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Octahedrality and Gâteaux smoothness

Ch. Cobollo [a,*], P. Hájek [b]

[a] *Instituto Universitario de Matemática Pura y Aplicada. Universitat Politècnica de València Spain*
[b] *Department of Mathematics, Faculty of Electrical Engineering. Czech Technical University in Prague Czech Republic*

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ABSTRACT

We prove that every Banach space admitting a Gâteaux smooth norm and containing a complemented copy of ℓ_1 has an equivalent renorming which is simultaneously Gâteaux smooth and octahedral. This is a partial solution to an open problem from the early nineties.

1. Introduction

Octahedral norms were introduced by Godefroy and Maurey in the unpublished work [10], and appeared for the first time in the literature at [8], where it was proved that a Banach space X admits an octahedral norm if and only if it contains a copy of ℓ_1. Octahedrality can be viewed as a strong non-differentiability—in the sense of Fréchet—condition. Octahedrality has been extensively studied in recent years, see e.g. [1,3,5,17,20] and many others, in connection with various dentability conditions on a Banach space. Through its relation with the containment of ℓ_1 and thanks to Rosenthal's ℓ_1 Theorem—see [21]—, octahedrality also has interesting applications in connection with the weak sequential completeness of the Banach space—see also [6, §III.4 and Theorem 3.7 in §III.3]—, and it is naturally connected with the problem of preserved Gâteaux smoothness points too,—see [8,22] and the paragraph below Theorem 1.1.

The coexistence of Gâteaux smoothness and octahedrality for a single norm has, therefore, been a known problem since the early nineties—see [11, Problem 197], [13, Problem 7] and [22,23]). Curiously, up to date, the only example of a Banach space X which admits such a norm is the Hardy space $H_1(D)$, as shown (using deep results from harmonic analysis) in the monograph [6, p. 120]. This space is a separable subspace

* Corresponding author.
E-mail addresses: chcogo@upv.es (Ch. Cobollo), hajekpe8@fel.cvut.cz (P. Hájek).

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of L_1 containing a complemented copy of ℓ_1. To find such a renorming, even in the basic case $X = \ell_1$, was open. Our main result in this note is the following.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space admitting a Gâteaux smooth (equivalent) norm and having a complemented subspace isomorphic to ℓ_1. Then, X admits a renorming $|||\cdot|||$ which is simultaneously octahedral and Gâteaux smooth.*

By [13, Lemma 28], a given element $x \in S_X$ is a *very smooth* point if and only if x is a point of Gâteaux smoothness in the bidual X^{**}. By [9, Lemma 9.1], for a separable Banach space X, its norm $\|\cdot\|$ being octahedral is equivalent to the existence of an element $x^{**} \in X^{**}$ such that

$$\|x + x^{**}\| = \|x\| + \|x^{**}\|^{**}, \quad \text{for every } x \in X.$$

This property is a stronger condition of the norm that implies octahedrality and also prevents the existence of preserved smooth points—see [22]. Although the existence of such a point x^{**} is a consequence of octahedrality in the separable case by the aforementioned lemma, the implication does not hold when removing the separability assumption—see [18, Theorem 3.2]. In particular, for separable spaces X, our main result provides a norm $|||\cdot|||$ where every non-zero point is Gâteaux smooth but not very smooth. Notice that the notion of very smoothness coincides with the one of *strong Gâteaux smoothness*—see [13].

Our proof relies on a new method of construction based on controlled directional estimates of the norm on a dense subspace, which passes to the completion. It is somewhat subtle, and uses the complementability of ℓ_1 heavily. To some extent, this is inevitable, as octahedral norms cannot have a rotund dual norm, which is the standard condition in order to obtain a Gâteaux smooth norm. Indeed, octahedral spaces contain an asymptotically isometric ℓ_1-sequence ([2]), and spaces with such a sequence cannot have a dual rotund norm ([19]). The proof would be no simpler if we just assumed that $X = \ell_1$, but it is not clear if there is a simple formal argument in our case, using the special case of ℓ_1, together with the complementability of ℓ_1 in X.

We are inclined to believe that the complementability condition in Theorem 1.1 is redundant (so, the containment of ℓ_1 should be sufficient and, of course, also necessary), but our method of proof does not cover this case.

The rest of the text is devoted to the proof of the main Theorem 1.1 through the construction of a renorming $|||\cdot|||$ being simultaneously Gâteaux smooth and octahedral. The document is organized as follows: the remaining part of this introductory section will contain preliminaries and notation. Section 2 consist of the inductive construction of the renorming and the proof of its elementary properties. Lemma 2.5 shows that it is an equivalent norm to the original one, and Proposition 2.6 contains the argument for octahedrality. The last Section 3 is completely dedicated to showing the Gâteaux smoothness of the final norm, which is the most delicate part of the proof. It consists of showing that the Gâteaux smoothness on the original construction—Proposition 3.1 and Corollary 3.2—is inherited to the whole space X. The argument depends on some suitable estimates of the directional derivatives and splitting in two cases, depending if there exists a Birkoff–James orthogonal relation between the point and the direction or not—Subsections 3.1 and 3.2, respectively.

1.1. Preliminaries and notation

We assume that our Banach space $(X, \|\cdot\|)$ has a Gâteaux smooth norm $\|\cdot\|$, and $X \cong X_0 \oplus \ell_1$, for some Banach space X_0. We will use $\{e_i\}_{i=1}^{\infty}$ to denote the canonical basis of ℓ_1. For every $n \in \mathbb{N}$, consider the linear subspace

$$X_n := X_0 + \mathrm{span}\{e_i : 1 \le i \le n\},$$

and put

$$Y := \bigcup_{n\in\mathbb{N}} X_n = \{x \in X : x \in X_n \text{ for some } n \in \mathbb{N}\}.$$

Thus, the whole space X is the completion of the subspace Y.

