, \mathcal{D}, K)$ be a b-metric space. Then, the triple $(\mathcal{S}, \mathcal{M}, K)$ is a modular b-metric space where $\mathcal{M}(t,x,y) = \mathcal{D}(x,y)/t$ for every $x, y \in \mathcal{S}$ and $t > 0$.*

Combining Examples 1 and 2, we have the following one.

Example 3. *Let $(\mathcal{S}, \varrho)$ be a metric space. Then, the triple $(\mathcal{S}, \mathcal{M}, 2^{C-1})$ is a modular b-metric space, where $c > 0$ and $C > 1$ are real constants, and $\mathcal{M}(t,x,y) = c(\varrho(x,y)^C)/t$ for every $x, y \in \mathcal{S}$ and $t > 0$.*

Regarding Remark 2, we point out that in contrast to what happens to modular metric spaces, there are modular b-metric spaces $(\mathcal{S}, \mathcal{M}, K)$ for which the function $t \to \mathcal{M}(t,x,y)$ is not nonincreasing on $(0,\infty)$, as the following example demonstrates (compare [33] (Lemma 5.9 (iv)).

Example 4. *Let us designate by $\mathbb{R}$ the set of real numbers and let $\mathcal{M}$ be the function from $(0,\infty) \times \mathbb{R} \times \mathbb{R}$ to $[0,\infty)$ given by*

$$\mathcal{M}(t,x,y) = |x-y|^2/t \text{ if } t \in (0,1) \quad \text{and} \quad \mathcal{M}(t,x,y) = 2|x-y|^2 \text{ if } t \geq 1.$$

We are going to show that $(\mathcal{M}, 4)$ is a modular b-metric on $\mathbb{R}$.

Indeed, conditions (Mbm0) and (Mbm1) are obviously fulfilled, while condition (Mbm2) is readily verified, taking into account the following relations for every $x, y, z \in \mathbb{R}$:

$$|x-y|^2 \leq 2[|x-z|^2 + |z-y|^2],$$

$$\frac{|x-y|^2}{t+s} \leq 2[\frac{|x-z|^2}{t} + \frac{|z-y|^2}{s}],$$

whenever $t, s > 0$ with $t + s < 1$, and

$$2|x-y|^2 \leq 4[\frac{|x-z|^2}{t} + \frac{|z-y|^2}{s}],$$

whenever $t, s \in (0,1)$ *with* $t+s \geq 1$.

However, for $x \neq y$, *we have* $\mathcal{M}(3/4, x, y) = 4|x-y|^2/3 < 2|x-y|^2 = \mathcal{M}(1, x, y)$.

This scenario motivates the following notion.

Definition 4. *A modular b-metric* $(\mathcal{M}, K)$ *on a set* $\mathcal{S}$ *is called strong if, for each* $x, y \in \mathcal{S}$, *the function* $t \to \mathcal{M}(t, x, y)$ *is nonincreasing on* $(0, \infty)$. *In such a case, we say that the modular b-metric space* $(\mathcal{S}, \mathcal{M}, K)$ *is a strong modular b-metric space.*

Remark 4. *Note that Examples 2 and 3 provide interesting instances of strong modular b-metric spaces.*

Now, for a given modular b-metric $(\mathcal{M}, K)$ on a set $\mathcal{S}$, we proceed to construct, in a natural fashion, a topology $\Gamma_{(\mathcal{M},K)}$ on $\mathcal{S}$. To this end, put $\mathcal{B}_{(\mathcal{M},K)}(x, \varepsilon, t) = \{y \in \mathcal{S} : \mathcal{M}(t, x, y) < \varepsilon\}$ for every $x \in \mathcal{S}$ and $\varepsilon, t > 0$, and define

$$\Gamma_{(\mathcal{M},K)} := \{\varnothing\} \cup \{\mathcal{O} \subseteq \mathcal{S} : \text{ for each } x \in \mathcal{O} \text{ there are } \varepsilon, t > 0 \text{ such that } B_{(\mathcal{M},K)}(x, \varepsilon, t) \subseteq \mathcal{O}\}.$$

We have the following.

Proposition 1. *Let* $(\mathcal{M}, K)$ *be a modular b-metric on a set* $\mathcal{S}$. *Then,* $\Gamma_{(\mathcal{M},K)}$ *is a topology on* $\mathcal{S}$.

Proof. Let $\mathcal{O}_1, ..., \mathcal{O}_n$ be a finite family of members of $\Gamma_{(\mathcal{M},K)}$. We show that $\bigcap_{j=1}^n \mathcal{O}_j \in \Gamma_{(\mathcal{M},K)}$. Indeed, let $x \in \bigcap_{j=1}^n \mathcal{O}_j$. Then, for every $j \in \{1, ..., n\}$, there exist $\varepsilon_j > 0$ and $t_j > 0$ such that $B_{(\mathcal{M},K)}(x, \varepsilon_j, t_j) \subseteq \mathcal{O}_j$. Choose $\varepsilon = \min\{\varepsilon_j : j = 1, ..., n\}$ and $t > 0$ such that $t < \min\{t_j : j = 1, ..., n\}$. Hence, for every $y \in B_{(\mathcal{M},K)}(x, \varepsilon/K, t)$ and $j \in \{1, ..., n\}$, we obtain, by Remark 2,

$$\mathcal{M}(t_j, x, y) \leq K\mathcal{M}(t, x, y) < \varepsilon \leq \varepsilon_j,$$

which implies that

$$B_{(\mathcal{M},K)}(x, \varepsilon/K, t) \subseteq \bigcap_{j=1}^n B_{(\mathcal{M},K)}(x, \varepsilon_j, t_j) \subseteq \bigcap_{j=1}^n \mathcal{O}_j.$$

Therefore, $\bigcap_{j=1}^n \mathcal{O}_j \in \Gamma_{(\mathcal{M},K)}$.

Finally, it is routine to check that the union of any family of members of $\Gamma_{(\mathcal{M},K)}$ belongs to $\Gamma_{(\mathcal{M},K)}$, which concludes the proof. □

Remark 5. *Note that, as in the case of b-metric spaces, if* $(\mathcal{M}, K)$ *and* $(\mathcal{M}, K')$ *are modular b-metrics on a set* $\mathcal{S}$, *then the topologies* $\Gamma_{(\mathcal{M},K)}$ *and* $\Gamma_{(\mathcal{M},K')}$ *agree on* $\mathcal{S}$. *For this reason, in the rest of the paper, we will write* $\Gamma_{\mathcal{M}}$ *instead of* $\Gamma_{(\mathcal{M},K)}$ *if there is no possibility of ambiguity.*

We are going to prove that for any modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$, the topology $\Gamma_{\mathcal{M}}$ is metrizable. To achieve it, the next result will be crucial.

Proposition 2. *Let* $(\mathcal{M}, K)$ *be a modular b-metric on a set* $\mathcal{S}$. *For each* $n \in \mathbb{N}$ *(the set of natural numbers), define*

$$U_n = \{(x, y) \in \mathcal{S} \times \mathcal{S} : \mathcal{M}(2^{-n}, x, y) < 2^{-n}K\}.$$

Then, the following properties hold:

(P1) $\{(x, x) : x \in \mathcal{S}\} \subseteq U_n$ *for every* $n \in \mathbb{N}$.
(P2) For every $x, y \in \mathcal{S}$ *with* $x \neq y$, *there is* $n \in \mathbb{N}$ *such that* $(x, y) \notin U_n$.
(P3) $U_n = U_n^{-1}$ *for every* $n \in \mathbb{N}$.

(P4) For every $n, m \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $U_j \circ U_j \subseteq U_n \cap U_m$.

(P5) For every $\varepsilon, t > 0$, one has that $U_n \subseteq \{(x,y) \in \mathcal{S} \times \mathcal{S} : \mathcal{M}(t,x,y) < \varepsilon\}$ whenever $2^{-n}K^2 < \min\{\varepsilon, t\}$.

Proof.

(P1): It is immediate by condition (Mbm0).

(P2): Since $x \neq y$, there is $t > 0$ such that $\mathcal{M}(t,x,y) > 0$ by condition (Mbm0). Pick $n \in \mathbb{N}$ satisfying $2^{-n} < t$ and $2^{-n}K^2 < \mathcal{M}(t,x,y)$. Then, $2^{-n}K^2 < K\mathcal{M}(2^{-n},x,y)$, by Remark 2. Therefore, $(x,y) \notin U_n$.

(P3): It is immediate by condition (Mbm1).

(P4): For $n, m \in \mathbb{N}$ given, choose $j \in \mathbb{N}$ verifying

$$j > \max\{n+1, m+1\} \text{ and } 2^{-(j-1)}K^2 < \min\{2^{-n}, 2^{-m}\}.$$

Now let $(x,y) \in U_j \circ U_j$. Then, there exists $z \in \mathcal{S}$ such that $(x,z) \in U_j$ and $(z,y) \in U_j$. Since $j > n+1$, we obtain $2^{-n} > 2^{-(j-1)}$, so, $\mathcal{M}(2^{-n},x,y) \leq K\mathcal{M}(2^{-(j-1)},x,y)$ by Remark 2.

Moreover, by our assumption that $(x,z) \in U_j$ and $(z,y) \in U_j$, we get

$$\mathcal{M}(2^{-j},x,z) < 2^{-j}K \text{ and } \mathcal{M}(2^{-j},z,y) < 2^{-j}K.$$

By applying condition (Mbm2) joint with the preceding two inequalities and the fact that $2^{-(j-1)}K^2 < 2^{-n}$, we have

$$\begin{aligned} \mathcal{M}(2^{-n},x,y) &\leq K^2[\mathcal{M}(2^{-j},x,z) + \mathcal{M}(2^{-j},z,y)] \\ &< 2^{-(j-1)}K^3 < 2^{-n}K, \end{aligned}$$

which implies that $(x,y) \in U_n$. Similarly, we show that $(x,y) \in U_m$.

(P5): Let $\varepsilon, t > 0$ and $n \in \mathbb{N}$ such that $2^{-n}K^2 < \min\{\varepsilon, t\}$. Given $(x,y) \in U_n$, we have $\mathcal{M}(2^{-n},x,y) < 2^{-n}K$, so

$$\mathcal{M}(t,x,y) \leq K\mathcal{M}(2^{-n},x,y) \leq 2^{-n}K^2 < \varepsilon.$$

□

From Proposition 2, we derive the next consequences for a given modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$:

(Con1) By properties (P1), (P2), (P3) and (P4)), the family $\{U_n : n \in \mathbb{N}\}$ constructed in Proposition 2 forms a countable base for a separated uniformity $\mathcal{U}$ on $\mathcal{S}$. Hence, there is a metric ϱ on $\mathcal{S}$ such that the topology $\Gamma_{\mathcal{U}}$ induced by $\mathcal{U}$ agrees with the topology Γ_ϱ induced by ϱ.