It is clear that there exists a unique decomposition of every $x \in X$ as

$$x = x_0 + \sum_{j=1}^{\infty} x_j e_j, \quad \text{where } x_0 \in X_0, \ \sum_{j=1}^{\infty} |x_j| < \infty.$$

For any $n \in \mathbb{N}$, consider the n-th canonical projection (or the canonical projection to X_n) as the map $P_n : X \to X_n$,

$$P_n(x) := x_0 + \sum_{j=1}^{n} x_j e_j.$$

To simplify the notation, we will also denote the n-th canonical projection of a given element x by the symbol

$$x^n := P_n(x).$$

The final norm $|||\cdot|||$ will be obtained through the construction of a sequence of compatible renormings $|||\cdot|||_n$ on the spaces X_n. Such a sequence has, of course, a unique extension to the whole space X. It will be easy to check that $|||\cdot|||$ is octahedral, as it will have the property on Y, and octahedrality passes to the completion X. The construction will also be Gâteaux smooth at all points of Y—see Section 3. The difficult part of the argument is to prove the Gâteaux smoothness for every $x \in X \setminus Y$. This is equivalent to the existence of all directional derivatives $\frac{\partial |||x|||}{\partial h}$, where $h \in Y$.

We refer to [6,7,12,14] for standard results and notation.

2. The construction of the norm

This section will contain the inductive construction of the norm $|||\cdot|||$, which will be octahedral and Gâteaux smooth. As said before, it will start through an inductive process of constructing norms on the spaces X_n. Roughly speaking, the main idea behind this is to add one more dimension and construct a new norm as the Minkowski functional of a new convex body, defined through homothetic copies of the previous unit ball. Through the assumptions of the function that indicates the homothetic factor depending on the height, we may achieve the new convex body being still smooth, and that on each step, the norm for the new vectors is “asymptotically” an ℓ_1-sum.

We will start by consider sequences of real numbers $\{z_n\}_{n\in\mathbb{N}}$, $\{l_n\}_{n\in\mathbb{N}}$, and $\{s_n\}_{n\in\mathbb{N}}$ such that:

- $0 < z_n < l_n < s_n < 1$;
- z_n strictly decreasing;
- l_n strictly decreasing, $\{l_n\}_{n\in\mathbb{N}} \in \ell_1$;
- s_n strictly increasing, $s_n \to 1$.

Also, take a sequence of continuous convex and real-valued functions $\{f_n\}_{n\in\mathbb{N}}$, $f_n : [0,1] \to \mathbb{R}$ with the following properties,

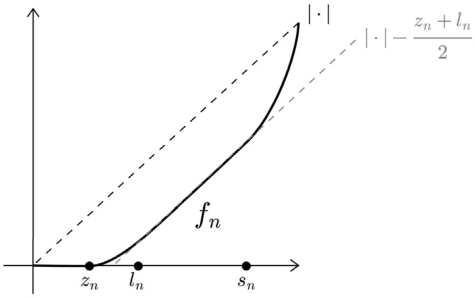

Fig. 1. Shape of the function f_n.

i. $f_n \equiv 0$ in $[0, z_n]$;
ii. f_n is smooth and strictly increasing in $[z_n, l_n]$;
iii. $f_n(t) := t - \frac{z_n + l_n}{2}$ when $t \in [l_n, s_n]$;
iv. f_n in $[s_n, 1]$ is strictly increasing, smooth, $f(1) = 1$ and $\lim_{t\to 1} f'(t) = \infty$.

Then, the following holds.

Proposition 2.1. *By the construction above, for every* $t \in [0, 1]$

$$t - \frac{z_n + l_n}{2} \leq f_n(t) \leq t.$$

In particular, $f_n \to |\cdot|$ *uniformly.*

Proof. Is clear from the construction. See also Fig. 1. □

From the result above we also have that for every $t \in [0, 1]$,

$$1 - t + \frac{z_n + l_n}{2} \geq 1 - f_n(t) \geq 1 - t.$$

Now, we are ready to start with the construction of the norms. For $n = 0$ just define $|||\cdot|||_0 := \|\cdot\|$ as the restriction of the Gâteaux smooth norm from X to X_0. For $n \geq 1$, we will define a (equivalent) norm $|||\cdot|||_n$ in X_n by the Minkowski functional of the set

$$B_n := \{x \in X_n : |||P_{n-1}x|||_{n-1} \leq 1 - f_n(|x_n|), x_n \in [-1, 1]\}. \tag{1}$$

Thus, $|||\cdot|||_n := \mu_{B_n}$, and so $B_{X_n} = B_n$ and

$$S_{X_n} = \{x \in X_n : |||P_{n-1}x|||_{n-1} = 1 - f_n(|x_n|), x_n \in [-1, 1]\}. \tag{2}$$

Lemma 2.2. *Let* $x \in Y$. *Then,*

$$|||P_{n-1}x|||_{n-1} = \left(1 - f_n\left(\frac{|x_n|}{|||P_nx|||_n}\right)\right)|||P_nx|||_n.$$

In particular, $|||P_{n-1}x|||_{n-1} \leq |||P_nx|||_n$.

Proof. We can assume without loss of generality that $|||P_nx|||_n > 0$—otherwise, the result is trivial. Then, $P_n\left(\frac{x}{|||P_nx|||_n}\right) \in S_{X_n}$, and by (2), the equality

$$\left|\!\left|\!\left| P_{n-1}\left(P_n\left(\frac{x}{|||P_n x|||_n}\right)\right)\right|\!\right|\!\right|_{n-1} = 1 - f_n\left(\frac{|x_n|}{|||P_n x|||_n}\right)$$

follows. From here, we deduce

$$|||P_{n-1}x|||_{n-1} = \left(1 - f_n\left(\frac{|x_n|}{|||P_n x|||_n}\right)\right)|||P_n x|||_n. \quad \square$$

For the previous description of the norm there are some easy consequences that follow naturally. We will state them for further reference.