(Con2) The topologies $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{U}}$ agree on $\mathcal{S}$: Indeed, let $\mathcal{O}$ be a $\Gamma_{\mathcal{M}}$ By property (P5), there is $n \in \mathbb{N}$ such that $U_n(x) \subseteq B_{\mathcal{M}}(x,\varepsilon,t)$, so, $\Gamma_{\mathcal{M}} \subseteq \Gamma_{\mathcal{U}}$. Now let $\mathcal{O}$ be a $\Gamma_{\mathcal{U}}$ -open set. For each $x \in \mathcal{O}$, there is $n_x \in \mathbb{N}$ such that $U_{n_x}(x) \subseteq \mathcal{O}$. Since $U_{n_x}(x) = B_{\mathcal{M}}(x, 2^{-n_x}, 2^{-n_x}K)$, we conclude that $\Gamma_{\mathcal{U}} \subseteq \Gamma_{\mathcal{M}}$.

(Con3) The proof of (Con2), also shows the following statement:
A sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{S}$ is $\Gamma_{\mathcal{M}}$-convergent to a point $x \in \mathcal{S}$ if and only if, for each $t > 0$, $\mathcal{M}(t,x,x_n) \to 0$ as $n \to \infty$.

Moreover, from consequences (Con1) and (Con2), we obtain the following promised result.

Theorem 1. *Let $(\mathcal{S}, \mathcal{M}, K)$ be a modular b-metric space. Then, the topological space $(\mathcal{S}, \Gamma_{\mathcal{M}})$ is metrizable.*

Remark 6. *In [32], Ege and Alaca introduced the following notion: a sequence* $\{x_n\}_{n\geq 1}$ *in a modular b-metric space* $(\mathcal{S}, \mathcal{M}, K)$ *is said to be convergent to a point* $x \in \mathcal{S}$ *if, for each* $t > 0, \mathcal{M}(t, x, x_n) \to 0$ *as* $n \to \infty$.

Note that consequence (Con3) provides a solid topological support to Ege and Alaca's concept.

Our next goal consists of constructing an explicitly formulated b-metric compatible with the topology $\Gamma_{\mathcal{M}}$ induced by any strong modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$. In this way, we extend both an important result on metrizability due to Chistyakov and the so-called topology of metric convergence [31] (Chapter 4).

Theorem 2. *Let* $(\mathcal{S}, \mathcal{M}, K)$ *be a modular b-metric space. Define a function* $\mathcal{E}_{\mathcal{M}}$ *from* $\mathcal{S} \times \mathcal{S}$ *to* $[0, \infty]$ *by*

$$\mathcal{E}_{\mathcal{M}}(x, y) = \infty \ \ if \ \ \mathcal{M}(t, x, y) = \infty \ \ for \ \ all \ \ t > 0,$$

and

$\mathcal{E}_{\mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(t, x, y) \leq Kt\}$ *otherwise, and, then, a function* $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ *from* $\mathcal{S} \times \mathcal{S}$ *to* $[0, 1]$ *by*

$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x, y) = \min\{1, \mathcal{E}_{\mathcal{M}}(x, y)\}$ *for every* $x, y \in \mathcal{S}$.

Then, the following statements hold:

(St1) $\mathcal{E}_{\mathcal{M}}$ *and* $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ *satisfy conditions (bm0) and (bm1).*

Furthermore, the next implications are satisfied for every $x, y \in \mathcal{S}$ *and* $\varepsilon \in (0, 1)$:

$$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x, y) < \varepsilon \Rightarrow \mathcal{M}(\varepsilon, x, y) \leq K\varepsilon \Rightarrow \mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x, y) \leq \varepsilon. \tag{1}$$

(St2) If $(\mathcal{M}, K)$ *is strong, we get that* $(\mathcal{E}_{\mathcal{M}}, K)$ *is an enlarged b-metric on* $\mathcal{S}$ *and* $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, K)$ *is a b-metric on* $\mathcal{S}$ *whose induced topology* $\Gamma_{\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}}$ *agrees with the topology* $\Gamma_{\mathcal{M}}$ *induced by* $(\mathcal{M}, K)$.

Proof. We first note that, indeed, $\mathcal{E}_{\mathcal{M}}$ is well-defined: let $x, y \in \mathcal{S}$ such that $\mathcal{M}(t_0, x, y) < \infty$ for some $t_0 > 0$. Then, there is $t_1 > 0$ such that $\mathcal{M}(t_0, x, y) < t_1$. If $t_1 < t_0$, we get $\mathcal{M}(t_0, x, y) < t_0 \leq Kt_0$; otherwise, we obtain $\mathcal{M}(t_1, x, y) \leq K\mathcal{M}(t_0, x, y) < Kt_1$.

Now we show the statement (St1) for $\mathcal{E}_{\mathcal{M}}$.

(bm0): Let $x, y \in \mathcal{S}$. If $x = y$, we have $\mathcal{M}(t, x, y) = 0$ for all $t > 0$, so $\mathcal{E}_{\mathcal{M}}(x, y) = 0$.
If $\mathcal{E}_{\mathcal{M}}(x, y) = 0$, we obtain $\mathcal{M}(t, x, y) \leq Kt$ for every $t > 0$. Let $t > 0$ be arbitrary. Put $s = t/K$. Thus, $\mathcal{M}(t, x, y) \leq K\mathcal{M}(s, x, y) \leq Kt$. Hence, $x = y$ by Remark 3.

(bm1): It is immediate by (Mbm1).

By virtue of its definition, we directly infer that $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ also verifies conditions (bm0) and (bm1).

Finally, the implications given in Equation (1) are a direct consequence of the definitions of $\mathcal{E}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$.

Next, we prove the statement (St2). To this end, we first check that $(\mathcal{E}_{\mathcal{M}}, K)$ is an enlarged modular b-metric on $\mathcal{S}$.

Let $x, y, z \in \mathcal{S}$.

If $\mathcal{E}_{\mathcal{M}}(x, z) = \infty$ or $\mathcal{E}_{\mathcal{M}}(z, y) = \infty$, condition (bm2) is trivially fulfilled.

Assume then that $\mathcal{E}_{\mathcal{M}}(x, z) < \infty$ and $\mathcal{E}_{\mathcal{M}}(z, y) < \infty$. Let $t, s > 0$ such that $\mathcal{M}(t, x, z) < \infty$ and $\mathcal{M}(s, z, y) < \infty$. Thus, by condition (Mbm2), $\mathcal{M}(t + s, x, y) < \infty$, so $\mathcal{E}_{\mathcal{M}}(x, y) < \infty$. Now, choose an arbitrary $\varepsilon > 0$. Then, there exist $t_\varepsilon, s_\varepsilon > 0$ such that $t_\varepsilon < \mathcal{E}_{\mathcal{M}}(x, z) + \varepsilon$, $\mathcal{M}(t_\varepsilon, x, z) \leq Kt_\varepsilon$, $s_\varepsilon < \mathcal{E}_{\mathcal{M}}(z, y) + \varepsilon$ and $\mathcal{M}(s_\varepsilon, z, y) \leq Ks_\varepsilon$. Hence,

$$\begin{aligned} \mathcal{M}(K(t_\varepsilon + s_\varepsilon), x, y) &\leq \mathcal{M}(t_\varepsilon + s_\varepsilon, x, y) \\ &\leq K[\mathcal{M}(t_\varepsilon, x, z) + \mathcal{M}(s_\varepsilon, z, y)] \leq K^2(t_\varepsilon + s_\varepsilon). \end{aligned}$$

By the definition of $\mathcal{E}_{\mathcal{M}}$, we deduce that $\mathcal{E}_{\mathcal{M}}(x,y) \leq K(t_\varepsilon + s_\varepsilon)$, so

$$\mathcal{E}_{\mathcal{M}}(x,y) < K[\mathcal{E}_{\mathcal{M}}(x,z) + \mathcal{E}_{\mathcal{M}}(z,y) + 2\varepsilon].$$

Since ε is arbitrary, we conclude that $\mathcal{E}_{\mathcal{M}}(x,y) \leq K[\mathcal{E}_{\mathcal{M}}(x,z) + \mathcal{E}_{\mathcal{M}}(z,y)]$. Hence, $(\mathcal{E}_{\mathcal{M}}, K)$ is an enlarged modular b-metric on $\mathcal{S}$, which clearly implies that $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, K)$ is a modular b-metric on $\mathcal{S}$.

Lastly, notice that the fact that the topologies $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}}$ agree on $\mathcal{S}$ is easily deduced combining the implications of Equation (1) joint with the next known equivalences for any sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{S}$ and any $x \in \mathcal{S}$ (in particular, equivalence (ii) below was obtained as the consequence (Con3) of Proposition 2):

(i) $\{x_n\}_{n\geq 1}$ is $\Gamma_{\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}}$-convergent to x if and only if $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x, x_n) \to 0$ as $n \to \infty$,

and

(ii) $\{x_n\}_{n\geq 1}$ is $\Gamma_{\mathcal{M}}$-convergent to x if and only if, for each $t > 0$, $\mathcal{M}(t, x, x_n) \to 0$ as $n \to \infty$.

□

In the rest of the paper, we will refer to $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, K)$ as the b-metric associated to the strong modular b-metric $(\mathcal{M}, K)$.

Example 5. *Let $(\mathcal{S}, \varrho)$ be a metric space and let $(\mathcal{S}, \mathcal{M}, 2^{C-1})$ be the strong modular b-metric space constructed in Example 3. Since $\mathcal{M}(t,x,y) = c(\varrho(x,y)^C)/t$ for every $x, y \in \mathcal{S}$ and $t > 0$, we infer, by the definition of $\mathcal{E}_{\mathcal{M}}$, that for the b-metric $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, 2^{C-1})$ associated to $(\mathcal{M}, 2^{C-1})$, one has*

$$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,y) = \min\{1, \frac{\varrho(x,y)^{C/2}}{\sqrt{2^{C-1}}}\},$$

for every $x, y \in \mathcal{S}$.