Corollary 2.3. *If $x \in S_{(X_n,|||\cdot|||_n)}$ and $|t| < z_{n+1}$, then*

$$|||x + te_{n+1}|||_{n+1} = |||x|||_n.$$

In particular,

$$X_n \cap B_{(X_{n+1},|||\cdot|||_{n+1})} = B_{(X_n,|||\cdot|||_n)}.$$

Proof. The formulae follow readily from the properties of f_n and the construction of $|||\cdot|||_n$. $\square$

Now, we are ready to define the final renorming $|||\cdot|||$, through the supremum of the already constructed $|||\cdot|||_n$, that is, for any $x \in X$

$$|||x||| := \sup_{n\in\mathbb{N}}\{|||P_n x|||_n\}.$$

We will prove that it is indeed an equivalent norm through being equivalent to the already equivalent norm $|||\cdot|||_0 \oplus_1 \|\cdot\|_1$ in X (the computation of $|||x_0|||_0 + \|(x_n)_{n=1}^{\infty}\|_1$ for any $x \in X$).

Proceeding by induction again, for X_0 take $|\cdot|_0 := |||\cdot|||_0$—the original Gâteaux norm in X_0. For $n \geq 1$, consider $|\cdot|_n := |||\cdot|||_{n-1} \oplus_1 |\cdot|$. It is straightforward to see that for $n \geq 1$ the unit ball associated to this norm is

$$B_{|\cdot|_n} := \{x \in X_n : |||P_{n-1}x|||_{n-1} \leq 1 - |x_n|\} = \mathrm{conv}\,(B_{n-1}, e_n).$$

Remark 2.4. *In the following, we will use the fact that $\prod_{n\in\mathbb{N}}(1 + \frac{z_n+l_n}{2})$ converges. This is due to the fact that $\{\frac{z_n+l_n}{2}\}_{n\in\mathbb{N}} \in \ell_1$, because in the construction of the functions f_n we took $\{l_n\}_{n\in\mathbb{N}} \in \ell_1$.*

Lemma 2.5. *For the constructed norms, it is satisfied that*

$$\frac{1}{1 + \frac{z_n+l_n}{2}}|\cdot|_n \leq |||\cdot|||_n \leq |\cdot|_n.$$

In particular, by considering the norm $|||\cdot|||_0 \oplus_1 \|\cdot\|_1$ on X, we have that

$$\frac{1}{\prod_{n\in\mathbb{N}}(1 + \frac{z_n+l_n}{2})}(|||\cdot|||_0 \oplus_1 \|\cdot\|_1) \leq |||\cdot||| \leq |||\cdot|||_0 \oplus_1 \|\cdot\|_1.$$

Proof. First, it is clear by Proposition 2.1 that

$$B_{|\cdot|_n} \subset B_n \subset \left(1 + \frac{z_n + l_n}{2}\right) B_{|\cdot|_n}.$$

Now, on the one hand, from the right-hand side inclusion, we have that for every n,

$$|||P_n x|||_n \le |P_n x|_n = |||P_{n-1}x|||_{n-1} + |x_n|.$$

Applying this inequation iteratively, we reach

$$|||P_n x|||_n \le |||x_0|||_0 + \sum_{i=1}^{n} |x_i| \le |||x_0|||_0 + \sum_{i=1}^{\infty} |x_i|.$$

In particular, by taking the supremum on n in the left side, we reach the first inequality $|||\cdot||| \le |||\cdot|||_0 \oplus_1 \|\cdot\|_1$.

On the other hand, by the left-hand side inclusion, we have

$$|||P_n x|||_n \ge \frac{1}{1+\frac{z_n+l_n}{2}} \|P_n x\|_n = \frac{1}{1+\frac{z_n+l_n}{2}} \left(|||P_{n-1}x|||_{n-1} + |x_n| \right).$$

Once again, applying this iteratively

$$|||x||| = \sup_{k\in\mathbb{N}} |||P_k x|||_k \ge |||P_n x|||_n \ge \left(\prod_{j=1}^{n} \frac{1}{1+\frac{z_j+l_j}{2}} \right) \left(|||x_0|||_0 + \sum_{i=1}^{n} |x_i| \right).$$

In particular, by taking suprema on $n \in \mathbb{N}$, we reach

$$|||\cdot||| \ge \frac{1}{\prod_{n\in\mathbb{N}} (1+\frac{z_n+l_n}{2})} (|||\cdot|||_0 \oplus_1 \|\cdot\|_1).$$

The proof is over. □

2.1. Octahedrality of the norm

Here, we will show the octahedrality of $|||\cdot|||$. Recall that a norm $\|\cdot\|$ of a Banach space X is said to be **octahedral** if for every $\varepsilon > 0$ and every finite-dimensional subspace F of X there exists $x \in S_X$ such that

$$\|y + \alpha x\| \ge (1-\varepsilon)(\|y\| + |\alpha|)$$

for every $y \in F$ and $\alpha \in \mathbb{R}$.

The core of the idea is that, in the set B_n, the element e_n is "close" to witnessing the octahedrality of the norm for any element in B_{n-1}, and the closeness is reduced with the increasing of the n.

Proposition 2.6. *Let $\varepsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that, for any $n \ge n_0$,*

$$|||P_{n-1}x + \alpha e_n|||_n \ge (1-\varepsilon)(|||P_{n-1}x|||_n + |\alpha|).$$

In particular, the norm $|||\cdot|||$ is octahedral.