Example 6. *Let $(\mathcal{S}, \mathcal{D}, K))$ be a b-metric space and let $c > 0$ be a constant. Define a function $\mathcal{M}$ from $(0,\infty) \times \mathcal{S} \times \mathcal{S}$ to $[0,\infty)$ by $\mathcal{M}(t,x,x) = 0$ for every $x \in \mathcal{S}$ and $t > 0$, and $\mathcal{M}(t,x,y) = \mathcal{D}(x,y) + c/t$ for every $x, y \in \mathcal{S}$ with $x \neq y$ and $t > 0$.*

Then, $(\mathcal{M}, K)$ is a strong modular b-metric on $\mathcal{S}$. To check it, it suffices to verify condition (Mbm2). Let $x, y, z \in \mathcal{S}$ and $t, s > 0$. We get

$$\begin{aligned} \mathcal{M}(t+s,x,y) &= \mathcal{D}(x,y) + \frac{c}{t+s} < K[\mathcal{D}(x,z) + \mathcal{D}(z,y)] + K(\frac{c}{t} + \frac{c}{s}) \\ &= K[\mathcal{M}(t,x,z) + \mathcal{M}(s,z,y)]. \end{aligned}$$

Taking into account the definition of $\mathcal{E}_{\mathcal{M}}$, an easy computation shows that the b-metric $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, K)$ associated to $(\mathcal{M}, K)$ is given by $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,x) = 0$ for every $x \in \mathcal{S}$ and

$$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,y) = \min\left\{1, \frac{\mathcal{D}(x,y) + \sqrt{(\mathcal{D}(x,y))^2 + 4cK}}{2K}\right\},$$

for every $x, y \in \mathcal{S}$ with $x \neq y$.

Since $\mathcal{M}(t,x,y) > c/t$ for every $x, y \in \mathcal{S}$, with $x \neq y$, and $t > 0$, we get that $\Gamma_{\mathcal{M}}$ is the discrete topology on $\mathcal{S}$. Therefore, the topologies $\Gamma_{\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}}$ and $\Gamma_{\mathcal{D}}$ agree on $\mathcal{S}$ if and only if $\Gamma_{\mathcal{D}}$ is the discrete topology on $\mathcal{S}$.

3. A Modular *b*-Metric Version of Caristi–Kirk's Theorem

In [38], Caristi proved his famous fixed point theorem that every Caristi mapping on a complete metric space has a fixed point. Kirk showed in [39] that Caristi's theorem characterizes the metric completeness. More precisely, we have the following important result, named Caristi–Kirk's theorem, where a self mapping $\mathcal{T}$ of a metric space $(\mathcal{S}, \varrho)$ is a

Caristi mapping on $(\mathcal{S}, \varrho)$ provided that there is a lower semicontinuous function f from $\mathcal{S}$ to $[0, \infty)$ such that $\varrho(x, \mathcal{T}x) \leq f(x) - f(\mathcal{T}x)$ for all $x \in \mathcal{S}$.

Theorem 3 ([38,39])**.** *A metric space is complete if and only if every Caristi mapping on it has a fixed point.*

This theorem was extended and generalized by numerous authors in different contexts (see, e.g. [40–46] and the references therein). In particular, a b-metric version of it has been obtained in [40] in the terms that are detailed below.

Definition 5 ([40])**.** *Let $(\mathcal{S}, \mathcal{D}, K)$ be a b-metric space. A function f from $\mathcal{S}$ to $[0, \infty)$ is 0-lower semicontinuous (0-lsc in short) provided that it fulfills the following condition: if $\{x_n\}_{n\geq 1}$ is a sequence in $\mathcal{S}$ that $\Gamma_{\mathcal{D}}$-converges to $x \in \mathcal{S}$ and verifies that $f(x_n) \to 0$ as $n \to \infty$, then $f(x) = 0$.*

Note that every lower semicontinuous function from $\mathcal{S}$ to $[0, \infty)$ is 0-lsc. However, the converse is not true in general [40] (Remark 2.3).

Definition 6 ([40])**.** *A self mapping $\mathcal{T}$ of a b-metric space $(\mathcal{S}, \mathcal{D}, K)$, with $K > 1$, is a b-Caristi mapping on $(\mathcal{S}, \mathcal{D}, K)$ provided that there exist a constant $r \in (1, K]$ and a 0-lsc function f such that for each $x \in \mathcal{S}$,*

$$\mathcal{D}(x, \mathcal{T}x) > 0 \implies \mathcal{D}(x, \mathcal{T}x) \leq f(x) - rf(\mathcal{T}x). \tag{2}$$

Then, it was proved in [40] (Theorem 2.9) the following.

Theorem 4 ([40])**.** *A b-metric space $(\mathcal{S}, \mathcal{D}, K)$, with $K > 1$, is complete if and only if every b-Caristi mapping on it has a fixed point.*

With the help of Theorems 2–4, we will characterize both complete modular metric spaces and complete strong modular b-metric spaces.

Let us recall [32] that a sequence $\{x_n\}_{n\geq 1}$ in a modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$ is a Cauchy sequence provided that for each $\varepsilon, t > 0$, there is an $n_{\varepsilon,t} \in \mathbb{N}$ such that $\mathcal{M}(t, x_n, x_m) < \varepsilon$ for every $n, m \geq n_{\varepsilon,t}$, and that a modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$ is complete provided that every Cauchy sequence is $\Gamma_{\mathcal{M}}$-convergent.