Proof. Take $n_0 \in \mathbb{N}$ such that $\frac{z_n+l_n}{z_n+l_n+2} \le \varepsilon$. Then, applying Lemmata 2.2 and 2.5, we get

$$|||P_{n-1}x + \alpha e_n|||_n \ge \frac{1}{1+\frac{z_n+l_n}{2}} \|P_{n-1}x + \alpha e_n\|_n$$

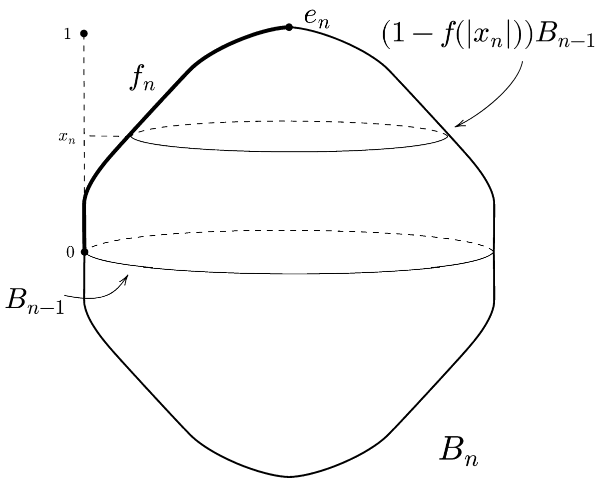

Fig. 2. Shape of the unit ball B_n. The slice of B_n at height x_n is a homotetic copy of B_{n-1} with scalar factor $1 - f_n(x_n)$.

$$= \frac{1}{1+\frac{z_n+l_n}{2}}\left(|||P_{n-1}x|||_{n-1} + |\alpha|\right)$$
$$= \frac{1}{1+\frac{z_n+l_n}{2}}\left(|||P_{n-1}x|||_{n} + |\alpha|\right),$$

and this last term is greater or equal than $(1-\varepsilon)(|||P_{n-1}x|||_n + |\alpha|)$ if and only if

$$1 - \frac{1}{1+\frac{z_n+l_n}{2}} = \frac{z_n+l_n}{z_n+l_n+2} \leq \varepsilon.$$

As this is satisfied because of the choice of $n \in \mathbb{N}$, we conclude the proof. □

3. Gâteaux smoothness of the norm

This final section is fully devoted to showing the Gâteaux smoothness of $|||\cdot|||$. It will require splitting the argument into several steps. First, the smoothness of the norm $|||\cdot|||_n$ in X_n. Geometrically, this is due to the properties on the functions f_n—see again Fig. 2. The smoothness of $|||\cdot|||_n$ in the points that belong to the previous $X_{n-1}\backslash\{0\}$ is due to the assumption $f_n \equiv 0$ in $[0, z_n]$, and the smoothness in $\pm e_n$ is achieved because $\lim_{t\to 1} f'(t) = \infty$.

Proposition 3.1. *The space $(X_n, |||\cdot|||_n)$ is Gâteaux smooth.*

Proof. By induction, for $n = 0$, we know that the norm $|||\cdot|||_0$ on X_0 is Gâteaux smooth by hypothesis. By equation (2) we know that $|||\cdot|||_n$ is the Minkowski functional of the 1-level set of the function

$$g(x) := |||P_{n-1}x|||_{n-1} + f_n(|x_n|).$$

By inductive assumption, $|||\cdot|||_{n-1}$ is Gâteaux smooth (except at the origin). Hence $g(x)$ is also a Gâteaux smooth convex function in its domain, except for the origin and possibly the point $\pm e_n$. Hence there is a unique tangent hyperplane to the graph of g at x and this immediately implies that there is also a unique tangent hyperplane to the Minkowski functional of S_{X_n}. In other words, $|||\cdot|||_n$ is Gâteaux differentiable at x. The remaining case when $x = \pm e_n$ is clear, since the only tangent hyperplane at this point is the kernel of the n-th coefficient functional on X, yielding Gâteaux smoothness again. □

Corollary 3.2. *The space* $(Y, |||\cdot|||)$ *is Gâteaux smooth.*

Proof. Given $x, h \in Y$, there exists $n \in \mathbb{N}$ so that both $x, h \in X_n$, so the result follows from the previous one. □

In the remaining part of the section, we will prove that $|||\cdot|||$ is Gâteaux smooth on the whole X. This is the most delicate part of the argument. Our Banach space $(X, |||\cdot|||)$ is the completion of the normed space $(Y, |||\cdot|||) = \cup_n (X_n, |||\cdot|||_n)$. To prove that the final norm is Gâteaux smooth, it suffices to prove that $|||\cdot|||$ has a directional derivative at any point $0 \neq x \in X$ with respect to a dense set of directions, in our case, for any $h \in Y$. If $x \in Y$ then this follows directly from Corollary 3.2, as both x and the direction $h \in Y$ are contained in some X_n and we know that $|||\cdot|||_n$ is Gâteaux smooth. It remains to deal with the delicate case $x \in X \setminus Y$.

In what follows, we assume without loss of generality that $x = x_0 + \sum_{j=1}^{\infty} x_j e_j$ (where of course $\sum_n |x_n| < \infty$), $|||x||| = 1$. Since $x^n = x_0 + \sum_{j=1}^{n} x_j e_j$, $|||x^n||| \leq 1$ for all $n \in \mathbb{N}$.

In order to prove the Gâteaux smoothness at a fixed point x we will obtain an estimate of the function (of parameter t)

$$\phi_{x,h}(t) := |||x + th||| - |||x|||$$

for every $h \in Y$. In the end this will lead to the desired conclusion because

$$\phi'_{x,h}(0) = \frac{\partial |||x|||}{\partial h}.$$

We start by collecting some simple observations concerning the functions $\phi_{x,h}$. The proof is omitted, as it is immediate.

Lemma 3.3. *For any* $\tau > 0$,

$$\begin{aligned}
\phi_{x,\tau h}(t) &= |||x + t\tau h||| - |||x||| = \phi_{x,h}(\tau t);\\
\phi_{\tau x,\tau h}(t) &= |||\tau x + t\tau h||| - |||\tau x||| = \tau \phi_{x,h}(t);\\
\phi_{\tau x,h}(t) &= \left|\left|\left|\tau x + t\tau(\frac{1}{\tau}h)\right|\right|\right| - |||\tau x||| = \tau \phi_{x,\frac{1}{\tau}h}(t) = \tau \phi_{x,h}(\frac{1}{\tau}t).
\end{aligned}$$

Also, for fixed $x \in X, h \in Y, t \in \mathbb{R}$, *we have*

$$\phi_{x,h}(t) = \lim_n \phi_{x^n,h}(t).$$

Our strategy is to show that the sequence of functions $\phi_{x^n,h}$ yields the estimates needed for Gâteaux smoothness at x.

We will split the argument into two cases through the well-known notion of orthogonality initially introduced by Birkhoff at [4] and studied by James in [15,16].

Definition 3.4. *Given a Banach space* $(X, \|\cdot\|)$, *and* $x, h \in X \backslash \{0\}$. *It is said that* x *is Birkhoff–James orthogonal to* h *if for every* $t \in \mathbb{R}$

$$\|x + th\| \geq \|x\|.$$

In our context, x is Birkhoff–James orthogonal to h if and only if $\phi_{x,h}(t) \geq 0$ for every $t \in \mathbb{R}$. Geometrically, the above definition allows us to think of h as a "tangent direction" on x, meaning that the vector $x + th$ belongs to a tangent hyperplane of the multiple of the unit ball $\|x\| B_X$ at x. Notice that if $\|\cdot\|$ is Gâteaux at x, then $\phi'_{x,h}(0) = 0$ for any h tangent direction at x. Finally, one of the key properties of the renorming is that the norms are inductively constructed so that the sliced of unit ball B_{n+1} in the direction of e_{n+1} are homothetic copies of the previous step B_n—recall Fig. 2. This implies that Birkoff–James orthogonality for a projection is preserved at further steps.

Corollary 3.5. *Let $x \in (X, |||\cdot|||)$ such that $x^n \neq 0$. If x^n is Birkoff–James orthogonal to a given direction $h \in X_n$, then $x^{n+1} \in X_{n+1}$ is Birkoff–James orthogonal to h.*

Proof. By Lemma 2.2,

$$\left(1 - f_{n+1}\left(\frac{|x_{n+1}|}{|||x^{n+1}|||}\right)\right) \phi_{x^{n+1},h}(t) = \phi_{x^n,h}(t),$$

and the result follows. $\square$

3.1. Case 1: x^n is Birkoff–James orthogonal to $h \in X_n$

In this part, we will assume that the direction h belongs to X_n for a given $n \in \mathbb{N}$, and furthermore, $x^n \in X_n$ is Birkoff–James orthogonal to h.

As x^n is a smooth point, $\phi'_{x^n,h}(0) = 0$, or equivalently

$$\phi_{x^n,h}(t) = o(t). \tag{3}$$

Let us now estimate $\phi_{x^{n+1},h}$.

From the construction of the renormings $|||\cdot|||_n$—see Lemma 2.2 and Corollary 2.3—we have that for every $\lambda > 0$ there is some $\tilde{\lambda} \leq \lambda$ such that

$$\lambda S_{X_{n+1}} \cap (X_n + x_{n+1}e_{n+1}) = \tilde{\lambda} S_{X_n} + x_{n+1}e_{n+1}.$$

i.e. for a fixed value x_{n+1} we have

$$|||x^n||| = \tilde{\lambda} \iff |||x^n + x_{n+1}e_{n+1}||| = \left|\left|\left|x^{n+1}\right|\right|\right| = \lambda.$$

Take $\tilde{P} := \frac{\tilde{\lambda}}{|||x^n + th|||}(x^n + th)$, the point on the ray from the origin to the point $x^n + th$ which has norm $\tilde{\lambda} = |||x^n|||$.

Now, put $P := \tilde{P} + x_{n+1}e_{n+1}$. Then $P_n(P) = \tilde{P}$ and $|||P||| = \lambda = \left|\left|\left|x^{n+1}\right|\right|\right|$.

Denote by

$$R := \frac{\lambda}{|||x^{n+1} + th|||}(x^{n+1} + th)$$

the point of intersection of the ray from zero to $(x^{n+1} + th)$ with $\lambda S_{X_{n+1}}$.

We claim that $|||P_n(R)||| \geq \tilde{\lambda}$ as, from a simple geometric argument, we deduce that R must project farther away from the origin than $\tilde{\mathrm{P}}$—see Fig. 3. Indeed, we may write

$$R = \frac{\left|\left|\left|x^{n+1}\right|\right|\right|}{|||x^{n+1} + th|||}\left(\left(\frac{|||x^n + th|||}{|||x^n|||} - 1\right)\tilde{P} + P\right),$$

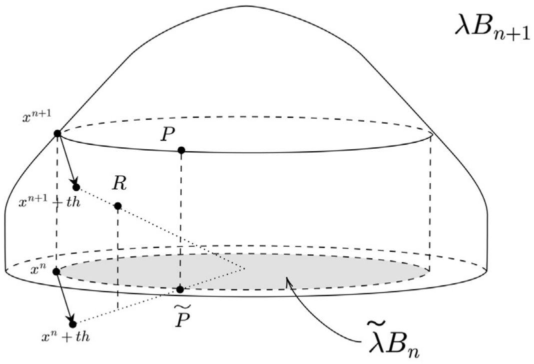

Fig. 3. Geometric interpretation of the claim.

so $R \in \operatorname{span}\{P, \tilde{P}\}$. Notice that $P, R \in \operatorname{span}\{P, \tilde{P}\} \cap \lambda S_{X_{n+1}}$, but $\left|\left|\left|\tilde{P}\right|\right|\right| \leq \lambda$. Then, for every $z \in \operatorname{conv}(\tilde{P}, P)$, it is satisfied that $|||z||| \leq \lambda$.