Remark 7. *It is well known that if $(\mathcal{S}, \varrho)$ is a complete metric space, then both the b-metric space constructed in Example 1 and the strong modular b-metric space constructed in Example 3 are complete.*

Proposition 3. *A strong modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$ is complete if and only if the b-metric space $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, K)$ is complete.*

Proof. It follows from the implications (1) in Theorem 2, St(1), and the fact that the topologies $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}}$ agree on $\mathcal{S}$ (Theorem 2, St(2)). □

Due to the different peculiarities that present the formulations of Theorems 3 and 4, we will split our study separating the case where $K = 1$ (i.e., the modular metric case) from the case where $K > 1$. In fact, when $K = 1$, we will write $(\mathcal{S}, \mathcal{M})$, $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_{\mathcal{M}}})$, $\mathcal{M}$ and $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ instead of $(\mathcal{S}, \mathcal{M}, 1)$, $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, 1)$, $(\mathcal{M}, 1)$ and $(\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}, 1)$, respectively.

Definition 7. *Let $(\mathcal{S}, \mathcal{M})$ be a modular metric space. We say that a self mapping $\mathcal{T}$ of $\mathcal{S}$ is a modular Caristi mapping if there exists a lower semicontinuous function f from $\mathcal{S}$ to $[0, \infty)$ such that for every $x \in \mathcal{S}$ and $t \in (0, 1)$ the following contraction condition holds:*

$$f(x) - f(\mathcal{T}x) < t \Longrightarrow \mathcal{M}(t, x, \mathcal{T}x) < t. \tag{3}$$

Proposition 4. *Let $(\mathcal{S}, \mathcal{M})$ be a modular metric space. Then, a self mapping of $\mathcal{S}$ is a modular Caristi mapping if and only if it is a Caristi mapping on the metric space $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}})$.*

Proof. First, we prove the "only if" part. Let $\mathcal{T}$ be a modular Caristi mapping of $\mathcal{S}$. Then, there exists a lower semicontinuous function f from $\mathcal{S}$ to $[0, \infty)$ such that for every $x \in \mathcal{S}$ and $t \in (0, 1)$ the contraction condition (3) holds. Since $\Gamma_\mathcal{M} = \Gamma_{\mathcal{D}_{\mathcal{E}_\mathcal{M}}}$, f is lower semicontinuous for the metric space $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}})$. Suppose that there is $x \in \mathcal{S}$ such that $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) > f(x) - f(\mathcal{T}x)$. Then, $f(x) - f(\mathcal{T}x) < \min\{1, \mathcal{E}_\mathcal{M}(x, \mathcal{T}x\}$. Hence, $\mathcal{M}(1, x, \mathcal{T}x) < 1$, which implies by the definition of $\mathcal{E}_\mathcal{M}$ that $\mathcal{E}_\mathcal{M}(x, \mathcal{T}x) \le 1$. Pick $\mu \in (0, \mathcal{E}_\mathcal{M}(x, \mathcal{T}x))$ such that $f(x) - f(\mathcal{T}x) < \mu$. It follows from condition (3) that $\mathcal{M}(\mu, x, \mathcal{T}x) < \mu$, so, $\mathcal{E}_\mathcal{M}(x, \mathcal{T}x) \le \mu$, which leads to a contradiction. Consequently, $\mathcal{T}$ is a Caristi mapping on $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}})$.

Now, we prove the "if" part. Suppose that $\mathcal{T}$ is a Caristi mapping on $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}})$. Then, there exists a lower semicontinuous function f from $\mathcal{S}$ to $[0, \infty)$ such that $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) \le f(x) - f(\mathcal{T}x)$ for every $x \in \mathcal{S}$. If $f(x) - f(\mathcal{T}x) < t$ with $t \in (0, 1)$, we infer that $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) \le f(x) - f(\mathcal{T}x)$, so $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) = \mathcal{E}_\mathcal{M}(x, \mathcal{T}x) < t < 1$. Choose any $s \in (\mathcal{E}_\mathcal{M}(x, \mathcal{T}x), t)$. Then, $\mathcal{M}(s, x, \mathcal{T}x) \le Ks$. Hence, $\mathcal{M}(t, x, \mathcal{T}x) \le Ks < Kt$. This finishes the proof. □

Combining Theorem 3 with Propositions 3 and 4, we obtain the following.

Theorem 5. *A modular metric space $(\mathcal{S}, \mathcal{M})$ is complete if and only if every modular Caristi mapping on it has a fixed point.*

Definition 8. *Let $(\mathcal{S}, \mathcal{M}, K)$ be a modular b-metric space with $K > 1$. We say that a self mapping $\mathcal{T}$ of $\mathcal{S}$ is a modular b-Caristi mapping if there exist a constant $r \in (1, K]$ and a 0-lsc function f from $\mathcal{S}$ to $[0, \infty)$ such that for every $x \in \mathcal{S}$ and $t \in (0, 1)$ the following contraction condition holds:*

$$f(x) - rf(\mathcal{T}x) < t \Longrightarrow \mathcal{M}(t, x, \mathcal{T}x) < Kt. \tag{4}$$

Although in the proof of the following result we employ a technique similar to that of Proposition 4, we will present it in detail both for completeness and to point out the importance of the property of strongness.

Proposition 5. *Let $(\mathcal{S}, \mathcal{M}, K)$ be a strong modular b-metric space with $K > 1$. Then, a self mapping of $\mathcal{S}$ is a modular b-Caristi mapping if and only if it is a b-Caristi mapping on the b-metric space $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}}, K)$.*

Proof. First, we prove the "only if" part. Let $\mathcal{T}$ be a modular b-Caristi mapping of $\mathcal{S}$. Then, there exist a constant $r \in (1, K]$ and a 0-lsc function f from $\mathcal{S}$ to $[0, \infty)$ such that for every $x \in \mathcal{S}$ and $t \in (0, 1)$, the contraction condition (4) holds. Since $\Gamma_\mathcal{M} = \Gamma_{\mathcal{D}_{\mathcal{E}_\mathcal{M}}}$, f is 0-lsc for the b-metric space $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}}, K)$. Suppose that there is $x \in \mathcal{S}$ such that $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) > 0$ and $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) > f(x) - rf(\mathcal{T}x)$. Then, $f(x) - rf(\mathcal{T}x) < \min\{1, \mathcal{E}_\mathcal{M}(x, \mathcal{T}x\}$. So $\mathcal{M}(1, x, \mathcal{T}x) < 1 \le K$, which implies by the definition of $\mathcal{E}_\mathcal{M}$ that $\mathcal{E}_\mathcal{M}(x, \mathcal{T}x) \le 1$. Pick $\mu \in (0, \mathcal{E}_\mathcal{M}(x, \mathcal{T}x)/K)$ such that $f(x) - rf(\mathcal{T}x) < \mu$. It follows from condition (4) that $\mathcal{M}(\mu, x, \mathcal{T}x) < K\mu$, so, $\mathcal{E}_\mathcal{M}$, $\mathcal{E}_\mathcal{M}(x, \mathcal{T}x) \le K\mu$, which leads to a contradiction. Consequently, $\mathcal{T}$ is a b-Caristi mapping on $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}}, K)$.

Now, we prove the "if" part. Suppose that $\mathcal{T}$ is a b-Caristi mapping on $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_\mathcal{M}}, K)$. Then, there exist a constant $r \in (1, K]$ and a 0-lsc function f from $\mathcal{S}$ to $[0, \infty)$ for which contraction condition (2) holds. If $f(x) - rf(\mathcal{T}x) < t$ with $t \in (0, 1)$, we infer that $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) \le f(x) - rf(\mathcal{T}x)$, so $\mathcal{D}_{\mathcal{E}_\mathcal{M}}(x, \mathcal{T}x) = \mathcal{E}_\mathcal{M}(x, \mathcal{T}x) < t < 1$. Choose any $s \in (\mathcal{E}_\mathcal{M}(x, \mathcal{T}x), t)$. Then, $\mathcal{M}(s, x, \mathcal{T}x) \le Ks$. Since $(\mathcal{M}, K)$ is strong, we deduce that $\mathcal{M}(t, x, \mathcal{T}x) \le Ks < Kt$. This finishes the proof. □

Combining Theorem 4 with Propositions 3 and 5, we obtain the following.

Theorem 6. *A strong modular b-metric space $(\mathcal{S}, \mathcal{M}, K)$, with $K > 1$, is complete if and only if every modular b-Caristi mapping on it has a fixed point.*

The following question remains open: are Theorems 2 and 6 valid in the framework of non-strong modular *b*-metric spaces?

We conclude the paper with two examples illustrating the results of this section.

Example 7. *Let $\mathcal{S} = (0,1]$ and ϱ be the usual metric on $\mathcal{S}$, i.e., $\varrho(x,y) = |x-y|$ for every $x, y \in \mathcal{S}$.*

Now let $\mathcal{T}$ be the self mapping of $\mathcal{S}$ given by $\mathcal{T}x = 1$ if $x \in [1/2, 1]$ and $\mathcal{T}x = 1/n$ if $x \in [1/(n+1), 1/n)$, $n \geq 2$.

Note that we cannot apply the "only if" part of Caristi–Kirk's theorem to this self mapping and the metric space $(\mathcal{S}, \varrho)$ because $(\mathcal{S}, \varrho)$ is not complete.

Define a function $\mathcal{M}$ from $(0,\infty) \times \mathcal{S} \times \mathcal{S}$ to $[0,\infty)$ by $\mathcal{M}(t,x,x) = 0$ for every $x \in \mathcal{S}$ and $t > 0$, and $\mathcal{M}(t,x,y) = |x-y| + 1/t$ for every $x, y \in \mathcal{S}$ with $x \neq y$ and $t > 0$. By applying Example 6 (with $K = c = 1$), we obtain that $\mathcal{M}$ is a modular metric on $\mathcal{S}$, whose associated metric $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ is given by $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,x) = 0$ for every $x \in \mathcal{S}$, and

$$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,y) = \min\left\{1, \frac{|x-y| + \sqrt{|x-y|^2 + 4}}{2}\right\},$$

for every $x, y \in \mathcal{S}$ with $x \neq y$.

Observe that

$$\frac{|x-y| + \sqrt{|x-y|^2 + 4}}{2} > 1,$$

for every $x, y \in \mathcal{S}$ with $x \neq y$, so $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}$ is the discrete metric on $\mathcal{S}$, i.e., $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x,y) = 1$ for every $x, y \in \mathcal{S}$ with $x \neq y$. Hence, $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_{\mathcal{M}}})$ is a complete metric space for which every function f from $\mathcal{S}$ to $[0,\infty)$ is lower semicontinuous.