Then, the intersection of $\operatorname{span}\{x^{n+1}+th\} \cap \operatorname{conv}(\tilde{P}, P)$ is a unique point S, that clearly satisfies $P_n(S) = \tilde{P}$ and $|||S||| \leq \lambda$. As both $R, S \in \operatorname{span}\{x^{n+1}+th\}$, but $|||R||| \geq |||S|||$, we finally deduce that

$$|||P_n(R)||| \geq |||P_n(S)||| = \left|\left|\left|\tilde{P}\right|\right|\right|.$$

So, the claim is proved.

But then, by the claim

$$\frac{\left|\left|\left|x^{n+1}\right|\right|\right|}{|||x^{n+1}+th|||} \frac{|||x^n+th|||}{|||x^n|||} \left|\left|\left|\tilde{P}\right|\right|\right| = |||P_n(R)||| \geq \left|\left|\left|\tilde{P}\right|\right|\right|.$$

So we just deduced

$$\frac{\left|\left|\left|x^{n+1}\right|\right|\right|}{|||x^n|||} \geq \frac{\left|\left|\left|x^{n+1}+th\right|\right|\right|}{|||x^n+th|||}.$$

Using Corollary 3.5 and this last inequality,

$$\begin{aligned} 0 \leq \phi_{x^{n+1},h}(t) &= \left|\left|\left|x^{n+1}+th\right|\right|\right| - \left|\left|\left|x^{n+1}\right|\right|\right| \\ &\leq \frac{\left|\left|\left|x^{n+1}\right|\right|\right|}{|||x^n|||} |||x^n+th||| - \left|\left|\left|x^{n+1}\right|\right|\right| \\ &= \frac{\left|\left|\left|x^{n+1}\right|\right|\right|}{|||x^n|||} \phi_{x^n,h}(t). \end{aligned}$$

Now, we may fix n large enough so that $h \in X_n$ and $|||x^n||| \geq \frac{1}{2}|||x|||$. Proceeding inductively as above, we get the estimate for every $m \geq n$:

$$\phi_{x^m,h}(t) \leq \frac{|||x^m|||}{|||x^n|||} \phi_{x^n,h}(t) \leq 2\phi_{x^n,h}(t). \tag{4}$$

Then, passing to a uniform limit for $m \to \infty$ this clearly implies that

$$\phi_{x,h}(t) \leq 2\phi_{x^n,h}(t),$$

which using (3) means that $\phi_{x,h}(t) = o(t)$ and so $\frac{\partial |||x|||}{\partial h} = 0$.

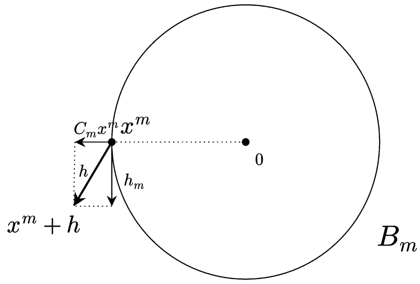

Fig. 4. Decomposition of a direction $h \in X_m$.

3.2. Case 2: no projection x^m is Birkoff–James orthogonal to the direction h

Let us pass to the general case. We may assume $|||x||| = |||h||| = 1$. For sufficiently large $N \in \mathbb{N}$, $h \in X_N$, and $|||x - x^N||| < \frac{1}{16}$. In what follows, we tacitly assume that $m \geq n > N$, i.e. without loss of generality, we assume that always $|||x^n||| > \frac{15}{16}$ and $|||x^n - x^m||| < \frac{1}{8}$. We may also consider $|t| < \frac{15}{2 \cdot 16}$.

For m, let $h = h_m + C_m x^m$ be a (unique) decomposition such that h_m is a tangent direction at x^m—see Fig. 4.

Notice that, considering $g_m \in S_{X_m^*}$, the (unique) norming functional for $x^m \in X_m$ (i.e., $\langle g_m, x^m \rangle = |||x^m|||$), it is well known that this functional g_m is the one describing the value of the directional derivatives at x^m, i.e., for any direction $y \in X_m \backslash \{0\}$

$$\frac{\partial |||x^m|||}{\partial y} = \langle g_m, y \rangle.$$

In particular, we have that $\langle g_m, h_m \rangle = 0$. Geometrically, the unique tangent hyperplane of $|||x^m||| B_{X_m}$ at x^m is exactly $x^m + \ker(g_m)$.

As $|\langle g_m, h \rangle| \leq 1$ and $\langle g_m, x^m \rangle = |||x^m||| > \frac{15}{16}$, we have

$$\frac{\partial |||x^m|||}{\partial h} = \langle g_m, h \rangle = C_m \langle g_m, x^m \rangle = C_m |||x^m|||,$$

from where we deduce that

$$|C_m| \leq \frac{16}{15}. \tag{5}$$

Recall that, by the construction of the norm $|||\cdot|||$, tangent directions at one dimension are preserved in further dimensions (see Corollary 2.3), so $h_n \in \ker g_m$, i.e. $\langle g_m, h_n \rangle = 0$, whenever $m \geq n$. Thus, as

$$g_m(h) = g_m(h_m + C_m x^m) = g_m(h_n + C_n x^n),$$

we deduce that $C_m g_m(x^m) = C_n g_m(x^n)$ and then we get

$$\begin{aligned} 0 &= C_m g_m(x^m) - C_n g_m(x^n) \\ &= (C_m - C_n) g_m(x^n) + C_m \sum_{i=n+1}^{m} x_i g_m(e_i). \end{aligned}$$

Now, if we combine this equation above with the lower bound

$$|g_m(x^n)| \geq |g_m(x^m)| - |||x^m - x^n||| > \frac{15}{16} - \frac{1}{8},$$

and using also the upper bound $|C_m| \leq \frac{16}{15}$ from (5), we achieve

$$|C_m - C_n| \leq \frac{|C_m|}{|g_m(x^n)|} \sum_{i=n+1}^{m} |x_i| < 2 \sum_{i=n+1}^{m} |x_i|$$

and, in particular,

$$|C_{n+1} - C_n| < 2|x_{n+1}|. \tag{6}$$

With this, we will get the estimation

$$|||h_{n+1} - h_n||| < 4|x_{n+1}|. \tag{7}$$

Indeed, using equations (5) and (6),

$$\begin{aligned} |||h_{n+1} - h_n||| &= \left|\left|\left|C_{n+1}x^{n+1} - C_n x^n\right|\right|\right| \\ &\leq \left|\left|\left|(C_{n+1} - C_n)x^{n+1} + C_n x^{n+1} - C_n x^n\right|\right|\right| \\ &\leq \left|\left|\left|(C_{n+1} - C_n)x^{n+1}\right|\right|\right| + \left|\left|\left|C_n x^{n+1} - C_n x^n\right|\right|\right| \\ &\leq 2|x_{n+1}| + C_n |||x_{n+1}e_{n+1}||| \\ &< 4|x_{n+1}|. \end{aligned}$$

Now, using Lemma 3.3,

$$\begin{aligned} \phi_{x^n,h}(t) &= |||x^n + th_n + tC_n x^n||| - |||x^n||| \\ &= |||(1 + tC_n)x^n + th_n||| - |||(1 + tC_n)x^n||| + |||(1 + tC_n)x^n||| - |||x^n||| \\ &= \phi_{(1+tC_n)x^n,h_n}(t) + tC_n|||x^n||| \\ &= (1 + tC_n)\phi_{x^n,h_n}\left(\frac{1}{1 + tC_n}t\right) + tC_n|||x^n|||. \end{aligned}$$

Notice that the above formula gives an expression of $\phi_{x^n,h}(t)$ that depends on the tangent direction h^n. By (re-)writing the formula for the $n+1$-dimension, we would get an expression depending on the next tangent direction h_{n+1}. The idea is that, thanks to (7), we can estimate the function $\phi_{x^{n+1},h}$, but still use the previous tangent direction h_n. In fact, by writing the formula and adding and subtracting the vector $v_n := \frac{1}{1+tC_{n+1}} th_n$, we get

$$\begin{aligned} \phi_{x^{n+1},h}(t) &= (1 + tC_{n+1})\phi_{x^{n+1},h_{n+1}}\left(\frac{1}{1 + tC_{n+1}}t\right) + tC_{n+1}\left|\left|\left|x^{n+1}\right|\right|\right| \\ &= (1 + tC_{n+1})\left(\left|\left|\left|x^{n+1} + \frac{1}{1 + tC_{n+1}}th_{n+1}\right|\right|\right| - \left|\left|\left|x^{n+1}\right|\right|\right|\right) \\ &\quad + tC_{n+1}\left|\left|\left|x^{n+1}\right|\right|\right| \\ &= (1 + tC_{n+1})\left(\left|\left|\left|x^{n+1} + v_n + \frac{1}{1 + tC_{n+1}}th_{n+1} - v_n\right|\right|\right| - \left|\left|\left|x^{n+1}\right|\right|\right|\right) \\ &\quad + tC_{n+1}\left|\left|\left|x^{n+1}\right|\right|\right|. \end{aligned}$$

We may now use the triangular inequality on the last step of the formula above (where the vectors v_n are introduced). This means that, for a certain error $E_n(t)$, we are able to express

$$\phi_{x^{n+1},h}(t) = \phi_{(1+tC_{n+1})x^{n+1},h_n}(t) + tC_{n+1}|||x^{n+1}||| + E_n(t) \tag{8}$$

where the error is estimated by

$$|E_n(t)| \le |t||||h_{n+1} - h_n||| < 4|t||x_{n+1}|. \tag{9}$$

Notice that comparing the two formulas that we achieved for $\phi_{x^{n+1},h}$, we have that the error is exactly

$$E_n(t) = \phi_{(1+tC_{n+1})x^{n+1},h_{n+1}}(t) - \phi_{(1+tC_{n+1})x^{n+1},h_n}(t). \tag{10}$$

We can then use (8), getting

$$\begin{aligned}\phi_{x^{n+k},h}(t) - \phi_{x^n,h}(t) &= \sum_{j=0}^{k-1}(\phi_{x^{n+j+1},h}(t) - \phi_{x^{n+j},h}(t)) \\ &= A + B + C, \end{aligned} \tag{11}$$

where we have the three terms

$$\begin{aligned} A &:= \sum_{j=0}^{k-1}\left(\phi_{(1+tC_{n+j+1})x^{n+j+1},h_{n+j}}(t) - \phi_{(1+tC_{n+j})x^{n+j},h_{n+j}}(t)\right), \\ B &:= \sum_{j=0}^{k-1} t(C_{n+j+1} - C_{n+j})|||x^{n+j+1}||| + \sum_{j=0}^{k-1} tC_{n+j}\left(|||x^{n+j+1}||| - |||x^{n+j}|||\right), \\ C &:= E_{n+k}(t) - E_n(t). \end{aligned}$$

We can estimate separately the three of them. First, notice that using previous estimations on $|||\cdot|||$ and using inequalities (5) and (6) to control $|C_{n+j+1}|$ and $|C_{n+j+1} - C_{n+j}|$ respectively, we get the bound

$$|B| \le 2|t|\sum_{j=0}^{k-1}|x_{n+j+1}| + \frac{16}{15}|t|\sum_{j=0}^{k-1}|x_{n+j+1}|.$$