Choose f as follows: $f(1) = 1$ and $f(x) = n+1$ if $x \in [1/(n+1), 1/n)$, $n \in \mathbb{N}$.

We have $\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(1, \mathcal{T}1) = 0$. Moreover, for every $x \in \mathcal{S} \backslash \{1\}$, there is $n \in \mathbb{N}$ such that $x \in [1/(n+1), 1/n)$, and thus, $\mathcal{T}x = 1/n$. Therefore,

$$\mathcal{D}_{\mathcal{E}_{\mathcal{M}}}(x, \mathcal{T}x) = 1 = (n+1) - n = f(x) - f(\mathcal{T}x),$$

which implies that $\mathcal{T}$ is a Caristi mapping on $(\mathcal{S}, \mathcal{D}_{\mathcal{E}_{\mathcal{M}}})$. By Proposition 3, $(\mathcal{S}, \mathcal{M})$ is a complete modular metric space and by Proposition 4, $\mathcal{T}$ is a modular Caristi mapping on $(\mathcal{S}, \mathcal{M})$. Consequently, all conditions of the "only if" part of Theorem 5 are satisfied. In fact, $\mathcal{T}$ has a (unique) fixed point $x = 1$.

Example 8. *As usual, for $0 < p < 1$, define*

$$l_p = \{x := \{x_n\}_{n \geq 1} \text{ such that } x_n \in \mathbb{R} \text{ for every } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Denote by d_p the classical metric on l_p given by $d_p(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ for every $x, y \in l_p$.

Now, let $l_p^+ = \{x \in l_p : x_n \geq 0 \text{ for every } n \in \mathbb{N}\}$, and denote also by d_p the restriction of the metric d_p on l_p^+.

Since l_p^+ is a closed subset of the complete metric space (l_p, d_p), we infer that (l_p^+, d_p) is also a complete metric space.

By applying Example 3 and Remark 4, we get that the triple $(l_p^+, \mathcal{M}, 2^{(1/p)-1})$ is a strong modular b-metric space, where $\mathcal{M}(t,x,y) = (d_p(x,y))^{1/p}/t$ for every $x, y \in l_p^+$ and $t > 0$. Furthermore, it is complete because (l_p^+, d_p) is so (see Remark 7).

Put $\mathcal{Y} := \{x \in l_p^+ : x_n \geq 2^{-n}$ *for every* $n \in \mathbb{N}\}$*, and let* C *be a constant such that* $C \geq 2^{(1/p)-1}$.

We show that the self mapping $\mathcal{T}$ *of* l_p^+ *given by* $(\mathcal{T}x)_1 = 0$ *and*

$$(\mathcal{T}x)_{n+1} = \frac{x_n - 2^{-n}}{C},$$

for every $x \in \mathcal{Y}$*, and* $\mathcal{T}x = x$ *for every* $x \in l_p^+ \backslash \mathcal{Y}$*, is a modular b-Caristi mapping on* $(l_p^+, \mathcal{M}, 2^{(1/p)-1})$ *for* $r = 2^{(1/p)-1}$.

Let f *be the function from* l_p^+ *to* $[0, \infty)$ *given by* $f(x) = \sum_{n=1}^{\infty} x_n$ *for every* $x := \{x_n\}_{n\geq 1} \in l_p^+$ *(note that* f *is well-defined because* $\sum_{n=1}^{\infty}(x_n)^p < \infty$*, and thus,* $\sum_{n=1}^{\infty} x_n < \infty$*).*

It is clear that f *is lsc and, hence, 0-lsc on* $(l_p^+, \Gamma_{\mathcal{M}})$.

Now, for each $x \in \mathcal{Y}$*, we get*

$$f(x) - 2^{(1/p)-1} f(\mathcal{T}x) = (\sum_{n=1}^{\infty} x_n) - 2^{(1/p)-1}(\sum_{n=1}^{\infty} \frac{x_n - 2^{-n}}{C})$$
$$\geq (\sum_{n=1}^{\infty} x_n) - \sum_{n=1}^{\infty}(x_n - 2^{-n}) = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

and for $x \in l_p^+ \backslash \mathcal{Y}$, $\mathcal{M}(t, x, \mathcal{T}x) = 0$ *for every* $t > 0$*. So, the contraction condition (4) is trivially satisfied for every* $x \in l_p^+$*. Thus, all conditions of the "only if" part of Theorem 6 are fulfilled.*

4. Conclusions

We have equipped modular b-metric spaces with a topology that provides solid support to the idea of convergence proposed by Ege and Alaca. We have proved that this topology is metrizable, which is carried out by constructing a compatible uniformity with a countable base. We also introduced the notion of a strong modular b-metric space and showed that for this class of spaces, our topology can be explicitly formulated by means of a compatible b-metric, thereby extending an important metrization theorem by Chistyakov as well as the so-called topology of metric convergence. This approach allowed us to obtain a characterization of the completeness for this class of modular b-metric spaces that may be viewed as an offsetting of the celebrated Caristi–Kirk theorem to our context.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author thanks the reviewers for their detailed reading of the paper and several constructive comments and recommendations. In particular, Reviewer 1 suggested the wording of the last paragraph of Section 1 as presented.

Conflicts of Interest: The author declares no conflicts of interest.

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