Also, the bound on the error that we had at inequality (9) yields

$$|C| \le 4|t||x_{n+k}| + 4|t||x_n|.$$

So, with respect to B and C—and for the sake of simplicity—we might take the same bound,

$$|B| \le 4|t|\sum_{j=0}^{\infty}|x_{n+j}| \tag{12}$$

$$|C| \le 4|t|\sum_{j=0}^{\infty}|x_{n+j}| \tag{13}$$

Now, we only need to estimate A. Notice that we may split again

$$|A| \le |A_1| + |A2|,$$

where

$$A_1 := \phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t) - \phi_{(1+tC_n)x^n,h_n}(t) \tag{14}$$

$$A_2 := \sum_{j=0}^{k-2} \left|\phi_{(1+tC_{n+j+1})x^{n+j+1},h_{n+j}}(t) - \phi_{(1+tC_{n+j+1})x^{n+j+1},h_{n+j+1}}(t)\right| \tag{15}$$

For A_2, by the error expression at (10), we have for $j = 0, \ldots, k-1$

$$\begin{aligned}\left|\phi_{(1+tC_{n+j+1})x^{n+j+1},h_{n+j}}(t) - \phi_{(1+tC_{n+j+1})x^{n+j+1},h_{n+j+1}}(t)\right| &\leq |E_{n+j}(t)| \\ &\leq 4|t||x_{n+j+1}|. \end{aligned} \tag{16}$$

So, A_2 can be bounded as

$$A_2 \leq 4|t| \sum_{j=0}^{k-2} |x_{n+j+1}| = 4|t| \sum_{j=0}^{k-1} |x_{n+j}|. \tag{17}$$

For A_1, we may express the first of its two terms as

$$\begin{aligned}\phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t) = \big(&\phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t) - \phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\big) \\ &+ \phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\end{aligned} \tag{18}$$

For the first summand of (18)

$$\begin{aligned}\left|\phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t) - \phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| &\leq |t| |||h_{n+k-1} - h_n||| \\ &\leq 4|t| \sum_{j=1}^{k-1} |x_{n+j}|. \end{aligned} \tag{19}$$

And for the second term of (18), as x^n is Birkoff–James orthogonal to h_n, we can reduce the last term of the above formula to the previous Case 1 in Subsection 3.1, so applying (4) and re-writing through Lemma 3.3, we get

$$\left|\phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| \leq \left|(1+tC_{n+k})\frac{|||x^{n+k}|||}{|||x^n|||}\phi_{x^n,h_n}\left(\frac{1}{1+tC_{n+k}}t\right)\right|,$$

and recalling the estimates of $|C_m| \leq \frac{16}{15}$ in (5), and the initial assumptions on the norms $|||x^m|||$ and that we took $|t| < \frac{15}{2\cdot 16}$, we get that for any $k \in \mathbb{N}$

$$\begin{aligned}\left|\phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| &\leq \frac{16}{15}\left(1 + \frac{15}{2\cdot 16}\frac{16}{15}\right)\left|\phi_{x^n,h_n}\left(\frac{1}{1+tC_{n+k}}t\right)\right| \\ &\leq 2\phi_{x^n,h_n}\left(\frac{1}{1+tC_{n+k}}t\right), \end{aligned} \tag{20}$$

where for every $k \in \mathbb{N}$, $\frac{1}{2} \leq 1 + tC_{n+k} \leq \frac{3}{2}$ and

$$\frac{2|t|}{3} \leq \left|\frac{t}{1+tC_{n+k}}\right| \leq 2|t|. \tag{21}$$

So, by (18) and (19) we get the estimation on A_1,

$$\begin{aligned}|A_1| &= \left|\phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t) - \phi_{(1+tC_n)x^n,h_n}(t)\right| \\ &\le \left|\phi_{(1+tC_{n+k})x^{n+k},h_{n+k-1}}(t)\right| + \left|\phi_{(1+tC_n)x^n,h_n}(t)\right| \\ &\le 4|t|\sum_{j=1}^{k-1}|x_{n+j}| + \left|\phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| + \left|\phi_{(1+tC_n)x^n,h_n}(t)\right|. \end{aligned} \tag{22}$$

Combining the bounds of A_1 and A_2 (adding equations (22) and (17) respectively), we get the remaining estimation of A,

$$|A| \le 8|t|\sum_{j=1}^{k-1}|x_{n+j}| + \left|\phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| + \left|\phi_{(1+tC_n)x^n,h_n}(t)\right| \tag{23}$$

Combining together the bounds of A, B and C (equations (23), (12) and (13) respectively) and returning to (11), we finally achieve

$$\begin{aligned}\left|\phi_{x^{n+k},h}(t) - \phi_{x^n,h}(t)\right| &\le A + B + C \\ &\le (8+4+4)|t|\sum_{j=0}^{\infty}|x_{n+j}| \\ &\quad + \left|\phi_{(1+tC_{n+k})x^{n+k},h_n}(t)\right| + \left|\phi_{(1+tC_n)x^n,h_n}(t)\right| \end{aligned} \tag{24}$$

Finally, applying (20) to bound the last two terms in the equation above, we reach

$$\begin{aligned}\left|\phi_{x^{n+k},h}(t) - \phi_{x^n,h}(t)\right| \le 16|t|\sum_{j=0}^{\infty}|x_{n+j}| + 2\phi_{x^n,h_n}\left(\frac{1}{1+tC_{n+k}}t\right) \\ + 2\phi_{x^n,h_n}\left(\frac{1}{1+tC_n}t\right)\end{aligned}$$

And this last inequality implies that $\phi_{x,h}(t)$ is differentiable at $t = 0$—with derivative equal to $\lim_{n\to\infty} C_n$. Indeed, given any $\varepsilon > 0$ we may pick n large enough so that $\sum_{j=0}^{\infty}|x_{n+j}| < \varepsilon$. We know that $\phi_{x^n,h}(t)$ and $\phi_{x^n,h_n}(t)$ are differentiable at $t = 0$ by the previous case—with values being equal to C_n and 0, respectively. So,

$$\left|\phi_{x^{n+k},h}(t) - \phi_{x^n,h}(t)\right| \le 16\varepsilon|t| + o(t) \tag{25}$$

where the $o(t)$ estimate is independent of k because of (21). This finishes the argument and the proof of the main result.

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